1 Introduction

Modal logic extends classical propositional logic by introducing modal operators, most notably $\Box$ ("necessarily") and $\Diamond$ ("possibly"). In the standard Kripke semantics, these operators are interpreted over a set of possible worlds connected by an accessibility relation. An idea to generalize Kripke semantics was topology. The study of topological semantics in modal logic was initiated by McKinsey and Tarski in 1944 [3]. It is well known, that $S4$ is fully captured. Despite its rich expressiveness, it can not fully capture Kripke semantics. That is why Neighbourhood semantics (see [11] for an introduction) as a generalization of Kripke semantics for modal logic was introduced independently by Dana Scott [9] and Richard Montague [10]. Neighborhood semantics is more general than Kripke semantics and in the case of normal reflexive and transitive logics coincides with topological semantics. The original motivation for introducing these general models was to provide a semantics for non-normal modal logics. Beyond their theoretical motivation, interest in topological semantics and neighborhood frames has grown considerably, partly due to its applications in artificial intelligence.

Oftentimes, it is necessary to combine frames for different modal logics into a complex frame. The natural way of doing that is a product construction. For Kripke frames, the resulting product is the Cartesian product of the two frames with two accessibility relations. For topological semantics, the product of topogical spaces as bi-topological spaces with so-called horizontal and vertical topologies have been considered. In a similar fashion, the product of neighborhood frames was introduced by Sano in [8].

For our main question, we consider the neighborhood product additionally with the standard topology. We will answer whether the logic with three copies of T with the interaction axiom $\Box p \to \Box_1 p \wedge \Box_2 p$ can be fully characterized as the logic a product of neighbourhood frames with standard topology, where each frame validates T. The question was originally motivated by two papers where the authors showed in topological semantics a similar version of the question but for $S4$. In the other, it was proven that for any $L_1, L_2 \in \{D, D4, T, S4\}$, that their logic together can be characterized as the logic of a product of neighbourhood frames but without standard topology. In this work, we start with a brief recap of necessary notations and theorems in Kripke semantics and then introduce toplogical spaces and neighborhood frames. We will reprove the theorems from the authors in a more detailed way and then use similar techniques to answer our main question. Finally, we conclude with a discussion of potential directions for future research related to the topic. This thesis primarily builds upon the works "Multimodal Logics of Products of Topologies" [1] and "Modal Logic of Products of Neighborhood Frames" [2].

2 Preliminaries

This chapter is primarily devoted to the semantics of Kripke, topological spaces, and neighborhood frames. In addition, we define the notions of product frames and multimodal logic. We also present fundamental results in modal logic, such as Filtration Theorem and Sahlqvist Theorem which play a role in our later chapters. The content is mainly based on the Modal Logic book [7].

Definition 2.0.1. *Let prop be a set of variables. Then a formula ϕ is defined as follows:*

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \Box_i \phi$$

where $p \in Prop$ and $\Box_i$ is a modal operator. Other connectives are expressed through $\bot$ and $\rightarrow$ and dual modal operators $\Diamond_i$ as $\Diamond_i \phi = \neg\Box_i\neg\phi$

Definition 2.0.2. *Let $M = (W, R_1 ..., R_n, V)$ be a model and $w \in W$ a state in M. The notion of a formula being true at w is inductively defined as follows :*

$$\begin{aligned}
&M, w \Vdash p \text{ iff } w \in V(p) \\
&M, w \Vdash \bot \text{ never} \\
&M, w \Vdash \neg\phi \text{ iff not } M, w \Vdash \phi \\
&M, w \Vdash \phi \vee \psi \text{ iff } M, w \Vdash \phi \vee M, w \Vdash \psi \\
&M, w \Vdash \Box_i \phi \text{ iff } \forall v \in W : wRv \rightarrow M, v \Vdash \phi
\end{aligned}$$

Definition 2.0.3. *A set Σ is closed under subformulas, if for all formulas ϕ and ϕ' the following holds :*

1. *if $\neg\phi \in \Sigma$ then $\phi \in \Sigma$*
2. *if $\phi \vee \phi' \in \Sigma$ then $\phi, \phi' \in \Sigma$*
3. *if $\Box_i \phi \in \Sigma$ then $\phi \in \Sigma$*

Example : Suppose we have a multimodal logic with $\Box, \Box_1$ called $\mathcal{L}_{\Box,\Box_1}$ where $\phi = \Box p \rightarrow \Box_1 q$ and $\phi \in \Sigma$. Then $\Sigma = \{\phi, \neg\Box p, \Box_1 q, \Box p, p, q\}$ is closed under subformulas.

Definition 2.0.4. *Let $M = (W, R_1, ..., R_n, V)$ be a model and suppose Σ is a set of formulas. We define a relation $\equiv$ on W as follows :*

$$w \equiv v \text{ iff } \forall \phi \in \Sigma : M, w \Vdash \phi \Leftrightarrow M, v \Vdash \phi$$

It is well known that the $\equiv$-relation is an equivalence relation. We denote the equivalence class of a state $w \in W$ as $[w]_\Sigma = \{v \mid v \equiv w\}$. Furthermore W_Σ is the set of all equivalence classes, i.e $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$.

Definition 2.0.5. *Let $M = (W, R_1, ..., R_n, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes induced by $\equiv$. A model $M^f_\Sigma = (W^f, R^f_1, ..., R^f_n, V^f)$ is called filtration of M through Σ if the following holds :*

1. $W^f = W_\Sigma$
2. If $(w,v) \in R_i$ then $([w],[v]) \in R_i^f$
3. If $([w],[v]) \in R_i^f$ then for any $\Box_i\phi \in \Sigma$: if $M,w \Vdash \Box_i\phi$ then $M,v \Vdash \phi$
4. $V^f = \{[w] \mid M,w \Vdash p\}$, for all propositional variables $p \in \Sigma$

Theorem 2.0.6. *Consider $L_{\Box_1...\Box_n}$. Let $M^f = (W_\Sigma, R_1^f, ..., R_n^f, V)$ be a filtration of M through a subformula closed set Σ. Then for all formulas $\phi \in \Sigma$, and all nodes $w \in M$, we have*

$$M,w \Vdash \phi \text{ iff } M^f,[w] \Vdash \phi$$

Proof. By induction on ϕ. We will only show non-trivial and, for our purposes, necessary cases.

Case $\phi = p$: Left to right follows immediately from filtration defintion. Conversely, suppose $M^f,[w] \Vdash p$. This means $[w] \in V^f(p)$. But this means V(p) can not be empty. Pick any $v \in V(p)$. Obviously, $w \equiv v$ and $M,v \Vdash p$. Hence, $M,w \Vdash p$.

Case $\phi = \neg\psi$: Suppose ψ holds. Then we have : $M,w \Vdash \phi$ iff $M,w \nVdash \psi$. Applying induction hypothesis, we get : $M^f,[w] \nVdash \psi$. But then, we have $M^f,[w] \Vdash \phi$. Right to left is the same.

Case $\phi = \phi_1 \wedge \phi_2$: Suppose ϕ_1, ϕ_2 holds. Let $M,w \Vdash \phi$. That means $M,w \Vdash \phi_1$ and $M,w \Vdash \phi_2$. Applying induction hypothesis, we get $M^f,[w] \Vdash \phi_1$ and $M^f,[w] \Vdash \phi_2$. But then, $M^f,[w] \Vdash \phi_1 \wedge \phi_2 = \phi$. Right to left is similar.

Case $\phi = \Box_i\psi$: Left to right. Suppose ψ holds and $M,w \Vdash \Box_i\psi$. We need to show $M^f,[w] \Vdash \Box_i\psi$, this means $\forall[v] \in W_\Sigma : [w]R_i[v] \to M^f,[v] \Vdash \Box_i\psi$. Pick any $[v] \in W_\Sigma$ s.t $[w]R_i[v]$. By condition 3, w.r.t to the modal operator, we have $M,v \Vdash \psi$. By induction hypothesis, we get $M^f,[v] \Vdash \psi$. Because [v] was arbitrary it follows that $M^f,[w] \Vdash \Box_i\psi$.

Right to left. Suppose ψ holds and $M^f,[w] \Vdash \Box_i\psi$. Pick $v \in W$ s.t wR_iv. By condition 2, w.r.t to the modal operator, we have $[w]R_i^f[v]$. So, $M^f,[v] \Vdash \psi$. By induction hypothesis, we get $M,v \Vdash \psi$. Because v was arbitrary, we have $M,w \Vdash \Box_i\psi$.

□

Definition 2.0.7. *Let $M = (W, R_1, ..., R_n, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes. We define :*

$$R^s = \{[w],[v] \mid \exists w' \in [w], \exists v' \in [v] : w'R_iv'\}$$

Lemma 2.0.8. *Let $M = (W, R_1, ..., R_n, V)$ be a model, Σ is closed under subformlas and W_Σ the set of equivalence classes induced by $\equiv$ and V^f the standard valuation on W_Σ. Then $(W_\Sigma, R_1^s, ..., R_n^s, V^f)$ is a filtration of M through Σ.*

Proof. It suffices to show R_i^s fullfills the condition 2 and 3 w.r.t to the corresponding modal operator $\Box_i$. But R_i^s already satisfies condition 2. Let's check the other condition. Let $\Box_i\phi \in \Sigma$, $[w]R_i^s[v]$ and $M, w \Vdash \Box_i\phi$. Because of $[w]R_i^s[v]$ we pick a $w' \in [w]$ and $v' \in [v]$. By defintion, we have $w'R_iv'$. Because $w' \equiv w$, we get $M, w' \Vdash \Box_i\phi$. Hence, $M, v' \Vdash \phi$ and by $v' \equiv v$, we get $M, v \Vdash \phi$.

$\Box$

Proposition 2.0.9. *Let Σ be a finite subformula closed set of $L_{\Box_1,\ldots,\Box_n}$. For any model M, if M^f is a filtration through Σ, then M^f contains at most 2^n nodes (where n denotes the size of Σ).*

Proof. The states of M^f are the equivalence classes in W_Σ. Let $g : W_\Sigma \to P(\Sigma)$ defined by $g([w]) = \{\phi \in \Sigma \mid M, w \Vdash \phi\}$. g is well defined. Pick any u and v s.t $u \equiv v$. But then by defintion of $\equiv$, they fullfill the same subformulas. This means g([v]) = g([u]).
g is also injective. Pick any $[u], [v] \in W_\Sigma$ s.t g([u]) = g([v]). We show $[u] \subseteq [v]$. The other inclusion is similar. By assumption we have $u \equiv v$. Pick any $u' \in [u]$. Then we have $u' \equiv u \equiv v$. Hence, $u' \in [v]$. At the end, this means M^f contains at most 2^n nodes.

