Assignment 2 Solution

Ted Ladas

11/18/2020

Question 1

$F(y;\theta) = 1 - e^{-y^2/(2\theta)}$ and $y \geq 0, \theta \geq 0$ and \

$X_i = \begin{cases} Y_i & Y_i \leq C \\ C & Y_i > C \end{cases}$

$R_i = \begin{cases} 1 & Y_i \leq C \\ 0 & Y_i > C \end{cases}$

a) Calculation of $\hat{\theta}_{MLE}$

We need to calculate the pdf of the data given theta, the survival function of the upper censor point, and the Likelihood of the data given theta. In order to reach the conclusion of the exercise I will define first these.

$X_i = Y_i R_i + C(1 - R_i) \iff X_i^2 = Y_i^2 R_i^2 + 2Y_i R_i C(1 - R_i) + C^2(1 - R_i)^2$

We know that $R_i = 0 \ \vee \ R_i = 1$ therefore $2Y_i R_i C(1 - R_i) = 0$

If $R_i = 0$ then $R_i^2 = R_i \ \ and \ \ (1 - R_i)^2 = (1 - R_i)$

If $R_i = 1$ then $R_i^2 = R_i \ \ and \ \ (1 - R_i)^2 = (1 - R_i)$

Therefore $R_i^2 = R_i \ \ and \ \ (1 - R_i)^2 = (1 - R_i)$

So, Finally: $X_i^2 = Y_i^2 R_i + C^2(1 - R_i)$

We also have to have an expression for the Survival function:

$S(C;\theta) = 1 - F(y;\theta) \iff S(C;\theta) = e^{-y^2/(2\theta)}$

And finally we need to derive the pdf of the data

$f(y;\theta) = \frac{\partial}{\partial y} F(y;\theta) \iff f(y;\theta) = \frac{1}{\theta} y e^{-y^2/(2\theta)}$

So we can start:

$L(\theta) = \prod_i \{(f(y;\theta))^{r_i} (S(C;\theta))^{(1-r_i)}\} = \prod_i \{(\frac{1}{\theta} y e^{-y^2/(2\theta)})^{r_i} (e^{-C^2/(2\theta)})^{(1-r_i)}\} =$

$\theta^{-\sum r_i} \sum y_i^{r_i} e^{-\sum y_i^2 r_i + C^2(1-r_i)} = \theta^{-\sum r_i} \sum y_i^{r_i} e^{-\sum X_i^2}$

$\ln L(\theta) = \ell(\theta) = -\sum r_i \ln\theta - \frac{1}{2\theta} \sum X_i^2 + \ln \sum y_i^{r_i}$

$\frac{\partial}{\partial\theta} \ell(\theta) = 0 \iff -\frac{\sum r_i}{\theta} + \frac{\sum X_i^2}{2\theta^2} + 0 = 0 \iff \frac{-2\theta \sum r_i + \sum X_i^2}{2\theta^2} = 0$

and since $\theta \geq 0$

$-2\theta \sum r_i + \sum X_i^2 = 0 \iff \hat{\theta_{MLE}} = \frac{\sum X_i^2}{2 \sum r_i}$

b) We know that $I(\theta) = -\mathbb{E}\left[\frac{d^2 l(\theta)}{d\theta^2}\right]$

and also that $\mathbb{E}[R_i] = 1P(R_i = 1) + 0P(R_i = 0) = P(Y \leq C) = F(C;\theta) = 1 - e^{-C^2/(2\theta)}$ and therefore $\mathbb{E}[1 - R_i] = 1 - \mathbb{E}[R_i] = e^{-C^2/(2\theta)}$

so if we decompose $\sum X_i^2$ again we have:

$I(\theta) = -\frac{\sum \mathbb{E}[r_i]}{\theta^2} + \frac{\sum \mathbb{E}[Y_i^2 r_i]}{\theta^3} + \frac{\sum \mathbb{E}[C^2(1-r_i)]}{\theta^3} =$

$n/\theta^2(1 - e^{-C^2/(2\theta)})n/\theta^3(-C^2 e^{-C2/2\theta} + 2\theta(1 - e^{-C^2/(2\theta)}) - e^{-C^2/(2\theta)}))$

$I(\theta) = \frac{n}{\theta^2}(1 - e^{-C^2/(2\theta)})$

c) Since we know that $\hat{\theta_{MLE}} \sim N(\theta, I(\theta)^{-1})$ we can Normalize $\hat{\theta_{MLE}}$ and produce the 95% confidence interval as follows

$Z = \frac{\hat{\theta_{MLE}} - \theta}{\sqrt{I(\theta)^{-1}}} \sim N(0,1)$

$P(z_{a/2} \leq Z \leq z_{a/2}) = 1 - a$

$a = 0.05$ hence $z_{-a/2} = -1.959964, z_{a/2} = 1.959964$

$[z_{-a/2}\sqrt{I(\theta)^{-}1}, z_{a/2}\sqrt{I(\theta)^{-}1}]$

Question 2