1 Introduction

The study of topological semantics in modal logic was initiated by McKinsey and Tarski in 1944 [?]. The idea was to generalize Kripke frames using tools from topology. Neighbourhood semantics [?] as a generalization of Kripke semantics for modal logic were invented independently by Dana Scott [?] and Richard Montague [?]. Neighborhood semantics is more general than Kripke semantics and in the case of normal reflexive and transitive logics coincides with topological semantics. The original motivation for introducing was to provide a semantics for non-normal modal logics. But in recent years, interest in topological semantics and neighborhood frames has grown considerably, partly due to its applications in artificial intelligence.
Oftentimes, it is necessary to combine frames for different modal logics into a complex frame. The natural way of doing that is a product construction. For Kripke frames, the resulting product is the Cartesian product of the two frames with two accessibility relations. For topological semantics, the product of topogical spaces as bi-topological spaces with so-called horizontal and vertical topologies have been considered. In a similar fashion, the product of neighborhood frames was introduced by Sano in [?].
Now, let L_1 and L_2 be two modal logics. We say $L_1 \otimes L_2$ (called fusion) is the minimal modal logic containing L_1 and L_2', where L_2' is the logic L_2 after renaming all modalities. Furthermore, we say $L_1 \times_n L_2$ is the logic (i.e the set of all valid formulas) of the class products of neighbourhood frames $N_1 \times_n N_2$ such that L_i is valid in N_i for i = 1,2. It was proven in [?] that for any two logics $L_1, L_2 \in \{D4, D, T, S4\} : L_1 \times_n L_2 = L_1 \otimes L_2$. In [?], the authors studied a product of two spaces with three topologies : horizontal, vertical and classic product topology. They proved that the logic of such spaces is $S4 \otimes S4 \otimes S4 + \Box p \to \Box_1 p \wedge \Box_2 p$ where $\Box$ corresponds to the product topology.
The following work will present a detailed proof of the shown results. Additionally, we will show that $D \otimes D \otimes D + \Box p \to \Box_1 p \wedge \Box_2 p$ (we abbreviate it with DNL) $= D\, x_n^+\, D$. The proof ideas here are inspired by the shown results.

2 Preliminaries

Definition 2.0.1. *Let prop be a set of variables. Then a formula ϕ is defined as follows:*

$$\phi ::= p \mid \bot \mid \phi \mid \phi \to \phi \mid \Box_i \phi$$

where $p \in Prop$ and $\Box_i$ is a modal operator. Other connectives are expressed through $\bot$ and $\to$ and dual modal operators $\diamond_i$ as $\diamond_i \phi = \neg \Box_i \neg \phi$

Definition 2.0.2. *A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \to \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms*

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$ Let L be a logic and let Γ be a set of formulas. Then L+Γ denotes the minimal logic containing L and Γ

Definition 2.0.3. *Let L_1 and L_2 be two modal logic with one modality $\Box$. Then the fusion of these logics are defined as follows :*

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

The following logics may be important

$$D = K + \Box p \to \Diamond p$$

$$T = K + \Box p \to p$$

$$D4 = D + \Box p \to \Box\Box p$$

$$S4 = T + \Box p \to \Box\Box p$$

Now we introduce some sepcial kind of frames, which we will use through this work.

Definition 2.0.4. *Let A be a nonempty set.*

$$A^* = \{a_1 ... a_k \mid a_i \in A\}$$

is the set of all finite sequences of elements from A, including the empty sequence Λ. Elements from A^* will be denoted as $\vec{a}$. The length of a sequence $\vec{a} = a_1 ... a_k$ is k (also $l(\vec{a}) = k$) and the length of Λ is 0 ($l(\Lambda) = 0$). Concatenation is denoted by " $\cdot$ " : $(a_1 ... a_k) \cdot (b_1 ... b_l) = \vec{a} \cdot \vec{b} = a_1 ... a_k b_1 ... b_l$.

Definition 2.0.5. *Let A be a nonempty set. We define an infinite frame $F_{in}[A] = (A^*, R)$ s.t for $\vec{a}, \vec{b} \in A^*$*

$$\vec{a} R \vec{b} \Leftrightarrow \exists x \in A \, (\vec{b} = \vec{a} \cdot x)$$

Furthermore we define :

$$F_{rn}[A] = (A^*, R^r), \text{ where } R^r = R \cup Id \text{ (reflexive closure)}$$

$$F_{it}[A] = (A^*, R^*), \text{ where } R^* = \bigcup_{i=1}^{\infty} R^i \text{ (transitive closure)}$$

$$F_{rt}[A] = (A^*, R^{r*})$$

where "t" stands for transtive, "n" for non-tarnsitive, "r" for reflexive and "i" for irreflexive.
For now, we will use the following notion to generalize : $F_{\xi\eta}$ *where* $\xi \in \{i, r\}$ *and* $\eta \in \{t, n\}$