$\Box$

Theorem 2.0.10. *Let ϕ be a formula of $L_{\Box_1,\ldots,\Box_n}$. If ϕ is satisfiable, then it is satisfiable on a finite model containing at most 2^v nodes, where v is the number of subformulas of ϕ.*

Proof. Assume that ϕ is satisfiable on a model on M. Take any filtration of M through the set of subformulas of ϕ. By Filtration Theorem, we get that ϕ is satisfied in the filtration model M^f. Furthermore, it is bounded by 2^v.

$\Box$

Now we define special kind of formulas called Sahlqvist formulas. Sahlqvist formulas possess important properties, which are guaranteed by the Sahlqvist Theorem. It says that, when given a normal modal logic K and a set of Sahlqvist formulas, the resulting logic is complete w.r.t to the class of frames, which satisfies the corresponding first-order formula of the Sahlqvist formulas. This also holds for multimodal logic. We will not prove it here, but we will use it later in this work.

Definition 2.0.11. *A modal formula ϕ is positive if all variables occurs in the scope of an even number of negations. In the other hand, a formula is negative, if all variables occurs in the scope of an odd number of negations. A boxed atom is a modal formula of the form $\Box^n p$ for some $n \in \mathbb{N}$, where p is a propositional variable and $\Box^n p$ is defined as follows : $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$.*

Furthermore, a Sahlqvist antecedent is built from $\bot, \top$, negative formulas and boxed atoms by applying $\Diamond$ and $\wedge$. A Sahlqvist implication is a modal formula of the form $\phi \to \psi$, where ϕ is a Sahlqvist antecedent and ψ a positive formula.
Now, a Sahlqvist formula is built from Sahlqvist implications by applying $\Box$ and $\vee$.

Examples for Sahlqvist formulas:

$$\Box\Box p \to \Box p$$

$$\Diamond\neg p \to p$$

$$\Diamond\Box\Box\Box\Box\Box p \to \Box\Diamond\Box\Diamond p$$

$$\Box\Box\Box\Box(\Diamond\Box p \to p) \vee \Box\Box p \to \Box p$$

Non Sahlqvist Formulas :

$$\Box\Diamond p \to \Diamond\Box p$$

$$\Diamond\Box p \to \Box\neg p$$

We can extend this defintion for multimodal logics. We say a boxed atom can be $\Box_i^n p$. A Sahlqvist antecedent can also be build by applying $\Diamond_i$. A Sahlqvist formula can be build by Sahlqvist implications by applying additionally $\Box_i$.

Definition 2.0.12. *Let $F = (W, R_1, R_2, ...)$ and $F' = (W', R'_1, R'_2, ...)$ be two frames. A bounded morphism from F to F' is a function $f : W \to W'$ satisfying the following conitions:*

$$\text{If } (u, v) \in R_i \text{ then } (f(u), f(v)) \in R'_i$$

$$\text{If } (f(w), v') \in R'_i \text{ then } \exists v \in W \text{ s.t } (w, v) \in R'_i \text{ and } f(v) = v'$$

We say F' is a bounded morphic image of F, if there is a surjective bounded morphism from F to F'.

Proposition 2.0.13. *Let ϕ be a formula in $L_{\Box_1,...,\Box_n}$, $F = (W, R_1, ..., R_n)$ and $F' = (W', R'_1, ..., R'_n)$ be two frames and F to F' a surjective bounded morphism. Then the following holds :*

$$\text{If } F \Vdash \phi \text{ then } F' \Vdash \phi$$

Proof. This can be shown by structural induction on the length of the formula. $\Box$

Corollary 2.0.14. *If F' is a bounded morphic image of F, then we have $Log(F) \subseteq Log(F')$*

Proof. The proof is by structural induction.

$\Box$

In this work, we deal with products of frames and multimodal logics of product of frames. The study of products of Kripke frames and their modal logics was first initiated by Segerberg [4] and Shehtman [5]. In the following, we will define neccessary notions for such logics.

Definition 2.0.15. *Let $F = (W, R_1)$ and $G = (W, R_2)$. Then we define the Kripke product on $W \times V$ as follows :*

$$(w, v)R'_1(w', v') \text{ iff } wR_1w' \text{ and } v = v'$$

$$(w, v)R'_2(w', v') \text{ iff } w = w' \text{ and } vR_2v'$$

R'_1 is also called horizontal and R'_2 vertical relation.

Definition 2.0.16. *A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \to \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms*

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$. Let L be a logic and let Γ be a set of formulas. Then L+Γ denotes the minimal logic containing L and Γ

Definition 2.0.17. *Let L_1 and L_2 be two modal logic with one modality $\Box$. Then the fusion of these logics are defined as follows :*

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

The following logics may be important

$$D = K + \Box p \to \Diamond p$$

$$T = K + \Box p \to p$$

$$D4 = D + \Box p \to \Box\Box p$$

$$S4 = T + \Box p \to \Box\Box p$$

2.1 Topological semantics, product of topological spaces

Topological space opens a new area of expressiveness. It is well known that such logic is $S4$. It has some crucial applications for example in spatial reasoning or epistemic logic.

Definition 2.1.1. *A topological space is a pair (X, τ) where τ is a collection of subsets of X (elements of τ are also called open sets) such that :*

1. *the empty set $\emptyset$ and X are open*
2. *the union of an arbitrary collection of open sets is open*
3. *the intersection of finite collection of open sets is open*

The space is called Alexandroff, if we allow the intersection of infinite collection of open sets. A topological model is a structure $M = (X, \tau, v)$ where (X, τ) is a topological space and v is a valuation assigining subsets of X to propositional variables.

Example 2.1.2. *Let $X = \{1, 2, 3\}$. We consider different topologies on X:*

- *If $\tau = \{\emptyset, X\}$, then (X, τ) is called the trivial topological space.*
- *If $\tau = \mathcal{P}(X)$, then (X, τ) is called the discrete topological space.*
- *If $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, then (X, τ) is topological space.*

Remark 2.1.3. *There is an equivalent defintion for open sets. Let (X,τ) be a topological space and U a set. U is open iff $\forall x \in U\ \exists V \subseteq U : V$ is open and $x \in V$. This is true because, the union of open sets is an open set.*

Definition 2.1.4. *Let $M = (X, \tau, v)$ be a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows:*

$$
\begin{array}{ll}
M, x \vDash p & \text{iff } x \in v(p) \\
M, x \vDash \bot & \text{never} \\
M, x \vDash \neg\phi & \text{iff } M, x \nvDash \phi \\
M, x \vDash \phi \vee \psi & \text{iff } M, x \vDash \phi \text{ or } M, x \vDash \psi \\
M, x \vDash \Box\phi & \text{iff there exists } U \in \tau \text{ such that } x \in U \text{ and } \forall u \in U,\ M, u \vDash \phi
\end{array}
$$

Definition 2.1.5. *Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \upsilon)$ be topological spaces and $N \subseteq X \times Y$.*

We define

Horizontally open: *N is horizontally open iff*

$$\forall (x, y) \in N\ \exists U \in \chi \text{ such that } x \in U \text{ and } U \times \{y\} \subseteq N.$$

Vertically open: *N is vertically open iff*

$$\forall (x, y) \in N\ \exists V \in \upsilon \text{ such that } y \in V \text{ and } \{x\} \times V \subseteq N.$$

(a) vertically (b) horizontally

Figure 1: Each rectangle represents N and the redlines are subsets of N

If N is H-open and V-open, then we call it HV-open.
We denote τ_1 is the set of all H-open subsets of $X \times Y$ and τ_2 is the set of all V-open subsets of $X \times Y$. We say τ is the standard product of subsets $X \times Y$ s.t. $X \in \chi$ and $Y \in \upsilon$.

Example 2.1.6. *Let $\mathcal{X} = (\mathbb{R}, \chi)$ and $Y = (\mathbb{R}, \upsilon)$ with standard topology, that means the basis $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} \mid a < b\}$. Now consider $\mathcal{X} \times \mathcal{Y}$. The horizontal topology is generated by $\mathcal{B}_1 = \{U \times \{y\} \mid U \in \chi$ and $y \in \mathbb{R}\}$ and vertical topology by $\mathcal{B}_2 = \{\{x\} \times V \mid x \in \mathbb{R}$ and $V \in \upsilon\}$*

Definition 2.1.7. *Let $A = (X, \chi)$ and $B = (Y, \upsilon)$ be topological spaces and $(x, y) \in X \times Y$. The truth in $M = (X \times Y, \tau_1, \tau_2, \tau, \upsilon)$ is similar as in Definition 2.1.4. For $\Box$, $\Box_1$ and $\Box_2$ we interpret this as follows :*

$M, (x, y) \vDash \Box_1\phi$ *iff* $\exists U \in \tau_1 : (x, y) \in U$ *and* $\forall (x', y') \in U : (x', y') \vDash \phi$
$M, (x, y) \vDash \Box_2\phi$ *iff* $\exists V \in \tau_2 : (x, y) \in V$ *and* $\forall (x', y') \in V : (x', y') \vDash \phi$
$M, (x, y) \vDash \Box\phi$ *iff* $\exists U \in \chi \exists V \in \upsilon : (x, y) \in U \times V$ *and* $\forall (x', y') \in U \times V : (x', y') \vDash \phi$

Definition 2.1.8. *Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \upsilon)$ be two topological spaces. Then the full product of these spaces is defined as follows :*

$$\mathcal{X} \times_t^+ \mathcal{Y} = (X \times Y, \tau_1, \tau_2, \tau)$$

with horizontal, vertical and standard topologies.

Definition 2.1.9. *For two unimodal logics L_1 and L_2 we define the full t-product of them as follows :*

$$L_1 \times_t^+ L_2 = Log(\{\mathcal{X} \times \mathcal{Y} \mid \mathcal{X} \vDash L_1 \text{ and } \mathcal{Y} \vDash L_2\})$$

where X, Y are topological spaces with horizontal, vertical and standard topology.

Definition 2.1.10. *A topo-bisimulation between two topological models $M = (X, \tau, \upsilon)$ and $M' = (X', \tau', \upsilon')$ is a nonempty relation $\leftrightarroweq \subseteq X \times X'$ s.t. if $x \leftrightarroweq x'$ then*

1. Base: $x \in V(p)$ iff $x' \in \upsilon'(p)$ for any propositional variable p
2. Forth: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ s.t. $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \leftrightarroweq y'$
3. Back: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ s.t. $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \leftrightarroweq y'$

Proposition 2.1.11. *Let $M = (X, \tau, \upsilon)$ and $M' = (X', \tau', \upsilon')$. Assume $w \in M'$ and $w' \in M'$ are topo-bisimilar points. Then for each formula ϕ we have*

$$M, w \vDash \phi \text{ iff } M', w' \vDash \phi$$

Proof. The proof is by structural induction. □

Definition 2.1.12. *Let X and Y be topological spaces and $f : X \to Y$ a function. We call f continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X. We say f is open if for each open set $V \subseteq X$ the set f[V] is open in Y.*

2.2 Product and semantics of neighborhood frames

Neighborhood frames are more flexible than topological space. Its logic is not $S4$ in general. It is well known, that neighborhood frames fully captures the semantics of Kripke. Moreover, neighborhood frames are expressive enough to characterize non-normal modal logics, which makes them a powerful tool.

