

# 1 Introduction

Modal logic extends classical propositional logic by introducing modal operators, most notably  $\Box$  ("necessarily") and  $\Diamond$  ("possibly"). In the standard Kripke semantics, these operators are interpreted over a set of possible worlds connected by an accessibility relation. An idea to generalize Kripke semantics was topology. The study of topological semantics in modal logic was initiated by McKinsey and Tarski in 1944 [?]. It is well known, that  $S4$  is fully captured. Despite its rich expressiveness, it can not fully capture Kripke semantics. That is why Neighbourhood semantics [?] as a generalization of Kripke semantics for modal logic was introduced independently by Dana Scott [?] and Richard Montague [?]. Neighborhood semantics is more general than Kripke semantics and in the case of normal reflexive and transitive logics coincides with topological semantics. The original motivation for introducing these general models was to provide a semantics for non-normal modal logics. Beyond their theoretical motivation, interest in topological semantics and neighborhood frames has grown considerably, partly due to its applications in artificial intelligence.

Oftentimes, it is necessary to combine frames for different modal logics into a complex frame. The natural way of doing that is a product construction. For Kripke frames, the resulting product is the Cartesian product of the two frames with two accessibility relations. For topological semantics, the product of topological spaces as bi-topological spaces with so-called horizontal and vertical topologies have been considered. In a similar fashion, the product of neighborhood frames was introduced by Sano in [?].

For our main question, we consider the neighborhood product additionally with the standard topology. We will answer whether the logic with three copies of  $T$  with the interaction axiom  $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$  can be fully characterized as the logic a product of neighbourhood frames with standard topology, where each frame validates  $T$ . The question was originally motivated by two papers where the authors showed in topological semantics a similar version of the question but for  $S4$ . In the other, it was proven that for any  $L_1, L_2 \in \{D, D4, T, S4\}$ , that their logic together can be characterized as the logic of a product of neighbourhood frames but without standard topology. In this work, we start with a brief recap of necessary notations and theorems in Kripke semantics and then introduce topological spaces and neighborhood frames. We will reprove the theorems from the authors in a more detailed way and then use similar techniques to answer our main question. Finally, we conclude with a discussion of potential directions for future research related to the topic. This thesis primarily builds upon the works "Multimodal Logics of Products of Topologies" and "Modal Logic of Products of Neighborhood Frames."

## 2 Preliminaries

This chapter is primarily devoted to the semantics of Kripke, topological spaces, and neighborhood frames. In addition, we define the notions of product frames and multimodal logic. We also present fundamental results in modal logic, such as Filtration Theorem and Sahlqvist Theorem which plays a role in our later chapters.

**Definition 2.0.1.** *Let  $\text{Prop}$  be a set of variables. Then a formula  $\phi$  is defined as follows:*

$$\phi ::= p \mid \perp \mid \neg\phi \mid \phi \vee \psi \mid \Box_i \phi$$

where  $p \in \text{Prop}$  and  $\Box_i$  is a modal operator. Other connectives are expressed through  $\perp$  and  $\rightarrow$  and dual modal operators  $\Diamond_i$  as  $\Diamond_i \phi = \neg \Box_i \neg \phi$

**Definition 2.0.2.** *Let  $M = (W, R_1, \dots, R_n, V)$  be a model and  $w \in W$  a state in  $M$ . The notion of a formula being true at  $w$  is inductively defined as follows :*

$$\begin{aligned} M, w \Vdash p & \text{ iff } w \in V(p) \\ M, w \Vdash \perp & \text{ never} \\ M, w \Vdash \neg\phi & \text{ iff not } M, w \Vdash \phi \\ M, w \Vdash \phi \vee \psi & \text{ iff } M, w \Vdash \phi \vee M, w \Vdash \psi \\ M, w \Vdash \Box_i \phi & \text{ iff } \forall v \in W : w R_i v \rightarrow M, v \Vdash \phi \end{aligned}$$

**Definition 2.0.3.** *A set  $\Sigma$  is closed under subformulas, if for all formulas  $\phi$  and  $\phi'$  the following holds :*

1. if  $\neg\phi \in \Sigma$  then  $\phi \in \Sigma$
2. if  $\phi \vee \phi' \in \Sigma$  then  $\phi, \phi' \in \Sigma$
3. if  $\Box_i \phi \in \Sigma$  then  $\phi \in \Sigma$

*Example :* Suppose we have a multimodal logic with  $\Box, \Box_1$  called  $\mathcal{L}_{\Box, \Box_1}$  where  $\phi = \Box p \rightarrow \Box_1 q$  and  $\phi \in \Sigma$ . Then  $\Sigma = \{\phi, \neg\Box p, \Box_1 q, \Box p, p, q\}$  is closed under subformulas.

**Definition 2.0.4.** *Let  $M = (W, R_1, \dots, R_n, V)$  be a model and suppose  $\Sigma$  is a set of formulas. We define a relation  $\equiv$  on  $W$  as follows :*

$$w \equiv v \text{ iff } \forall \phi \in \Sigma : M, w \Vdash \phi \Leftrightarrow M, v \Vdash \phi$$

*It is well known that the  $\equiv$ -relation is an equivalence relation. We denote the equivalence class of a state  $w \in W$  as  $[w]_\Sigma = \{v \mid v \equiv w\}$ . Furthermore  $W_\Sigma$  is the set of all equivalence classes, i.e  $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$ .*

**Definition 2.0.5.** *Let  $M = (W, R_1, \dots, R_n, V)$  be a model,  $\Sigma$  is closed under subformulas and  $W_\Sigma$  the set of equivalence classes induced by  $\equiv$ . A model  $M_\Sigma^f = (W_\Sigma^f, R_1^f, \dots, R_n^f, V^f)$  is called filtration of  $M$  through  $\Sigma$  if the following holds :*

1.  $W^f = W_\Sigma$
2. If  $(w, v) \in R_i$  then  $([w], [v]) \in R_i^f$
3. If  $([w], [v]) \in R_i^f$  then for any  $\Box_i \phi \in \Sigma$  : if  $M, w \Vdash \Box_i \phi$  then  $M, v \Vdash \phi$
4.  $V^f = \{[w] \mid M, w \Vdash p\}$ , for all propositional variables  $p \in \Sigma$

**Theorem 2.0.6.** Consider  $L_{\Box_1 \dots \Box_n}$ . Let  $M^f = (W_\Sigma, R_1^f, \dots, R_n^f, V)$  be a filtration of  $M$  through a subformula closed set  $\Sigma$ . Then for all formulas  $\phi \in \Sigma$ , and all nodes  $w \in M$ , we have

$$M, w \Vdash \phi \text{ iff } M^f, [w] \Vdash \phi$$

*Proof.* By induction on  $\phi$ . We will only show non-trivial and, for our purposes, necessary cases.

Case  $\phi = p$ : Left to right follows immediately from filtration definition. Conversely, suppose  $M^f, [w] \Vdash p$ . This means  $[w] \in V^f(p)$ . But this means  $V(p)$  can not be empty. Pick any  $v \in V(p)$ . Obviously,  $w \equiv v$  and  $M, v \Vdash p$ . Hence,  $M, w \Vdash p$ .

Case  $\phi = \neg\psi$ : Suppose  $\psi$  holds. Then we have :  $M, w \Vdash \phi$  iff  $M, w \not\Vdash \psi$ . Applying induction hypothesis, we get :  $M^f, [w] \not\Vdash \psi$ . But then, we have  $M^f, [w] \Vdash \phi$ . Right to left is the same.

Case  $\phi = \phi_1 \wedge \phi_2$ : Suppose  $\phi_1, \phi_2$  holds. Let  $M, w \Vdash \phi$ . That means  $M, w \Vdash \phi_1$  and  $M, w \Vdash \phi_2$ . Applying induction hypothesis, we get  $M^f, [w] \Vdash \phi_1$  and  $M^f, [w] \Vdash \phi_2$ . But then,  $M^f, [w] \Vdash \phi_1 \wedge \phi_2 = \phi$ . Right to left is similar.

Case  $\phi = \Box_i \psi$  : Left to right. Suppose  $\psi$  holds and  $M, w \Vdash \Box_i \psi$ . We need to show  $M^f, [w] \Vdash \Box_i \psi$ , this means  $\forall [v] \in W_\Sigma : [w]R_i[v] \rightarrow M^f, [v] \Vdash \Box_i \psi$ . Pick any  $[v] \in W_\Sigma$  s.t  $[w]R_i[v]$ . By condition 3, w.r.t to the modal operator, we have  $M, v \Vdash \psi$ . By induction hypothesis, we get  $M^f, [v] \Vdash \psi$ . Because  $[v]$  was arbitrary it follows that  $M^f, [w] \Vdash \Box_i \psi$ .

Right to left. Suppose  $\psi$  holds and  $M^f, [w] \Vdash \Box_i \psi$ . Pick  $v \in W$  s.t  $wR_i v$ . By condition 2, w.r.t to the modal operator, we have  $[w]R_i^f[v]$ . So,  $M^f, [v] \Vdash \psi$ . By induction hypothesis, we get  $M, v \Vdash \psi$ . Because  $v$  was arbitrary, we have  $M, w \Vdash \Box_i \psi$ . □

**Definition 2.0.7.** Let  $M = (W, R_1, \dots, R_n, V)$  be a model,  $\Sigma$  is closed under subformulas and  $W_\Sigma$  the set of equivalence classes. We define :

$$R^s = \{[w], [v] \mid \exists w' \in [w], \exists v' \in [v] : w'R_i v'\}$$

**Lemma 2.0.8.** Let  $M = (W, R_1, \dots, R_n, V)$  be a model,  $\Sigma$  is closed under subformulas and  $W_\Sigma$  the set of equivalence classes induced by  $\equiv$  and  $V^f$  the standard valuation on  $W_\Sigma$ . Then  $(W_\Sigma, R_1^s, \dots, R_n^s, V^f)$  is a filtration of  $M$  through  $\Sigma$ .

*Proof.* It suffices to show  $R_i^s$  fullfills the condition 2 and 3 w.r.t to the corresponding modal operator  $\Box_i$ . But  $R_i^s$  already satisfies condition 2. Let's check the other condition.