Proposition 2.0.6. *Let* $F = F_{\xi\eta}[A] = (A^*, R)$ *then*

$$\vec{a}R(\vec{a} \cdot \vec{c}) \Leftrightarrow \Lambda R\vec{c}$$

Definition 2.0.7. *Let* $F_1 = F_{\xi_1\eta_1}[A] = (A^*, R_1)$ *and* $F_2 = F_{\xi_2\eta_2}[B] = (B^*, R_2)$*, where* $\xi_1, \xi_2 \in \{i, r\}$ *and* $\eta_1, \eta_2 \in \{t, n\}$*. Furthermore, we assume* $A = \{a_1, a_2, ...\}$ *and* $B = \{b_1, b_2, ...\}$ *with* $A \cap B = \emptyset$*. Then we define the frame* $F_1 \otimes F_2 = (W, R_1', R_2')$ *as follows :*

$$W = (A \cup B)^*$$

$$\vec{x}R_1'\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in A^* \text{such that } \Lambda R_1\vec{z}$$

$$\vec{x}R_2'\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in B^* \text{such that } \Lambda R_2\vec{z}$$

Proposition 2.0.8 ([?], [?])**.** *Let* F_1 *and* F_2 *be as in Defintion 2.0.7. Then*

$$Log(F_1 \otimes F_2) = Log(F_1) \otimes Log(F_2)$$

Proposition 2.0.9. *Let* $F_{in} = F_{in}[\mathbb{N}], F_{rn} = F_{rn}[\mathbb{N}], F_{it} = F_{it}[\mathbb{N}]$ *and* $F_{rt} = F_{rt}[\mathbb{N}]$*. Then the following holds:*

$$Log(F_{in}) = D$$
$$Log(F_{rn}) = T$$
$$Log(F_{it}) = D4$$
$$Log(F_{rt}) = S4$$

3 Completeness result for DNL

We introduce a frame called $T_{\omega,\omega,\omega[in]}$. We will show that DNL is sound and complete w.r.t $T_{\omega,\omega,\omega[in]}$ The idea is to pick a class of frame C s.t $Log(C) = Log(T_{\omega,\omega,\omega[in]})$ and then show that the class has FMP. In the end, we will use an unravelling technique to show completeness.

3.1 Filtration and Sahlqvist for multimodal logic

Definition 3.1.1. *Let $M = (W, R, V)$ be a model and $w \in W$ a state in M. The notion of a formula being true at w is inductively defined as follows :*

$$M, w \Vdash \Box\phi \text{ iff } \forall v \in W : wRv \to M, v \Vdash \phi$$

This defintion can be extended to a multimodal version, where the modal operators are interpreted the same way but with the respective relation.

Definition 3.1.2. *A set Σ is closed under subformulas, if for all formulas ϕ and ϕ' the following holds :*

1. *if $\neg\phi \in \Sigma$ then $\phi \in \Sigma$*
2. *if $\phi \vee \phi' \in \Sigma$ then $\phi, \phi' \in \Sigma$*
3. *if $\Box\phi \in \Sigma$ then $\phi \in \Sigma$*

We can define it similary for a multimodal logic. For every modal operator $\Box_n$, we extend this definition by adding a new condition similar to the third. Example : Suppose we have a multimodal logic with $\Box, \Box_1$ called $L_{\Box,\Box_1}$ and $\phi = \Box p \to \Box_1 q$ and $\phi \in \Sigma$. Then $\Sigma = \{\phi, \neg\Box p, \Box_1 q, \Box p, p, q\}$ is closed under subformulas.

Definition 3.1.3. *Let $M = (W, R, V)$ be a model and suppose Σ is a set of formulas. We define a relation $\equiv$ on W as follows :*

$$w \equiv v \text{ iff } \forall\phi \in \Sigma : M, w \Vdash \phi \Leftrightarrow M, v \Vdash \phi$$

It is well known that the $\equiv$-relation is an equivalence relation. We denote the equivalence class of a state $w \in W$ as $[w]_\Sigma = \{v \mid v \equiv w\}$. Furthermore W_Σ is the set of all equivalence classes, i.e $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$.