Definition 2.2.1. *Let X be a non-empty set. A function $\tau : X \to 2^{2^X}$ is called a neighbourhood function. A pair F = (X,τ) is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X,τ,v), where v assigns a subset of X to a variable*

Definition 2.2.2. *Let M = (X,τ,v) be a neighourhood model and $x \in X$. The truth of a formula is defined inductively as follows :*

$$M, x \Vdash p \text{ iff } x \in V(p)$$
$$M, x \Vdash \bot \text{ never}$$
$$M, x \Vdash \neg\phi \text{ iff } M, x \nVdash \phi$$
$$M, x \Vdash \phi \vee \psi \text{ iff } M, x \vDash \phi \vee M, x \vDash \psi$$
$$M, x \Vdash \Box\phi \text{ iff } \exists V \in N(x) \forall y \in V : M, y \models \phi$$

A formula is valid in a n-model M if it is valid at all points of M ($M \vDash \phi$). Formula is valid in a n-frame F if it is valid in all models based on F (notation $F \vDash \phi$). We write $F \vDash L$ if for any $\phi \in L, F \vDash \phi$. Logic of a class of n-frames C as $Log(C) = \{\phi \mid F \vDash \phi$ for some $F \in C\}$ We define $nV(L) = \{F \mid F$ is an n-frame and $F \models \phi\}$.

Definition 2.2.3. *Let X be a non-empty set and τ neighborhood function. We call τ is a filter if for each $x \in X$ the collection $\tau(x)$ satisfies the following conditions :*

1. $\emptyset \notin \tau(x)$
2. *If $U \in \tau(x)$ and $U \subseteq V$ then $V \in \tau(x)$ (upward closed)*
3. *If $U, V \in \tau(x)$, then $U \cap V \in \tau(x)$*

Definition 2.2.4. *Let X = (X, τ_1,...) and Y = (Y, σ_1,...) be n-frames. Then the function f: $X \to Y$ is called bounded morphism if*

1. *f is surjective*
2. $\forall x \in X \, \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X \, \forall V \in \sigma_i(f(x)) \, \exists U \in \tau_i(x) : f(U) \subseteq V$

Corollary 2.2.5. *Let X = (X, τ_1,...) and Y = (Y, σ_1,...) be n-frames and $f : X \to Y$ a bounded morphism. Then*

$$Log(X) \subseteq Log(Y)$$

Proof. The proof is by structural induction. $\square$

In topology, we defined horizontal and vertical topologies for products of topological spaces. In the following, we will define a similar construction for neighborhood frames.

Definition 2.2.6. *Let $\mathcal{X} = (X, \tau_1)$ and $\mathcal{Y} = (Y, \tau_2)$ be two n-frames. Then the product of these two frames is a n-2-frame and is defined as follows :*

$$\mathcal{X} \times_n \mathcal{Y} = (X \times Y, \tau_1', \tau_2')$$
$$\tau_1'(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$
$$\tau_2'(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

Additionally, we say the full product of n-frames $\mathcal{X} \times_n^+ \mathcal{Y}$ is :

$$\mathcal{X} \times_n^+ \mathcal{Y} = (X \times Y, \tau_1', \tau_2', \tau) \textit{ where}$$

$$\tau(x, y) = \{U \subseteq X \times Y \mid \exists W \in \tau_1(x) \, \exists V \in \tau_2(y) : W \times V \subseteq U\}$$

Definition 2.2.7. *For two unimodal logics L_1 and L_2 we define the n-product of them as follows :*

$$L_1 \times_n L_2 = Log(\{\mathcal{X} \times \mathcal{Y} \mid \mathcal{X} \in nV(L_1) \textit{ and } \mathcal{Y} \in nV(L_2)\})$$

In similar way, we can define $L_1 \times_n^+ L_2$ with three topologies.

Proposition 2.2.8 ([8])**.** *For two unimodal logics L_1 and L_2 it holds :*

$$L_1 \otimes L_2 \subseteq L_1 \times_n L_2$$

After introducing the necessary notations, we can now formally state our main research question as follows: Does the following equivalence hold?

$$T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p = T \times_t^+ T$$

We will answer this in the following chapters. From now on, we also abbreviate

$$T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$$

as *TNL*.

3 Completeness result for TNL

We introduce a frame called $T_{\omega,\omega,\omega[rn]}$. We will show TNL is sound and complete w.r.t $T_{\omega,\omega,\omega[rn]}$. The idea is to pick a class of frame C s.t $Log(C) = Log(T_{\omega,\omega,\omega[rn]})$ and then show the class has FMP. In the end, we will use an unravelling technique to show completeness.

Definition 3.0.1. *We say for a modal logic Λ has the finite model property (FMP) if for every formula ϕ that is not provable in Λ, is falsifiable in a finite model.*

Proposition 3.0.2. *The logic $T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$ has FMP.*

Proof. The idea is to pick a class of frame $\mathcal{F}$, where TNL is sound and complete with respect to and then show by filtration the FMP.

Let $\mathcal{C} = \{F \mid F \Vdash TNL\}$. Obviously, TNL is sound w.r.t $\mathcal{C}$. Completeness can be shown by using the Sahlqvist Theorem. We remember T = K + $\Box_i p \to p$ for $i \in \{1, 2, \epsilon\}$. These axioms and $\Box p \to \Box_1 p \wedge \Box_2 p$ are Sahlqvist formulas. By Sahlqvist, we have TNL is complete w.r.t $\{F \mid F \Vdash \forall x\, R_i(x, x)$ and $F \Vdash \forall x \forall y(R_1(x, y) \vee R_2(x, y)) \to R(x, y)\}$.

Now assume a formula ϕ is not derivable from TNL. By completeness we get ϕ is falsifiable in a model M and a world w. Hence, $M, w \Vdash \neg\phi$. Now we build the set subf($\neg\phi$) which denotes the closed subformulas set of $\neg\phi$. Let $M^s = (W_\Sigma, R^s, R^s_1, R^s_2, V^s)$ be the smallest filtration of M through subf($\neg\phi$) and V^s is the standard valuation on W_Σ. By filtration Theorem, they preserve truth. It remains to show $F^s = (W_\Sigma, R^s, R^s_1, R^s_2) \in \mathcal{C}$. For that, every point must have an edge to itself in every relation and it must hold that $R^s_1, R^s_2 \subseteq R^s$. For the first one, we show this for R^s because the rest is similar. Pick $w' \in [w] \in W_\Sigma$. Because M is based on a frame $F \in \mathcal{C}$, we have $w'Rw'$. By the defintion of smallest filtration, we have that $[w']R^s[w']$. For the second one, we show only for R^s_1 because it is the same for R^s_2. Pick [w],[v] s.t $[w]R^s_1[v]$. By defintion of smallest filtration there are points $w' \in [w]$ and $v' \in [v]$ s.t $w'R_1v'$. Furthermore, it holds $R_1 \subseteq R$, so $w'Rv'$. By smallest filtration, we get $[w']R^s[v']$. Because $w' \equiv w$ and $v' \equiv v$, we have $[w'] = [w]$ and $[v'] = [v]$. It follows $[w]R^s[v]$.

$\square$

Definition 3.0.3. *Let $T_{\omega[rn]}$ (r = reflexive, n = non-transitiv) denote the infinite branching and infinite depth tree, which is reflexive and non-transitive. Formally the tree can be defined as : $T_{\omega[rn]} = (W, R)$ where $W = \mathbb{N}^*$ and sRt iff $\exists u \in \mathbb{N} \cup \{\epsilon\} : s \cdot u = t$ ("$\cdot$" is the concatenation operator)*

The $T_{\omega,\omega,\omega[rn]}$ tree has three relations with infinite branching and infinite depth and we have $R_1, R_2 \subseteq R$. Before characterizing it, we say $\mathbb{N}_1*$ is the set of finite number combinations which has a subscript "1" to denote that these numbers relate to R_1 (examples : $2_1, 4231123_1$, $32_1\epsilon 45_1 \epsilon 9_1 = 32459_1$).

Now let $T_{\omega,\omega,\omega[rn]} = (W, R, R_1, R_2)$ *where* $W = (\mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2)^*$,

$$sRt \text{ iff } \exists u \in \mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_1t \text{ iff } \exists u \in \mathbb{N}_1 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_2t \text{ iff } \exists u \in \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

where $s, t \in W$. *Again, the "·" operator acts here as a concatenation operator.*

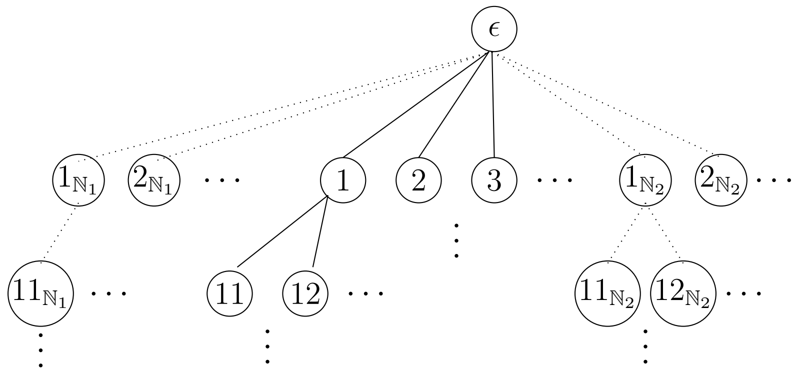

Figure 2: This is $T_{\omega,\omega,\omega[rn]}$. The dotted lines on the left are from R_1 and in the right R_2. The whole tree is R.

Proposition 3.0.4. *T is sound and complete w.r.t* $T_{\omega[rn]}$.

Proof. Sound is clear. For completeness, we use the well known fact that T has FMP. This means, $T = Log\{F \mid F \Vdash T\}$ where F is a finite frame. We can pick such a finite frame F and it suffices to find a surjective bounded morphism f from T_w to F. This would imply $Log(T_{w[rn]}) \subseteq T$.
Now, let $F = (W', R')$ be such a finite rooted frame with root w. We define inductively an assignment of nodes of F to the nodes of T_ω. For the base case, we assign w to the root of T_ω. The induction step looks like the following : Assume a point $x \in T_\omega$ has been assigned to a point $u \in F$ but the successors of x has no assignment. Let $s_1, s_2, ..., s_k$ be successors of u (k denotes amount of successors and $k \geq 1$ because reflexivity guarantees us at least one successor). For $n \geq 1, n \in \mathbb{N}, i \in \{1, ..., k\}$ we assign s_i to the $(n * i)$th-successor of x. This means we are assigining the successors alternatingly.
Now we check for f the conditions of bounded morphism. First condition : Let $x, y \in T_\omega$ s.t xRy and $f(x) = s$. But then, y will be assigned to a successor point of s. Hence, $f(x)R'f(y)$. Second condition : Suppose $f(x)R't$ and $f(x) = s$. Since t is a successor of s and $f(x) = s$, then a successor of x, say y, gets the assignment t.