Let  $\Box_i \phi \in \Sigma$ ,  $[w]R_i^s[v]$  and  $M, w \Vdash \Box_i \phi$ . Because of  $[w]R_i^s[v]$  we pick a  $w' \in [w]$  and  $v' \in [v]$ . By definition, we have  $w'R_i v'$ . Because  $w' \equiv w$ , we get  $M, w' \Vdash \Box_i \phi$ . Hence,  $M, v' \Vdash \phi$  and by  $v' \equiv v$ , we get  $M, v \Vdash \phi$ .  $\square$

**Proposition 2.0.9.** *Let  $\Sigma$  be a finite subformula closed set of  $L_{\Box_1, \dots, \Box_n}$ . For any model  $M$ , if  $M^f$  is a filtration through  $\Sigma$ , then  $M^f$  contains at most  $2^n$  nodes (where  $n$  denotes the size of  $\Sigma$ ).*

*Proof.* The states of  $M^f$  are the equivalence classes in  $W_\Sigma$ . Let  $g : W_\Sigma \rightarrow P(\Sigma)$  defined by  $g([w]) = \{\phi \in \Sigma \mid M, w \Vdash \phi\}$ .  $g$  is well defined. Pick any  $u$  and  $v$  s.t  $u \equiv v$ . But then by definition of  $\equiv$ , they fullfill the same subformulas. This means  $g([v]) = g([u])$ .  $g$  is also injective. Pick any  $[u], [v] \in W_\Sigma$  s.t  $g([u]) = g([v])$ . We show  $[u] \subseteq [v]$ . The other inclusion is similar. By assumption we have  $u \equiv v$ . Pick any  $u' \in [u]$ . Then we have  $u' \equiv u \equiv v$ . Hence,  $u' \in [v]$ . At the end, this means  $M^f$  contains at most  $2^n$  nodes.  $\square$

**Theorem 2.0.10.** *Let  $\phi$  be a formula of  $L_{\Box_1, \dots, \Box_n}$ . If  $\phi$  is satisfiable, then it is satisfiable on a finite model containing at most  $2^v$  nodes, where  $v$  is the number of subformulas of  $\phi$ .*

*Proof.* Assume that  $\phi$  is satisfiable on a model on  $M$ . Take any filtration of  $M$  through the set of subformulas of  $\phi$ . By Filtration Theorem, we get that  $\phi$  is satisfied in the filtration model  $M^f$ . Furthermore, it is bounded by  $2^v$ .  $\square$

Now we define special kind of formulas called Sahlqvist formulas. Sahlqvist formulas possess important properties, which are guaranteed by the Sahlqvist Theorem. It says that, when given a normal modal logic  $K$  and a set of Sahlqvist formulas, the resulting logic is complete w.r.t to the class of frames, which satisfies the corresponding first-order formula of the Sahlqvist formulas. This also holds for multimodal logic. We will not prove it here, but we will use it later in this work.

**Definition 2.0.11.** *A modal formula  $\phi$  is positive if all variables occurs in the scope of an even number of negations. In the other hand, a formula is negative, if all variables occurs in the scope of an odd number of negations. A boxed atom is a modal formula of the form  $\Box^n p$  for some  $n \in \mathbb{N}$ , where  $p$  is a propositional variable and  $\Box^n p$  is defined as follows :  $\Box^0 p = p$ ,  $\Box^1 p = \Box p$ ,  $\Box^{n+1} p = \Box(\Box^n p)$ .*

*Furthermore, a Sahlqvist antecedent is built from  $\perp, \top$ , negative formulas and boxed atoms by applying  $\Diamond$  and  $\wedge$ . A Sahlqvist implication is a modal formula of the form  $\phi \rightarrow \psi$ , where  $\phi$  is a Sahlqvist antecedent and  $\psi$  a positive formula.*

*Now, a Sahlqvist formula is built from Sahlqvist implications by applying  $\Box$  and  $\vee$ .*

*Examples for Sahlqvist formulas:*

$$\Box \Box p \rightarrow \Box p$$

$$\begin{aligned}
& \Diamond \neg p \rightarrow p \\
& \Diamond \Box \Box \Box \Box p \rightarrow \Box \Diamond \Box p \\
& \Box \Box \Box (\Diamond \Box p \rightarrow p) \vee \Box \Box p \rightarrow \Box p
\end{aligned}$$

*Non Sahlqvist Formulas :*

$$\begin{aligned}
& \Box \Diamond p \rightarrow \Diamond \Box p \\
& \Diamond \Box p \rightarrow \Box \neg p
\end{aligned}$$

We can extend this definition for multimodal logics. We say a boxed atom can be  $\Box_i^n p$ . A Sahlqvist antecedent can also be build by applying  $\Diamond_i$ . A Sahlqvist formula can be build by Sahlqvist implications by applying additionally  $\Box_i$ .

**Definition 2.0.12.** Let  $F = (W, R_1, R_2, \dots)$  and  $F' = (W', R'_1, R'_2, \dots)$  be two frames. A bounded morphism from  $F$  to  $F'$  is a function  $f : W \rightarrow W'$  satisfying the following conditions:

$$\begin{aligned}
& \text{If } (u, v) \in R_i \text{ then } (f(u), f(v)) \in R'_i \\
& \text{If } (f(w), v') \in R'_i \text{ then } \exists v \in W \text{ s.t } (w, v) \in R_i \text{ and } f(v) = v'
\end{aligned}$$

We say  $F'$  is a bounded morphic image of  $F$ , if there is a surjective bounded morphism from  $F$  to  $F'$ .

**Proposition 2.0.13.** Let  $\phi$  be a formula in  $L_{\Box_1, \dots, \Box_n}$ ,  $F = (W, R_1, \dots, R_n)$  and  $F' = (W', R'_1, \dots, R'_n)$  be two frames and  $F$  to  $F'$  a surjective bounded morphism. Then the following holds :

$$\text{If } F \Vdash \phi \text{ then } F' \Vdash \phi$$

*Proof.* This can be shown by structural induction on the length of the formula. □

**Corollary 2.0.14.** If  $F'$  is a bounded morphic image of  $F$ , then we have  $\text{Log}(F) \subseteq \text{Log}(F')$

*Proof.* The proof is by structural induction. □

In this work, we deal with products of frames and multimodal logics of product of frames. The study of products of Kripke frames and their modal logics was first initiated by Segerberg [1] and Shehtman [2]. In the following, we will define necessary notions for such logics.

**Definition 2.0.15.** Let  $F = (W, R_1)$  and  $G = (W, R_2)$ . Then we define the Kripke product on  $W \times V$  as follows :

$$\begin{aligned}
& (w, v)R'_1(w', v') \text{ iff } wR_1w' \text{ and } v = v' \\
& (w, v)R'_2(w', v') \text{ iff } w = w' \text{ and } vR_2v'
\end{aligned}$$

$R'_1$  is also called horizontal relation and  $R'_2$  is called vertical relation.

**Definition 2.0.16.** A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ( $\frac{\phi(p_i)}{\phi(\psi)}$ ), Modus Ponens ( $\frac{\phi, \phi \rightarrow \psi}{\psi}$ ), Generalization rules ( $\frac{\phi}{\Box_i \phi}$ ) and the following axioms

$$\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$$

$K_n$  denotes the minimal normal modal logic with  $n$  modalities and  $K = K_1$ . Let  $L$  be a logic and let  $\Gamma$  be a set of formulas. Then  $L + \Gamma$  denotes the minimal logic containing  $L$  and  $\Gamma$

**Definition 2.0.17.** Let  $L_1$  and  $L_2$  be two modal logic with one modality  $\Box$ . Then the fusion of these logics are defined as follows :

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \rightarrow \Box_1)} L_{2(\Box \rightarrow \Box_2)}$$

The following logics may be important

$$D = K + \Box p \rightarrow \Diamond p$$

$$T = K + \Box p \rightarrow p$$

$$D4 = D + \Box p \rightarrow \Box \Box p$$

$$S4 = T + \Box p \rightarrow \Box \Box p$$

## 2.1 Topological semantics, product of topological spaces

Topological space opens a new area of expressiveness. It is well known that such logic is S4. It has some crucial applications for example in spatial reasoning or epistemic logic.

**Definition 2.1.1.** A topological space is a pair  $(X, \tau)$  where  $\tau$  is a collection of subsets of  $X$  (elements of  $\tau$  are also called open sets) such that :

1. the empty set  $\emptyset$  and  $X$  are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

The space is called Alexandroff, if we allow the intersection of infinite collection of open sets. A topological model is a structure  $M = (X, \tau, v)$  where  $(X, \tau)$  is a topological space and  $v$  is a valuation assigning subsets of  $X$  to propositional variables.

**Remark 2.1.2.** There is an equivalent definition for open sets. Let  $(X, \tau)$  be a topological space and  $U$  a set.  $U$  is open iff  $\forall x \in U \exists V \subseteq U : V$  is open and  $x \in V$ . This is true because, the union of open sets is an open set.

**Definition 2.1.3.** Let  $M = (X, \tau, v)$  be a topological model and  $x \in X$ . The satisfaction of a formula at the point  $x$  in  $M$  is defined inductively as follows:

$$\begin{aligned}
 M, x \models p & \quad \text{iff } x \in v(p) \\
 M, x \models \perp & \quad \text{never} \\
 M, x \models \neg \phi & \quad \text{iff } M, x \not\models \phi \\
 M, x \models \phi \vee \psi & \quad \text{iff } M, x \models \phi \text{ or } M, x \models \psi \\
 M, x \models \Box \phi & \quad \text{iff there exists } U \in \tau \text{ such that } x \in U \text{ and } \forall u \in U, M, u \models \phi
 \end{aligned}$$

**Definition 2.1.4.** Let  $A = (X, \chi)$  and  $B = (Y, v)$  be topological spaces and  $N \subseteq X \times Y$ .

$N$  is horizontally open iff  $\forall (x, y) \in N \exists U \in \chi : x \in U \text{ and } U \times \{y\} \subseteq N$ .

$N$  vertically open iff  $\forall (x, y) \in N \exists V \in v : y \in V \text{ and } \{x\} \times V \subseteq N$ .

If  $N$  is  $H$ -open and  $V$ -open, then we call it  $HV$ -open.

We denote  $\tau_1$  is the set of all  $H$ -open subsets of  $X \times Y$  and  $\tau_2$  is the set of all  $V$ -open subsets of  $X \times Y$ . We say  $\tau$  is the standard product of subsets  $X \times Y$  s.t.  $X \in \chi$  and  $Y \in v$ .

**Definition 2.1.5.** Let  $A = (X, \chi)$  and  $B = (Y, v)$  be topological spaces and  $(x, y) \in X \times Y$ . The truth in  $M = (X \times Y, \tau_1, \tau_2, \tau, v)$  is similar as in Definition 2.1.3. For  $\Box$ ,  $\Box_1$  and  $\Box_2$  we interpret this as follows :

$$\begin{aligned}
 M, (x, y) \models \Box_1 \phi & \quad \text{iff } \exists U \in \tau_1 : (x, y) \in U \text{ and } \forall (x', y') \in U : (x', y') \models \phi \\
 M, (x, y) \models \Box_2 \phi & \quad \text{iff } \exists V \in \tau_2 : (x, y) \in V \text{ and } \forall (x', y') \in V : (x', y') \models \phi \\
 M, (x, y) \models \Box \phi & \quad \text{iff } \exists U \in \chi \exists V \in v : (x, y) \in U \times V \text{ and } \forall (x', y') \in U \times V : (x', y') \models \phi
 \end{aligned}$$

**Definition 2.1.6.** For two unimodal logics  $L_1$  and  $L_2$  we define the full  $t$ -product of them as follows :

$$L_1 \times_t^+ L_2 = \text{Log}(\{X \times Y \mid X \models L_1 \text{ and } Y \models L_2\})$$

where  $X, Y$  are topological spaces with horizontal, vertical and standard topology.

