

1 Definition

1.1 Defintion

Let $prop$ be a set of variables. Then a formula ϕ is defined as follows :

$$\phi ::= p \mid \perp \mid \phi \mid \phi \rightarrow \phi \mid \Box_i \phi$$

where $p \in Prop$ and \Box_i is a modal operator. Other connectives are expressed through \perp and \rightarrow and dual modal operators \Diamond_i as $\Diamond_i \phi = \neg \Box_i \neg \phi$

1.2 Defintion

A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \rightarrow \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms

$$\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$ Let L be a logic and let Γ be a set of formulas. Then $L+\Gamma$ denotes the minimal logic containing L and Γ

1.3 Definition

Let $L1$ and $L2$ be two modal logic with one modality \Box . Then the fusion of these logics are defined as follows :

$$L1 \otimes L2 = K2 + L1(\Box \rightarrow \Box_1)L2(\Box \rightarrow \Box_2)$$

The follow logics may be important

$$D = K + \Box p \rightarrow \Diamond p$$

$$T = K + \Box p \rightarrow p$$

$$D4 = D + \Box p \rightarrow \Box \Box p$$

$$S4 = T + \Box p \rightarrow \Box \Box p$$

2 Kripke Definition

To show $D \otimes D \otimes D + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ has the finite model property, we need some important definitions.

2.1 Definition

Let $M = (W, R, V)$ be a model and $w \in W$ a state in M . The notion of a formula being true at w is inductively defined as follows :

$$M, w \Vdash \Box \phi \text{ iff } \forall v \in W : wRv \rightarrow M, v \Vdash \phi$$

This definition can be extended to a multimodal version, where the modal operators are interpreted the same way but with the respective relation.

2.2 Definition

A set Σ is closed under subformulas, if for all formulas ϕ and ϕ' the following holds :

1. if $\neg \phi \in \Sigma$ then $\phi \in \Sigma$
2. if $\phi \vee \phi' \in \Sigma$ then $\phi, \phi' \in \Sigma$
3. if $\Box \phi \in \Sigma$ then $\phi \in \Sigma$

We can define it similarly for a multimodal logic. For every modal operator \Box_n , we extend this definition by adding a new condition similar to the third. Example : Suppose we have a multimodal logic with \Box, \Box_1 called L_{\Box, \Box_1} and $\phi = \Box p \rightarrow \Box_1 q$ and $\phi \in \Sigma$. Then $\Sigma = \{\phi, \neg \Box p, \Box_1 q, \Box p, p, q\}$ is closed under subformulas.

2.3 Definition

Let $M = (W, R, V)$ be a model and suppose Σ is a set of formulas. We define a relation \equiv on W as follows :

$$w \equiv v \text{ iff } \forall \phi \in \Sigma : M, w \Vdash \phi \Leftrightarrow M, v \Vdash \phi$$

It is well known that the \equiv -relation is an equivalence relation. We denote the equivalence class of a state $w \in W$ as $[w]_\Sigma = \{v \mid v \equiv w\}$. Furthermore W_Σ is the set of all equivalence classes, i.e $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$.

2.4 Defintion

Let $M = (W, R, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes induced by \equiv . A model $M_\Sigma^f = (W^f, R^f, V^f)$ is called filtration of M through Σ if the following holds :

1. $W^f = W_\Sigma$
2. If $(w, v) \in R$ then $([w], [v]) \in R^f$
3. If $([w], [v]) \in R^f$ then for any $\Box\phi \in \Sigma$: if $M, w \Vdash \Box\phi$ then $M, v \Vdash \phi$
4. $V^f = \{[w] \mid M, w \Vdash p\}$, for all propositional variables $p \in \Sigma$

In our case, we are in a multimodal logic with three modal operators \Box, \Box_1, \Box_2 . We need to extend this defintion for L_{\Box, \Box_1, \Box_2} . This means our model looks like: $M = (W, R, R_1, R_2, V)$. We extend the conditions as follows :

If $(w, v) \in R_i$ then $([w], [v]) \in R_i^f$
 If $([w], [v]) \in R_i^f$ then for any $\Box_i\phi \in \Sigma$: if $M, w \Vdash \Box_i$ then $M, v \Vdash \phi$, where $i \in \{1, 2\}$

2.5 Filtration Theorem

Consider L_{\Box, \Box_1, \Box_2} . Let $M^f = (W_\Sigma, R^f, R_1^f, R_2^f, V)$ be a filtration of M through a subformula closed set Σ . Then for all formulas $\phi \in \Sigma$, and all nodes $w \in M$, we have

$$M, w \Vdash \phi \text{ iff } M^f, [w] \Vdash \phi$$

Proof. By induction on ϕ . We will only show non-trivial and, for our purposes, necessary cases.