$\square$

Proposition 3.0.5. *TNL is sound and complete w.r.t* $T_{\omega,\omega,\omega[rn]}$.

Proof. For soundness, we have that $T_{\omega,\omega,\omega[rn]} \Vdash \Box p \to \Box_1 p \land \Box_2 p$, because by defintion we have $R_1, R_2 \subseteq R$. The rest is clear. For completeness, we use the fact *TNL* has FMP. Let $F = (W', R', R'_1, R'_2)$ be a finite rooted frame with root w and $F \Vdash TNL$. We define inductively an assignment similar to 2.17. We assign w to the root of $T_{\omega,\omega,\omega[rn]}$. For induction step we start by only assigining points from R_1 to R'_1 and R_2 to R'_2. After that, the remaining points will be assigned to the points in R'. The procedure works similar as described previously.

We will only check the conditions for R and R'. For R_1, R'_1 and R_2, R'_2, we can argue as before. We pick any xRy with $f(x) = s$. If we also have xR_1y or xR_2y, then it follows $f(x)R'f(y)$ because $R_1, R_2 \subseteq R$ and $R'_1, R'_2 \subseteq R'$. Else, the successor of s was assigned to y, so $f(x)R'f(y)$.

Let $f(x)R't$ and $f(x) = s$. If we have $f(x)R'_1t$ or $f(x)R'_2t$, then the second condition follows because $R_1, R_2 \subseteq R$. Else, we have that t is a successor of $f(x) = s$, and a successor of x was assigned to t. $\square$

4 Logic of product of topological spaces with three topologies

In this section, we will reprove the logic of the product of two topological spaces with horizontal, vertical and classic product topology is $S4 \otimes S4 \otimes S4 + \Box p \to \Box_1 p \wedge \Box_2 p$ (we abbreviate the logic $S4 \otimes S4 \otimes ...$ as TPL). First we will show a version for one modality because it is easier to understand and the proof idea is based on this version.

Definition 4.0.1. *Let T_2 be the infinite binary tree with reflexive and transitive descendant relation.*
Formally it is defined as follows : $T_2 = (W, R)$ where $W = \{0,1\}^*$ and sRt iff $\exists u \in W^* : s \cdot u = t$.
The $T_{6,2,2}$ tree is the infinite six branching tree, where all nodes of $T_{6,2,2}$ is R-related, the first two R_1-related and the last two R_2-related. Formally we can define this tree as follows : $T_{6,2,2} = (W, R, R_1, R_2)$, where $W = \{0,1,2,3,4,5\}^*$,

$$sRt \text{ iff } \exists u \in \{0,1,2,3,4,5\}^* : s \cdot t = u$$

$$sR_1t \text{ iff } \exists u \in \{0,1\}^* : s \cdot t = u$$

$$sR_2t \text{ iff } \exists u \in \{5,6\}^* : s \cdot t = u$$

where $s, t \in W$ and "$\cdot$" is the concatenation-operator.

Theorem 4.0.2. *S4 is complete with respect to T_2.*

Proof. The idea is to use the fact S4 has finite model property. By that we can pick a finite rooted S4-frame and then show that this frame is the bounded morphic image of T_2. For details see [Goldblatt]. □

Theorem 4.0.3. *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to $\mathbb{Q}$.*

Proof. For a proof see e.g [?, Page 217, Theorem 2]. □

Our strategy is as follows. We use completeness of $S4$ w.r.t. T_2, view T_2 as an Alexandroff space, define a dense subset X of $\mathbb{Q}$ without endpoints, and then establish a topo-bisimulation between X and T_2. This will allow us to transfer counterexamples from T_2 to X, which by Cantor's theorem is order-isomorphic and hence homeomorphic to $\mathbb{Q}$.
Now let us define X as $X = \bigcup_{n\in\mathbb{N}} X_n$ where $X_0 = \{0\}$ and

$$X_{n+1} = X_n \cup \{x - \frac{1}{3^n}, x + \frac{1}{3^n} \mid x \in X_n\}$$

Lemma 4.0.4. *For $n > 0$ and $x, y \in X_n, x \neq y$ we have : $|x - y| \geq \frac{1}{3^{n-1}}$*

Proof. The proof is by induction on n. For the base case, if $n = 1$, then $X = 0, 1, -1$. For induction step, assume $u, v \in X_{n-1}$ with $u \neq v$. By induction hypothesis, it holds $|u - v| \geq \frac{1}{3^{n-2}}$. Suppose we pick $x = u + \frac{1}{3^{n-1}}$ and $y = v - \frac{1}{3^{n-1}}$. Hence, $x, y \in X_n$. But then, $|x - y| = \left|(u + \frac{1}{3^{n-1}}) - (v - \frac{1}{3^{n-1}})\right| = \left|u - v + \frac{2}{3^{n-1}}\right| \geq \frac{1}{3^{n-1}}$. If $u - v$ is positive, then the inequality follows immediately. Now assume $u - v$ is negative. By indcution hypothesis we get $u - v \leq -\frac{1}{3^{n-2}}$. So the worst case is, when $u - v$ gets $-\frac{1}{3^{n-2}}$. But then, $\left|-\frac{1}{3^{n-2}} + \frac{2}{3^{n-1}}\right| = \frac{1}{3^{n-1}}$. $\square$

It follows from Lemma 4.0.4 that $(X, <)$ is a countable, dense, and linear ordered without endpoints. By Cantor's Theorem, X is homeomorphic to $\mathbb{Q}$. It also follows that for each $x \in X$ with $x \neq 0$ there exists an index n_x with $x \in X_{n_x}$ but $x \notin X_{n_x-1}$. But also, we can find a unique $y \in X_{n_x-1}$ with $x = y - \frac{1}{3^{n_x-1}}$ or $x = y + \frac{1}{3^{n_x-1}}$. Furthermore, the basis for the open intervals for any $x \in X$ will be $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ where n_x is the index where x gets produced.

Now we define f from X onto T_2 by recursion. If $x = 0$, then we let $f(0)$ be the root r of T_2. If $x \neq 0$, then $x \in X_{n_x} - X_{n_x-1}$ and we let

$$f(x) = \begin{cases} \text{the left successor of f(y)} & \text{if } x = y - \frac{1}{3^{n_x-1}} \\ \text{the right successor of f(y)} & \text{if } x = y + \frac{1}{3^{n_x-1}} \end{cases}$$

Proposition 4.0.5. *We claim, f is open and continuous.*

Proof. We define the basis for the Alexandroff topology on T_2 as $\{B_t\}_{t \in T_2}$ where $B_t = \{s \in T_2 \mid tRs\}$. To show f is open, we pick an X-interval $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ and show $f((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})) = B_{f(x)}$.
$\subseteq$: If we pick $x \in ((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}))$, then obviously $f(x)Rf(x)$. Assume we pick a point $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ but $y \neq x$. Then we have $n_y > n_x$ and hence $f(x)Rf(y)$.
$\supseteq$: Assume $f(x)Rt$. We will show by induction on the length k between $f(x)$ and t s.t there exists $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ s.t $f(y) = t$. Base case : If $k = 0$, then we have $t = f(x)$ and we can pick $y = x$. For induction step, assume it holds for $f(x)Rt'$ with length n, $f(y) = t'$ and $t'Rt$. For y, there must exists an index $n_y > n_x$ s.t. $y \in X_{n_y} - X_{n_y-1}$. Now let $y' = y - \frac{1}{3^{n_y}}$ or $y' = y + \frac{1}{3^{n_y}}$. It holds that, $n'_y > n_y > n_x$, so $y' \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$. By defintion of f, $f(y')$ is either left or right successor of $f(y)$ and by $f(y) = t'Rt$, we get $f(y') = t$. It follows f is open.
To show f is continuous, it suffices to show that for each $t \in T_2$, the f-inverse image of B_t is open. Let $x \in f^{-1}(B_t)$. Then $tRf(x)$. As argued before, we can find an open interval in X s.t. $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)}$. By $tRf(x)$, we get $B_{f(x)} \subseteq B_t$. Hence, we found an open interval $I = (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$, s.t. $I = f^{-1}(B_{f(x)}) \subseteq f^{-1}(B_t)$. This holds because we can pick $y \in I$. By applying f, we get $f(y) \in B_{f(x)} \subseteq B_t$. But then $y \in f^{-1}(B_t)$. By that fact, we can follow with Remark 2.2.8, that $f^{-1}(B_t)$ is open and hence f is continuous.

$\square$

To complete the proof, if $S4 \nvdash \phi$, then by Theorem 4.0.2, there is a valuation v on T_2 s.t. $(T_2, v), r \nVdash \phi$. We define a valuation ξ on X by $\xi(p) = f^{-1}(v(p))$. Since f is continuous and open, $f(0) = r$ and by the choice of the valuation, we have that 0 and r are topo-bisimilar. By Proposition 2.2.7 it follows $(X, \xi), 0 \nVdash \phi$. Because X is homeomorphic to $\mathbb{Q}$, we obtain ϕ refutable on $\mathbb{Q}$.

Definition 4.0.6. *Let $\mathcal{L}_{\Box,\Box_1,\Box_2}$ be a modal language with three modal operators $\Box, \Box_1$ and $\Box_2$. We define the topological product logic TPL as the least set of formulas in $\mathcal{L}_{\Box,\Box_1,\Box_2}$ containing all axioms $S4 \otimes S4 \otimes S4 + \Box p \to \Box_1 p \wedge \Box_2 p$, and closed under modus ponens, substitution and $\Box$-, $\Box_1$- and $\Box_2$-neccessitation.*

Theorem 4.0.7. *TPL is complete w.r.t. $T_{6,2,2}$.*

Proof. A generalization of Theorem 4.0.2 can be seen on [?]. □

Now we are ready to show TPL is complete w.r.t. $\mathbb{Q} \times \mathbb{Q}$ with three topologies. The idea is similar as previous shown. We view $T_{6,2,2}$ as equipped with three Alexandroff topologies defined from R, R_1 and R_2. It suffices to show that there exists a 3-topo-bisimulation between $T_{6,2,2}$ and $X \times X = X'$ where each X is as defined in Theorem 4.0.3.