Definition 3.1.4. *Let $M = (W, R, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes induced by $\equiv$. A model $M^f_\Sigma = (W^f, R^f, V^f)$ is called filtration of M through Σ if the following holds :*

1. *$W^f = W_\Sigma$*
2. *If $(w, v) \in R$ then $([w], [v]) \in R^f$*
3. *If $([w], [v]) \in R^f$ then for any $\Box\phi \in \Sigma$: if $M, w \Vdash \Box\phi$ then $M, v \Vdash \phi$*

4. $V^f = \{[w] \mid M, w \Vdash p\}$, *for all propositional variables* $p \in \Sigma$

In our case, we are in a multimodal logic with three modal operators $\Box, \Box_1, \Box_2$. *We need to extend this defintion for* $L_{\Box,\Box_1,\Box_2}$. *This means our model looks like:* $M = (W, R, R_1, R_2, V)$. *We extend the conditions as follows :*

If $(w, v) \in R_i$ *then* $([w], [v]) \in R_i^f$
If $([w], [v]) \in R_i^f$ *then for any* $\Box_i\phi \in \Sigma$: *if* $M, w \Vdash \Box_i$ *then* $M, v \Vdash \phi$, *where* $i \in \{1, 2\}$

Theorem 3.1.5. *Consider* $L_{\Box,\Box_1,\Box_2}$. *Let* $M^f = (W_\Sigma, R^f, R_1^f, R_2^f, V)$ *be a filtration of* M *through a subformula closed set* Σ. *Then for all formulas* $\phi \in \Sigma$, *and all nodes* $w \in M$, *we have*

$$M, w \Vdash \phi \text{ iff } M^f, [w] \Vdash \phi$$

Proof. By induction on ϕ. We will only show non-trivial and, for our purposes, necessary cases.

Case $\phi = p$: Left to right follows immediately from filtration defintion. Conversely, suppose $M^f, [w] \Vdash p$. This means $[w] \in V^f(p)$. But this means V(p) can not be empty. Pick any $v \in V(p)$. Obviously, $w \equiv v$ and $M, v \Vdash p$. Hence, $M, w \Vdash p$.

Case $\phi = \neg\psi$: Suppose ψ holds. Then we have : $M, w \Vdash \phi$ iff $M, w \nVdash \psi$. Applying induction hypothesis, we get : $M^f, [w] \nVdash \psi$. But then, we have $M^f, [w] \Vdash \phi$. Right to left is the same.

Case $\phi = \phi_1 \wedge \phi_2$: Suppose ϕ_1, ϕ_2 holds. Let $M, w \Vdash \phi$. That means $M, w \Vdash \phi_1$ and $M, w \Vdash \phi_2$. Applying induction hypothesis, we get $M^f, [w] \Vdash \phi_1$ and $M^f, [w] \Vdash \phi_2$. But then, $M^f, [w] \Vdash \phi_1 \wedge \phi_2 = \phi$. Right to left is similar.

Case $\phi = \Box_i\psi$ ($i \in \{1, 2, \epsilon\}, \Box_\epsilon = \Box$): Left to right. Suppose ψ holds and $M, w \Vdash \Box_i\psi$. We need to show $M^f, [w] \Vdash \Box_i\psi$, this means $\forall[v] \in W_\Sigma : [w]R_i[v] \to M^f, [v] \Vdash \Box_i\psi$. Pick any $[v] \in W_\Sigma$ s.t $[w]R_i[v]$. By condition 3, w.r.t to the modal operator, we have $M, v \Vdash \psi$. By induction hypothesis, we get $M^f, [v] \Vdash \psi$. Because [v] was arbitrary it follows that $M^f, [w] \Vdash \Box_i\psi$.

Right to left. Suppose ψ holds and $M^f, [w] \Vdash \Box_i\psi$. Pick $v \in W$ s.t wR_iv. By condition 2, w.r.t to the modal operator, we have $[w]R_i^f[v]$. So, $M^f, [v] \Vdash \psi$. By induction hypothesis, we get $M, v \Vdash \psi$. Because v was arbitrary, we have $M, w \Vdash \Box_i\psi$.

□

Now define the smallest filter for $L_{\Box,\Box_1,\Box_2}$ and show that this is a filter. We denote this as R^s.

Definition 3.1.6. *Let* $M = (W, R, V)$ *be a model,* Σ *is closed under subformulas and* W_Σ *the set of equivalence classes. We define :*

$$R^s = \{[w], [v] \mid \exists w' \in [w], \exists v' \in [v] : w'R_iv'\}$$

where $i \in \{1, 2, \epsilon\}$.

Lemma 3.1.7. *Let* $M = (W, R, V)$ *be a model,* Σ *is closed under subformlas and* W_Σ *the set of equivalence classes induced by* $\equiv$ *and* V^f *the standard valuation on* W_Σ*. Then* $(W_\Sigma, R^s, R_1^s, R_2^s, V^f)$ *is a filtration of* M *through* Σ*.*

Proof. It suffices to show R_i^s fullfills the condition 2 and 3 w.r.t to the corresponding modal operator $\Box_i$. But R_i^s already satisfies condition 2. Let's check the other condition. Let $\Box_i\phi \in \Sigma$, $[w]R_i^s[v]$ and $M, w \Vdash \Box_i\phi$ where $i \in \{1, 2, \epsilon\}$. Because of $[w]R_i^s[v]$ we pick a $w' \in [w]$ and $v' \in [v]$. By defintion, we have $w'R_iv'$. Because $w' \equiv w$, we get $M, w' \Vdash \Box_i\phi$. Hence, $M, v' \Vdash \phi$ and by $v' \equiv v$, we get $M, v \Vdash \phi$.