**Definition 2.1.7.** A topo-bisimulation between two topological models  $M = (X, \tau, v)$  and  $M' = (X', \tau', v')$  is a nonempty relation  $\Leftrightarrow \subseteq X \times X'$  s.t. if  $x \Leftrightarrow x'$  then

1. Base:  $x \in V(p)$  iff  $x' \in v'(p)$  for any propositional variable  $p$
2. Forth:  $x \in U \in \tau$  implies that there exists  $U' \in \tau'$  s.t.  $x' \in U'$  and for every  $y' \in U'$  there is  $y \in U$  with  $y \Leftrightarrow y'$
3. Back:  $x' \in U' \in \tau'$  implies that there exists  $U \in \tau$  s.t.  $x \in U$  and for every  $y \in U$  there is  $y' \in U'$  with  $y \Leftrightarrow y'$

**Proposition 2.1.8.** Let  $M = (X, \tau, v)$  and  $M' = (X', \tau', v')$ . Assume  $w \in M'$  and  $w' \in M'$  are topo-bisimilar points. Then for each formula  $\phi$  we have

$$M, w \models \phi \text{ iff } M', w' \models \phi$$

The proof is by induction.

**Definition 2.1.9.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a function. We call  $f$  continuous if for each open set  $U \subseteq Y$  the set  $f^{-1}(U)$  is open in  $X$ . We say  $f$  is open if for each open set  $V \subseteq X$  the set  $f[V]$  is open in  $Y$ .

## 2.2 Product and semantics of neighborhood frames

Neighborhood frames are more flexible than topological space. Its logic is not  $S4$  in general. It is well known, that neighborhood frames fully captures the semantics of Kripke. Moreover, neighborhood frames are expressive enough to characterize non-normal modal logics, which makes them a powerful tool.

**Definition 2.2.1.** Let  $X$  be a non-empty set. A function  $\tau : X \rightarrow 2^{2^X}$  is called a neighbourhood function. A pair  $F = (X, \tau)$  is called a neighbourhood frame (or  $n$ -frame). A model based on  $F$  is a tuple  $(X, \tau, v)$ , where  $v$  assigns a subset of  $X$  to a variable

**Definition 2.2.2.** Let  $M = (X, \tau, v)$  be a neighbourhood model and  $x \in X$ . The truth of a formula is defined inductively as follows :

$$\begin{aligned} M, x \Vdash p & \text{ iff } x \in V(p) \\ M, x \Vdash \perp & \text{ never} \\ M, x \Vdash \neg \phi & \text{ iff } M, x \nVdash \phi \\ M, x \Vdash \phi \vee \psi & \text{ iff } M, x \Vdash \phi \vee M, x \Vdash \psi \\ M, x \Vdash \Box \phi & \text{ iff } \exists V \in N(x) \forall y \in V : M, y \Vdash \phi \end{aligned}$$



A formula is valid in a  $n$ -model  $M$  if it is valid at all points of  $M$  ( $M \models \phi$ ). Formula is valid in a  $n$ -frame  $F$  if it is valid in all models based on  $F$  (notation  $F \models \phi$ ). We write  $F \models L$  if for any  $\phi \in L$ ,  $F \models \phi$ . Logic of a class of  $n$ -frames  $C$  as  $\text{Log}(C) = \{\phi \mid F \models \phi \text{ for some } F \in C\}$ . We define  $nV(L) = \{F \mid F \text{ is an } n\text{-frame and } F \models \phi\}$ .

**Definition 2.2.3.** Let  $X$  be a non-empty set and  $\tau$  neighborhood function. We call  $\tau$  is a filter if for each  $x \in X$  the collection  $\tau(x)$  satisfies the following conditions :

1.  $\emptyset \notin \tau(x)$
2. If  $U \in \tau(x)$  and  $U \subseteq V$  then  $V \in \tau(x)$  (upward closed)
3. If  $U, V \in \tau(x)$ , then  $U \cap V \in \tau(x)$

**Definition 2.2.4.** Let  $X = (X, \tau_1, \dots)$  and  $Y = (Y, \sigma_1, \dots)$  be  $n$ -frames. Then the function  $f: X \rightarrow Y$  is called bounded morphism if

1.  $f$  is surjective
2.  $\forall x \in X \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3.  $\forall x \in X \forall V \in \sigma_i(f(x)) \exists U \in \tau_i(x) : f(U) \subseteq V$

**Corollary 2.2.5.** Let  $X = (X, \tau_1, \dots)$  and  $Y = (Y, \sigma_1, \dots)$  be  $n$ -frames and  $f: X \rightarrow Y$  a bounded morphism. Then

$$\text{Log}(X) \subseteq \text{Log}(Y)$$

.

*Proof.* The proof is by structural induction. □

**Definition 2.2.6.** Let  $X = (X, \tau_1)$  and  $Y = (Y, \tau_2)$  be two  $n$ -frames. Then the product of these two frames is an  $n$ -2-frame and is defined as follows :

$$\begin{aligned} X \times_n Y &= (X \times Y, \tau'_1, \tau'_2) \\ \tau'_1(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\} \\ \tau'_2(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\} \end{aligned}$$

Additionally, we say the full product of  $n$ -frames  $X \times_n^+ Y$  is :

$$\begin{aligned} X \times_n^+ Y &= (X \times Y, \tau'_1, \tau'_2, \tau) \text{ where} \\ \tau(x, y) &= \{U \subseteq X \times Y \mid \exists W \in \tau_1(x) \exists V \in \tau_2(y) : W \times V \subseteq U\} \end{aligned}$$

**Definition 2.2.7.** For two unimodal logics  $L_1$  and  $L_2$  we define the  $n$ -product of them as follows :

$$L_1 \times_n L_2 = \text{Log}(\{X \times Y \mid X \in nV(L_1) \text{ and } Y \in nV(L_2)\})$$

In similar way, we can define  $L_1 \times_n^+ L_2$  with three topologies.

**Proposition 2.2.8.** ([?]). *For two unimodal logics  $L_1$  and  $L_2$  it holds :*

$$L_1 \otimes L_2 \subseteq L_1 \times_n L_2$$

After introducing the necessary notations, we can now formally state our main research question as follows: Does the following equivalence hold?

$$T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p = T \times_t^+ T$$

We will answer this in the following chapters. From now on, we also abbreviate  $T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$  as  $TNL$ .

### 3 Completeness result for $TNL$

We introduce a frame called  $T_{\omega,\omega,\omega[rn]}$ . We will show  $TNL$  is sound and complete w.r.t  $T_{\omega,\omega,\omega[rn]}$ . The idea is to pick a class of frame  $C$  s.t  $Log(C) = Log(T_{\omega,\omega,\omega[rn]})$  and then show the class has FMP. In the end, we will use an unravelling technique to show completeness.

**Definition 3.0.1.** *We say for a modal logic  $\Lambda$  has the finite model property (FMP) if for every formula  $\phi$  that is not provable in  $\Lambda$ , is falsifiable in a finite model.*

**Proposition 3.0.2.** *The logic  $T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$  has FMP.*

*Proof.* The idea is to pick a class of frame  $\mathcal{F}$ , where  $TNL$  is sound and complete with respect to and then show by filtration the FMP.

Let  $\mathcal{C} = \{F \mid F \models TNL\}$ . Obviously,  $TNL$  is sound w.r.t  $\mathcal{C}$ . Completeness can be shown by using the Sahlqvist Theorem. We remember  $T = K + \Box_i p \rightarrow p$  for  $i \in \{1, 2, \epsilon\}$ . These axioms and  $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$  are Sahlqvist formulas. By Sahlqvist, we have  $TNL$  is complete w.r.t  $\{F \mid F \models \forall x R_i(x, x) \text{ and } F \models \forall x \forall y (R_1(x, y) \vee R_2(x, y)) \rightarrow R(x, y)\}$ .

Now assume a formula  $\phi$  is not derivable from  $TNL$ . By completeness we get  $\phi$  is falsifiable in a model  $M$  and a world  $w$ . Hence,  $M, w \models \neg\phi$ . Now we build the set  $\text{subf}(\neg\phi)$  which denotes the closed subformulas set of  $\neg\phi$ . Let  $M^s = (W_\Sigma, R^s, R_1^s, R_2^s, V^s)$  be the smallest filtration of  $M$  through  $\text{subf}(\neg\phi)$  and  $V^s$  is the standard valuation on  $W_\Sigma$ . By filtration Theorem, they preserve truth. It remains to show  $F^s = (W_\Sigma, R^s, R_1^s, R_2^s) \in \mathcal{C}$ . For that, every point must have an edge to itself in every relation and it must hold that  $R_1^s, R_2^s \subseteq R^s$ . For the first one, we show this for  $R^s$  because the rest is similar. Pick  $w' \in [w] \in W_\Sigma$ . Because  $M$  is based on a frame  $F \in \mathcal{C}$ , we have  $w' R w'$ . By the definition of smallest filtration, we have that  $[w'] R^s [w']$ . For the second one, we show only for  $R_1^s$  because it is the same for  $R_2^s$ . Pick  $[w], [v]$  s.t  $[w] R_1^s [v]$ . By definition of smallest filtration there are points  $w' \in [w]$  and  $v' \in [v]$  s.t  $w' R_1 v'$ . Furthermore, it holds  $R_1 \subseteq R$ , so  $w' R v'$ . By smallest filtration, we get  $[w'] R^s [v']$ . Because  $w' \equiv w$  and  $v' \equiv v$ , we have  $[w'] = [w]$  and  $[v'] = [v]$ . It follows  $[w] R^s [v]$ . □

**Definition 3.0.3.** *Let  $T_{\omega[rn]}$  ( $r = \text{reflexive}$ ,  $n = \text{non-transitive}$ ) denote the infinite branching and infinite depth tree, which is reflexive and non-transitive. Formally the tree can be defined as :  $T_{\omega[rn]} = (W, R)$  where  $W = \mathbb{N}^*$  and  $s R t$  iff  $\exists u \in \mathbb{N} \cup \{\epsilon\} : s \cdot u = t$  (" $\cdot$ " is the concatenation operator)*

*The  $T_{\omega,\omega,\omega[rn]}$  tree has three relations with infinite branching and infinite depth and we have  $R_1, R_2 \subseteq R$ . Before characterizing it, we say  $\mathbb{N}_1^*$  is the set of finite number combinations which has a subscript "1" to denote that these numbers relate to  $R_1$  (examples :  $2_1, 4231123_1, 32_1\epsilon 45_1\epsilon 9_1 = 32459_1$ ).*

Now let  $T_{\omega, \omega, \omega[rn]} = (W, R, R_1, R_2)$  where  $W = (\mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2)^*$ ,

$$sRt \text{ iff } \exists u \in \mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_1t \text{ iff } \exists u \in \mathbb{N}_1 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_2t \text{ iff } \exists u \in \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

where  $s, t \in W$ . Again, the  $\cdot$  operator acts here as a concatenation operator.

