1 Definition

1.1 Defintion

Let Prop be a set of variables. Then a formula ϕ is defined as follows :

$$\phi ::= p \mid \bot \mid \phi \mid \phi \to \phi \mid \Box_i \phi$$

where $p \in Prop$ and $\Box_i$ is a modal operator. Other connectives are expressed through $\bot$ and $\to$ and dual modal operators $\diamond_i$ as $\diamond_i \phi = \neg \Box_i \neg \phi$

1.2 Defintion

A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \to \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms

$$\Box_i (p \to q) \to (\Box_i p \to \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$ Let L be a logic and let Γ be a set of formulas. Then L+Γ denotes the minimal logic containing L and Γ

1.3 Definition

Let L1 and L2 be two modal logic with one modality $\Box$. Then the fusion of these logics are defined as follows :

$$L1 \otimes L2 = K2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

The follow logics may be important

$$D = K + \Box p \to \Diamond p$$

$$T = K + \Box p \to p$$

$$D4 = D + \Box p \to \Box \Box p$$

$$S4 = T + \Box p \to \Box \Box p$$

2 Topological Space Defintion

2.1 Defintion

A topological space is a pair (X, τ) where τ is a collection of subsets of X (elements of τ are also called open sets) such that :

1. the empty set $\emptyset$ and X are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

A topological model is a structure M = (X,τ,v) where (X,τ) is a topological space and v is a valuation assigining subsets of X to propositional variables.

2.2 Defintion

Let M = (X,τ,v) a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows :$M, x \models \Box\phi$ iff ,$\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$
$M, x \models \Box\phi$ iff ,$\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$
$M, x \models \Diamond\phi$ iff ,$\forall U \in \tau$ s.t $x \in U$ and $\exists u \in U : M, u \models \phi$

2.3 Defintion

Let $A = (X, \chi)$ and B =(Y, υ) be topological spaces. The standard product topology τ is the set of subsets of $X \times Y$ such that $X \in \chi$ and $Y \in \upsilon$.
Let $N \subseteq X \times Y$. We call N horizontally open if $\forall(x, y) \in N\ \exists U \in \chi : x \in U$ and $U \times \{y\} \subseteq N$.
We call N vertically open if $\forall(x, y) \in N\ \exists V \in \upsilon : y \in V$ and $\{x\} \times V \subseteq N$
If N is H-open and V-open, then we call it HV-open.
We denote τ_1 is the set of all H-open subsets of $X \times Y$ and τ_2 is the set of all V-open subsets of $X \times Y$

3 Neighbourhood

3.1 Defintion

Let X be a non-empty set. A function $\tau : X \to 2^{2^X}$ is called a neighbourhood function. A pair F = (X,τ) is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X,τ,v), where v assigns a subset of X to a variable

3.2 Defintion

Let M =(X,τ,v) be a neighourhood model and x $\in$ X. The truth of a formula is defined inductively as follows :

$$M, x \models \Box\phi \text{ iff } \exists V \in N(x) \forall y \in V : M, y \models \phi$$

A formula is valid in a n-model M if it is valid at all points of M ($M \models \phi$). Formula is valid in a n-frame F if it is valid in all models based on F (notation $F \models \phi$). For Logic L we write $F \models L$, if for any $\phi \in L, F \models \phi$. We define nV(L) = $\{F \mid F$ is an n-frame and $F \models \phi\}$.

3.3 Defintion

Let F = (W,R) be a Kripke frame. We define an n-frame N(F) = (W, τ) as follows. For any $w \in W$ we have :

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

3.4 Defintion

Let X = (X, τ_1,...) and Y = (Y, σ_1,...) be n-frames. Then the function f: $X \to Y$ is called bounded morphism if

1. f is surjective
2. $\forall x \in X\, \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X\, \forall V \in \sigma_i(f(x))\, \exists U \in \tau_i(x) : f(U) \subseteq V$

3.5 Defintion