Assignment1

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Assessing Systematic Risk In The S&P500 Index Between 2000 And 2011: A Bayesian Nonparametric Approach

▼ Introduction

This paper studied the evolution of systemic and idiosyncratic risks in the U.S. economy by focusing on the behavior of the S&P500 index and it's sector components.

▼ S&P500 Definition

S&P500 is a stock market index which works as a market-value weighted average of 500 publicly traded companies, from 10 economics sectors. In addition to the overall Standard & Poor's publishes separate indexes for teach of these sectors.

The most widely used tool to characterize idiosyncratic risks is the Capital Asset Pricing Model (CAPM)

▼ CAPM Definition

Model where agents maximize their expected utility derived from the investment, which is a function of the expected return on risky activities and their associated variance as well as those of the market as a whole. Parameters are then estimated using Linear Regression and measure the sensitivity of the expected excess asset returns to the expected excess market returns. Large coefficients mean large idiosyncratic risk.

▼ CAPM Assumption

Too simplistic because it assumes that deviation from the model follows a Gaussian distribution. This is not resealable as the financial crisis of 2007 - 2008 showed that the behavior of markets during the calm periods is dramatically different during periods of distress.

This paper created a novel approach to estimate systemic and idiosyncratic risks, with a focus on the tail, meaning risk associated with rare extreme events. More specifically the paper models the time of appearance of extreme losses in each market, using a superposition of two Poisson process (one for systemic and one for idiosyncratic risk).

The Poisson process has a parameter that is called intensity, so it a function of the subsector of the S&P500. In order to capture changes in the risk structure over time, this intensity functions are modeled using a Dirichlet Process.

- **Poisson Process**

  The Poisson Process is a continuous time version of the Bernoulli Process.

  Time is continuous and we might have arrivals of an event (e.g. a crisis) at any point.

  - Assumption

    - Independence at each disjoint time intervals.
    - Probability of having k arrivals in a time interval τ is the same no matter the interval (time homogeneity)
    - During a very small δ interval of time there is a zero probability of more than one arrival.

    λ is the arrival rate (intensity). Not sum of arrivals.

    Expected number of arrivals during the interval a,b ⇒ $E_{PP}[a,b] = \lambda(|a-b|)$

- **Dirichlet Process.**

  Probabilistic Model over a number of clusters. Dirichlet Process are distributions over distributions α, G.

  α: Concentration Parameter.

  G: Base Distribution.

G ⇒ Dirichlet Process ⇒ G'. α controls how similar G and G' are

so: $x \sim DP(G, \alpha)$

The model does not rely on the assumption that return arise from a Gaussian distribution. The main goal of the analysis is to describe the the structure of different risks and explanation of that, rather than prediction.

▼ Model

▼ General

Negative log returns for the ten S&P500 sector indexes.
$x_{i,j} = -100\ \log(\frac{S_{i,j}}{S_{i-1,j}})$

$S_{i,j}$: value of the index for sector j at time i. ⇒ for risk management purposes we are interested in large $x_{i,j}$ values that indicate big drops in the prices.

For a given threshold u we focus our attention on the collection of times $t_{j,k}$ (date associated with the appearance of the kth negative log return in sector j) that is larger than u.

- ▼ We regard the collection of that times where exceedances occur as a realization from a **point process** $N_j(t) = \sum_{k-1}^{n_j} I_{[t_{j,k},T]}$.
  - ▼ Each $N_j(t)$ is constructed as the superposition of two independent nonhomogenous Poisson processes, one for systematic and the other for idiosyncratic risk, with Cumulative **intensity functions** Λ_0^* and Λ_j^* respectively.

    So because of the Poisson process we know that
    $\Lambda_j(t) = \Lambda_0^*(t) + \Lambda_j^*(t)$ and
    $\lambda_j(t) = \lambda_0^*(t) + \lambda_j^*(t)$, where these are the Poisson process cumulative intensity functions. (when we integrate $\lambda_j^*(t)$ over the support of t we get $\Lambda_j^*(T)$.

    - ▼ So we can write: $\Lambda_j^*(t) = \gamma_j^* F_j^*(t)$,
      where $\gamma_j^* = \Lambda_j^*(T)$ (idiosyncratic test) and
      $F_j^* = \Lambda_j^*(t)\Lambda_j^*(T)$.

$F_j^*(t)$, is a distribution that controls how the exceedance are distributed over time $[0,T]$.

so: $\Lambda_j(t) = \gamma_j F_j(t) = [\gamma_o^* + \gamma_j^*][\frac{\gamma_0^*}{\gamma_0^*+\gamma_j^*} + \frac{\gamma_j^*}{\gamma_0^*+\gamma_j^*} F_j^*(t)]$

Sector specific exceedance rate γ_j is the sum of the systematic and idiosyncratic rates

Sector specific distribution F_j is a mixture of the systemic and idiosyncratic distribution functions.

If we write: $\epsilon_j = \frac{\gamma_0^*}{\gamma_0^*+\gamma_j^*}$, this represents the proportion of exceedance in sector j that are associated with the systematic component.

- ▼ **Modeling of density function f_0^* and $f_1^*, ..., f_J^*$ and intensity functions λ_0^* $\lambda_1^*, ..., \lambda_J^*$**
  - ▼ **Density functions**
    - ▼ $f_j^*(t) = \int \psi(t|\mu,\tau)dG_j^*(\mu)$ ← DP mixture

      $\psi(t|\mu,\tau)$: kernel density on $[0,T]$

      $G_j^*(\mu)$: discreate mixing distribution ← prior
    - ▼ Prior G_j^* needs to have full support → DP prior with precision parameter α_j and centering distribution H, common for all G_j^*.

