1 Definition

1.1 Defintion

Let Prop be a set of variables. Then a formula ϕ is defined as follows :

$$\phi ::= p \mid \bot \mid \phi \mid \phi \to \phi \mid \Box_i \phi$$

where $p \in Prop$ and $\Box_i$ is a modal operator. Other connectives are expressed through $\bot$ and $\to$ and dual modal operators $\diamond_i$ as $\diamond_i \phi = \neg \Box_i \neg \phi$

1.2 Defintion

A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \to \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms

$$\Box_i (p \to q) \to (\Box_i p \to \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$ Let L be a logic and let Γ be a set of formulas. Then L+Γ denotes the minimal logic containing L and Γ

1.3 Definition

Let L1 and L2 be two modal logic with one modality $\Box$. Then the fusion of these logics are defined as follows :

$$L1 \otimes L2 = K2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

The follow logics may be important

$$D = K + \Box p \to \Diamond p$$

$$T = K + \Box p \to p$$

$$D4 = D + \Box p \to \Box\Box p$$

$$S4 = T + \Box p \to \Box\Box p$$

2 Topological Space Defintion

2.1 Defintion

A topological space is a pair (X, τ) where τ is a collection of subsets of X (elements of τ are also called open sets) such that :

1. the empty set $\emptyset$ and X are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

The space is called Alexandroff, if we allow the intersection of infinite collection. A topological model is a structure M = (X,τ,v) where (X,τ) is a topological space and v is a valuation assigining subsets of X to propositional variables.

2.2 Defintion

Let M = (X,τ,v) a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows :$M, x \models \Box\phi$ iff ,$\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$
$M, x \models \Box\phi$ iff ,$\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$
$M, x \models \Diamond\phi$ iff ,$\forall U \in \tau$ s.t $x \in U$ and $\exists u \in U : M, u \models \phi$

2.3 Defintion

Let $A = (X, \chi)$ and B =(Y, υ) be topological spaces. The standard product topology τ is the set of subsets of $X \times Y$ such that $X \in \chi$ and $Y \in \upsilon$.
Let $N \subseteq X \times Y$. We call N horizontally open if $\forall(x, y) \in N$ $\exists U \in \chi : x \in U$ and $U \times \{y\} \subseteq N$.
We call N vertically open if $\forall(x, y) \in N$ $\exists V \in \upsilon : y \in V$ and $\{x\} \times V \subseteq N$
If N is H-open and V-open, then we call it HV-open.
We denote τ_1 is the set of all H-open subsets of $X \times Y$ and τ_2 is the set of all V-open subsets of $X \times Y$

2.4 Defintion

Let X and Y be topological spaces and $f : X \to Y$ a function. We call f continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X. We say f is open if for each open set $V \subseteq X$ the set f[V] is open in Y.

2.5 Remark

There is an alternative defintion for open sets. Let (X,τ) be a topological space and U a set. U is open iff $\forall x \in U \; \exists V \subseteq U$: V is open and $x \in V$. This is true because, the union of open sets is an open set.

Now we define some Kripke frames, which we will use through this chapter.

2.6 Defintion

Let T_2 be the infinite binary tree with reflexive and transitive descendant relation.
Formally it is defined as follows : $T_2 = (W, R)$ where W = {0,1}* and sRt iff $\exists u \in W$: $s * u = t$.
The $T_{6,2,2}$ tree is the infinite six branching tree, where all nodes of $T_{6,2,2}$ is R-related, the first two R1-related and the last two R2-related. Formally we can define this tree as follows : $T_{6,2,2} = (W, R, R_1, R_2)$, where W = {0,1,2,3,4,5}*,

$$sRt \text{ iff } \exists u \in \{0,1,2,3,4,5\}^* : s * t = u$$

$$sR_1t \text{ iff } \exists u \in \{0,1\}^* : s * t = u$$

$$sR_2t \text{ iff } \exists u \in \{5,6\}^* : s * t = u$$

where s and t are elements of the set where the element u can come from, w.r.t to the relation. For example in the case sRt, s and t are elements of {0,1,2,3,4,5}*.

