Linear Programming 5

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Problem 1

This is already in Standard Equality Form, so adding auxiliary variables u_1 and u_2 and redefining the objective function to $-u_1 - u_2$ begins the first phase. The tableau gives us a basic feasible solution:

$$\begin{array}{c|ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 3 \\ 0 & 2 & 3 & 3 & 1 & 1 & 0 & 1 & 7 \end{array}$$

This gives us $x^* = [0,0,0,0,0,3,7]^T$, a feasible basic solution under basis $B = \{6,7\}$ to the auxiliary LP with

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 2 & 3 & 3 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

Using the revised simplex method, $y = (A_B^T)^{-1} c_B$, so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, so $y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Checking $c_k - [-1,-1]A_k$ for $k = 1$ through 5, we select $k = 4$, for which $0 - [-1,-1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 > 0$. Then, $d = A_B^- 1 A_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since every entry of d is greater than or equal to 0, we may continue: t cannot be $\frac{x_6^*}{d_6} = \frac{3}{0}$, so it must be $\frac{x_7^*}{d_7} = \frac{7}{1}$, making $r = 7$. So $B_{new} = B \backslash 7 \cup 4 = \{4,6\}$, and $x_{new}^* = [0,0,0,7,0,3,0]^T$.

With this, we perform the process again: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so $y = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Now we check $c_k - t^T A_k$ for $k = 1,2,3,5$ and 7 and choose $k = 1$, for $c_1 - y^t A_k = 0 - \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 - (-1) = 1 > 0$. Then, $d = A_B^{-1} A_k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then $d_4, d_6 > 0$, so we may continue:
$\frac{x_4^*}{d_4} = 0$, $\frac{x_6^*}{d_6} = \frac{3}{1} = 3$. So choose $t = 3$, $r = 6$. The new basis is $B_{new} = B \backslash 1 \cup 6 = \{1,4\}$, and $x_{new}^* = [3,0,0,7-3\times 2,0,0,0]^T = [3,0,0,1,0,0,0]^T$.
To check if this is valid, try to perform another step: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then no $c_k - y^T A_k > 0$, so this solution is a solution for the first half of the revised simplex method!

Now, working on the LP $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 2 & 3 & 3 & 1 & 1 \end{bmatrix}$, $c^T = [5, 8, 4, 2, 3]$, $b^T = [3, 7]$. Using the basis $B = \{1, 4\}$ and feasible point $x^{*T} = [3, 0, 0, 1, 0]$.

Now, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} y = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. For $k = 2, 3, 5$, we choose $c_2 - y^T A_2 = 8 - [1, 2] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 8 - 7 = 1$. Then, $d = A_B^{-1} A_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $t = min\{\frac{x_1^*}{d_1}, \frac{x_4^*}{d_4}\} = min\{\frac{3}{1}, \frac{1}{1}\} = 1$. $r = 4$.

Using this, we find a new basis and feasible point: $B_{new} = B \backslash 4 \cup 2 = \{1, 2\}$. $x_{new}^* = [3-1, 1, 0, 0] = [2, 1, 0, 0, 0]$.

Once more, $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} y = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, so $y = \begin{bmatrix} -1 & 3 \end{bmatrix}$. Checking $k = 3, 4, 5$, we select $c_5 - y^T A_5 = 3 - [-1, 3] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5 - (-1 + 3) = 3 > 0$. Then $d = A_B^{-1} A_k = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Then t must be $\frac{2}{2} = 1$, and $r = 1$ is leaving.

Finally, we have the optimal solution $x_{new}^* = [0, 2, 0, 0, 1]^T$ for basis $B_{new} = B \backslash 1 \cup 5 \{2, 5\}$. This must be optimal because

$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} y = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$, so $y = \begin{bmatrix} \frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$. Then, for no value of k is $c_k - y^T A_k$ positive.