Case $\phi = p$: Left to right follows immediately from filtration defintion. Conversely, suppose $M^f, [w] \Vdash p$. This means $[w] \in V^f(p)$. But this means $V(p)$ can not be empty. Pick any $v \in V(p)$. Obviously, $w \equiv v$ and $M, v \Vdash p$. Hence, $M, w \Vdash p$.

Case $\phi = \neg\psi$: Suppose ψ holds. Then we have : $M, w \Vdash \phi$ iff $M, w \not\Vdash \psi$. Applying induction hypothesis, we get : $M^f, [w] \not\Vdash \psi$. But then, we have $M^f, [w] \Vdash \phi$. Right to left is the same.

Case $\phi = \phi_1 \wedge \phi_2$: Suppose ϕ_1, ϕ_2 holds. Let $M, w \Vdash \phi$. That means $M, w \Vdash \phi_1$ and $M, w \Vdash \phi_2$. Applying induction hypothesis, we get $M^f, [w] \Vdash \phi_1$ and $M^f, [w] \Vdash \phi_2$. But then, $M^f, [w] \Vdash \phi_1 \wedge \phi_2 = \phi$. Right to left is similar.

Case $\phi = \Box_i\psi$ ($i \in \{1, 2, \epsilon\}$, $\Box_\epsilon = \Box$): Left to right. Suppose ψ holds and $M, w \Vdash \Box_i\psi$. We need to show $M^f, [w] \Vdash \Box_i\psi$, this means $\forall [v] \in W_\Sigma : [w]R_i[v] \rightarrow M^f, [v] \Vdash \Box_i\psi$. Pick any $[v] \in W_\Sigma$ s.t $[w]R_i[v]$. By condition 3, w.r.t to the modal operator, we have $M, v \Vdash \psi$. By induction hypothesis, we get $M^f, [v] \Vdash \psi$. Because $[v]$ was arbitrary it

follows that $M^f, [w] \Vdash \Box_i \psi$.

Right to left. Suppose ψ holds and $M^f, [w] \Vdash \Box_i \psi$. Pick $v \in W$ s.t $wR_i v$. By condition 2, w.r.t to the modal operator, we have $[w]R_i^f[v]$. So, $M^f, [v] \Vdash \psi$. By induction hypothesis, we get $M, v \Vdash \psi$. Because v was arbitrary, we have $M, w \Vdash \Box_i \psi$. \square

Now define the smallest filter for L_{\Box, \Box_1, \Box_2} and show that this is a filter. We denote this as R^s .

2.6 Defintion

Let $M = (W, R, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes. We define :

$$R^s = \{[w], [v] \mid \exists w' \in [w], \exists v' \in [v] : w'R_i v'\}$$

where $i \in \{1, 2, \epsilon\}$.

2.7 Lemma

Let $M = (W, R, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes induced by \equiv and V^f the standard valuation on W_Σ . Then $(W_\Sigma, R^s, R_1^s, R_2^s, V^f)$ is a filtration of M through Σ .

Proof. It suffices to show R_i^s fullfills the condition 2 and 3 w.r.t to the corresponding modal operator \Box_i . But R_i^s already satisfies condition 2. Let's check the other condition. Let $\Box_i \phi \in \Sigma$, $[w]R_i^s[v]$ and $M, w \Vdash \Box_i \phi$ where $i \in \{1, 2, \epsilon\}$. Because of $[w]R_i^s[v]$ we pick a $w' \in [w]$ and $v' \in [v]$. By definition, we have $w'R_i v'$. Because $w' \equiv w$, we get $M, w' \Vdash \Box_i \phi$. Hence, $M, v' \Vdash \phi$ and by $v' \equiv v$, we get $M, v \Vdash \phi$. \square

2.8 Proposition

Let Σ be a finite subformula closed set of L_{\Box, \Box_1, \Box_2} . For any model M , if M^f is a filtration through Σ , then M^f contains at most 2^n nodes (where n denotes the size of Σ).