$\Box$

Proposition 3.1.8. *Let* Σ *be a finite subformula closed set of* $L_{\Box,\Box_1,\Box_2}$*. For any model M, if* M^f *is a filtration through* Σ*, then* M^f *contains at most* 2^n *nodes (where n denotes the size of* Σ*).*

Proof. The states of M^f are the equivalence classes in W_Σ. Let $g : W_\Sigma \to P(\Sigma)$ defined by $g([w]) = \{\phi \in \Sigma \mid M, w \Vdash \phi\}$. g is well defined. Pick any u and v s.t $u \equiv v$. But then by defintion of $\equiv$, they fullfill the same subformulas. This means g([v]) = g([u]).
g is also injective. Pick any $[u], [v] \in W_\Sigma$ s.t g([u]) = g([v]). We show $[u] \subseteq [v]$. The other inclusion is similar. By assumption we have $u \equiv v$. Pick any $u' \in [u]$. Then we have $u' \equiv u \equiv v$. Hence, $u' \in [v]$. At the end, this means M^f contains at most 2^n nodes.

$\Box$

Theorem 3.1.9. *Let* ϕ *be a formula of* $L_{\Box,\Box_1,\Box_2}$*. If* ϕ *is satisfiable, then it is satisfiable on a finite model containing at most* 2^n *nodes, where n is the number of subformulas in* ϕ*.*

Proof. Assume that ϕ is satisfiable on a model on M. Take any filtration of M through the set of subformulas of ϕ. By Filtration Theorem, we get that ϕ is satisfied in the filtration model M^f. Furthermore, it is bounded by 2^n.

$\Box$

Now we define Sahlqvist formulas for our purposes.

Definition 3.1.10. *A modal formula* ϕ *is positive if all variables occurs without negation. In the other hand, a formula is negative, if all variables occurs with negation. A boxed atom is a modal formula of the form* $\Box^n p$ *for some* $n \in \mathbb{N}$*, where p is a propositional variable and* $\Box^n p$ *is defined as follows :* $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$*.*

Furthermore, a Sahlqvist antecedent is built from $\bot, \top$*, negative formulas and boxed atoms by applying* $\Diamond$ *and* $\wedge$*. A Sahlqvist implication is a modal formula of the form* $\phi \to \psi$*, where* ϕ *is a Sahlqvist antecedent and* ψ *a positive formula.*
Now, a Sahlqvist formula is built from Sahlqvist implications by applying $\Box$ *and* $\vee$*.*

Examples for Sahlqvist formulas:

$$\Box\Box p \to \Box p$$

$$\Diamond\neg p \to p$$

$$\Diamond\Box\Box\Box\Box\Box p \to \Box\Diamond\Box\Diamond p$$

$$\Box\Box\Box\Box(\Diamond\Box p \to p) \lor \Box\Box p \to \Box p$$

Non Sahlqvist Formulas :

$$\Box\Diamond p \to \Diamond\Box p$$

$$\Diamond\Box p \to \Box\neg p$$

We can extend this defintion for our logic. We say a boxed atom can be $\Box_1^n p$ and $\Box_2^n p$. A Sahlqvist antecedent can also be build by applying $\Diamond_1$ and $\Diamond_2$. A Sahlqvist formula can be build by Sahlqvist implications by applying additionally $\Box_1$ and $\Box_2$.

Sahlqvist formulas possess important properties, which are guaranteed by the Sahlqvist Theorem. It says that, when given a normal modal logic K and a set of Sahlqvist formulas, the resulting logic is complete w.r.t to the class of frames, which satisfies the corresponding first-order formula of the Sahlqvist formulas. This also holds for multimodal logic. We will not prove it here, but we will use this for our logic to show completeness w.r.t to a suitable class of frames.

Definition 3.1.11. *We say for a modal logic Λ has the finite model property (FMP) if for every formula ϕ that is not provable in Λ, is falsifiable in a finite model.*

Proposition 3.1.12. *The logic $D \otimes D \otimes D + \Box p \to \Box_1 p \land \Box_2 p$ has FMP.*

Proof. The idea is to pick a class of frame $\mathcal{F}$, where DNL is sound and complete with respect to and then show by filtration the FMP.