**Proposition 3.0.4.** *T is sound and complete w.r.t  $T_{\omega[rn]}$ .*

*Proof.* Sound is clear. For completeness, we use the well known fact that T has FMP. This means,  $T = \text{Log}\{F \mid F \models T\}$  where  $F$  is a finite frame. We can pick such a finite frame  $F$  and it suffices to find a surjective bounded morphism  $f$  from  $T_w$  to  $F$ . This would imply  $\text{Log}(T_{w[rn]}) \subseteq T$ .

Now, let  $F = (W', R')$  be such a finite rooted frame with root  $w$ . We define inductively an assignment of nodes of  $F$  to the nodes of  $T_w$ . For the base case, we assign  $w$  to the root of  $T_w$ . The induction step looks like the following : Assume a point  $x \in T_w$  has been assigned to a point  $u \in F$  but the successors of  $x$  has no assignment. Let  $s_1, s_2, \dots, s_k$  be successors of  $u$  ( $k$  denotes amount of successors and  $k \geq 1$  because reflexivity guarantees us at least one successor). For  $n \geq 1, n \in \mathbb{N}, i \in \{1, \dots, k\}$  we assign  $s_i$  to the  $(n * i)$ th-successor of  $x$ . This means we are assigning the successors alternatingly.

Now we check for  $f$  the conditions of bounded morphism. First condition : Let  $x, y \in T_w$  s.t  $xRy$  and  $f(x) = s$ . But then,  $y$  will be assigned to a successor point of  $s$ . Hence,  $f(x)R'f(y)$ . Second condition : Suppose  $f(x)R't$  and  $f(x) = s$ . Since  $t$  is a successor of  $s$  and  $f(x) = s$ , then a successor of  $x$ , say  $y$ , gets the assignment  $t$ .

□

**Proposition 3.0.5.** *TNL is sound and complete w.r.t  $T_{\omega, \omega, \omega[rn]}$ .*

*Proof.* For soundness, we have that  $T_{\omega, \omega, \omega[rn]} \models \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ , because by definition we have  $R_1, R_2 \subseteq R$ . The rest is clear. For completeness, we use the fact TNL has FMP. Let  $F = (W', R', R'_1, R'_2)$  be a finite rooted frame with root  $w$  and  $F \models \text{TNL}$ . We define inductively an assignment similar to 2.17. We assign  $w$  to the root of  $T_{\omega, \omega, \omega[rn]}$ . For induction step we start by only assigning points from  $R_1$  to  $R'_1$  and  $R_2$  to  $R'_2$ . After that, the remaining points will be assigned to the points in  $R'$ . The procedure works similar as described previously.

We will only check the conditions for  $R$  and  $R'$ . For  $R_1, R'_1$  and  $R_2, R'_2$ , we can argue as before. We pick any  $xRy$  with  $f(x) = s$ . If we also have  $xR_1y$  or  $xR_2y$ , then it follows  $f(x)R'f(y)$  because  $R_1, R_2 \subseteq R$  and  $R'_1, R'_2 \subseteq R'$ . Else, the successor of  $s$  was assigned to  $y$ , so  $f(x)R'f(y)$ .

Let  $f(x)R't$  and  $f(x) = s$ . If we have  $f(x)R'_1t$  or  $f(x)R'_2t$ , then the second condition follows because  $R_1, R_2 \subseteq R$ . Else, we have that  $t$  is a successor of  $f(x) = s$ , and a successor of  $x$  was assigned to  $t$ .

□

## 4 Logic of product of topological spaces with three topologies

In this section, we will reprove the logic of the product of two topological spaces with horizontal, vertical and classic product topology is  $S4 \otimes S4 \otimes S4 + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$  (we abbreviate the logic  $S4 \otimes S4 \otimes \dots$  as TPL). First we will show a version for one modality because it is easier to understand and the proof idea is based on this version.

**Definition 4.0.1.** *Let  $T_2$  be the infinite binary tree with reflexive and transitive descendant relation.*

*Formally it is defined as follows :  $T_2 = (W, R)$  where  $W = \{0, 1\}^*$  and  $sRt$  iff  $\exists u \in W^* : s \cdot u = t$ .*

*The  $T_{6,2,2}$  tree is the infinite six branching tree, where all nodes of  $T_{6,2,2}$  is  $R$ -related, the first two  $R_1$ -related and the last two  $R_2$ -related. Formally we can define this tree as follows :  $T_{6,2,2} = (W, R, R_1, R_2)$ , where  $W = \{0, 1, 2, 3, 4, 5\}^*$ ,*

$$sRt \text{ iff } \exists u \in \{0, 1, 2, 3, 4, 5\}^* : s \cdot t = u$$

$$sR_1t \text{ iff } \exists u \in \{0, 1\}^* : s \cdot t = u$$

$$sR_2t \text{ iff } \exists u \in \{5, 6\}^* : s \cdot t = u$$

*where  $s, t \in W$  and  $\cdot$  is the concatenation-operator.*

**Theorem 4.0.2.**  *$S4$  is complete with respect to  $T_2$ .*

*Proof.* The idea is to use the fact  $S4$  has finite model property. By that we can pick a finite rooted  $S4$ -frame and then show that this frame is the bounded morphic image of  $T_2$ . For details see [Goldblatt].  $\square$

**Theorem 4.0.3.** *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to  $\mathbb{Q}$ .*

*Proof.* For a proof see e.g [?, Page 217, Theorem 2].  $\square$

Our strategy is as follows. We use completeness of  $S4$  w.r.t.  $T_2$ , view  $T_2$  as an Alexandroff space, define a dense subset  $X$  of  $\mathbb{Q}$  without endpoints, and then establish a topo-bisimulation between  $X$  and  $T_2$ . This will allow us to transfer counterexamples from  $T_2$  to  $X$ , which by Cantor's theorem is order-isomorphic and hence homeomorphic to  $\mathbb{Q}$ .

Now let us define  $X$  as  $X = \bigcup_{n \in \mathbb{N}} X_n$  where  $X_0 = \{0\}$  and

$$X_{n+1} = X_n \cup \left\{ x - \frac{1}{3^n}, x + \frac{1}{3^n} \mid x \in X_n \right\}$$

**Lemma 4.0.4.** *For  $n > 0$  and  $x, y \in X_n, x \neq y$  we have :  $|x - y| \geq \frac{1}{3^{n-1}}$*

*Proof.* The proof is by induction on  $n$ . For the base case, if  $n = 1$ , then  $X = 0, 1, -1$ . For induction step, assume  $u, v \in X_{n-1}$  with  $u \neq v$ . By induction hypothesis, it holds  $|u - v| \geq \frac{1}{3^{n-2}}$ . Suppose we pick  $x = u + \frac{1}{3^{n-1}}$  and  $y = v - \frac{1}{3^{n-1}}$ . Hence,  $x, y \in X_n$ . But then,  $|x - y| = |(u + \frac{1}{3^{n-1}}) - (v - \frac{1}{3^{n-1}})| = |u - v + \frac{2}{3^{n-1}}| \geq \frac{1}{3^{n-1}}$ . If  $u - v$  is positive, then the inequality follows immediately. Now assume  $u - v$  is negative. By induction hypothesis we get  $u - v \leq -\frac{1}{3^{n-2}}$ . So the worst case is, when  $u - v$  gets  $-\frac{1}{3^{n-2}}$ . But then,  $|\frac{1}{3^{n-2}} - \frac{2}{3^{n-1}}| = \frac{1}{3^{n-1}}$ .  $\square$

It follows from Lemma 4.0.4 that  $(X, <)$  is a countable, dense, and linear ordered without endpoints. By Cantor's Theorem,  $X$  is homeomorphic to  $\mathbb{Q}$ . It also follows that for each  $x \in X$  with  $x \neq 0$  there exists an index  $n_x$  with  $x \in X_{n_x}$  but  $x \notin X_{n_x-1}$ . But also, we can find a unique  $y \in X_{n_x-1}$  with  $x = y - \frac{1}{3^{n_x-1}}$  or  $x = y + \frac{1}{3^{n_x-1}}$ . Furthermore, the basis for the open intervals for any  $x \in X$  will be  $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$  where  $n_x$  is the index where  $x$  gets produced.

Now we define  $f$  from  $X$  onto  $T_2$  by recursion. If  $x = 0$ , then we let  $f(0)$  be the root  $r$  of  $T_2$ . If  $x \neq 0$ , then  $x \in X_{n_x} - X_{n_x-1}$  and we let

$$f(x) = \begin{cases} \text{the left successor of } f(y) & \text{if } x = y - \frac{1}{3^{n_x-1}} \\ \text{the right successor of } f(y) & \text{if } x = y + \frac{1}{3^{n_x-1}} \end{cases}$$

**Proposition 4.0.5.** *We claim,  $f$  is open and continuous.*

*Proof.* We define the basis for the Alexandroff topology on  $T_2$  as  $\{B_t\}_{t \in T_2}$  where  $B_t = \{s \in T_2 \mid tRs\}$ . To show  $f$  is open, we pick an  $X$ -interval  $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$  and show  $f((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})) = B_{f(x)}$ .

$\subseteq$  : If we pick  $x \in ((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}))$ , then obviously  $f(x)Rf(x)$ . Assume we pick a point  $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$  but  $y \neq x$ . Then we have  $n_y > n_x$  and hence  $f(x)Rf(y)$ .

$\supseteq$  : Assume  $f(x)Rt$ . We will show by induction on the length  $k$  between  $f(x)$  and  $t$  s.t. there exists  $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$  s.t.  $f(y) = t$ . Base case : If  $k = 0$ , then we have  $t = f(x)$  and we can pick  $y = x$ . For induction step, assume it holds for  $f(x)Rt'$  with length  $n$ ,  $f(y) = t'$  and  $t'Rt$ . For  $y$ , there must exist an index  $n_y > n_x$  s.t.  $y \in X_{n_y} - X_{n_y-1}$ . Now let  $y' = y - \frac{1}{3^{n_y}}$  or  $y' = y + \frac{1}{3^{n_y}}$ . It holds that,  $n'_y > n_y > n_x$ , so  $y' \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ . By definition of  $f$ ,  $f(y')$  is either left or right successor of  $f(y)$  and by  $f(y) = t'Rt$ , we get  $f(y') = t$ . It follows  $f$  is open.

