Linear Programming 4

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Part 1A

First, transform into an LP in Standard Equality Form: Max $-3x_1 - x_2 + 2x_3$ subject to

$$x_1 + 2x_2 - x_3 + x_4 = 1$$

$$-x_1 + x_2 + x_5 = 4$$

$$-2x_1 + x_2 + 2x_3 + x_6 = 6$$

$$x_1, \ldots, x_6 \geq 0$$

Use this to make a tableaux:

$$\begin{array}{c|cccccc|c} 1 & 3 & 1 & \text{-}2 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & \text{-}1 & 1 & 0 & 0 & 1 \\ 0 & \text{-}1 & 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & \text{-}2 & 1 & 2 & 0 & 0 & 1 & 6 \end{array}$$

Get rid of the negative element in the top row by making the third column into the third basis vector (as the leading entry is that third column's only positive entry). Apply row operations accordingly:

$$\begin{array}{c|cccccc|c} 1 & 1 & 2 & 0 & 0 & 0 & 1 & 6 \\ \hline 0 & 0 & \frac{5}{2} & 0 & 1 & 0 & \frac{1}{2} & 4 \\ 0 & \text{-}1 & 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & \text{-}1 & \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & 3 \end{array}$$

The first, second, and sixth columns are not basis vectors, so $x_1, x_2, x_6 = 0$. For the others, $x_3 = 3$, $x_4 = 4$, and $x_5 = 4$. Reporting just the first three entries of this answer, we find that $(0, 0, 3)$ is an optimal solution to the LP, with the objective value equal to 6. With this, and complementary slackness, we have, in the dual,

$$y_1 + y_2 + 6y_3 = 6$$

$$-y_1 + 2y_3 = 2$$

So, an optimal point in the dual is $(0, 0, 1)^T$.

Part 1B

Transform into an LP in Standard Equality Form: Max $2x_1 - 3x_2 + x_3 - x_4$ subject to

$$-x_1 + x_2 - x_3 + x + 5 = 4$$

$$x_1 - x_2 + x_3 - x_4 + x_6 = 2$$

$$-x_1 + x_2 - x_3 + 2x_4 + x_7 = 2$$

$$x_1, \ldots, x_7 \geq 0$$

Use this to make a tableaux:

1	-2	3	-1	1	0	0	0	0
0	-1	1	-1	0	1	0	0	4
0	1	-1	1	-1	0	1	0	2
0	-1	1	-1	2	0	0	1	2

Make the first column (which has the most negative top entry) into the the basis vector $(0, 0, 1, 0)^T$:

1	0	1	1	-1	0	2	0	4
0	0	0	0	-1	1	1	0	6
0	1	-1	1	-1	0	1	0	2
0	0	0	0	1	0	1	1	4

Now make the fourth column into the basis vector $(0, 0, 0, 1)^T$:

1	0	1	1	0	0	3	1	8
0	0	0	0	0	1	2	1	10
0	1	-1	1	0	0	2	1	6
0	0	0	0	1	0	1	1	4

Choosing columns 1, 4, and 5 to be the basis, we find the optimal solution $(6, 0, 0, 4)^T$, with optimal value 8. Then, using this solution and complementary slackness,

$$4y_1 + 2y_2 + 2y_3 = 8$$

$$-y_1 + y_2 - y_2 = 2$$

$$-y_2 + 2y_3 = -1$$

yielding $(0, 3, 1)^T$ as an optimal solution to the dual.

Part 1C

Make this into an LP in Standard Equality Form: Max $x_1 + 5x_2 + 5x_3 + 5x_4$ subject to

$$x_1 + 4x_2 + 3x_3 + 4x_4 + x_5 = 17$$

$$x_2 + x_3 + x_4 + x_6 = 4$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 + x_7 = 10$$

$$x_1, \ldots, x_7 \geq 0$$

Make this into a tableaux:

1	-1	-5	-5	-5	0	0	0	0
0	1	4	3	4	1	0	0	17
0	0	1	1	1	0	1	0	4
0	1	2	2	3	0	0	1	10

Using Dantzig's rule, we decide to make the third column into the basic vector $(0, 0, 1, 0)^T$ using row operations:

1	-1	0	0	0	0	5	0	20
0	1	1	0	1	1	-3	0	5
0	0	1	1	1	0	1	0	4
0	1	0	0	1	0	-2	1	2

Then, making the first column into $(0,0,0,1)$, we have

1	0	0	0	1	0	3	1	22
0	0	1	0	0	1	-1	-1	3
0	0	1	1	1	0	1	0	4
0	1	0	0	1	0	-2	1	2

Giving us the optimal solution $(2,0,4,0)$, with objective value 22. With this solution and complementary slackness, we find

$$17y_1 + 4y_2 + 10y_3 = 22$$
$$y_1 + y_3 = 1$$
$$3y_1 + y_2 + 2y_3 = 5$$

yielding optimal solution $(0,3,1)^T$ for the dual.

Part 2

Put the LP into SEF and call it (P): max $5x_1 - 2x_2 + x_3$ such that

$$x_1 + 4x_2 + 3x_3 + s_1 = 6$$
$$2x_1 + x_2 + 3x_3 - s_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Define a new LP (P'): max $-u_1 - u_2$ such that

$$x_1 + 4x_2 + 3x_3 + s_1 + u_1 = 6$$
$$2x_1 + x_2 + 3x_3 - s_2 + u_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2, u_1, u_2 \geq 0$$

Put this into a tableaux:

1	0	0	0	0	0	1	1	0
0	1	4	3	1	0	1	0	6
0	2	1	3	0	-1	0	1	2

The 1's in the top row should be 0's, so we subtract the first and second rows from the top row:

1	-3	-5	-6	-1	1	0	0	-8
0	1	4	3	1	0	1	0	6
0	2	1	3	0	-1	0	1	2

We should now have a legitimate tableaux, finishing the first phase, and we solve as a normal simplex. The -6 is the most negative value in the top row, so the third column is the new pivot. Its bottom entry is its leading variable, as $\frac{2}{3} < \frac{6}{3}$:

1	-1	3	0	1	1	2	0	4
0	-1	3	0	1	1	1	-1	4
0	$\frac{2}{3}$	$\frac{1}{3}$	1	0	-$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$

This is almost optimized! Make the first column into the basis vector $(0,0,1)^T$:

1	0	$\frac{7}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	2	$\frac{1}{2}$	5
0	0	$\frac{7}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	1	-$\frac{1}{2}$	5
0	1	$\frac{1}{2}$	$\frac{3}{2}$	0	-$\frac{1}{2}$	0	$\frac{1}{2}$	1

Finally, we have $(1,0,0,5,0,0,0)$ as an optimal solution for (P'), so $(1,0,0,5,0)$ is an optimal solution of (P) in SEF, while $(1,0,0)$ is an optimal solution of the LP as written on the assignment, with objective value 5.

Part 3

We seek to maximize $Price = 1x_1 + 2x_2 + 3x_3 + 2x_4$, where x_i represents the number of grams of ore type i used (which must, of course, be positive). This has the constraints

$$1x_0 + 2x_2 + 2x_3 + 1x_4 \leq 6 \text{ (less than 6mg lead)}$$

$$1x_2 + 1x_3 + 2x_4 \leq 10 \text{ (less than 10mg cobalt)}$$

Plugging this into the solver at Comnuan.com, we find an optimal solution: use $\frac{2}{3}$ grams of ore 3 and $4 + \frac{2}{3}$ grams of ore 4, so the value of the ring is $11 + \frac{1}{3}$. This uses 6mg of lead and 10mg of cobalt, perfectly satisfying the constraints.