Let $\mathcal{C} = \{F \mid F \Vdash DNL\}$. Obviously, DNL is sound w.r.t $\mathcal{C}$. Completeness can be shown by using the Sahlqvist Theorem. We remember D = K + $\Box_i p \to \Diamond_i p$ for $i \in \{1, 2, \epsilon\}$. These axioms and $\Box p \to \Box_1 p \land \Box_2 p$ are Sahlqvist formulas. By Sahlqvist, we have that DNL is complete w.r.t $\{F \mid F \Vdash \forall x \exists y\, R_i(x, y)$ and $F \Vdash \forall x \forall y(R_1(x, y) \lor R_2(x, y)) \to R(x, y)\}$.

Now assume a formula ϕ is not derivable from DNL. By completeness we get ϕ is falsifiable in a model M and a world w. Hence, $M, w \Vdash \neg\phi$. Now we build the set subf($\neg\phi$) which denotes the closed subformulas set of $\neg\phi$. Let $M^s = (W_\Sigma, R^s, R_1^s, R_2^s, V^s)$ be the smallest filtration of M through subf($\neg\phi$) and V^s is the standard valuation on W_Σ. By filtration Theorem, they preserve truth. It remains to show $F^s = (W_\Sigma, R^s, R_1^s, R_2^s) \in \mathcal{C}$.

For that, every point must have at least one successor in every relation and it must hold that $R_1^s, R_2^s \subseteq R^s$. For the first one, we show this for R because the rest is similar. Pick $[w] \in W_\Sigma$. Because M is based on a frame $F \in \mathcal{C}$, there is a point v s.t wRv. By the defintion of smallest filtration, we have that $[w]R^s[v]$. For the second one, we show only for R_1^s because it is the same for R_2^s. Pick [w],[v] s.t $[w]R_1^s[v]$. By defintion of smallest filtration there are points $w' \in [w]$ and $v' \in [v]$ s.t $w'R_1v'$. Furthermore, it holds $R_1 \subseteq R$, so $w'Rv'$. By smallest filtration, we get $[w']R^s[v']$. Because $w' \equiv w$ and $v' \equiv v$, we have $[w'] = [w]$ and $[v'] = [v]$. It follows $[w]R^s[v]$.

□

Definition 3.1.13. *Let $T_{\omega[in]}$ (i = irreflexiv, n = non-transitiv) denote the infinite branching and infinite depth tree, which is irreflexiv and non-transitive. Formally the tree can be defined as : $T_{\omega[in]} = (W, R)$ where $W = \mathbb{N}^*$ and sRt iff $\exists u \in \mathbb{N} : s * u = t$ (the '*' is the concatenation operator)*

The $T_{\omega,\omega,\omega[in]}$ tree is similary defined as the $T_{6,2,2}$ tree but with infinite branching and infinite depth. Before characterizing it, we say $\mathbb{N}_{R1}{}^*$ is the set of finite number combinations which has a subscript R_1 to denote that these numbers relate to R_1 (examples : $0_{R1}, 0123_{R1}$). $\mathbb{N}^+_{R_1}$ is the set $\mathbb{N}_{R_1}{}^*$- $\{\epsilon\}$. $\mathbb{N}^+_R$, $\mathbb{N}^+_{R_2}$ are defined similar.

Now let $T_{\omega,\omega,\omega[in]} = (W, R, R_1, R_2)$ where $W = \mathbb{N}^+_R \cup \mathbb{N}^+_{R_1} \cup \mathbb{N}^+_{R_2} \cup \{\epsilon\}$,

$$sRt \textit{ iff } \exists u \in \mathbb{N}_R \cup \mathbb{N}_{R_1} \cup \mathbb{N}_{R_2} : s * u = t$$

$$sR_1t \textit{ iff } \exists u \in \mathbb{N}_{R_1} : s * u = t$$

$$sR_2t \textit{ iff } \exists u \in \mathbb{N}_{R_2} : s * u = t$$

where s,t are elements of the positive closure set where the element u can come from (additionally s can be ϵ), w.r.t to the relation. For example if we consider sR_1t, then $s, t \in \mathbb{N}^+_{R_1}$ but also $s = \epsilon$. Of course the '' operator acts here again as a concatenation operator.*

Definition 3.1.14. *Let $F = (W, R_1, R_2, ...)$ and $F' = (W', R'_1, R'_2, ...)$ be two frames. A bounded morphism from F to F' is a function $f : W \to W'$ satisfying the following conitions:*

$$\textit{If } (u, v) \in R_i \textit{ then } (f(u), f(v)) \in R'_i$$

$$\textit{If } (f(w), v') \in R'_i \textit{ then } \exists v \in W \textit{ s.t } (w, v) \in R'_i \textit{ and } f(v) = v'$$

We say that F' is a bounded morphic image of F, if there is a surjective bounded morphism from F to F'.