To show  $f$  is continuous, it suffices to show that for each  $t \in T_2$ , the  $f$ -inverse image of  $B_t$  is open. Let  $x \in f^{-1}(B_t)$ . Then  $tRf(x)$ . As argued before, we can find an open interval in  $X$  s.t.  $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)}$ . By  $tRf(x)$ , we get  $B_{f(x)} \subseteq B_t$ . Hence, we found an open interval  $I = (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ , s.t.  $I = f^{-1}(B_{f(x)}) \subseteq f^{-1}(B_t)$ . This holds because we can pick  $y \in I$ . By applying  $f$ , we get  $f(y) \in B_{f(x)} \subseteq B_t$ . But then  $y \in f^{-1}(B_t)$ . By that fact, we can follow with Remark 2.2.8, that  $f^{-1}(B_t)$  is open and hence  $f$  is continuous.  $\square$

To complete the proof, if  $S4 \not\models \phi$ , then by Theorem 4.0.2, there is a valuation  $v$  on  $T_2$  s.t.  $(T_2, v), r \not\models \phi$ . We define a valuation  $\xi$  on  $X$  by  $\xi(p) = f^{-1}(v(p))$ . Since  $f$  is continuous and open,  $f(0) = r$  and by the choice of the valuation, we have that  $0$  and  $r$  are topo-bisimilar. By Proposition 2.2.7 it follows  $(X, \xi), 0 \not\models \phi$ . Because  $X$  is homeomorphic to  $\mathbb{Q}$ , we obtain  $\phi$  refutable on  $\mathbb{Q}$ .

**Definition 4.0.6.** Let  $\mathcal{L}_{\Box, \Box_1, \Box_2}$  be a modal language with three modal operators  $\Box, \Box_1$  and  $\Box_2$ . We define the topological product logic  $TPL$  as the least set of formulas in  $\mathcal{L}_{\Box, \Box_1, \Box_2}$  containing all axioms  $S4 \otimes S4 \otimes S4 + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ , and closed under modus ponens, substitution and  $\Box$ -,  $\Box_1$ - and  $\Box_2$ -necessitation.

**Theorem 4.0.7.**  $TPL$  is complete w.r.t.  $T_{6,2,2}$ .

*Proof.* A generalization of Theorem 4.0.2 can be seen on [?].  $\square$

Now we are ready to show  $TPL$  is complete w.r.t.  $\mathbb{Q} \times \mathbb{Q}$  with three topologies. The idea is similar as previous shown. We view  $T_{6,2,2}$  as equipped with three Alexandroff topologies defined from  $R, R_1$  and  $R_2$ . It suffices to show that there exists a 3-topo-bisimulation between  $T_{6,2,2}$  and  $X \times X = X'$  where each  $X$  is as defined in Theorem 4.0.3.

**Theorem 4.0.8.**  $TPL$  is complete w.r.t.  $\mathbb{Q} \times \mathbb{Q}$ .

*Proof.* We define  $h$  from  $X \times X$  onto  $T_{6,2,2}$  by recursion following the inductive definition of  $X$ . If  $(x, y) = (0, 0)$ , then  $h(0, 0)$  is the root  $r$  of  $T_{6,2,2}$ . If  $(x, y) \neq (0, 0)$ , then as argued in Lemma 4.0.4, there exists  $n_{(x,y)}$  with  $(x, y) \in X'_{n_{(x,y)}} - X'_{n_{(x,y)}-1}$  and that there exists a unique  $(u, v) \in X'_{n_{(x,y)}-1}$  such that  $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v)$  or  $(x, y) = (u, v \pm \frac{1}{3^{n_{(x,y)}-1}})$  or  $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v \pm \frac{1}{3^{n_{(x,y)}-1}})$ .

As visualized in the picture, we can see as an example  $X'_0, X'_1$  and an excerpt of  $X'_2$ . For clarity, we illustrated some points of  $X'_1$  as a line. We clearly can see, that any point has his own "square", where the points within it can be only build from his center point. This is because, the square size gets exponentially smaller with each iteration, due to its construction. That means, in each iteration they can not intersect.

Now we define  $h$  as

$$h(x, y) = \begin{cases} \text{The left } R_1\text{-successor of } h(u, v) & \text{if } (x, y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{The right } R_1\text{-successor of } h(u, v) & \text{if } (x, y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{The left } R_2\text{-successor of } h(u, v) & \text{if } (x, y) = (u, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The right } R_2\text{-successor of } h(u, v) & \text{if } (x, y) = (u, v + \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The first remaining successor of } h(u, v) & \text{if } (x, y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v + \frac{1}{3^{n_{(x,y)}-1}}) \\ & \text{or } (x, y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The last remaining successor of } h(u, v) & \text{if } (x, y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ & \text{or } (x, y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v + \frac{1}{3^{n_{(x,y)}-1}}) \end{cases}$$

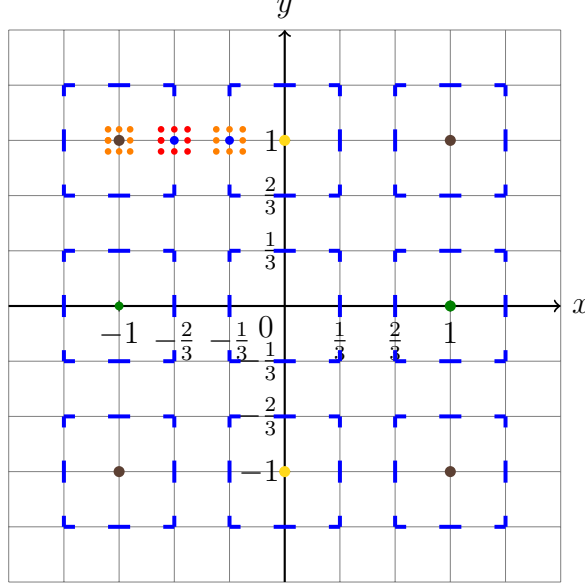


Figure 1:

It is still left to prove, that  $h$  is open and continuous w.r.t. all three topologies. We will show it for  $\tau_1$  and  $\tau$ . For  $\tau_2$  it is similar as for  $\tau_1$ .

First we observe the basis for  $\tau_1$  is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \mid (x, y) \in X \times X\}$$

and for  $\tau$  is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}}) \mid (x, y) \in X \times X\}$$

We also observe that a basis for the topology on  $T_{6,2,2}$  from  $R_1$  is  $\{B_t^1\}_{t \in T_{6,2,2}}$  where  $B_t^1 = \{s \in T_{6,2,2} \mid tR_1s\}$  and from  $R$  is  $\{B_t\}_{t \in T_{6,2,2}}$  where  $B_t = \{s \in T_{6,2,2} \mid tRs\}$ . The following arguments are carried over from Proposition 4.0.5.

Now, to see that  $g$  is open w.r.t.  $\tau_1$ , we pick an open set  $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \in \tau_1$ . In similar way, we can show  $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B_{g(x,y)}^1$ . Thus  $g$  is open. To show  $g$  is continuous, it suffices to show that for each  $t \in T_{6,2,2}$ , the  $g$ -inverse image of  $B_t^1$  belongs to  $\tau_1$ . Let  $(x, y) \in g^{-1}(B_t^1)$ . Then  $tR_1g(x, y)$ . Hence, it holds for  $(x, y)$  that  $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B_{g(x,y)}^1$ . By  $tR_1g(x, y)$ , we get  $B_{g(x,y)}^1 \subseteq B_t^1$ . Thus, we found a neighborhood  $U$  of  $(x, y)$  s.t.  $U = ((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) \subseteq g^{-1}(B_t^1)$ , implying  $g$  is continuous.

To see  $g$  is open w.r.t.  $\tau$ , we pick an open set  $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})$ . A similar argument as in Proposition 4.0.5, yields us  $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{g(x,y)}$ .



To show  $g$  is continuous, it suffices to show for each  $t \in T_{6,2,2}$  that  $g^{-1}(B_t) \in \tau$ . Let  $(x, y) \in g^{-1}(B_t)$ . Then  $tRg(x, y)$ . But then, it holds that :  $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{g(x,y)} \subseteq B_t$ . Thus, we found a  $U \in \tau$  s.t.  $U = (x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}}) \subseteq g^{-1}(B_t)$ .  $\square$

## 5 Products of neighbourhood frames, completeness results

We now turn our focus to neighbourhood semantics, and in particular, we revisit the main theorem from the paper "Modal Logic of Products of Neighbourhood Frames". This work established that, for any  $L_1, L_2 \in \{D, D4, T, S4\}$  we have  $L_1 \otimes L_2 = L_1 \times_n L_2$ . Our goal is to reprove this theorem within our framework and then adapt its key ideas to prove our theorem.

### 5.1 Kripke tree frames, natural neighborhood version and their logic

We introduce some trees as Kripke frames which are essential for the proof.

**Definition 5.1.1.** *Let  $A$  be a nonempty set.*

$$A^* = \{a_1 \dots a_k \mid a_i \in A\}$$

*is the set of all finite sequences of elements from  $A$ , including the empty sequence  $\Lambda$ . Elements from  $A^*$  will be denoted as  $\vec{a}$ . The length of a sequence  $\vec{a} = a_1 \dots a_k$  is  $k$  (also  $l(\vec{a}) = k$ ) and the length of  $\Lambda$  is 0 ( $l(\Lambda) = 0$ ). Concatenation is denoted by " $\cdot$ " :  $(a_1 \dots a_k) \cdot (b_1 \dots b_l) = \vec{a} \cdot \vec{b} = a_1 \dots a_k b_1 \dots b_l$ .*

**Definition 5.1.2.** *Let  $A$  be a nonempty set. We define an infinite frame  $F_{in}[A] = (A^*, R)$  s.t for  $\vec{a}, \vec{b} \in A^*$*

$$\vec{a}R\vec{b} \Leftrightarrow \exists x \in A (\vec{b} = \vec{a} \cdot x)$$

*Furthermore we define :*

$$F_{rn}[A] = (A^*, R^r), \text{ where } R^r = R \cup Id \text{ (reflexive closure)}$$

$$F_{it}[A] = (A^*, R^*), \text{ where } R^* = \bigcup_{i=1}^{\infty} R^i \text{ (transitive closure)}$$

$$F_{rt}[A] = (A^*, R^{r*})$$

*where "t" stands for transitive, "n" for non-transitive, "r" for reflexive and "i" for ir-reflexive.*

*For now, we will use the following notion to generalize :  $F_{\xi\eta}$  where  $\xi \in \{i, r\}$  and  $\eta \in \{t, n\}$*