Theorem 4.0.8. *TPL is complete w.r.t. $\mathbb{Q} \times \mathbb{Q}$.*

Proof. We define h from $X \times X$ onto $T_{6,2,2}$ by recursion following the inducitve definition of X. If $(x, y) = (0, 0)$, then $h(0, 0)$ is the root r of $T_{6,2,2}$. If $(x, y) \neq (0, 0)$, then as argued in Lemma 4.0.4, there exists $n_{(x,y)}$ with $(x, y) \in X'_{n_{(x,y)}} - X'_{n_{(x,y)}-1}$ and that there exists a unique $(u, v) \in X'_{n_{(x,y)}-1}$ such that $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v)$ or $(x, y) = (u, v \pm \frac{1}{3^{n_{(x,y)}-1}})$ or $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v \pm \frac{1}{3^{n_{(x,y)}-1}})$.
As visiualized in the picture, we can see as an example X'_0, X'_1 and an excerpt of X'_2. For clarity, we illustrated some points of X'_1 as a line. We cleary can see, that any point has his own "square", where the points within it can be only build from his center point. This is because, the square size gets exponentially smaller with each iteration, due to its construction. That means, in each iteration they can not intersect.
Now we define h as

$$h(x,y) = \begin{cases} \text{The left } R_1\text{-successor of } h(u,v) & \text{if } (x,y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{The right } R_1\text{-successor of } h(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{The left } R_2\text{-successor of } h(u,v) & \text{if } (x,y) = (u, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The right } R_2\text{-successor of } h(u,v) & \text{if } (x,y) = (u, v + \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The first remaining successor of } h(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v + \frac{1}{3^{n_{(x,y)}-1}}) \\ & \text{or } (x,y) = (u - \frac{1}{3^{(x,y)-1}}, v - \frac{1}{3^{(x,y)-1}}) \\ \text{The last remaining successor of } h(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ & \text{or } (x,y) = (u - \frac{1}{3^{(x,y)-1}}, v + \frac{1}{3^{(x,y)-1}}) \end{cases}$$

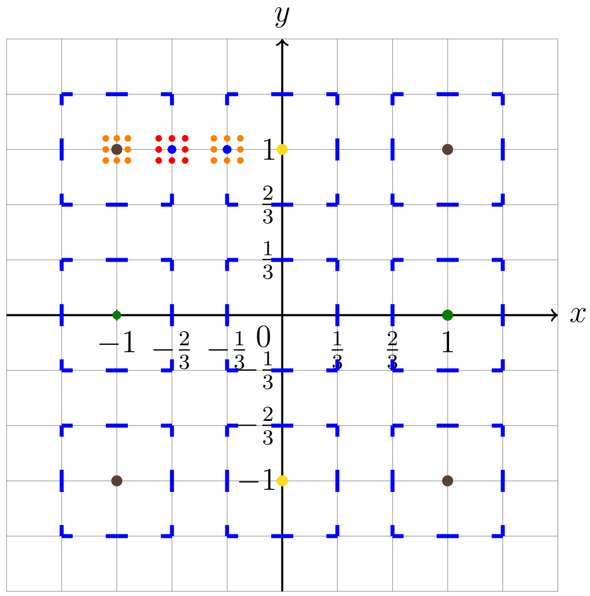

Figure 3

It is still left to prove, that h is open and continuous w.r.t. all three topologies. We will show it for τ_1 and τ. For τ_2 it is similar as for τ_1.

First we observe the basis for τ_1 is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \mid (x, y) \in X \times X\}$$

and for τ is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}}) \mid (x, y) \in X \times X\}$$

We also observe that a basis for the topology on $T_{6,2,2}$ from R_1 is $\{B_t^1\}_{t \in T_{6,2,2}}$ where $B_t^1 = \{s \in T_{6,2,2} \mid tR_1s\}$ and from R is $\{B_t\}_{t \in T_{6,2,2}}$ where $B_t = \{s \in T_{6,2,2} \mid tRs\}$. The following arguments are carried over from Proposition 4.0.5.
Now, to see that g is open w.r.t. τ_1, we pick an open set $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \in \tau_1$. In similar way, we can show $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)}$. Thus g is open. To show g is continuous, it suffices to show that for each $t \in T_{6,2,2}$, the g-inverse image of B_t^1 belongs to τ_1. Let $(x, y) \in g^{-1}(B_t^1)$. Then $tR_1g(x, y)$. Hence, it holds for (x, y) that $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)}$. By $tR_1g(x, y)$, we get $B^1_{g(x,y)} \subseteq B_t^1$. Thus, we found a neighborhood U of (x, y) s.t. $U = ((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) \subseteq g^{-1}(B_t^1)$, implying g is continuous.
To see g is open w.r.t. τ, we pick pick an open set $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})$. A similar argument as in Proposition 4.0.5, yields us $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{g(x,y)}$.

To show g is continuous, it suffices to show for each $t \in T_{6,2,2}$ that $g^{-1}(B_t) \in \tau$. Let $(x,y) \in g^{-1}(B_t)$. Then $tRg(x,y)$. But then, it holds that : $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{g(x,y)} \subseteq B_t$. Thus, we found a $U \in \tau$ s.t. $U = (x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}}) \subseteq g^{-1}(B_t)$.

$\square$

To complete the proof, if $TPL \nvdash \phi$, then by Theorem 4.0.8, there is a valuation υ on $T_{6,2,2}$ such that $(T_{6,2,2}, \upsilon), r \nvDash \phi$. Now we define a valuation ξ on $X \times X$ by $\xi(p) = h^{-1}(\upsilon(p))$. Since h is continuous and open w.r.t. all three topologies and $h(0,0) = r$, we have that $(0,0)$ and r are 3-topo-bisimilar. Therefore, $(X \times X, \xi), (0,0) \nvDash \phi$. Similar as in Theorem 4.0.3, we argue that $X \times X$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ w.r.t. all three topologies, it follows that ϕ is refutable on $\mathbb{Q} \times \mathbb{Q}$.

Corollary 4.0.9. *In the language $L_{\Box,\Box_1,\Box_2}$, TPL is the logic of products of arbitrary topologies.*

Proof. Let X and Y be arbitrary topological spaces. By the interpretation of $\Box$, it follows $Log(X), Log(Y) \supseteq S4$. Let's consider the full product of the spaces $X \times_t^+ Y$. Hence, with theorem 4.0.8, it is easy to see that $Log(X \times_t^+ Y) \supseteq S4 \times_t^+ S4 = TPL$.

$\square$

5 Products of neighbourhood frames, completeness results

We now turn our focus to neighbourhood semantics, and in particular, we revisit the main theorem from the paper "Modal Logic of Products of Neighbourhood Frames". This work established that, for any $L_1, L_2 \in \{D, D4, T, S4\}$ we have $L_1 \otimes L_2 = L_1 \times_n L_2$. Our goal is to reprove this theorem within our framework and then adapt its key ideas to prove our theorem.

5.1 Kripke tree frames, natural neighborhood version and their logic

We introduce some trees as Kripke frames which are essential for the proof.

Definition 5.1.1. *Let A be a nonempty set.*

$$A^* = \{a_1 ... a_k \mid a_i \in A\}$$

is the set of all finite sequences of elements from A, including the empty sequence Λ. Elements from A^* will be denoted as $\vec{a}$. The length of a sequence $\vec{a} = a_1 ... a_k$ is k (also $l(\vec{a}) = k$) and the length of Λ is 0 ($l(\Lambda) = 0$). Concatenation is denoted by " $\cdot$ " : $(a_1 ... a_k) \cdot (b_1 ... b_l) = \vec{a} \cdot \vec{b} = a_1 ... a_k b_1 ... b_l$.

Definition 5.1.2. *Let A be a nonempty set. We define an infinite frame $F_{in}[A] = (A^*, R)$ s.t for $\vec{a}, \vec{b} \in A^*$*

$$\vec{a}R\vec{b} \Leftrightarrow \exists x \in A\,(\vec{b} = \vec{a} \cdot x)$$

Furthermore we define :

$$F_{rn}[A] = (A^*, R^r),\ \text{where } R^r = R \cup Id \text{ (reflexive closure)}$$

$$F_{it}[A] = (A^*, R^*),\ \text{where } R^* = \bigcup_{i=1}^{\infty} R^i \text{ (transitive closure)}$$

$$F_{rt}[A] = (A^*, R^{r*})$$

where "t" stands for transtive, "n" for non-transitive, "r" for reflexive and "i" for irreflexive.
For now, we will use the following notion to generalize : $F_{\xi\eta}$ where $\xi \in \{i, r\}$ and $\eta \in \{t, n\}$

Proposition 5.1.3. *Let $F = F_{\xi\eta}[A] = (A^*, R)$ then*

$$\vec{a}R(\vec{a} \cdot \vec{c}) \Leftrightarrow \Lambda R\vec{c}$$

Definition 5.1.4. *Let $F_1 = F_{\xi_1\eta_1}[A] = (A^*, R_1)$ and $F_2 = F_{\xi_2\eta_2}[B] = (B^*, R_2)$, where $\xi_1, \xi_2 \in \{i, r\}$ and $\eta_1, \eta_2 \in \{t, n\}$. Furthermore, we assume $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$ with $A \cap B = \emptyset$. Then we define the frame $F_1 \otimes F_2 = (W, R'_1, R'_2)$ as follows :*

$$W = (A \cup B)^*$$

$$\vec{x}R'_1\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in A^* \text{such that } \Lambda R_1\vec{z}$$

$$\vec{x}R'_2\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in B^* \text{such that } \Lambda R_2\vec{z}$$

Proposition 5.1.5 ([?], [?])**.** *Let F_1 and F_2 be as in Defintion 2.1.7. Then*

$$Log(F_1 \otimes F_2) = Log(F_1) \otimes Log(F_2)$$

Proposition 5.1.6. *Let $F_{in} = F_{in}[\mathbb{N}], F_{rn} = F_{rn}[\mathbb{N}], F_{it} = F_{it}[\mathbb{N}]$ and $F_{rt} = F_{rt}[\mathbb{N}]$. Then the following holds:*

$$Log(F_{in}) = D$$

$$Log(F_{rn}) = T$$

$$Log(F_{it}) = D4$$

$$Log(F_{rt}) = S4$$

Definition 5.1.7. *Let $F = (W, R)$ be a Kripke frame. We define an n-frame $N(F) = (W, \tau)$ as follows. For any $w \in W$ we have :*

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

Lemma 5.1.8. *Let $F = (W, R)$ be a Kripke frame. Then*

$$Log(F) = Log(N(F))$$

Proof. The proof is by structural induction. □

5.2 Main Construction

In the following, we construct a useful neighborhood frame called $N_\omega[F]$ based on a frame F where F is a tree frame. We will show that the construction has the same logic as the orignal frame. At the end, we construct a bounded morphism from the product of $N_\omega[F]$ frames to the fusion of frames F.