Proposition 3.1.15. *Let ϕ be a formula in $L_{\Box,\Box_1,\Box_2}$, $F = (W, R, R_1, R_2)$ and $F' = (W', R', R'_1, R'_2)$ be two frames and F to F' a surjective bounded morphism. Then the following holds :*

$$\textit{If } F \Vdash \phi \textit{ then } F' \Vdash \phi$$

This can be shown by structural induction on the length of the formula.

Corollary 3.1.16. *If F' is a bounded morphic image of F, then we have $Log(F) \subseteq Log(F')$*

Proposition 3.1.17. *D is sound and complete w.r.t $T_{\omega[in]}$.*

Proof. Sound is clear. For completeness, we use the well known fact that D has FMP. This means, $D = Log\{F \mid F \Vdash D\}$ where F is a finite frame. We can pick such a finite frame F and it suffices to find a surjective bounded morphism f from T_w to F. This would imply $Log(T_w) \subseteq D$.
Now, let $F = (W', R')$ be such a finite rooted frame with root w.(We can pick such because $F, w \Vdash D$ and then generate a subframe by w). We define inductively an assignment of nodes of F to the nodes of T_ω. For the base case, we assign w to the root of T_ω. The induction step looks like the following : Assume a point $x \in T_\omega$ has been assigned to a point $u \in F$ but the successors of x has no assignment. Let $s_1, s_2, ..., s_k$ be successors of u (k denotes amount of successors and $k \geq 1$ because seriality guarantees us at least one successor). For $n \geq 1, n \in \mathbb{N}$ we assign s_i to the $(n * i)$th-successor of x. This means we are assigining the successors alternatingly.
Now we check for f the conditions of bounded morphism. First condition : Let x,y $\in T_\omega$ s.t xRy and f(x) = s. But then, y will be assigned to a successor point of s. Hence, $f(x)R'f(y)$. Second condition : Suppose $f(x)R't$ and $f(x) = s$. Since t is a successor of s and f(x) = s, then a successor of x, say y, gets the assignment t.

□

Proposition 3.1.18. *DNL is sound and complete w.r.t $T_{\omega,\omega,\omega[in]}$.*

Proof. For soundness, we have that $T_{\omega,\omega,\omega[in]} \Vdash \Box p \to \Box_1 p \land \Box_2 p$, because by defintion we have $R_1, R_2 \subseteq R$. The rest is clear. For completeness, we use the fact that DNL has FMP. Let $F = (W', R', R'_1, R'_2)$ be a finite rooted frame with root w and $F \Vdash DNL$. We define inductively an assignment similar to 2.17. We assign w to the root of $T_{\omega,\omega,\omega[in]}$. For induction step we start by only assigining points from R_1 to R'_1 and R_2 to R'_2. After that, the remaining points will be assigned to R'. The procedure works similar as described previously.
Now we check the conditions for R_1 and R'_1. Let xR_1y and $f(x) = s$. We assigned y to a successor of s. So, $f(x)R'_1f(y)$. For the second condition, the same argument holds as before. It works the same for R_2 and R'_2. For R and R' we pick any xRy with $f(x) = s$. If we also have xR_1y, then it follows $f(x)Rf(y)$, because we showed that condtion for R_1 and R'_1. The same argument holds if xR_2y. Else, the successor of s was assigned to y, so $f(x)R'f(y)$. Let $f(x)R't$ and $f(x) = s$. If we have $f(x)R'_1t$ and x a point in R_1, then the second condition follows. The same holds for $f(x)R'_2t$ and x in R_2. Else, we have that t is a successor of f(x) = s, and a successor of x was assigned to t. □

4 Topological Space Defintion

Definition 4.0.1. *A topological space is a pair (X, τ) where τ is a collection of subsets of X (elements of τ are also called open sets) such that :*

1. *the empty set $\emptyset$ and X are open*
2. *the union of an arbitrary collection of open sets is open*
3. *the intersection of finite collection of open sets is open*

The space is called Alexandroff, if we allow the intersection of infinite collection of open sets. A topological model is a structure $M = (X,\tau,v)$ where (X,τ) is a topological space and v is a valuation assigining subsets of X to propositional variables.