      α_j controls the relative weight of the mixture components.

      H controls the location of the mixture prior.
    - ▼ Kernel $\psi(t|\mu,\tau)$ is a rescaled beta density.

    That way, the model is continuous on the space $[0,T]$, as long as the prior H and the scale parameter of the Beta, τ are selected to provide full support.
  - ▼ **Intensity functions**

    This requires prior specification for the rate parameters $\gamma_0^*, \gamma_1^*, ..., \gamma_j^*$ (definition)

Which are going to be Gamma priors. If $\gamma_j^* = 0 \Rightarrow \epsilon_j = 1.$ Hence, this prior allows us to formally test for the presence of idiosyncratic risks. Idiosyncratic test.

Posterior Simulation and Prior Elicitation

MCMC

The joint posterior distribution for our model can be written as:

$$p(\{\gamma_j^*\}, \{\nu_{j,l}\}, \{\mu_{j,l}\}, \tau, \{\alpha_j\}, \pi|\text{data}) \propto$$
$$\propto \prod_{j=1}^{J}(\gamma_0^* + \gamma_j^*)^{n_j} \exp\{-(\gamma_0^* + \gamma_j^*)\}$$
$$\times \prod_{j=1}^{J}\prod_{k=1}^{n_j}(\frac{\gamma_0^*}{\gamma_0^*+\gamma_j^*}\sum\{\nu_{0,l}\prod_{s<l}(1-\nu_{0,s})\}\psi(t_{j,k}|\mu_{0,l},\tau))$$
$$+ \frac{\gamma_j^*}{\gamma_0^*+\gamma_j^*}\sum\{\nu_{j,l}\prod_{s<l}(1-\nu_{j,s})\}\psi(t_{j,k}|\mu_{j,l},\tau)$$
$$\times p(\pi)p(\tau)p(\gamma_0^*)\prod_j p(\gamma_j^*|\pi)\prod_j\prod_l h(\mu_{j,l}Beta(\nu_{j,l}|1,a_j)\prod_j p(a_j)$$

because of the Poisson process and $p(\pi), p(\tau), p(\gamma_0^*), p(\gamma_j^*|\pi)$ are the priors defined above.

Essentially this is a Poisson, Dirichlet Process.

It is impossible to compute the posterior analytically, so an MCMC method is required.

Blocked Gibbs Sampling

Hyperparameter elicitation

Historical and Expert information is used to elicit the model's hyperparameters.

$\gamma_0^*, \gamma_1^*, ..., \gamma_{10}^*$

Total number of exceedance in each sector, are being modeled for each sector j, by assuming that $E[\gamma_0^* + \gamma_j^*]$ are normally distributed. The mean and the variance of the distribution comes from historical averages.

f_0^* and $f_1^*, ..., f_J^*$.

Extreme returns tend to cluster over time.

precision $\alpha_0, \alpha_1, ..., \alpha_J$

Should support a large values

The prior mean for α_j can be elicited from a rough estimate of the frequency at which distress periods arise in secotr j

- scale τ

  Rough estimate of the length of distress periods in each sector j

- In the absence of information

  Improper Uniform Prior should be selected

Analysis of US market

General

Data information

Negative daily log returns above $u = 2\%$ in each of the ten sectors, from January 1, 2000 - December 31, 2011.

Sample size ranging from: 85 - 387 in sectors.

Source Bloomberg financial Services.

u has been cross validated without any quantitative differences.

Simulation information

Inference reported are based on an effective sample size of $3,000$ from a burn in period of $20,000$ and thinning every 50 iterations.

Convergence was monitored with Trace plots and four independent chains, according to the Gelman and Rubin methodology.

No lack of convergence was evident.

The algorithm was executed in a typical laptop.

$N = 60$ was sufficient as the sensitivity analysis showed that for bigger N there was no significant difference in the results.

Priors

- $E[\pi] = 0.2$ with high probability on values of π close to zero
- $\gamma_0^*, \gamma_1^*, ..., \gamma_{10}^*$ have zero mean and 18% annualized volatility with the prior belief that 85% of the observed exceedance with 99%

probability that at least 50% of them arise from systematic component of the model.

- Dirichlet Process priors: H, τ, a_j are expected to have high multimodal intensity but we don't know where the modes lie. Hence, $E[a_j] = 12$, $H \sim Unif$ that favors large values of τ.
- Estimates of the intensities $\lambda_1(t), \lambda_2(t), ..., \lambda_{10}(t)$ are historical averages.
- f_0^* and $f_1^*, ..., f_J^*$ systemic and idiosyncratic risk intensities. The majority of the sectors have at least an 80% of systematic risk while the other sectors are between $40\% - 80\%$. Idiosyncratic risk is vastly different in each sector and is estimated by historical averages.

▼ **Model Validation**

▼ out-of-sample k fold cross validation with 80-20 split and $k = 10$.

▼ In sample goodness of fit using QQ plots. → the fit appears acceptable although there is some evidence of poor fits in some specific sectors.

▼ **Prior sensitivity analysis**

- τ 3 different priors that didn't affect posterior inference.
- $\alpha_0, ..., \alpha_{10}$ 4 different priors that didn't affect posterior inference.
- H, a $Beta(3, 3)$ was tested instead of a $Unif$ but the was no change in posterior inference.
- $\gamma_0^*, \gamma_1^*, ..., \gamma_{10}^*$ very little impact on posterior inference.
- π negligible effect on posterior inference

▼ **Conclusions**

Advantages of this model over the others.

- *nonparametric nature*
- *focus on extreme returns rather than average returns*

- *interpretability of the components of the model*

Although the model has used data from the equity market, it has potential application in the debt market as well.