Problem 2

Beginning the revised method, $y = (A_B^T)^{-1} c_B$, so $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$. Then, $y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Let us choose $k = 6$, for which $c_6 - y^T A_6 = 4 - [1, 0, 0, 0] \begin{bmatrix} 2 \\ 1 \\ -2 \\ 4 \end{bmatrix} = 4 - 2 = 2 > 0$. Then $d = A_B^{-1} A_6 = \begin{bmatrix} -3 \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$.

Take t to be the minimum of $\{\frac{1}{\frac{1}{2}}, \frac{1}{\frac{1}{2}}, \frac{1}{\frac{3}{2}}\}$, or $\frac{2}{3}$, so we we know $r = 4$ is entering:

Start again with basis $B_{new} = B \backslash 4 \cup 6 = \{1, 2, 3, 6\}$, $x_{new}^{*T} = [3, \frac{2}{3}, \frac{2}{3}, 0, 0, \frac{2}{3}, 0]$. Then $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & -2 & 4 \end{bmatrix} y = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$, so $y = \begin{bmatrix} 3 \\ 2 \\ -2 \\ -2 \end{bmatrix}$.

Choose $k = 5$, such that $c_5 - y^T A_5 = 6 - y^T \begin{bmatrix} 2 \\ 1 \\ -2 \\ 4 \end{bmatrix} = 4 > 0$. Then $d = A_B^{-1} A_k = \begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \end{bmatrix}$. t must be $\frac{x_3^*}{d_3} = \frac{3}{\frac{2}{3}}$, so $r = 3$ is leaving the basis.

This provides optimal solution $x_{new}^* = [3, \frac{24}{3}, 0, 0, \frac{4}{2}, \frac{2}{3}, 0]^T$, for basis $B_{new} = B \backslash 3 \cup 5 = \{1, 2, 5, 6\}$. We know this is optimal because now $y = \begin{bmatrix} 7 \\ \frac{14}{3} \\ -\frac{14}{3} \\ -6 \end{bmatrix}$. For no value of k is $c_k - y^T A_k > 0$.

Problem 3A

The last three entries of x^* must be 0, as they are not basic entries. Then, using A and b to make a system of equations, we see $x_1 + 2x_2 = 11$, $x_2 + x_3 = 6$, and $x_1 + 2x_2 + x_3 = 13$. Solving, we find feasible point $x^* = [3, 4, 2, 0, 0, 0]^T$. This is optimal, because

$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} y = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$, so $y = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. Then $6 - y^T \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = 6 - (-3 + 6 + 3) = 6 - 6 = 0$, $-10 - y^T \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = -10 - (-4 - 1) = -5 < 0$, and $5 - y^T \begin{bmatrix} -6 \\ -1 \\ -5 \end{bmatrix} = 5 - (18 - 2 - 5) = -6 < 0$. So, no k can be chosen to continue the revised simplex method. x^* must be optimal.

Problem 3B

x^* is still feasible (the constraints are the same), so it's optimal if the top entry of the column representing x_4 in the tableau is non-negative. Since this entry's value is $-[c_4 - y^T A_4] = -[6 + \theta - [-3, 2, 1] \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}] = -6 - \theta + (-3 + 6 + 3) = -\theta$. So x^* should be optimal if θ is 0 or below.

Problem 3C

In this case, B is still a basis and $x^* = [3, 4, 2, 0, 0, 0, 0]^T$ (note the extra 0) is still feasible. It is optimal if $-[c_7 - y^T A_7]$ is non-negative. This is $-[\theta - [-3, 2, 1] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}] = -\theta + (-3 + 2) = -\theta - 1$. Therefore, x^* is optimal if $\theta < -1$.

Problem 3D

As in part $3B$, x^* is still feasible, so it is optimal if the entry above x_1 in the tableau is non-negative. This means $0 < -[c_1 - y^T A_1] = -[-2 + \theta - [-3, 2, 1] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}] = 2 - \theta + (-3 + 1) = 2 - \theta - 2$. So, for x^* to be optimal, θ should be non-positive.