Proof. The states of M^f are the equivalence classes in W_Σ . Let $g : W_\Sigma \rightarrow P(\Sigma)$ defined by $g([w]) = \{\phi \in \Sigma \mid M, w \Vdash \phi\}$. g is well defined. Pick any u and v s.t $u \equiv v$. But then by definition of \equiv , they fullfill the same subformulas. This means $g([v]) = g([u])$. g is also injective. Pick any $[u], [v] \in W_\Sigma$ s.t $g([u]) = g([v])$. We show $[u] \subseteq [v]$. The other inclusion is similar. By assumption we have $u \equiv v$. Pick any $u' \in [u]$. Then we have $u' \equiv u \equiv v$. Hence, $u' \in [v]$. At the end, this means M^f contains at most 2^n nodes. \square

2.9 Finite Model Property - via Filtrations

Let ϕ be a formula of L_{\Box, \Box_1, \Box_2} . If ϕ is satisfiable, then it is satisfiable on a finite model containing at most 2^n nodes, where n is the number of subformulas in ϕ .

Proof. Assume that ϕ is satisfiable on a model on M . Take any filtration of M through the set of subformulas of ϕ . By Filtration Theorem, we get that ϕ is satisfied in the filtration model M^f . Furthermore, it is bounded by 2^n . □

Now we define Sahlqvist formulas for our purposes.

2.10 Definition

A modal formula ϕ is positive if all variables occurs without negation. In the other hand, a formula is negative, if all variables occurs with negation. A boxed atom is a modal formula of the form $\Box^n p$ for some $n \in \mathbb{N}$, where p is a propositional variable and $\Box^n p$ is defined as follows : $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$.

Furthermore, a Sahlqvist antecedent is built from \perp , \top , negative formulas and boxed atoms by applying \Diamond and \wedge . A Sahlqvist implication is a modal formula of the form $\phi \rightarrow \psi$, where ϕ is a Sahlqvist antecedent and ψ a positive formula.

Now, a Sahlqvist formula is built from Sahlqvist implications by applying \Box and \vee .

Examples for Sahlqvist formulas:

$$\begin{aligned} & \Box \Box p \rightarrow \Box p \\ & \Diamond \neg p \rightarrow p \\ & \Diamond \Box \Box \Box \Box p \rightarrow \Box \Diamond \Box \Diamond p \\ & \Box \Box \Box \Box (\Diamond \Box p \rightarrow p) \vee \Box \Box p \rightarrow \Box p \end{aligned}$$

Non Sahlqvist Formulas :

$$\begin{aligned} & \Box \Diamond p \rightarrow \Diamond \Box p \\ & \Diamond \Box p \rightarrow \Box \neg p \end{aligned}$$

We can extend this definition for our logic. We say a boxed atom can be $\Box_1^n p$ and $\Box_2^n p$. A Sahlqvist antecedent can also be build by applying \Diamond_1 and \Diamond_2 . A Sahlqvist formula can be build by Sahlqvist implications by applying additionally \Box_1 and \Box_2 .

Sahlqvist formulas possess important properties, which are guaranteed by the Sahlqvist Theorem. It says that, when given a normal modal logic K and a set of Sahlqvist formulas, the resulting logic is complete w.r.t to the class of frames, which satisfies the

corresponding first-order formula of the Sahlqvist formulas. This also holds for multimodal logic. We will not prove it here, but we will use this for our logic to show completeness w.r.t to a suitable class of frames.

2.11 Defintion

We say for a modal logic Λ has the finite model property (FMP) if for every formula ϕ that is not provable in Λ , is falsifiable in a finite model.

2.12 Proposition

The logic $D \otimes D \otimes D + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ has FMP.

Proof. The idea is to pick a class of frame \mathcal{F} , where DPL is sound and complete with respect to and then show by filtration the FMP.

Let $\mathcal{C} = \{F \mid F \Vdash DPL\}$. Obviously, DPL is sound w.r.t \mathcal{C} . Completeness can be shown by using the Sahlqvist Theorem. We remember $D = K + \Box_i p \rightarrow \Diamond_i p$ for $i \in \{1, 2, \epsilon\}$. These axioms and $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$ are Sahlqvist formulas. By Sahlqvist, we have that DPL is complete w.r.t $\{F \mid F \Vdash \forall x \exists y R_i(x, y) \text{ and } F \Vdash \forall x \exists y (R_1(x, y) \wedge R_2(x, y)) \rightarrow R(x, y)\}$.