Definition 4.0.2. *Let $M = (X,\tau,v)$ a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows :*
$M, x \models \Box\phi$ *iff* $\exists U \in \tau$ *s.t* $x \in U$ *and* $\forall u \in U : M, u \models \phi$
$M, x \models \Diamond\phi$ *iff* $\forall U \in \tau$ *s.t* $x \in U$ *and* $\exists u \in U : M, u \models \phi$

Definition 4.0.3. *Let $A = (X, \chi)$ and $B =(Y, \upsilon)$ be topological spaces. The standard product topology τ is the set of subsets of $X \times Y$ such that $X \in \chi$ and $Y \in \upsilon$.*
Let $N \subseteq X \times Y$. We call N horizontally open if $\forall (x, y) \in N\ \exists U \in \chi : x \in U$ and $U \times \{y\} \subseteq N$.
We call N vertically open if $\forall (x, y) \in N\ \exists V \in \upsilon : y \in V$ and $\{x\} \times V \subseteq N$
If N is H-open and V-open, then we call it HV-open.
We denote τ_1 is the set of all H-open subsets of $X \times Y$ and τ_2 is the set of all V-open subsets of $X \times Y$

Definition 4.0.4. *Let X and Y be topological spaces and $f : X \to Y$ a function. We call f continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X. We say f is open if for each open set $V \subseteq X$ the set f[V] is open in Y.*

Remark 4.0.5. *There is an alternative defintion for open sets. Let (X,τ) be a topological space and U a set. U is open iff $\forall x \in U\ \exists V \subseteq U$: V is open and $x \in V$. This is true because, the union of open sets is an open set.*

Now we define some Kripke frames, which we will use through this chapter.

Definition 4.0.6. *Let T_2 be the infinite binary tree with reflexive and transitive descendant relation.*
Formally it is defined as follows : $T_2 = (W, R)$ where $W = \{0,1\}^*$ and sRt iff $\exists u \in W$: $s * u = t$.
The $T_{6,2,2}$ tree is the infinite six branching tree, where all nodes of $T_{6,2,2}$ is R-related, the first two R1-related and the last two R2-related. Formally we can define this tree as follows : $T_{6,2,2} = (W, R, R_1, R_2)$, where $W = \{0,1,2,3,4,5\}^*$,

$$sRt \text{ iff } \exists u \in \{0,1,2,3,4,5\}^* : s * t = u$$

$$sR_1t \ \textit{iff} \ \exists u \in \{\mathit{0,1}\}^*: s * t = u$$

$$sR_2t \ \textit{iff} \ \exists u \in \{\mathit{5,6}\}^*: s * t = u$$

where s and t are elements of the set where the element u can come from, w.r.t to the relation. For example in the case sRt, s and t are elements of $\{\mathit{0,1,2,3,4,5}\}^*$.

5 Neighborhood

5.1 Neighborhood frames

Definition 5.1.1. *Let X be a non-empty set. A function $\tau : X \to 2^{2^X}$ is called a neighbourhood function. A pair F = (X,τ) is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X,τ,v), where v assigns a subset of X to a variable*

Definition 5.1.2. *Let M =(X,τ,v) be a neighourhood model and $x \in X$. The truth of a formula is defined inductively as follows :*

$$M, x \models \Box\phi \text{ iff } \exists V \in N(x) \forall y \in V : M, y \models \phi$$

A formula is valid in a n-model M if it is valid at all points of M ($M \models \phi$). Formula is valid in a n-frame F if it is valid in all models based on F (notation $F \models \phi$). For Logic L we write $F \models L$, if for any $\phi \in L, F \models \phi$. We define $nV(L) = \{F \mid F$ is an n-frame and $F \models \phi\}$.

Definition 5.1.3. *Let X be a non-empty set and τ neighborhood function. We call τ is a filter if for each $x \in X$ the collection $\tau(x)$ satisfies the following conditions :*

1. $\emptyset \notin \tau(x)$
2. If $U \in \tau(x)$ and $U \subseteq V$ then $V \in \tau(x)$ (upward closed)
3. If $U, V \in \tau(x)$, then $U \cap V \in \tau(x)$

Definition 5.1.4. *Let F = (W,R) be a Kripke frame. We define an n-frame $N(F)$ = (W, τ) as follows. For any $w \in W$ we have :*

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

Lemma 5.1.5. *Let $F = (W, R)$ be a Kripke frame. Then*

$$Log(F) = Log(N(F))$$

The proof is by structural induction.

Definition 5.1.6. *Let X = $(X, \tau_1,...)$ and Y = $(Y, \sigma_1,...)$ be n-frames. Then the function f: $X \to Y$ is called bounded morphism if*

1. f is surjective
2. $\forall x \in X\, \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X\, \forall V \in \sigma_i(f(x))\, \exists U \in \tau_i(x) : f(U) \subseteq V$

Definition 5.1.7. *Let $X = (X, \tau_1)$ and $Y = (Y, \tau_2)$ be two n-frames. Then the product of these two frames is an n-2-frame and is defined as follows :*

$$X \times_n Y = (X \times Y, \tau_1', \tau_2')$$
$$\tau_1'(x,y) = \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$
$$\tau_2'(x,y) = \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