Definition 5.2.1. *Let $F = (A^*, R) = F_{\xi\eta}[A]$ and $0 \notin A$. We define "pseudo-infinite" sequences*

$$X = \{a_1a_2a_3... \mid a_i \in A \cup \{0\} \textit{ and } \exists N \forall k \geq N : a_k = 0\}$$

Furthermore, we define $f_F : X \to A^*$ to be the function, that deletes all zeros.

Example : Say $12034002340^\omega \in X$ *(0^ω denotes infinitely many zeros). Then* $f_F(12034002340^\omega) = 1234234$

Definition 5.2.2. *Let $F = (A^*, R) = F_{\xi\eta}[A]$ and $0 \notin A$. Assume the function f_F and the set X as defined before. For $\alpha \in X$ such that $\alpha = a_1a_2...$ we define*

$$st(a) = min\{N \mid \forall k \geq N : a_k = 0\}$$
$$a \mid_k = a_1a_2...a_k$$
$$U_k(\alpha) = \{\beta \mid f_F(\alpha)Rf_F(\beta) \textit{ and } \alpha \mid_m = \beta \mid_m, \textit{ where } m = max((k, st(\alpha))\}$$

Remark : Let $\alpha \in X$ with $st(\alpha) = n$. Then we have that $U_n(\alpha) = U_j(\alpha)$ for any $j \leq n$

Lemma 5.2.3. *$U_k(\alpha) \subseteq U_m(\alpha)$, whenever $k \geq m$.*

Proof. Let $\beta \in U_k(\alpha)$. Since $\alpha \mid_k = \beta \mid_k$ and $k \geq m$, we have $\alpha \mid_m = \beta \mid_m$. It follows, $\beta \in U_m(\alpha)$. □

Definition 5.2.4. *Due to Lemma 5.2.3 the sets $U_n(\alpha)$ forms a filter base. So we can define :*

$$\tau(\alpha) \textit{ is a filter with base } \{U_n(\alpha) \mid n \in \mathbb{N}\}$$

$$N_\omega = (X, \tau) \textit{ is the n-frame based on } F$$

Lemma 5.2.5. *Let $F = (A^*, R) = F_{\xi\eta}[A]$. Based on that, let $N_\omega(F) = (X, \tau)$, $N(F) = (A^*, \sigma)$ and $f_F : N\omega(F) \to N(F)$. Then for any $m \in \mathbb{N}$ and $x \in X$ with $x = a_1 a_2 ...$ we have*

$$f_F(U_m(x)) = R(f_F(x))$$

Proof. $\subseteq$: Let $f_F(\alpha) \in f_F(U_m(x))$ with $\alpha \in U_m(x)$. By defintion of $U_m(x)$, we get $f_F(\alpha) \in R(f_F(x))$.

For the other direction, we pick $\vec{a} \in R(f_F(x))$. We have to find $\beta \in U_m(x)$ s.t $f_F(\beta) = \vec{a}$. We assume R is irreflexive and non-transitive. The other cases are similar.
Because $\vec{a} \in R(f_F(x))$, there must exists $c \in A$ such that $\vec{a} = f_F(x) * c$. We construct $\beta = x \mid_m \cdot\, c\, 0^\omega$. Hence, $f(\beta) = f_F(x \mid_m) * f_F(c)$ and because $0 \notin A$ we get $f_F(x \mid_m) * c = \vec{a}$. □

Lemma 5.2.6. *Let $F = (A^*, R) = F_{\xi\eta}[A]$. Then $f_F : N_\omega(F) \to N(F)$ is a bounded morphism.*

Proof. From now on this proof we will omit the subindex in f_F. Let $N_\omega(F) = (X, \tau)$ and $N(F) = (A^*, \sigma)$.
For surjectivity, we pick any $\vec{x} \in A^*$. But then, $\vec{x}\, 0^\omega \in X$. Hence, $f(\vec{x}\, 0^\omega) = \vec{x}$.
For the next condition, assume that $x \in X$ and $U \in \tau(x)$. We need to prove that $f(U) \in \sigma(f(x))$. That means $R(f(x)) \subseteq f(U)$. Because $U \in \tau(x)$, there is a m such that $U_m(x) \subseteq U$. By Lemma 5.2.5 we have $f(U_m(x)) = R(f(x))$. It follows,

$$R(f(x)) = f(U_m(x)) \subseteq f(U)$$

Assume $x \in X$ and V is a neighborhood of x, i.e $R(f(x)) \subseteq V$. We need to prove that there exists $U \in \tau(x)$, such that $f(U) \subseteq V$. As U we $U_m(x)$ for any $m \in \mathbb{N}$. By Lemma 5.2.5 we get $f(U_m(x)) = R(f(x))$. Hence,

$$f(U_m(x)) = R(f(x)) \subseteq V$$

□

Corollary 5.2.7. *For frame $F = F_{\xi\eta}[A]$ we have $Log(N_\omega(F)) \subseteq Log(F)$.*

Proof. It follows from Lemma 5.1.5, Corollary 5.1.7 and Lemma 5.2.6

$$Log(N_\omega(F)) \subseteq Log(N(F)) = Log(F)$$

□

Proposition 5.2.8. *Let* $F_{in} = F_{in}[\mathbb{N}], F_{rn} = F_{rn}[\mathbb{N}], F_{it} = F_{it}[\mathbb{N}]$ *and* $F_{rt} = F_{rt}[\mathbb{N}]$. *Then*

$$Log(N_\omega(F_{in})) = D$$
$$Log(N_\omega(F_{rn})) = T$$
$$Log(N_\omega(F_{it})) = D4$$
$$Log(N_\omega(F_{rt})) = S4$$

Proof. In all these cases, the inclusion from left to right follows from Proposition 2.1.9 and Corollary 5.2.7. Now the converse direction. Assume $X = (X, \tau)$.
It is easy to check that $X \vDash D$ iff for each $x \in X : \emptyset \notin \tau(x)$. For $N_\omega(F_{in})$ and $N_\omega(F_{it})$ this holds.
It is easy to check that $X \vDash T$ iff we have $x \in U \in \tau(x)$ for any x and U. For $N_\omega(F_{rn})$ and $N_\omega(F_{rt})$ this holds.
Now we check $X \vDash 4$ iff for each $U \in \tau(x) : \{y \mid U \in \tau(y)\} \in \tau(x)$.
$\supseteq$: Let $x \in X$ and assume $X, x \vDash \Box p$. That means there exists $U \in \tau(x)$ s.t $U \subseteq V(p)$. By assumption we have $S = \{y \mid U \in \tau(y)\} \in \tau(x)$. But then $X, x \vDash \Box\Box p$ because we can just pick the set S.
$\subseteq$: By contradiction, assume there exists a U s.t $\{y \mid U \in \tau(y)\} \notin \tau(x)$. Let $X, x \vDash \Box p$ where $V(p) = U$. If $X, x \vDash \Box\Box p$ then it must be the case that $S = \{y \in X \mid X, y \vDash \Box p\} \in \tau(x)$. $X, y \vDash \Box p$ means $U \in \tau(y)$. That means $S = \{y \in X \mid U \in \tau(y)\}$. But by assumption $S \notin \tau(x)$. Hence, $X, x \nvDash \Box\Box p$. But thats a contradiction. This also holds for $N_\omega(F_{it})$ and $N_\omega(F_{rt})$ because we have for any $y \in U_m(x)$ and $k \geq m : U_k(y) \subseteq U_m(x)$.

$\square$

Assume $F_1 = (A^*, R_1) = F_{\eta_1\xi_1}[A]$ and $F_2 = (B^*, R_2) = F_{\eta_2\xi_2}[B]$ with $A \cap B = \emptyset$, $A = \{a_1, a_2, a_3, ...\}$ and $B = \{b_1, b_2, b_3, ...\}$. Consider the product of n-frames $F_1' = (X_1, \tau_1) = N_\omega(F_1)$ and $F_2' = (X_2, \tau_2) = N_\omega(F_2)$ is

$$X = (X_1 \times X_2, \tau_1', \tau_2') = N_\omega(F_1) \times_n N_\omega(F_2)$$

Furthermore, we have $F_1 \otimes F_2 = ((A \cup B)^*, R_1', R_2')$ as defined in Defintion 2.0.7. We consider the neighborhood version

$$N(F_1 \otimes F_2) = ((A \cup B)^*, \sigma_1', \sigma_2')$$

Now we define $g : X_1 \times X_2 \to (A \cup B)^*$ as follows. For $(\alpha, \beta) \in X_1 \times X_2$ with $\alpha = a_1a_2...$ and $\beta = b_1b_2...$ we define $g(\alpha, \beta)$ to be the finite sequence which we get after eliminating all zeros from the infinite sequence $a_1b_1a_2b_2...$

Example : Let $\alpha = 012340^\omega$ and $\beta = 0ab00e0^\omega$. Then $g(\alpha, \beta) = 1a2b34e$.

Lemma 5.2.9. *Let X and $N(F_1 \otimes F_2)$ be as defined before and $(\alpha,\beta) \in X_1 \times X_2$. Then for any $m > max\{st(\alpha), st(\beta)\}$ we have*

$$R_1'(g(\alpha,\beta)) = g(U_m(\alpha) \times \{\beta\})$$

.

Proof. We will show both direction by assuming $F_1 = F_{in}[A]$. For $F_{it}[A], F_{rt}[A]$ and $F_{rn}[A]$ is it similar.
$\subseteq$: Let $\vec{w} \in R_1'(g(\alpha,\beta))$. By definition we get, there exists $\vec{c} \in A^*$ where $\vec{w} = g(\alpha,\beta) \cdot \vec{c}$ and $\Lambda R_1 \vec{c}$. Because R_1 is irreflexive and non-transitive, we get $\vec{c} \in A$. We construct (ζ,β) where $\zeta \in U_m(\alpha)$ and $g(\zeta,\beta) = \vec{w}$. For that, we can take $\zeta = \alpha \mid_m \cdot \vec{c}0^\omega$. Obviously, $\zeta \in U_m(\alpha)$. Because $m \in max\{st(\alpha), st(\beta)\}$, we have $g(\alpha \mid_m, \beta \mid_m) = g(\alpha,\beta)$. Hence, $g(\zeta,\beta) = g(\alpha \mid_m, \beta \mid_m) \cdot g(\vec{c}0^\omega, 0^\omega) = g(\alpha \mid_m, \beta \mid_m) \cdot \vec{c} = \vec{w}$.

$\supseteq$: Assume $\zeta \in U_m(\alpha)$. We have to show $g(\zeta,\beta) \in R_1'(g(\alpha,\beta))$. Because R_1 is irreflexive and non-transitive, it suffices to find a $\vec{c} \in A$ s.t. $g(\alpha,\beta) \cdot \vec{c} = g(\zeta,\beta)$. By choosing m is maximal and $\zeta \mid_m = \alpha \mid_m$, we have $g(\zeta_m, \beta_m) = g(\alpha,\beta)$. We also know $f_F(\alpha) R_1 f_F(\zeta)$, that means there exists a $\vec{d} \in A$ s.t. $f_F(\alpha) \cdot \vec{c} = f_F(\zeta)$. This $\vec{d}$ must appear at a point after ζ_m. We can follow $g(\zeta,\beta) = g(\alpha,\beta) \cdot \vec{d}$. Hence, $g(\zeta,\beta) \in R_1'(g(\alpha,\beta))$.