3 Neighbourhood

3.1 Defintion

Let X be a non-empty set. A function $\tau : X \to 2^{2^X}$ is called a neighbourhood function. A pair F = (X,τ) is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X,τ,v), where v assigns a subset of X to a variable

3.2 Defintion

Let M =(X,τ,v) be a neighourhood model and x $\in$ X. The truth of a formula is defined inductively as follows :

$$M, x \models \Box\phi \text{ iff } \exists V \in N(x)\forall y \in V : M, y \models \phi$$

A formula is valid in a n-model M if it is valid at all points of M ($M \models \phi$). Formula is valid in a n-frame F if it is valid in all models based on F (notation $F \models \phi$). For Logic L we write $F \models L$, if for any $\phi \in L, F \models \phi$. We define nV(L) = $\{F \mid F \text{ is an n-frame and } F \models \phi\}$.

3.3 Defintion

Let F = (W,R) be a Kripke frame. We define an n-frame N(F) = (W, τ) as follows. For any $w \in W$ we have :

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

3.4 Defintion

Let X = (X, τ_1,...) and Y = (Y, σ_1,...) be n-frames. Then the function f: $X \to Y$ is called bounded morphism if

1. f is surjective
2. $\forall x \in X\, \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X\, \forall V \in \sigma_i(f(x))\, \exists U \in \tau_i(x) \ : f(U) \subseteq V$

3.5 Defintion

Let X = (X, τ_1) and Y = (Y, τ_2) be two n-frames. Then the product of these two frames is an n-2-frame and is defined as follows :

$$X \times Y = (\mathrm{X} \times \mathrm{Y}, \tau_1', \tau_2')$$
$$\tau_1'(x,y) = \{U \subseteq \mathrm{X} \times \mathrm{Y} \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$
$$\tau_2'(x,y) = \{U \subseteq \mathrm{X} \times \mathrm{Y} \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

3.6 Defintion

For two unimodal logics L_1 and L_2 we define the n-product of them as follows :

$$L_1 \times_n L_2 = Log(\{X \times Y \mid X \in nV(L_1) \text{ and } Y \in nV(L_2)\})$$

Now we define some Kripke frames we need for this chapter.

3.7 Defintion

Let $T_{\omega[in]}$ (i = irreflexiv, n = non-transitiv) denote the infinite branching and infinite depth tree, which is irreflexiv and non-transitive. Formally the tree can be defined as : $T_{\omega[in]} = (W, R)$ where $W = \mathbb{N}^*$ and sRt iff $\exists u \in \mathbb{N} : s * u = t$ (the '*' is the concatenation operator)

The $T_{\omega,\omega,\omega[in]}$ tree is similary defined as the $T_{6,2,2}$ tree but with infinite branching and infinite depth. Before characterizing it, we say $\mathbb{N}_{R1}{}^*$ is the set of finite number combinations which has a subscript R_1 to denote that these numbers relate to R_1 (examples : $0_{R1}, 0123_{R1}$). $\mathbb{N}^+_{R_1}$ is the set $\mathbb{N}_{R_1}{}^*$- $\{\epsilon\}$. $\mathbb{N}^+_R$, $\mathbb{N}^+_{R_2}$ are defined similar.

Now let $T_{\omega,\omega,\omega[in]} = (W, R, R_1, R_2)$ where $W = \mathbb{N}^+_R \cup \mathbb{N}^+_{R_1} \cup \mathbb{N}^+_{R_2} \cup \{\epsilon\}$,

$$sRt \text{ iff } \exists u \in \mathbb{N}_R \cup \mathbb{N}_{R_1} \cup \mathbb{N}_{R_2} : s * u = t$$

$$sR_1t \text{ iff } \exists u \in \mathbb{N}_{R_1} : s * u = t$$
$$sR_2t \text{ iff } \exists u \in \mathbb{N}_{R_2} : s * u = t$$

where s,t are elements of the positive closure set where the element u can come from (additionally s can be ϵ), w.r.t to the relation. For example if we consider sR_1t, then $s, t \in \mathbb{N}^+_{R_1}$ but also $s = \epsilon$. Of course the '*' operator acts here again as a concatenation operator.