Linear Programming 2

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April 15, 2017

Part 1

Let B, F, E, P, D, and L represent the week we begin to built the foundation, build the wooden frame, do the electrical work, do the plumbing, put in the drywall, and perform the landscaping, respectively. Let a new variable y be constrained by the following: $B - y \leq -3$, $F - y \leq -2$, $E - y \leq -3$, $P - y \leq -4$, $D - y \leq -1$, and $L - y \leq -2$. Then, declaring the objective of the Linear Program to be minimizing y (or, in standard form for a Primal program, maximizing $-y$), y can be understood as the overall completion time for the house. To encode the order of the tasks into the program, use the following constraints:

$$B - F \leq -3 \text{ (begin frame after foundation)}$$
$$B - L \leq -3 \text{ (begin landscaping after foundation)}$$
$$F - E \leq -2 \text{ (electricity after frame)}$$
$$F - P \leq -2 \text{ (plumbing after frame)}$$
$$E - D \leq -3 \text{ (drywall after electrical)}$$
$$P - D \leq -4 \text{ (drywall after plumbing)}$$

Of course, all variables are ≥ 0, as they represent week numbers starting at 0. Plugging this into the LP solver at Comnuan.com, I find the vector $(0, 3, 6, 5, 9, 3, 10)$. We begin with the foundation in week 0. After three weeks, we build the frame and begin landscaping simultaneously, ending on week five. Then we start the plumbing, and then the electrical work a week later, finishing both of them on week 9. By week 10, we build the drywall and flooring.

Part 2

The pool must be guarded for nine hours. Denote by x_i the hourly salary of the lifeguard during hour i (as each salary is an integer, this is an IP). Our goal is to minimize $\sum_{i=1}^{9} x_i$ (or, in standard format, maximize $\sum_{i=1}^{9} -x_i$). These constraints ensure each variable is non-negative, and that the pool is guarded every hour:

$$-x_1 \leq -18 \text{ (lowest salary at hour 1)}$$
$$-x_2 \leq -18$$
$$-x_3 \leq -18$$
$$-x_4 \leq -21$$
$$-x_5 \leq -21$$
$$-x_6 \leq -20$$
$$-x_7 \leq -20$$
$$-x_8 \leq -9$$
$$-x_9 \leq -9$$

Part 3

For integers $1 \leq i, j \leq 8$, let $x_{i,j} = 0$ if there is not a queen on the i^{th} row, j^{th} column of our chessboard, and $x_{i,j} = 1$ if there is. We want to maximize the sum of all the $x_{i,j}$s, that is, place as many queens on the board as possible. To ensure a safe board state, no queens can share a row, column, or diagonal, so:

$$\sum_{j=1}^{8} x_{i,j} \leq 1 \text{ (one queen per row)}$$

$$\sum_{i=1}^{8} x_{i,j} \leq 1 \text{ (one queen per column)}$$

$$\sum_{i,j|i+j=k} x_{i,j} \leq 1 \text{ for } k = 2, \ldots, 16 \text{ (one queen per downwards diagonal)}$$

$$\sum_{i,j|i-j=k} x_{i,j} \leq 1 \text{ for } k = -7, \ldots, 7 \text{ (one queen per upwards diagonal)}$$

Solving this LP would tell us exactly where to place queens on the chessboard to optimize their quantity. I hope there are at least 8 of them!

Part 4

Maximize $-2x_1 - x_2 + 3x_3$, subject to

$$x_1 + x_2 + x_3 - s_1 = 2$$
$$x_1 - x_2 = 1$$
$$x_1 - x_3 + s_3 = 2$$
$$x_1, x_3, s_1, s_3 \geq 0$$

Part 5

If an LP is in Standard Inequality Form, it seeks to maximize the value of the objective function $c^T x$, with each $x_i \in x$ being non-negative. Each of its remaining constraints uses a $\leq$ rather than $\geq$, but for this problem, imagine there are no constraints except $x_i \geq 0$. If a line is completely inside the feasible area of such an LP, then every point along that line with location vector x has $x_i \geq 0$ for all $x_i \in x$. Such a line is impossible, which we now prove by induction on i, the number of variables in the LP:

For an LP in Standard Inequality Form with only one variable, the graph of the feasible area is a one-dimensional space. Any whole line in a 1-dimensional space consists of all points in the space, because lines are one-dimensional. The one constraint, $x_1 \geq 0$, removes half the space from the feasible area. Therefore, no whole line exists entirely within the feasible area of such an LP.