Remark : If $m \leq max\{st(\alpha), st(\beta)\}$, then we cannot gurantee the equality. Assume R_1 as before and $\alpha = 123000^\omega$ and $\beta = d0b0a0^\omega$. Lets pick $m = 3$ and $\zeta = 123100^\omega$. Then we have $g(\alpha,\beta) = 1d23ba$ and $g(\zeta,\beta) = 1d23b1a$. Obviously, we don't have $g(\zeta,\beta) \in R_1'(g(\alpha,\beta))$.
Furthermore, we can show $R_2'(g(\alpha,\beta)) = g(\alpha \times U_m(\beta))$ similar as above.

□

Lemma 5.2.10. *Function $g : X \to N(F_1 \otimes F_2)$ is a bounded morphism.*

Proof. Let $\vec{z} = z_1 z_2 ... z_n \in (A \cup B)^*$. Define for $i \leq n$:

$$x_i = \begin{cases} z_i, & \text{if } z_i \in A; \\ 0, & \text{if } z_i \notin A. \end{cases} \qquad y_i = \begin{cases} z_i, & \text{if } z_i \in B; \\ 0, & \text{if } z_i \notin B. \end{cases}$$

Let $\alpha = x_1 x_2 ... x_n \, 0^\omega$ and $\beta = y_1 y_2 ... y_n \, 0^\omega$. Then $g(\alpha,\beta) = \vec{z}$. Hence, g is surjective.

For the next conditions we check only for τ_1' and σ_1. The other case is similar. Assume $(\alpha,\beta) \in X_1 \times X_2$ and $U \in \tau_1'(\alpha,\beta)$. We have to show $g(U) \in \sigma_1(g(\alpha,\beta))$. That means $R_1'(g(\alpha,\beta)) \subseteq g(U)$. Pick a $m > max\{st(\alpha), st(\beta)\}$ s.t. $U_m(\alpha) \times \{\beta\} \subseteq U$. We can pick such m, because $U \in \tau_1'(\alpha,\beta)$. So there exists $U_k(\alpha) \in \tau(\alpha)$ s.t. $U_k(\alpha) \times \{\beta\} = U$. If $k > max\{st(\alpha), st(\beta)\}$ then we are done. Else, by Lemma 5.2.3, we have for any $n \geq k : U_n(\alpha) \subseteq U_k(\alpha)$. So we can lift the k til we reach m. Then we use Lemma 5.2.11 to get the following :

$$R_1'(g(\alpha,\beta)) = g(U_m(\alpha) \times \{\beta\}) \subseteq g(U)$$

For the last condition we assume $(\alpha, \beta) \in X_1 \times X_2$ and $V \in \sigma_1(g(\alpha, \beta))$ (or rather $R_1'(g(\alpha, \beta)) \subseteq V$). We need to prove there exists $U \in \tau_1'(\alpha, \beta)$, such that $g(U) \subseteq V$. As U we take $U_m(\alpha) \times \{\beta\}$ for some $m > max\{st(\alpha), st(\beta)\}$. Hence, by Lemma 5.2.11 we get :

$$g(U_m(\alpha) \times \{\beta\}) = R_1'(g(\alpha, \beta)) \subseteq V$$

□

5.3 Adapting the key ideas

Now we will show $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \to N(T_{\omega,\omega,\omega[rn]})$ is a bounded morphism. Let $(T_{\omega[rn]})_1 = (\mathbb{N}_1, R_1)$ and $(T_{\omega[rn]})_2 = (\mathbb{N}_2, R_2)$ as defined in Defintion 3.2.1. Then we take the n-frames $N_\omega(T_{\omega[rn]})_1 = (X_1, \tau_1)$ and $N_\omega(T_{\omega[rn]})_2 = (X_2, \tau_2)$. For the proof, we will omit the subscripts after the frame.

Let's consider the full product of the n-frames :

$$N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) = (X_1 \times X_2, \tau_1', \tau_2', \tau)$$

We say $N(T_{\omega,\omega,\omega[rn]}) = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*, \sigma_1, \sigma_2, \sigma)$ where the tree $T_{\omega,\omega,\omega[rn]} = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^* R_1', R_2', R)$ is as defined in Def.3.2.1

In order to define the bounded morphism, we have to fix a bijection first. Let $h : \mathbb{N}_1 \times \mathbb{N}_2 \to \mathbb{N}$ be a bijection.
Next, we define function $k : (\mathbb{N}_1 \cup \{0\}) \times (\mathbb{N}_2 \cup \{0\}) \to \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{0\}$ as follows :

$$f(a, b) = \begin{cases} a, & \text{if } b = 0; \\ b, & \text{if } a = 0; \\ h(a, b), & \text{otherwise.} \end{cases}$$

Let $(\alpha, \beta) \in X_1 \times X_2$ with $\alpha = a_1 a_2 ...$ and $\beta = b_1 b_2 ....$ We define

$$g'(\alpha, \beta) = f(a_1, b_1) f(a_2, b_2) ...$$

At last, we define $g(\alpha, \beta)$ as $g'(\alpha, \beta)$ but removing all zeros.

Lemma 5.3.1. *Function $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \to N(T_{\omega,\omega,\omega[rn]})$ is a bounded morphism.*

Proof. Let $\vec{z} = z_1 z_2 ... z_n \in (\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*$. We define for $i \leq n$

$$x_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_1; \\ a_1, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases} \qquad y_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_2; \\ b_2, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases}$$

Let $\alpha = x_1x_2...x_n\,0^\omega$ and $\alpha = y_1y_2...y_n\,0^\omega$. But then $g(\alpha,\beta) = \vec{z}$. Hence, g is surjective. For the next conditions we show it for τ and σ. For τ_1', σ_1 and τ_2', σ_2 we can argue similar as in Lemma 5.2.11 and 5.2.12.
First we need to show, $R(g(\alpha,\beta)) = g(U_m(\alpha) \times U_m(\beta))$ for $m > max\{st(\alpha), st(\beta)\}$. The proof is similar as in Lemma 5.2.1.
$\supseteq$: Pick $\zeta \in U_m(\alpha), \upsilon \in U_m(\beta)$. Show $g(\zeta,\upsilon) \in R(g(\alpha,\beta))$. By definition, we need to find a $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$ s.t. $g(\alpha,\beta) \cdot \vec{c} = g(\zeta,\upsilon)$. By choosing $m > max\{st(\alpha), st(\beta)\}$, $\alpha\ |_m = \zeta\ |_m$ and $\beta\ |_m = \upsilon\ |_m$, we get $g(\alpha,\beta) = g(\zeta_m, \upsilon_m)$. Furthermore, we have $f_F(\alpha) R_1 f_F(\zeta)$ and $f_F(\beta) R_2 f_F(\upsilon)$, that means $\exists \vec{d} \in \mathbb{N}_1 \cup \{\epsilon\}$ and $\exists \vec{e} \in \mathbb{N}_2 \cup \{\epsilon\}$ s.t. $f_F(\alpha) \cdot \vec{d} = f_F(\zeta)$ and $f_F(\beta) \cdot \vec{e} = f_F(\upsilon)$. So, $\vec{d}$ must appear at a point after $\zeta\ |_m$. The same holds for $\vec{e}$ in υ. But then we have $g(\zeta,\upsilon) = g(\alpha,\beta) * g(\vec{d},\vec{e})$.
Now there are four cases :

$$\begin{cases} g(\vec{d},\vec{e}) \in \mathbb{N}_1, & \text{if } \vec{d} \in \mathbb{N}_1 \ \&\ \vec{e} = \epsilon \\ g(\vec{d},\vec{e}) \in \mathbb{N}_2, & \text{if } \vec{d} = \epsilon \ \&\ \vec{e} \in \mathbb{N}_2 \\ g(\vec{d},\vec{e}) \in \{\epsilon\}, & \text{if } \vec{d} = \vec{e} = \epsilon \\ g(\vec{d},\vec{e}) \in \mathbb{N}, & \text{other} \end{cases}$$

Hence, $g(\zeta,\upsilon) \in R(g(\alpha,\beta))$.

$\subseteq$: Pick $\vec{w} \in R(g(\alpha,\beta))$. We will construct $\zeta \in U_m(\alpha)$ and $\upsilon \in U_m(\beta)$ s.t. $g(\zeta,\upsilon) = \vec{w}$. By definition there exists a $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$ s.t. $g(\alpha,\beta) \cdot \vec{c} = \vec{w}$. Depending on the $\vec{c}$, we build $\zeta = \alpha\ |_m \cdot \vec{d}\,0^\omega$ and $\upsilon = \beta\ |_m \cdot \vec{e}\,0^\omega$ where

$$\begin{cases} \vec{d} = \vec{c}, \vec{e} = 0, & \text{if } \vec{c} \in \mathbb{N}_1; \\ \vec{d} = 0, \vec{e} = \vec{c}, & \text{if } \vec{c} \in \mathbb{N}_2; \\ \vec{d} = \vec{e} = 0 & \text{if } \vec{c} \in \{\epsilon\} \\ \vec{d} = \vec{\alpha}, \vec{e} = \vec{b} \text{ where } h(\vec{\alpha},\vec{\beta}) = \vec{c} & \text{if } \vec{c} \in \mathbb{N}\,(\vec{\alpha} \in \mathbb{N}_1, \vec{\beta} \in \mathbb{N}_2); \end{cases}$$

We have $\zeta \in U_m(\alpha)$ and $\upsilon \in U_m(\beta)$. Furthermore, because of $g(\zeta\ |_m, \upsilon\ |_m) = g(\alpha,\beta)$ it holds that $g(\zeta,\upsilon) = g(\alpha,\beta) \cdot g(\vec{d},\vec{e}) = g(\alpha,\beta) \cdot \vec{c} = \vec{w}$.

Now to check the conditions, we assume $x \in X_1$ and $y \in X_2$ and $U \in \tau(x,y)$. We need to prove $g(U) \in \sigma(g(x,y))$ In other words, show $R(g(x,y)) \subseteq g(U)$. We pick a $m > max\{st(x), st(y)\}$ s.t. $U_m(x) \times U_m(y) \subseteq U$. But then we get

$$R(g(x,y)) = g(U_m(x) \times U_m(y)) \subseteq g(U)$$

Assume $x \in X_1$ and $x \in X_2$ and $R(g(x,y)) \subseteq V$. We need to find a $U \in \tau(x,y)$, s.t. $f(U) \subseteq V$. For U we pick $U_m(x) \times U_m(y)$ for some $m > \{st(x), st(y)\}$. But then

$$g(U_m(x) \times U_m(y)) = R(g(x,y)) \subseteq V$$

$\square$

5.4 Completeness Results

Lemma 5.4.1. *Let $T_{\omega,\omega,\omega[rn]}$ be as in Definition 3.2.1. Then*

$$Log(T_{\omega,\omega,\omega[rn]}) = Log(N(T_{\omega,\omega,\omega[rn]}))$$

The proof is by structural induction.

