Matrix methods in algebraic topology

He Wang
(joint work with Alex Suciu)

University of Nevada, Reno

The 8th International Conference on Matrix Analysis and Applications
(ICMAA 2019)

University of Nevada, Reno

July 17, 2019

Topology

- **Topological spaces:** Geometric objects like lines, planes, Euclidean Spaces $\mathbb{R}^n$, spheres, torus, Mobiüs band, Klein bottle, manifolds, metric spaces, etc.

Topology

- **Topological spaces:** Geometric objects like lines, planes, Euclidean Spaces $\mathbb{R}^n$, spheres, torus, Mobiüs band, Klein bottle, manifolds, metric spaces, etc.
- **Goal:** Classification of all topological spaces up to homeomoprhism.

Topology

- **Topological spaces:** Geometric objects like lines, planes, Euclidean Spaces $\mathbb{R}^n$, spheres, torus, Mobiüs band, Klein bottle, manifolds, metric spaces, etc.
- **Goal:** Classification of all topological spaces up to homeomoprhism.
- **Example:** Let M be a smooth compact n-manifold homotopy-equivalent to S^n. Then M is homeomorphic to S^n.

Topology

- **Topological spaces:** Geometric objects like lines, planes, Euclidean Spaces $\mathbb{R}^n$, spheres, torus, Mobiüs band, Klein bottle, manifolds, metric spaces, etc.
- **Goal:** Classification of all topological spaces up to homeomoprhism.
- **Example:** Let M be a smooth compact n-manifold homotopy-equivalent to S^n. Then M is homeomorphic to S^n. (Generalized Poincáre conjecture)

  Three Fields Medals were awarded: Smale (1966), Freedman (1986), and Perelman (2006) for solving the problem for $n \geq 5$, $n = 4$ and $n = 3$.

Topology

- **Topological spaces:** Geometric objects like lines, planes, Euclidean Spaces $\mathbb{R}^n$, spheres, torus, Mobiüs band, Klein bottle, manifolds, metric spaces, etc.
- **Goal:** Classification of all topological spaces up to homeomoprhism.
- **Example:** Let M be a smooth compact n-manifold homotopy-equivalent to S^n. Then M is homeomorphic to S^n. (Generalized Poincáre conjecture)

  Three Fields Medals were awarded: Smale (1966), Freedman (1986), and Perelman (2006) for solving the problem for $n \geq 5$, $n = 4$ and $n = 3$.
- **An "easier" goal:** Classification of all topological spaces up to homotopy equivalence.

Topology

- **Topological spaces:** Geometric objects like lines, planes, Euclidean Spaces $\mathbb{R}^n$, spheres, torus, Mobiüs band, Klein bottle, manifolds, metric spaces, etc.
- **Goal:** Classification of all topological spaces up to homeomoprhism.
- **Example:** Let M be a smooth compact n-manifold homotopy-equivalent to S^n. Then M is homeomorphic to S^n. (Generalized Poincáre conjecture)

  Three Fields Medals were awarded: Smale (1966), Freedman (1986), and Perelman (2006) for solving the problem for $n \geq 5$, $n = 4$ and $n = 3$.
- **An "easier" goal:** Classification of all topological spaces up to homotopy equivalence.
- **An even "easier" goal:** Classification of all topological spaces up to rational homotopy equivalence.

Algebraic Topology

A basic goal in algebraic topology:

"Find algebraic invariants that classify topological spaces up to homeomorphism, or up to (rational) homotopy equivalence."

Algebraic Topology

A basic goal in algebraic topology:

"Find algebraic invariants that classify topological spaces up to homeomorphism, or up to (rational) homotopy equivalence."

Let X be a connected CW-complex (topological space).

- Fundamental group $\pi_1(X)$.
- Homology groups $H_*(X;\mathbb{Q}) \Longrightarrow$ Betti numbers $b_i(X)$.
- Cohomology algebra $H^*(X;\mathbb{Q})$.

Algebraic Topology

A basic goal in algebraic topology:

"Find algebraic invariants that classify topological spaces up to homeomorphism, or up to (rational) homotopy equivalence."

Let X be a connected CW-complex (topological space).