**Proposition 5.1.3.** *Let  $F = F_{\xi\eta}[A] = (A^*, R)$  then*

$$\vec{a}R(\vec{a} \cdot \vec{c}) \Leftrightarrow \Lambda R\vec{c}$$

**Definition 5.1.4.** *Let  $F_1 = F_{\xi_1\eta_1}[A] = (A^*, R_1)$  and  $F_2 = F_{\xi_2\eta_2}[B] = (B^*, R_2)$ , where  $\xi_1, \xi_2 \in \{i, r\}$  and  $\eta_1, \eta_2 \in \{t, n\}$ . Furthermore, we assume  $A = \{a_1, a_2, \dots\}$  and  $B = \{b_1, b_2, \dots\}$  with  $A \cap B = \emptyset$ . Then we define the frame  $F_1 \otimes F_2 = (W, R'_1, R'_2)$  as follows :*

$$W = (A \cup B)^*$$

$$\vec{x}R'_1\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in A^* \text{ such that } \Lambda R_1\vec{z}$$

$$\vec{x}R'_2\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in B^* \text{ such that } \Lambda R_2\vec{z}$$

**Proposition 5.1.5** ( $[?], [?]$ ). *Let  $F_1$  and  $F_2$  be as in Definition 2.1.7. Then*

$$\text{Log}(F_1 \otimes F_2) = \text{Log}(F_1) \otimes \text{Log}(F_2)$$

**Proposition 5.1.6.** *Let  $F_{in} = F_{in}[\mathbb{N}]$ ,  $F_{rn} = F_{rn}[\mathbb{N}]$ ,  $F_{it} = F_{it}[\mathbb{N}]$  and  $F_{rt} = F_{rt}[\mathbb{N}]$ . Then the following holds:*

$$\text{Log}(F_{in}) = D$$

$$\text{Log}(F_{rn}) = T$$

$$\text{Log}(F_{it}) = D4$$

$$\text{Log}(F_{rt}) = S4$$

**Definition 5.1.7.** *Let  $F = (W, R)$  be a Kripke frame. We define an  $n$ -frame  $N(F) = (W, \tau)$  as follows. For any  $w \in W$  we have :*

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

**Lemma 5.1.8.** *Let  $F = (W, R)$  be a Kripke frame. Then*

$$\text{Log}(F) = \text{Log}(N(F))$$

*Proof.* The proof is by structural induction. □

## 5.2 Main Construction

In the following, we construct a useful neighborhood frame called  $N_\omega[F]$  based on a frame  $F$  where  $F$  is a tree frame. We will show that the construction has the same logic as the original frame. At the end, we construct a bounded morphism from the product of  $N_\omega[F]$  frames to the fusion of frames  $F$ .

**Definition 5.2.1.** Let  $F = (A^*, R) = F_{\xi\eta}[A]$  and  $0 \notin A$ . We define "pseudo-infinite" sequences

$$X = \{a_1 a_2 a_3 \dots \mid a_i \in A \cup \{0\} \text{ and } \exists N \forall k \geq N : a_k = 0\}$$

Furthermore, we define  $f_F : X \rightarrow A^*$  to be the function, that deletes all zeros.

*Example :* Say  $12034002340^\omega \in X$  ( $0^\omega$  denotes infinitely many zeros). Then  $f_F(12034002340^\omega) = 1234234$

**Definition 5.2.2.** Let  $F = (A^*, R) = F_{\xi\eta}[A]$  and  $0 \notin A$ . Assume the function  $f_F$  and the set  $X$  as defined before. For  $\alpha \in X$  such that  $\alpha = a_1 a_2 \dots$  we define

$$st(\alpha) = \min\{N \mid \forall k \geq N : a_k = 0\}$$

$$\alpha \upharpoonright_k = a_1 a_2 \dots a_k$$

$$U_k(\alpha) = \{\beta \mid f_F(\alpha) R f_F(\beta) \text{ and } \alpha \upharpoonright_m = \beta \upharpoonright_m, \text{ where } m = \max((k, st(\alpha)))\}$$

*Remark :* Let  $\alpha \in X$  with  $st(\alpha) = n$ . Then we have that  $U_n(\alpha) = U_j(\alpha)$  for any  $j \leq n$

**Lemma 5.2.3.**  $U_k(\alpha) \subseteq U_m(\alpha)$ , whenever  $k \geq m$ .

*Proof.* Let  $\beta \in U_k(\alpha)$ . Since  $\alpha \upharpoonright_k = \beta \upharpoonright_k$  and  $k \geq m$ , we have  $\alpha \upharpoonright_m = \beta \upharpoonright_m$ . It follows,  $\beta \in U_m(\alpha)$ .  $\square$

**Definition 5.2.4.** Due to Lemma 5.2.3 the sets  $U_n(\alpha)$  forms a filter base. So we can define :

$$\tau(\alpha) \text{ is a filter with base } \{U_n(\alpha) \mid n \in \mathbb{N}\}$$

$$N_\omega = (X, \tau) \text{ is the } n\text{-frame based on } F$$

**Lemma 5.2.5.** Let  $F = (A^*, R) = F_{\xi\eta}[A]$ . Based on that, let  $N_\omega(F) = (X, \tau)$ ,  $N(F) = (A^*, \sigma)$  and  $f_F : N_\omega(F) \rightarrow N(F)$ . Then for any  $m \in \mathbb{N}$  and  $x \in X$  with  $x = a_1 a_2 \dots$  we have

$$f_F(U_m(x)) = R(f_F(x))$$

*Proof.*  $\subseteq$  : Let  $f_F(\alpha) \in f_F(U_m(x))$  with  $\alpha \in U_m(x)$ . By definition of  $U_m(x)$ , we get  $f_F(\alpha) \in R(f_F(x))$ .

For the other direction, we pick  $\vec{a} \in R(f_F(x))$ . We have to find  $\beta \in U_m(x)$  s.t  $f_F(\beta) = \vec{a}$ . We assume  $R$  is irreflexive and non-transitive. The other cases are similar.

Because  $\vec{a} \in R(f_F(x))$ , there must exist  $c \in A$  such that  $\vec{a} = f_F(x) * c$ . We construct  $\beta = x \mid_m \cdot c 0^\omega$ . Hence,  $f(\beta) = f_F(x \mid_m) * f_F(c)$  and because  $0 \notin A$  we get  $f_F(x \mid_m) * c = \vec{a}$ .  $\square$

**Lemma 5.2.6.** *Let  $F = (A^*, R) = F_{\xi\eta}[A]$ . Then  $f_F : N_\omega(F) \rightarrow N(F)$  is a bounded morphism.*

*Proof.* From now on this proof we will omit the subindex in  $f_F$ . Let  $N_\omega(F) = (X, \tau)$  and  $N(F) = (A^*, \sigma)$ .

For surjectivity, we pick any  $\vec{x} \in A^*$ . But then,  $\vec{x} 0^\omega \in X$ . Hence,  $f(\vec{x} 0^\omega) = \vec{x}$ .

For the next condition, assume that  $x \in X$  and  $U \in \tau(x)$ . We need to prove that  $f(U) \in \sigma(f(x))$ . That means  $R(f(x)) \subseteq f(U)$ . Because  $U \in \tau(x)$ , there is a  $m$  such that  $U_m(x) \subseteq U$ . By Lemma 5.2.5 we have  $f(U_m(x)) = R(f(x))$ . It follows,

$$R(f(x)) = f(U_m(x)) \subseteq f(U)$$

Assume  $x \in X$  and  $V$  is a neighborhood of  $x$ , i.e.  $R(f(x)) \subseteq V$ . We need to prove that there exists  $U \in \tau(x)$ , such that  $f(U) \subseteq V$ . As  $U = U_m(x)$  for any  $m \in \mathbb{N}$ . By Lemma 5.2.5 we get  $f(U_m(x)) = R(f(x))$ . Hence,

$$f(U_m(x)) = R(f(x)) \subseteq V$$

$\square$

**Corollary 5.2.7.** *For frame  $F = F_{\xi\eta}[A]$  we have  $\text{Log}(N_\omega(F)) \subseteq \text{Log}(F)$ .*

*Proof.* It follows from Lemma 5.1.5, Corollary 5.1.7 and Lemma 5.2.6

$$\text{Log}(N_\omega(F)) \subseteq \text{Log}(N(F)) = \text{Log}(F)$$

$\square$

**Proposition 5.2.8.** *Let  $F_{in} = F_{in}[\mathbb{N}]$ ,  $F_{rn} = F_{rn}[\mathbb{N}]$ ,  $F_{it} = F_{it}[\mathbb{N}]$  and  $F_{rt} = F_{rt}[\mathbb{N}]$ . Then*

$$\text{Log}(N_\omega(F_{in})) = D$$

$$\text{Log}(N_\omega(F_{rn})) = T$$

$$\text{Log}(N_\omega(F_{it})) = D4$$

$$\text{Log}(N_\omega(F_{rt})) = S4$$

*Proof.* In all these cases, the inclusion from left to right follows from Proposition 2.1.9 and Corollary 5.2.7. Now the converse direction. Assume  $X = (X, \tau)$ .

It is easy to check that  $X \models D$  iff for each  $x \in X : \emptyset \notin \tau(x)$ . For  $N_\omega(F_{in})$  and  $N_\omega(F_{it})$  this holds.

It is easy to check that  $X \models T$  iff we have  $x \in U \in \tau(x)$  for any  $x$  and  $U$ . For  $N_\omega(F_{rn})$  and  $N_\omega(F_{rt})$  this holds.

Now we check  $X \models 4$  iff for each  $U \in \tau(x) : \{y \mid U \in \tau(y)\} \in \tau(x)$ .

$\supseteq$ : Let  $x \in X$  and assume  $X, x \models \Box p$ . That means there exists  $U \in \tau(x)$  s.t  $U \subseteq V(p)$ . By assumption we have  $S = \{y \mid U \in \tau(y)\} \in \tau(x)$ . But then  $X, x \models \Box \Box p$  because we can just pick the set  $S$ .

$\subseteq$ : By contradiction, assume there exists a  $U$  s.t  $\{y \mid U \in \tau(y)\} \notin \tau(x)$ . Let  $X, x \models \Box p$  where  $V(p) = U$ . If  $X, x \models \Box \Box p$  then it must be the case that

$S = \{y \in X \mid X, y \models \Box p\} \in \tau(x)$ .  $X, y \models \Box p$  means  $U \in \tau(y)$ . That means  $S = \{y \in X \mid U \in \tau(y)\}$ . But by assumption  $S \notin \tau(x)$ . Hence,  $X, x \not\models \Box \Box p$ . But that's a contradiction. This also holds for  $N_\omega(F_{it})$  and  $N_\omega(F_{rt})$  because we have for any  $y \in U_m(x)$  and  $k \geq m : U_k(y) \subseteq U_m(x)$ .

