Linear Programming 3

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Part 1

Farkas Lemma states if A is an $m \times n$ matrix and $b \in \mathbb{R}^n$, then exactly one of the following is true:

There is an $x \in \mathbb{R}^n$ with all entries greater than or equal to 0 such that $Ax = b$, or

There is a $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $y^T b < 0$.

To prove Farka's Lemma, we first note that the two conditions cannot apply simultaneously, as then

$$0 > y^T b = b^T y = (Ax)^T y = (x^T)(A^T y) > 0.$$

If just the first condition is true, we are done. If the first condition is not true, then the LP

$$\begin{aligned} \max\ & e^T x \\ s.t.: Ax &= b \\ x &\geq 0 \end{aligned}$$

is infeasible. The dual of this LP should therefore be infeasible or unbounded by complementary slackness—but $y = 0$ would be a feasible point of the dual, so we know it *must* be unbounded. Therefore, there exists a $y \in \mathbb{R}^m$ with $A^T y > 0$. If we take the specific case where b is the 0-vector, this solves the problem.

Part 2A

$x^T = [0, 0, 0, 0, \frac{4}{3}, \frac{7}{6}, \frac{9}{6}]$ is a solution.

Part 2B

For $y^T = [\frac{4}{9}, \frac{4}{3}, \frac{1}{4}]$, $A^T y \geq 0$ and $b^T y > 0$.

Part 3

The basic solutions are those whose non-zero entries correspond to linearly independent columns of A, where in this case,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Part 3A

$[2, 2, 0, 0, 0]^T$ does not solve the constrains (as $2 - 2 + 0 + 0 + 0 \neq 1$). It is not even a regular solution.

Part 3B

$[0, 1, 2, 1, 1]^T$ has four non-zero entries, corresponding to the columns of A $[1, -1, 1]^T$, $[1, 0, 1]^T$, $[1, 1, 0]^T$, and $[0, 1, 1]^T$. Four columns in $\mathbb{R}^3$ cannot be linearly independent. Therefore, this cannot be a basic solution (thought it is a solution).

Part 3C

$[2, 1, 1, 0, 0]^T$ refers to the first three columns of A, $[1, 1, 1]^T$, $[1, -1, 1]^T$, and $[1, 0, 1]^T$. Unfortunately,

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

In other words, these columns are not linearly independent. This is not a basic solution.

Part 3D

$[0, -1, 5, 0, 0]^T$ uses the second and third columns of A, $[1, -1, 1]^T$ and $[1, 0, 1]^T$. These columns are linearly independent, so this is a degenerate basic solution.

Part 3E

$[0, 0, \frac{7}{2}, \frac{1}{2}, \frac{1}{2}]^T$ refers to the last three columns of A, $[1, 0, 1]^T$, $[1, 1, 0]^T$, and $[0, 1, 1]$. These are linearly independent, so this is a basic solution.

Part 4A

The feasible region F of a Linear Program in SIF consists of the set of points x satisfying $x \geq 0$ and $Ax \leq b$. Suppose the points $x^{(1)}$ and $x^{(2)}$ are in F.

Any point with the form $\lambda x^{(1)} + (1-\lambda)x^{(2)}$ with $\lambda \in (0, 1)$ has all its elements greater than or equal to 0, because $x^{(1)}$ and $x^{(2)}$ had that property. Moreover, $A(\lambda x^{(1)} + (1-\lambda)x^{(2)}) = \lambda Ax^{(1)} + (1-\lambda)Ax^{(2)} \leq \lambda b + (1-\lambda)b = b$. Therefore, any points between $x^{(1)}$ and $x^{(2)}$ are in F, so F must be convex.

Part 4B

First, the number of optimal solutions is clearly a non-negative integer, as solutions are whole points and we cannot have a negative or non-integer number of points. Next, we show that an LP that has two optimal solutions has an infinite number of optimal solutions, completing the proof:

Let $x^{(1)}$ and $x^{(2)}$ be distinct optimal solutions to a linear program. By the result of Part A, all points between $x^{(1)}$ and $x^{(2)}$ are in the feasible region, because the feasible region is convex. Now we show that each of these points should be optimal, and therefore the LP has infinitely many solutions:

For the objective function of the LP, c^T, we know $c^T x^{(1)} = c^T x^{(2)}$ (if one were larger than the other, it would contradict the assumption that both are optimal). Call this optimal value V. Any point between $x^{(1)}$ and $x^{(2)}$ can be written as $\lambda x^{(1)} + (1-\lambda)x^{(2)}$, with $\lambda \in (0, 1)$. At this point, the value of the objective function should be

$$c^T\left(\lambda x^{(1)} + (1-\lambda)x^{(2)}\right) = \lambda c^T x^{(1)} + (1-\lambda)c^T x^{(2)} = \lambda V + (1-\lambda)V = V$$

So any point between $x^{(1)}$ and $x^{(2)}$ should also be an optimal point. There any infinitely many such points (because there are infinitely many possible values of λ between 0 and 1), so if there is more than one solution, there are infinitely many solutions. This completes the proof.

Part 5A

The point $[0, 3, 0, 0]^T$ is in F, because $0 + 3 + 0 + 0 = 3$ and $0 + 2(3) + 0 + 0 = 6$. This is a basic solution, because only one non-zero entry exists, and a single column of a matrix is always linearly independent with itself. As a basic solution, it necessarily lies on a corner, or extreme point.

Part 5B

$[1, 1, 0, 0]^T$ satisfies the constraints, as $2(1) + 1 + 0 + 0 = 3$ and $4(1) + 2(1) + 0 + 0 = 6$, and therefore the point is in F. However, this is not a corner point or extreme point, which we may prove by finding two more points in F on either side of $[1, 1, 0, 0]^T$. Two such points are

$$[1.5, 0, 0, 0]^T \quad \text{and} \quad [0, 3, 0, 0]^T$$

(both of which satisfy the conditionts, as $2(1.5) = 3$, $4(1.5) = 6$, $3 = 3$, and $2(3) = 6$). Since

$$\frac{2}{3}\begin{bmatrix} 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \left(1 - \frac{2}{3}\right)\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

we understand that $[1, 1, 0, 0]^T$ is in the interior of F.