# 1 Definition

## 1.1 Defintion

Let Prop be a set of variables. Then a formula $\phi$ is defined as follows :

$$\phi ::= p \mid \bot \mid \phi \mid \phi \to \phi \mid \Box_i \phi$$

where $p \in Prop$ and $\Box_i$ is a modal operator. Other connectives are expressed through $\bot$ and $\to$ and dual modal operators $\diamond_i$ as $\diamond_i \phi = \neg \Box_i \neg \phi$

## 1.2 Defintion

A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \to \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms

$$\Box_i (p \to q) \to (\Box_i p \to \Box_i q)$$

$K_n$ denotes the minimal normal modal logic with n modalities and $K = K_1$ Let L be a logic and let $\Gamma$ be a set of formulas. Then L+$\Gamma$ denotes the minimal logic containing L and $\Gamma$

## 1.3 Definition

Let L1 and L2 be two modal logic with one modality $\Box$. Then the fusion of these logics are defined as follows :

$$L1 \otimes L2 = K2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

The follow logics may be important

$$D = K + \Box p \to \Diamond p$$

$$T = K + \Box p \to p$$

$$D4 = D + \Box p \to \Box \Box p$$

$$S4 = T + \Box p \to \Box \Box p$$

# 2 Topological Space Defintion

## 2.1 Defintion

A topological space is a pair $(X, \tau)$ where $\tau$ is a collection of subsets of X (elements of $\tau$ are also called open sets) such that :

1. the empty set $\emptyset$ and X are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

A topological model is a structure M = (X,$\tau$,v) where (X,$\tau$) is a topological space and v is a valuation assigining subsets of X to propositional variables.

## 2.2 Defintion

Let M = (X,$\tau$,v) a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows :$M, x \models \Box\phi$ iff ,$\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$
$M, x \models \Box\phi$ iff ,$\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$
$M, x \models \Diamond\phi$ iff ,$\forall U \in \tau$ s.t $x \in U$ and $\exists u \in U : M, u \models \phi$

## 2.3 Defintion

Let $A = (X, \chi)$ and B =(Y, $\upsilon$) be topological spaces. The standard product topology $\tau$ is the set of subsets of $X \times Y$ such that $X \in \chi$ and $Y \in \upsilon$.
Let $N \subseteq X \times Y$. We call $N$ horizontally open if $\forall (x, y) \in N\ \exists U \in \chi : x \in U$ and $U \times \{y\} \subseteq N$.
We call $N$ vertically open if $\forall (x, y) \in N\ \exists V \in \upsilon : y \in V$ and $\{x\} \times V \subseteq N$
If $N$ is H-open and V-open, then we call it HV-open.
We denote $\tau_1$ is the set of all H-open subsets of $X \times Y$ and $\tau_2$ is the set of all V-open subsets of $X \times Y$

## 2.4 Defintion

Let X and Y be topological spaces and $f : X \to Y$ a function. We call f continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X. We say f is open if for each open set $V \subseteq X$ the set f[V] is open in Y.

## 2.5 Remark

There is an alternative defintion for open sets. Let (X,$\tau$) be a topological space and U a set. U is open iff $\forall x \in U \ \exists V \subseteq U$ : V is open and $x \in V$. This is true because, the union of open sets is an open set.

# 3 Neighbourhood

## 3.1 Defintion

Let X be a non-empty set. A function $\tau : X \to 2^{2^X}$ is called a neighbourhood function. A pair F = (X,$\tau$) is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X,$\tau$,v), where v assigns a subset of X to a variable

## 3.2 Defintion

Let $M$ =(X,$\tau$,v) be a neighourhood model and x $\in$ X. The truth of a formula is defined inductively as follows :

$$M, x \models \Box\phi \text{ iff } \exists V \in N(x) \forall y \in V : M, y \models \phi$$

A formula is valid in a n-model M if it is valid at all points of $M$ ($M \models \phi$). Formula is valid in a n-frame $F$ if it is valid in all models based on $F$ (notation $F \models \phi$). For Logic L we write $F \models L$, if for any $\phi \in L, F \models \phi$. We define nV(L) = $\{F \mid F \text{ is an n-frame and } F \models \phi\}$.

## 3.3 Defintion

Let F = (W,R) be a Kripke frame. We define an n-frame $N$(F) = (W, $\tau$) as follows. For any $w \in W$ we have :

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

## 3.4 Defintion

Let $X$ = (X, $\tau_1$,...) and $Y$ = (Y, $\sigma_1$,...) be n-frames. Then the function f: $X \to Y$ is called bounded morphism if

1. f is surjective
2. $\forall x \in X \, \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X \, \forall V \in \sigma_i(f(x)) \, \exists U \in \tau_i(x) \, : f(U) \subseteq V$

## 3.5 Defintion

Let $X = (\mathrm{X}, \tau_1)$ and $Y = (\mathrm{Y}, \tau_2)$ be two n-frames. Then the product of these two frames is an n-2-frame and is defined as follows :

$$X \times Y = (\mathrm{X} \times \mathrm{Y}, \tau_1', \tau_2')$$
$$\tau_1'(x, y) = \{U \subseteq \mathrm{X} \times \mathrm{Y} \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$
$$\tau_2'(x, y) = \{U \subseteq \mathrm{X} \times \mathrm{Y} \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

## 3.6 Defintion

For two unimodal logics $L_1$ and $L_2$ we define the n-product of them as follows :

$$L_1 \times_n L_2 = Log(\{X \times Y \mid X \in nV(L_1) \text{ and } Y \in nV(L_2)\})$$