- Fundamental group $\pi_1(X)$.
- Homology groups $H_*(X;\mathbb{Q}) \Longrightarrow$ Betti numbers $b_i(X)$.
- Cohomology algebra $H^*(X;\mathbb{Q})$.
- Higher (rational) homotopy groups $\pi_*(X) \otimes \mathbb{Q}$.
- Homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$.
- ......

Algebraic Topology

A basic goal in algebraic topology:

"Find algebraic invariants that classify topological spaces up to homeomorphism, or up to (rational) homotopy equivalence."

Let X be a connected CW-complex (topological space).

- Fundamental group $\pi_1(X)$.
- Homology groups $H_*(X;\mathbb{Q}) \Longrightarrow$ Betti numbers $b_i(X)$.
- Cohomology algebra $H^*(X;\mathbb{Q})$.
- Higher (rational) homotopy groups $\pi_*(X) \otimes \mathbb{Q}$.
- Homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$.
- ......
- Lie algebras/ varieties in this talk.
- Higher products on cohomology in a current project with Chris Rogers.

Relation with group theory

Groups: $\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Z}/p\mathbb{Z}$, S_n, F_n, ...

Goal: Classification of all groups up to isomorphism.

Relation with group theory

Groups: $\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Z}/p\mathbb{Z}$, S_n, F_n, ...

Goal: Classification of all groups up to isomorphism.

- X: a connected CW-complex with finitely many 1-cells.
- $G := \pi_1(X)$, **a finitely generated group**.

Relation with group theory

Groups: $\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Z}/p\mathbb{Z}$, S_n, F_n, ...

Goal: Classification of all groups up to isomorphism.

- X: a connected CW-complex with finitely many 1-cells.
- $G := \pi_1(X)$, **a finitely generated group**.
- $K(G,1)$: the Eilenberg-MacLane space.

$$G \overset{1-1}{\longleftrightarrow} K(G,1)$$

Relation with group theory

Groups: $\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Z}/p\mathbb{Z}$, S_n, F_n, ...

Goal: Classification of all groups up to isomorphism.

- X: a connected CW-complex with finitely many 1-cells.
- $G := \pi_1(X)$, **a finitely generated group**.
- $K(G, 1)$: the Eilenberg-MacLane space.

$$G \overset{1-1}{\longleftrightarrow} K(G,1)$$

- The *cohomology algebra* of G, $H^*(G;\mathbb{Q}) := H^*(K(G,1);\mathbb{Q})$.

Am matrix group example

Example (Heisenberg group)

- The Heisenberg manifold, $M = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$.
- Then $\mathcal{H} = \pi_1(M) = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}$, and $M = K(\mathcal{H}, 1)$.

Am matrix group example

Example (Heisenberg group)

- The Heisenberg manifold, $M = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$.
- Then $\mathcal{H} = \pi_1(M) = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}$, and $M = K(\mathcal{H}, 1)$.
- $\mathcal{H} = \langle x, y, z \mid xyx^{-1}y^{-1}z^{-1},\ xzx^{-1}z^{-1},\ yzy^{-1}z^{-1} \rangle$.

Am matrix group example

Example (Heisenberg group)

- The Heisenberg manifold, $M = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$.
- Then $\mathcal{H} = \pi_1(M) = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}$, and $M = K(\mathcal{H}, 1)$.
- $\mathcal{H} = \langle x, y, z \mid xyx^{-1}y^{-1}z^{-1},\ xzx^{-1}z^{-1},\ yzy^{-1}z^{-1} \rangle$.
- Rational cohomology: $H^1 = \mathbb{Q}^2$ with basis $\{u_1, u_2\}$; $H^2 = \mathbb{Q}^2$ with basis $\{v_1, v_2\}$; $H^3 = \mathbb{Q}$ with basis $\{w\}$.
- Nontrivial cup products: $u_1 \cup v_1 = u_2 \cup v_2 = w$.

References

Alexander I. Suciu and He Wang, *Cup products, lower central series, and holonomy Lie algebras*, J. Pure. Appl. Algebra **223** (2019), no. 8, 3359–3385.

Alexander I. Suciu and He Wang, *Formality properties of finitely generated groups and Lie algebras*, Forum Math. **31** (2019), no. 4, 867–905

Alexander I. Suciu and He Wang, *Chen ranks and resonance varieties of the upper McCool groups*, to appear in Advances in Applied Mathematics.

Thank You!