□

Assume  $F_1 = (A^*, R_1) = F_{\eta_1 \xi_1}[A]$  and  $F_2 = (B^*, R_2) = F_{\eta_2 \xi_2}[B]$  with  $A \cap B = \emptyset$ ,  $A = \{a_1, a_2, a_3, \dots\}$  and  $B = \{b_1, b_2, b_3, \dots\}$ . Consider the product of n-frames  $F'_1 = (X_1, \tau_1) = N_\omega(F_1)$  and  $F'_2 = (X_2, \tau_2) = N_\omega(F_2)$  is

$$X = (X_1 \times X_2, \tau'_1, \tau'_2) = N_\omega(F_1) \times_n N_\omega(F_2)$$

Furthermore, we have  $F_1 \otimes F_2 = ((A \cup B)^*, R'_1, R'_2)$  as defined in Defintion 2.0.7. We consider the neighborhood version

$$N(F_1 \otimes F_2) = ((A \cup B)^*, \sigma'_1, \sigma'_2)$$

Now we define  $g : X_1 \times X_2 \rightarrow (A \cup B)^*$  as follows. For  $(\alpha, \beta) \in X_1 \times X_2$  with  $\alpha = a_1 a_2 \dots$  and  $\beta = b_1 b_2 \dots$  we define  $g(\alpha, \beta)$  to be the finite sequence which we get after eliminating all zeros from the infinite sequence  $a_1 b_1 a_2 b_2 \dots$

Example : Let  $\alpha = 012340^\omega$  and  $\beta = 0ab00e0^\omega$ . Then  $g(\alpha, \beta) = 1a2b34e$ .

**Lemma 5.2.9.** *Let  $X$  and  $N(F_1 \otimes F_2)$  be as defined before and  $(\alpha, \beta) \in X_1 \times X_2$ . Then for any  $m > \max\{st(\alpha), st(\beta)\}$  we have*

$$R'_1(g(\alpha, \beta)) = g(U_m(\alpha) \times \{\beta\})$$

.

*Proof.* We will show both direction by assuming  $F_1 = F_{in}[A]$ . For  $F_{it}[A]$ ,  $F_{rt}[A]$  and  $F_{rn}[A]$  is it similar.

$\subseteq$  : Let  $\vec{w} \in R'_1(g(\alpha, \beta))$ . By definition we get, there exists  $\vec{c} \in A^*$  where  $\vec{w} = g(\alpha, \beta) \cdot \vec{c}$  and  $\Lambda R_1 \vec{c}$ . Because  $R_1$  is irreflexive and non-transitive, we get  $\vec{c} \in A$ . We construct  $(\zeta, \beta)$  where  $\zeta \in U_m(\alpha)$  and  $g(\zeta, \beta) = \vec{w}$ . For that, we can take  $\zeta = \alpha \upharpoonright_m \cdot \vec{c} 0^\omega$ . Obviously,  $\zeta \in U_m(\alpha)$ . Because  $m \in \max\{st(\alpha), st(\beta)\}$ , we have  $g(\alpha \upharpoonright_m, \beta \upharpoonright_m) = g(\alpha, \beta)$ . Hence,  $g(\zeta, \beta) = g(\alpha \upharpoonright_m, \beta \upharpoonright_m) \cdot g(\vec{c} 0^\omega, 0^\omega) = g(\alpha \upharpoonright_m, \beta \upharpoonright_m) \cdot \vec{c} = \vec{w}$ .

$\supseteq$  : Assume  $\zeta \in U_m(\alpha)$ . We have to show  $g(\zeta, \beta) \in R'_1(g(\alpha, \beta))$ . Because  $R_1$  is ir-reflexive and non-transitive, it suffices to find a  $\vec{c} \in A$  s.t.  $g(\alpha, \beta) \cdot \vec{c} = g(\zeta, \beta)$ . By choosing  $m$  is maximal and  $\zeta \upharpoonright_m = \alpha \upharpoonright_m$ , we have  $g(\zeta_m, \beta_m) = g(\alpha, \beta)$ . We also know  $f_F(\alpha)R_1f_F(\zeta)$ , that means there exists a  $\vec{d} \in A$  s.t.  $f_F(\alpha) \cdot \vec{d} = f_F(\zeta)$ . This  $\vec{d}$  must appear at a point after  $\zeta_m$ . We can follow  $g(\zeta, \beta) = g(\alpha, \beta) \cdot \vec{d}$ . Hence,  $g(\zeta, \beta) \in R'_1(g(\alpha, \beta))$ .

Remark : If  $m \leq \max\{st(\alpha), st(\beta)\}$ , then we cannot gurantee the equality. Assume  $R_1$  as before and  $\alpha = 123000^\omega$  and  $\beta = d0b0a0^\omega$ . Lets pick  $m = 3$  and  $\zeta = 123100^\omega$ . Then we have  $g(\alpha, \beta) = 1d23ba$  and  $g(\zeta, \beta) = 1d23b1a$ . Obviously, we don't have  $g(\zeta, \beta) \in R'_1(g(\alpha, \beta))$ .

Furthermore, we can show  $R'_2(g(\alpha, \beta)) = g(\alpha \times U_m(\beta))$  similar as above. □

**Lemma 5.2.10.** *Function  $g : X \rightarrow N(F_1 \otimes F_2)$  is a bounded morphism.*

*Proof.* Let  $\vec{z} = z_1z_2\dots z_n \in (A \cup B)^*$ . Define for  $i \leq n$  :

$$x_i = \begin{cases} z_i, & \text{if } z_i \in A; \\ 0, & \text{if } z_i \notin A. \end{cases} \quad y_i = \begin{cases} z_i, & \text{if } z_i \in B; \\ 0, & \text{if } z_i \notin B. \end{cases}$$

Let  $\alpha = x_1x_2\dots x_n 0^\omega$  and  $\beta = y_1y_2\dots y_n 0^\omega$ . Then  $g(\alpha, \beta) = \vec{z}$ . Hence,  $g$  is surjective.

For the next conditions we check only for  $\tau'_1$  and  $\sigma_1$ . The other case is similar. Assume  $(\alpha, \beta) \in X_1 \times X_2$  and  $U \in \tau'_1(\alpha, \beta)$ . We have to show  $g(U) \in \sigma_1(g(\alpha, \beta))$ . That means  $R'_1(g(\alpha, \beta)) \subseteq g(U)$ . Pick a  $m > \max\{st(\alpha), st(\beta)\}$  s.t.  $U_m(\alpha) \times \{\beta\} \subseteq U$ . We can pick such  $m$ , because  $U \in \tau'_1(\alpha, \beta)$ . So there exists  $U_k(\alpha) \in \tau(\alpha)$  s.t.  $U_k(\alpha) \times \{\beta\} = U$ . If  $k > \max\{st(\alpha), st(\beta)\}$  then we are done. Else, by Lemma 5.2.3, we have for any  $n \geq k : U_n(\alpha) \subseteq U_k(\alpha)$ . So we can lift the  $k$  til we reach  $m$ . Then we use Lemma 5.2.11 to get the following :

$$R'_1(g(\alpha, \beta)) = g(U_m(\alpha) \times \{\beta\}) \subseteq g(U)$$

For the last condition we assume  $(\alpha, \beta) \in X_1 \times X_2$  and  $V \in \sigma_1(g(\alpha, \beta))$  (or rather  $R'_1(g(\alpha, \beta)) \subseteq V$ ). We need to prove there exists  $U \in \tau'_1(\alpha, \beta)$ , such that  $g(U) \subseteq V$ . As  $U$  we take  $U_m(\alpha) \times \{\beta\}$  for some  $m > \max\{st(\alpha), st(\beta)\}$ . Hence, by Lemma 5.2.11 we get :

$$g(U_m(\alpha) \times \{\beta\}) = R'_1(g(\alpha, \beta)) \subseteq V$$

□

### 5.3 Adapting the key ideas

Now we will show  $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \rightarrow N(T_{\omega, \omega, \omega[rn]})$  is a bounded morphism. Let  $(T_{\omega[rn]})_1 = (\mathbb{N}_1, R_1)$  and  $(T_{\omega[rn]})_2 = (\mathbb{N}_2, R_2)$  as defined in Defintion 3.2.1. Then we

take the n-frames  $N_\omega(T_{\omega[rn]})_1 = (X_1, \tau_1)$  and  $N_\omega(T_{\omega[rn]})_2 = (X_2, \tau_2)$ . For the proof, we will omit the subscripts after the frame.

Let's consider the full product of the n-frames :

$$N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) = (X_1 \times X_2, \tau'_1, \tau'_2, \tau)$$

We say  $N(T_{\omega, \omega, \omega[rn]}) = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*, \sigma_1, \sigma_2, \sigma)$  where the tree  $T_{\omega, \omega, \omega[rn]} = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^* R'_1, R'_2, R)$  is as defined in Def.3.2.1

In order to define the bounded morphism, we have to fix a bijection first. Let  $h : \mathbb{N}_1 \times \mathbb{N}_2 \rightarrow \mathbb{N}$  be a bijection.

Next, we define function  $k : (\mathbb{N}_1 \cup \{0\}) \times (\mathbb{N}_2 \cup \{0\}) \rightarrow \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{0\}$  as follows :

$$f(a, b) = \begin{cases} a, & \text{if } b = 0; \\ b, & \text{if } a = 0; \\ h(a, b), & \text{otherwise.} \end{cases}$$

Let  $(\alpha, \beta) \in X_1 \times X_2$  with  $\alpha = a_1 a_2 \dots$  and  $\beta = b_1 b_2 \dots$ . We define

$$g'(\alpha, \beta) = f(a_1, b_1) f(a_2, b_2) \dots$$

At last, we define  $g(\alpha, \beta)$  as  $g'(\alpha, \beta)$  but removing all zeros.

**Lemma 5.3.1.** *Function  $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \rightarrow N(T_{\omega, \omega, \omega[rn]})$  is a bounded morphism.*

*Proof.* Let  $\vec{z} = z_1 z_2 \dots z_n \in (\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*$ . We define for  $i \leq n$

$$x_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_1; \\ a_1, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases} \quad y_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_2; \\ b_2, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases}$$

Let  $\alpha = x_1 x_2 \dots x_n 0^\omega$  and  $\beta = y_1 y_2 \dots y_n 0^\omega$ . But then  $g(\alpha, \beta) = \vec{z}$ . Hence,  $g$  is surjective. For the next conditions we show it for  $\tau$  and  $\sigma$ . For  $\tau'_1, \sigma_1$  and  $\tau'_2, \sigma_2$  we can argue similar as in Lemma 5.2.11 and 5.2.12.

First we need to show,  $R(g(\alpha, \beta)) = g(U_m(\alpha) \times U_m(\beta))$  for  $m > \max\{st(\alpha), st(\beta)\}$ . The proof is similar as in Lemma 5.2.1.