Additionally, we say the full product of n-frames $X \times_n^+ Y$ is :

$$X \times_n^+ Y = (X \times Y, \tau_1', \tau_2', \tau) \text{ where}$$

$$\tau(x,y) = \{U \subseteq X \times Y \mid \exists W \in \tau_1(x) \, \exists V \in \tau_2(y) : W \times V \subseteq U\}$$

Definition 5.1.8. *For two unimodal logics L_1 and L_2 we define the n-product of them as follows :*

$$L_1 \times_n L_2 = Log(\{X \times Y \mid X \in nV(L_1) \text{ and } Y \in nV(L_2)\})$$

5.2 Main Construction

In the following, we will construct a useful neighborhood frame called $N_\omega[F]$ based on a frame F. We will use it later, to show $L_1 \times_n L_2 = L_1 \otimes L_2$ where $L_1, L_2 \in \{D, S4, D4, T\}$

Definition 5.2.1. *Let $F = (A^*, R) = F_{\xi\eta}[A]$ and $0 \notin A$. We define "pseudo-infinite" sequences*

$$X = \{a_1 a_2 a_3 ... \mid a_i \in A \cup \{0\} \text{ and } \exists N \forall k \geq N : a_k = 0\}$$

Furthermore, we define $f_F : X \to A^*$ to be the function, that deletes all zeros.

Example : Say $12034002340^\omega \in X$ (0^ω denotes infinitely many zeros). Then $f_F(12034002340^\omega) =$ 1234234

Definition 5.2.2. *Let $F = (A^*, R) = F_{\xi\eta}[A]$ and $0 \notin A$. Assume the function f_F and the set X as defined before. For $\alpha \in X$ such that $\alpha = a_1 a_2 ...$ we define*

$$st(a) = min\{N \mid \forall k \geq N : a_k = 0\}$$
$$a \mid_k = a_1 a_2 ... a_k$$
$$U_k(\alpha) = \{\beta \mid f_F(\alpha) R f_F(\beta) \text{ and } \alpha \mid_m = \beta \mid_m, \text{ where } m = max((k, st(\alpha))\}$$

Remark : Let $\alpha \in X$ with $st(\alpha) = n$. Then we have that $U_n(\alpha) = U_j(\alpha)$ for any $j \leq n$

Lemma 5.2.3. *$U_k(\alpha) \subseteq U_m(\alpha)$, whenever $k \geq m$.*

Proof. Let $\beta \in U_k(\alpha)$. Since $\alpha \mid_k = \beta \mid_k$ and $k \geq m$, we have $\alpha \mid_m = \beta \mid_m$. It follows, $\beta \in U_m(\alpha)$. ☐

Definition 5.2.4. *Due to Lemma 5.2.3 the sets $U_n(\alpha)$ forms a filter base. So we can define :*

$$\tau(\alpha) \textit{ is a filter with base } \{U_n(\alpha) \mid n \in \mathbb{N}\}$$

$$N_\omega = (X, \tau) \textit{ is the n-frame based on } F$$

Lemma 5.2.5. *Let $F = (A^*, R) = F_{\xi\eta}[A]$. Based on that, let $N_\omega(F) = (X, \tau)$, $N(F) = (A^*, \sigma)$ and $f_F : N\omega(F) \to N(F)$. Then for any $m \in \mathbb{N}$ and $x \in X$ we have*

$$f_F(U_m(x)) = R(f_F(x))$$

Proof. $\subseteq$: Let $f_F(\alpha) \in f_F(U_m(x))$ with $\alpha \in U_m(x)$. By defintion of $U_m(x)$, we get $f_F(\alpha) \in R(f_F(x))$.

For the other direction, we pick $\vec{a} \in R(f_F(x))$. We have to find $\beta \in U_m(x)$ s.t $f_F(\beta) = \vec{a}$. We construct $\beta = x \mid_m \cdot \vec{c} 0^\omega$ where $\vec{c} = \vec{a} - f_F(x)$ and it is the path from $f_F(x)$ to $\vec{a}$ in R. Then we get

□

Lemma 5.2.6. *Let $F = (A^*, R) = F_{\xi\eta}[A]$. Then $f_F : N_\omega(F) \to N(F)$ is a bounded morphism.*

Proof. From now on this proof we will omit the subindex in f_F. Let $N_\omega(F) = (X, \tau)$ and $N(F) = (A^*, \sigma)$.
For surjectivity, we pick any $\vec{x} \in A^*$. But then, $\vec{x}\, 0^\omega \in X$. Hence, $f(\vec{x}\, 0^\omega) = \vec{x}$.
For the next condition, assume that $x \in X$ and $U \in \tau(x)$. We need to prove that $f(U) \in \sigma(f(x))$. That means $R(f(x)) \subseteq f(U)$.

□