For the inductive step, assume that no LP in SIF with n variables contains a whole line in its feasible area. Create an LP in SIF with $n + 1$ variables, with the only constraints being $x_1, \ldots, x_{n+1} \geq 0$. A line in the feasible area of this LP must satisfy the constraints at each point, implying, for instance, that it does not intersect the n-dimensional hyperplane dividing positive values of x_{n+1} from negative values of x_{n+1}. In order to never intersect that hyperplane, the line must be parallel to it, having constant value for x_{n+1}. This reduces the problem to the n-variable case, and the inductive step is complete.

As adding more constraints can only decrease the size of the feasible area further, we understand that no LP in SIF can contain a whole line in its feasible area. As Standard Equality Form demands each variable be non-negative, as well, no LP in Standard Equality form can hold a whole line, either; the proof is identical.

Part 6

The dual is: minimize $4y_1 - 2y_2$, subject to

$$y_1 + y_2 \geq 3$$
$$2y_1 - y_2 \geq 1$$
$$2y_1 + y_2 \geq 4$$
$$y_1 - y_2 \geq 1$$
$$y_1, y_2 \geq 0$$

A. $[0, 1, 0, 2]^T$ is feasible, because $2(1) + 2 = 4 \leq 4$ and $-(1) - (2) = -3 \leq -2$, satisfying the constraints of the primal. The non-zero values tell us to solve the system of linear equations related to the second and fourth constraint of the dual, $2y_1 - y_2 = 1$ and $y_1 - y_2 = 1$. Solving shows that $y_1 = 0$ and $y_2 = -1$. However, these values do not satisfy the other constraints of the dual, as $0 + (-1) \leq 3$, so this is not an optimal solution.

B. $[1, 0, 0, 3]^T$ is feasible, because $1 + 3 = 4 \leq 4$ and $1 - 3 = -2 \leq -2$, satisfying the constraints of the primal. The non-zero values tell us to solve the system of linear equations related to the first and fourth constraint of the dual, $y_1 + y_2 = 3$ and $y_1 - y_2 = 1$. Solving shows that $y_1 = 2$ and $y_2 = 1$. As these values satisfy the constraints of the dual, $[1, 0, 0, 3]$ is an optimal point for the primal LP by Complementary Slackness.

Part 7

The dual is: minimize $3y_1 + 2y_2 + 7y_3 + 4y_4$, subject to

$$y_1 + 2y_3 \geq 1$$
$$y_2 + 2y_4 \geq 5$$
$$y_1 + y_2 - y_3 \geq 3$$
$$3y_1 + y_2 - y_3 + y_4 \geq 6$$
$$3y_3 + 2y_4 \geq 6$$
$$y_1, y_2, y_3, y_4 \geq 0$$

The complementary slackness conditions state that, given a primal-dual pair (P)-(D), and feasible points $x*$ for (P) and $y*$ for (D), then these points are optimal if either $y_i* = 0$ or the i^{th} constraint of (P) is an equality, for all i.

Since $y* = [1, 2, 0, 3]^T$ is stated to be optimal, there is an optimal solution $x*$ for the primal such that the first, second, and fourth constraint in the primal are equalities. Then,

$$x_1 + x_3 + 3x_4 = 3$$
$$x_2 + x_3 + x_4 = 2$$
$$2x_2 + x_4 + 2x_5 = 4$$

Notably, this is solved by $x* = [0, 1, 0, 1, \frac{1}{2}]^T$, which also satisfies the other conditions of the primal. Given the complementary slackness conditions, this should be an optimal point, making the objective function equal to 14.