$\supseteq$  : Pick  $\zeta \in U_m(\alpha), v \in U_m(\beta)$ . Show  $g(\zeta, v) \in R(g(\alpha, \beta))$ . By definition, we need to find a  $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$  s.t.  $g(\alpha, \beta) \cdot \vec{c} = g(\zeta, v)$ . By choosing  $m > \max\{st(\alpha), st(\beta)\}$ ,  $\alpha \upharpoonright_m = \zeta \upharpoonright_m$  and  $\beta \upharpoonright_m = v \upharpoonright_m$ , we get  $g(\alpha, \beta) = g(\zeta_m, v_m)$ . Furthermore, we have  $f_F(\alpha) R_1 f_F(\zeta)$  and  $f_F(\beta) R_2 f_F(v)$ , that means  $\exists \vec{d} \in \mathbb{N}_1 \cup \{\epsilon\}$  and  $\exists \vec{e} \in \mathbb{N}_2 \cup \{\epsilon\}$  s.t.  $f_F(\alpha) \cdot \vec{d} = f_F(\zeta)$  and  $f_F(\beta) \cdot \vec{e} = f_F(v)$ . So,  $\vec{d}$  must appear at a point after  $\zeta \upharpoonright_m$ . The

same holds for  $\vec{e}$  in  $v$ . But then we have  $g(\zeta, v) = g(\alpha, \beta) * g(\vec{d}, \vec{e})$ .

Now there are four cases :

$$\begin{cases} g(\vec{d}, \vec{e}) \in \mathbb{N}_1, & \text{if } \vec{d} \in \mathbb{N}_1 \text{ \& } \vec{e} = \epsilon \\ g(\vec{d}, \vec{e}) \in \mathbb{N}_2, & \text{if } \vec{d} = \epsilon \text{ \& } \vec{e} \in \mathbb{N}_2 \\ g(\vec{d}, \vec{e}) \in \{\epsilon\}, & \text{if } \vec{d} = \vec{e} = \epsilon \\ g(\vec{d}, \vec{e}) \in \mathbb{N}, & \text{other} \end{cases}$$

Hence,  $g(\zeta, v) \in R(g(\alpha, \beta))$ .

$\subseteq$  : Pick  $\vec{w} \in R(g(\alpha, \beta))$ . We will construct  $\zeta \in U_m(\alpha)$  and  $v \in U_m(\beta)$  s.t.  $g(\zeta, v) = \vec{w}$ . By definition there exists a  $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$  s.t.  $g(\alpha, \beta) \cdot \vec{c} = \vec{w}$ . Depending on the  $\vec{c}$ , we build  $\zeta = \alpha \mid_m \cdot \vec{d} 0^\omega$  and  $v = \beta \mid_m \cdot \vec{e} 0^\omega$  where

$$\begin{cases} \vec{d} = \vec{c}, \vec{e} = 0, & \text{if } \vec{c} \in \mathbb{N}_1; \\ \vec{d} = 0, \vec{e} = \vec{c}, & \text{if } \vec{c} \in \mathbb{N}_2; \\ \vec{d} = \vec{e} = 0 & \text{if } \vec{c} \in \{\epsilon\} \\ \vec{d} = \vec{\alpha}, \vec{e} = \vec{\beta} \text{ where } h(\vec{\alpha}, \vec{\beta}) = \vec{c} & \text{if } \vec{c} \in \mathbb{N} (\vec{\alpha} \in \mathbb{N}_1, \vec{\beta} \in \mathbb{N}_2); \end{cases}$$

We have  $\zeta \in U_m(\alpha)$  and  $v \in U_m(\beta)$ . Furthermore, because of  $g(\zeta \mid_m, v \mid_m) = g(\alpha, \beta)$  it holds that  $g(\zeta, v) = g(\alpha, \beta) \cdot g(\vec{d}, \vec{e}) = g(\alpha, \beta) \cdot \vec{c} = \vec{w}$ .

Now to check the conditions, we assume  $x \in X_1$  and  $y \in X_2$  and  $U \in \tau(x, y)$ . We need to prove  $g(U) \in \sigma(g(x, y))$ . In other words, show  $R(g(x, y)) \subseteq g(U)$ . We pick a  $m > \max\{st(x), st(y)\}$  s.t.  $U_m(x) \times U_m(y) \subseteq U$ . But then we get

$$R(g(x, y)) = g(U_m(x) \times U_m(y)) \subseteq g(U)$$

Assume  $x \in X_1$  and  $x \in X_2$  and  $R(g(x, y)) \subseteq V$ . We need to find a  $U \in \tau(x, y)$ , s.t.  $f(U) \subseteq V$ . For  $U$  we pick  $U_m(x) \times U_m(y)$  for some  $m > \{st(x), st(y)\}$ . But then

$$g(U_m(x) \times U_m(y)) = R(g(x, y)) \subseteq V$$

□

## 5.4 Completeness Results

**Lemma 5.4.1.** *Let  $T_{\omega, \omega, \omega[rn]}$  be as in Definition 3.2.1. Then*

$$\text{Log}(T_{\omega, \omega, \omega[rn]}) = \text{Log}(N(T_{\omega, \omega, \omega[rn]}))$$

*The proof is by structural induction.*



**Corollary 5.4.2.** Let  $F_1 = (A^*, R_1) = F_{\xi_1, \eta_1}[A]$  and  $F_2 = (B^*, R_2) = F_{\xi_2, \eta_2}[B]$ . Then

$$\text{Log}(N_\omega(F_1) \times_n N_\omega(F_2)) \subseteq \text{Log}(F_1) \otimes \text{Log}(F_2)$$

. It follows from Proposition 2.1.8, Lemma 5.1.5, Corollary 5.1.7 and Lemma 5.1.12.

**Corollary 5.4.3.** Let  $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$ . Then

$$\text{Log}(N_\omega(F_1) \times_n N_\omega(F_2)) = \text{Log}(F_1) \otimes \text{Log}(F_2)$$

*Proof.* The left to right inclusion follows from Corollary 5.3.1

To prove right to left, we know due to Proposition 2.1.9 and Proposition 5.2.8 we have  $\text{Log}(N_\omega(F_i)) = \text{Log}(F_i)$  ( $i \in \{1, 2\}$ ). Due to Proposition 5.1.10 we get  $\text{Log}(N_\omega(F_1)) \otimes \text{Log}(N_\omega(F_2)) \subseteq \text{Log}(N_\omega(F_1)) \times_n \text{Log}(N_\omega(F_2))$ . Because  $N_\omega(F_1) \times_n N_\omega(F_2)$  is a frame in the n-product, we get  $\text{Log}(N_\omega(F_1)) \times_n \text{Log}(N_\omega(F_2)) \subseteq \text{Log}(N_\omega(F_1) \times_n N_\omega(F_2))$ .  $\square$

**Corollary 5.4.4.** Let  $N_\omega(T_{\omega[rn]})_1 = F_1$  and  $N_\omega(T_{\omega[rn]})_2 = F_2$  as defined before. Then

$$T \times_n^+ T \subseteq \text{Log}(T_{\omega, \omega, \omega[rn]})$$

It follows from Corollary 5.1.7, Proposition 5.2.8, Lemma 5.2.9, Lemma 5.2.12

*Proof.* First, it is the case that  $\text{Log}(F_1) \times_n^+ \text{Log}(F_2) \subseteq \text{Log}(F_1 \times_n^+ F_2)$ . We can argue as before. By Lemma 5.2.12 it holds  $\text{Log}(F_1) \times_n^+ \text{Log}(F_2) \subseteq \text{Log}(N(T_{\omega, \omega, \omega[rn]})) = \text{Log}(T_{\omega, \omega, \omega[rn]})$ . But by Proposition 5.2.8 we have  $\text{Log}(F_1) = \text{Log}(F_2) = T$ . Hence,  $\text{Log}(F_1) \times_n^+ \text{Log}(F_2) = T \times_n^+ T \subseteq \text{Log}(T_{\omega, \omega, \omega[rn]})$ .  $\square$

**Theorem 5.4.5.** Let  $L_1, L_2 \in \{S4, D4, D, T\}$ . Then

$$L_1 \times_n L_2 = L_1 \otimes L_2$$

*Proof.* Right to left is by Proposition 5.1.10.

For left to right, assume  $L_1 = \text{Log}(F_1)$  and  $L_2 = \text{Log}(F_2)$  for some  $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$ .

It follows from Corollary 5.3.2

$$L_1 \times_n L_2 = \text{Log}(N_\omega(F_1)) \times_n^+ \text{Log}(N_\omega(F_2)) \subseteq \text{Log}(N_\omega(F_1) \times_n^+ N_\omega(F_2)) = \text{Log}(F_1 \otimes F_2) = \text{Log}(F_1) \otimes \text{Log}(F_2) = L_1 \otimes L_2$$

$\square$

**Theorem 5.4.6.** Let  $N_\omega(T_{\omega[rn]})_1 = F'_1$  and  $N_\omega(T_{\omega[rn]})_2 = F'_2$  as defined before. Then

$$T \times_n^+ T = T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

*Proof.* For left to right it is easy to check the axioms. We pick any frame  $F_1$  and  $F_2$  s.t.  $F_1 \models T$  and  $F_2 \models T$ . We consider  $F_1 \times_n^+ F_2$  the full product and let  $(x, y) \in F_1 \times_n^+ F_2$ . Assume  $F_1 \times_n^+ F_2, (x, y) \models \Box_1 p$ . By definition, there exists a  $U \in \tau'_1(x, y)$  s.t.  $U \subseteq V(p)$ . But we have  $U \supseteq V \times \{y\}$  for some  $V \in \tau(x)$ . By assumption,  $F_1 \models T$ , so we can follow  $x \in V$ . Hence,  $(x, y) \in V(p)$ . For  $\Box_2 p \rightarrow p$  it's done similar.

Let's check  $\Box p \rightarrow p$ . Assume  $F_1 \times_n^+ F_2, (x, y) \models \Box p$ . By definition there exists a  $U \in \tau(x, y)$  s.t.  $U \subseteq V(p)$ . But  $U \supseteq V \times W$  for some  $V \in \tau_1(x)$  and  $W \in \tau_2(y)$ . Because of the assumption, we must have  $x \in V$  and  $y \in W$ . Hence,  $(x, y) \in V(p)$ . It follows  $F_1 \times_n^+ F_2, (x, y) \models p$ .

Now we check the extra axiom. Assume  $F_1 \times_n^+ F_2, (x, y) \Vdash \Box p$ . By definition, there exists a  $U \subseteq V(p)$ , where  $U \supseteq V \times W$  with  $V \in \tau_1(x)$  and  $W \in \tau_2(y)$ . By assumption, we get  $y \in W$ . That means  $U \in \tau'_1(x, y)$  because  $U \supseteq V \times \{y\}$ . The same argument we can apply to  $\tau'_2(x, y)$ . Hence,  $F_1 \times_n^+ F_2, (x, y) \models \Box_1 p \wedge \Box_2 p$ . The rest is clear.

Now for the other direction, we apply Corollary 5.3.3 to get  $T \times_n^+ T \subseteq \text{Log}(T_{\omega, \omega, \omega[rn]})$ . By Proposition 3.2.6 we have  $\text{Log}(T_{\omega, \omega, \omega[rn]}) = TNL$ . Hence,

$$T \times_n^+ T \subseteq T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

.

□