Corollary 5.4.2. *Let $F_1 = (A^*, R_1) = F_{\xi_1,\eta_1}[A]$ and $F_2 = (B^*, R_2) = F_{\xi_2,\eta_2}[B]$. Then*

$$Log(N_\omega(F_1) \times_n N_\omega(F_2)) \subseteq Log(F_1) \otimes Log(F_2)$$

. It follows from Proposition 2.1.8, Lemma 5.1.5, Corollary 5.1.7 and Lemma 5.1.12.

Corollary 5.4.3. *Let $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$. Then*

$$Log(N_\omega(F_1) \times_n N_\omega(F_2)) = Log(F_1) \otimes Log(F_2)$$

Proof. The left to right inclusion follows from Corollary 5.3.1

To prove right to left, we know due to Proposition 2.1.9 and Proposition 5.2.8 we have $Log(N_\omega(F_i)) = Log(F_i)$ $(i \in \{1, 2\})$. Due to Proposition 5.1.10 we get $Log(N_\omega(F_1)) \otimes Log(N_\omega(F_2)) \subseteq Log(N_\omega(F_1)) \times_n Log(N_\omega(F_2))$. Because $N_\omega(F_1) \times_n N_\omega(F_2)$ is a frame in the n-product, we get $Log(N_\omega(F_1)) \times_n Log(N_\omega(F_2)) \subseteq Log(N_\omega(F_1) \times_n N_\omega(F_2))$. □

Corollary 5.4.4. *Let $N_\omega(T_{\omega[rn]})_1 = F_1$ and $N_\omega(T_{\omega[rn]})_2 = F_2$ as defined before. Then*

$$T \times_n^+ T \subseteq Log(T_{\omega,\omega,\omega[rn]})$$

It follows from Corollary 5.1.7, Proposition 5.2.8, Lemma 5.2.9, Lemma 5.2.12

Proof. First, it is the case that $Log(F_1) \times_n^+ Log(F_2) \subseteq Log(F_1 \times_n^+ F_2)$. We can argue as before. By Lemma 5.2.12 it holds $Log(F_1) \times_n^+ Log(F_2) \subseteq Log(N(T_{\omega,\omega,\omega[rn]})) = Log(T_{\omega,\omega,\omega[rn]})$. But by Proposition 5.2.8 we have $Log(F_1) = Log(F_2) = T$. Hence, $Log(F_1) \times_n^+ Log(F_2) = T \times_n^+ T \subseteq Log(T_{\omega,\omega,\omega[rn]})$. □

Theorem 5.4.5. *Let $L_1, L_2 \in \{S4, D4, D, T\}$. Then*

$$L_1 \times_n L_2 = L_1 \otimes L_2$$

Proof. Right to left is by Proposition 5.1.10.
For left to right, assume $L_1 = Log(F_1)$ and $L_2 = Log(F_2)$ for some $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$.
It follows from Corollary 5.3.2
$L_1 \times_n L_2 = Log(N_\omega(F_1)) \times_n^+ Log(N_\omega(F_2)) \subseteq Log(N_\omega(F_1) \times_n^+ N_\omega(F_2)) = Log(F_1 \otimes F_2) = Log(F_1) \otimes Log(F_2) = L_1 \otimes L_2$ □

Theorem 5.4.6. *Let $N_\omega(T_{\omega[rn]})_1 = F'_1$ and $N_\omega(T_{\omega[rn]})_2 = F'_2$ as defined before. Then*

$$T \times_n^+ T = T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$$

Proof. For left to right it is easy to check the axioms. We pick any frame F_1 and F_2 s.t. $F_1 \vDash T$ and $F_2 \vDash T$. We consider $F_1 \times_n^+ F_2$ the full product and let $(x, y) \in F_1 \times_n^+ F_2$. Assume $F_1 \times_n^+ F_2, (x, y) \vDash \Box_1 p$. By definition, there exists a $U \in \tau'_1(x, y)$ s.t $U \subseteq V(p)$. But we have $U \supseteq V \times \{y\}$ for some $V \in \tau(x)$. By assumption, $F_1 \vDash T$, so we can follow $x \in V$. Hence, $(x, y) \in V(p)$. For $\Box_2 p \to p$ it's done similar.
Let's check $\Box p \to p$. Assume $F_1 \times_n^+ F_2, (x, y) \vDash \Box p$. By definition there exists a $U \in \tau(x, y)$ s.t. $U \subseteq V(p)$. But $U \supseteq V \times W$ for some $V \in \tau_1(x)$ and $W \in \tau_2(y)$. Because of the assumption, we must have $x \in V$ and $y \in W$. Hence, $(x, y) \in V(p)$. It follows $F_1 \times_n^+ F_2, (x, y) \vDash p$.
Now we check the extra axiom. Assume $F_1 \times_n^+ F_2, (x, y) \Vdash \Box p$. By definition, there exists a $U \subseteq V(p)$, where $U \supseteq V \times W$ with $V \in \tau_1(x)$ and $W \in \tau_2(y)$. By assumption, we get $y \in W$. That means $U \in \tau'_1(x, y)$ because $U \supseteq V \times \{y\}$. The same argument we can apply to $\tau'_2(x, y)$. Hence, $F_1 \times_n^+ F_2, (x, y) \vDash \Box_1 p \wedge \Box_2 p$. The rest is clear.
Now for the other direction, we apply Corollary 5.3.3 to get $T \times_n^+ T \subseteq Log(T_{\omega,\omega,\omega[rn]})$. By Proposition 3.2.6 we have $Log(T_{\omega,\omega,\omega[rn]}) = TNL$. Hence,

$$T \times_n^+ T \subseteq T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$$

.

□

Remark 5.4.7. *Let $L \in \{D, D4\}$. Then it is not guaranteed that*

$$L \times_n^+ L = L \otimes L \otimes L + \Box p \to \Box_1 p \wedge \Box_2 p$$

This can be shown by a counterexample. We will only do it for D because for $D4$ it's done similar. Assume $F_1 = N_\omega(T_{\omega[in]})_1 = (X, \tau_1)$ and $F_2 = N_\omega(T_{\omega[in]})_2 = (Y, \tau_2)$. By Proposition 5.2.8, it follows $Log(F_1) = Log(F_2) = D$. Let's consider the full product $X = F_1 \times_n^+ F_2 = (W \times V, \tau'_1, \tau'_2, \tau)$. Then we get $Log(X) \supseteq D \times_n^+ D$. Now let $(x, y) \in X$ $X, (x, y) \vDash \Box p$ and $V(p) = U \times V$ where $U \in \tau_1(x)$ and $V \in \tau_2(y)$. By construction of $N_\omega(T_{\omega[rn]})$, $x \notin U$ and $y \notin V$, because $f_F(x)$ and $f_F(y)$ don't have a relation to itself in $T_{\omega[rn]}$. In order to satisfy $\Box_1 p$ (similar for $\Box_2 p$), we have to find $U' \in \tau'_1(x, y)$ s.t. $U' \subseteq V(p)$. But this is not possible because any U' contains only elements, where the second coordinate is y. It follows $X, (x, y) \nvDash \Box p \to \Box_1 p \wedge \Box_2 p$. Hence, the extra axiom is not in $Log(X) \supseteq D \times_n^+ D$.

6 Conclusion

In this work, we reproved the theorems from "Multimodal Logics of Products of Topologies" and "Modal Logic of products of neighborhood frames". We showed the logic of the product of two arbitrary topological spaces is $S4 \otimes S4 \otimes S4 + \Box p \to \Box_1 p \wedge \Box_2 p$.

The idea was to find a 3-topo-bisimulation between $T_{6,2,2}$ and $X \times X$ which is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$. Additionally, by switching to neighborhood frames, it holds that for any $L_1, L_2 \in \{D, D4, T, S4\}$: $L_1 \otimes L_2 = L_1 \times_n L_2$. We constructed from a Kripke frame F a special neighborhood frame called $N_\omega(F)$ and then found a bounded morphim from the product of these constructed frame to the fusion of two frames. Inspired by these ideas, we first introduced a Kripke frame called $T_{\omega,\omega,\omega[rn]}$ where we proved FMP and used unravelling to show $Log(T_{\omega,\omega,\omega[rn]}) = TNL$. Afterwoods, we constructed a bounded morphism from $N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]})$ onto $N(T_{\omega,\omega,\omega[rn]})$ to finish the proof.

There are tons of ways to continue the research. We may consider the logic K and ask what is the logic of $K \times_n^+ K$? What's about different combinations like : $D \times_n^+ T$, $T \times_n^+ K$, $S4 \times_n^+ D4$ ...? We can also try to find out whether for logic Λ with $T \subseteq \Lambda \subseteq S4$ the following holds :

$$\Lambda \otimes \Lambda \otimes \Lambda + \Box p \to \Box_1 p \wedge \Box_2 p = \Lambda \times_n^+ \Lambda$$

References

[1] Johan van Benthem, Guram Bezhanishvili, Balder ten Cate, and Darko Sarenac. Multimodal logics of products of topologies. *Studia Logica*, 84:369–392, 2006.

[2] Andrei Kudinov. Modal logic of some products of neighbourhood frames. In *Advances in Modal Logic, Volume 9*, pages 286–294, London, 2012. College Publications.

[3] J.C.C. McKinsey and Alfred Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[4] K. Segerberg. Two-dimensional modal logic. *Journal of Philosophical Logic*, 2:77–96, 1973.

[5] V.B. Shehtman. Two-dimensional modal logics. *Mathematical Notices of the USSR Academy of Sciences*, 23:417–424, 1978. (Translated from Russian.)

[6] D. Sarenac. Modal logic and topological products. PhD Thesis, Stanford University, 2005.

[7] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, August 2002.

[8] Katsuhiko Sano. Axiomatizing hybrid products of monotone neighborhood frames. *Electronic Notes in Theoretical Computer Science*, 273:51–67, 2011.

[9] Dana Scott. Advice on modal logic. In *Philosophical Problems in Logic: Some Recent Developments*, pages 143–173. D. Reidel, 1970.

[10] Richard Montague. Universal grammar. *Theoria*, 36:373–398, 1970.

[11] Eric Pacuit. *Neighborhood Semantics for Modal Logic*. Short Textbooks in Logic. Springer Cham, 2017.