Now assume a formula ϕ is not derivable from DPL. By completeness we get ϕ is falsifiable in a model M and a world w . Hence, $M, w \Vdash \neg\phi$. Now we build the set $\text{subf}(\neg\phi)$ which denotes the closed subformulas set of $\neg\phi$. Let $M^s = (W_\Sigma, R^s, R_1^s, R_2^s, V^s)$ be the smallest filtration of M through $\text{subf}(\neg\phi)$ and V^s is the standard valuation on W_Σ . By filtration Theorem, they preserve truth. It remains to show $F^s = (W_\Sigma, R^s, R_1^s, R_2^s) \in \mathcal{C}$. For that, every point must have at least one successor in every relation and it must hold that $R_1^s, R_2^s \subseteq R^s$. For the first one, we show this for R because the rest is similar. Pick $[w] \in W_\Sigma$. Because M is based on a frame $F \in \mathcal{C}$, there is a point v s.t wRv . By the defintion of smallest filtration, we have that $[w]R^s[v]$. For the second one, we show only for R_1^s because it is the same for R_2^s . Pick $[w], [v]$ s.t $[w]R_1^s[v]$. By defintion of smallest filtration there are points $w' \in [w]$ and $v' \in [v]$ s.t $w'R_1v'$. Furthermore, it holds $R_1 \subseteq R$, so $w'Rv'$. By smallest filtration, we get $[w']R^s[v']$. Because $w' \equiv w$ and $v' \equiv v$, we have $[w'] = [w]$ and $[v'] = [v]$. It follows $[w]R^s[v]$.

□

2.13 Defintion

Let $T_{\omega[in]}$ (i = irreflexiv, n = non-transitiv) denote the infinite branching and infinite depth tree, which is irreflexiv and non-transitive. Formally the tree can be defined as :

$T_{\omega[in]} = (W, R)$ where $W = \mathbb{N}^*$ and sRt iff $\exists u \in \mathbb{N} : s * u = t$ (the '*' is the concatenation operator)

The $T_{\omega, \omega, \omega[in]}$ tree is similary defined as the $T_{6,2,2}$ tree but with infinite branching and infinite depth. Before characterizing it, we say $\mathbb{N}_{R_1}^*$ is the set of finite number combinations which has a subscript R_1 to denote that these numbers relate to R_1 (examples : $0_{R_1}, 0123_{R_1}$). $\mathbb{N}_{R_1}^+$ is the set $\mathbb{N}_{R_1}^* - \{\epsilon\}$. $\mathbb{N}_R^+, \mathbb{N}_{R_2}^+$ are defined similar.

Now let $T_{\omega, \omega, \omega[in]} = (W, R, R_1, R_2)$ where $W = \mathbb{N}_R^+ \cup \mathbb{N}_{R_1}^+ \cup \mathbb{N}_{R_2}^+ \cup \{\epsilon\}$,

$$sRt \text{ iff } \exists u \in \mathbb{N}_R \cup \mathbb{N}_{R_1} \cup \mathbb{N}_{R_2} : s * u = t$$

$$sR_1t \text{ iff } \exists u \in \mathbb{N}_{R_1} : s * u = t$$

$$sR_2t \text{ iff } \exists u \in \mathbb{N}_{R_2} : s * u = t$$

where s,t are elements of the positive closure set where the element u can come from (additionally s can be ϵ), w.r.t to the relation. For example if we consider sR_1t , then $s, t \in \mathbb{N}_{R_1}^+$ but also $s = \epsilon$. Of course the '*' operator acts here again as a concatenation operator.

2.14 Defintion

Let $F = (W, R_1, R_2, \dots)$ and $F' = (W', R'_1, R'_2, \dots)$ be two frames. A bounded morphism from F to F' is a function $f : W \rightarrow W'$ satisfying the following conitions:

$$\text{If } (u, v) \in R_i \text{ then } (f(u), f(v)) \in R'_i$$

$$\text{If } (f(w), v') \in R'_i \text{ then } \exists v \in W \text{ s.t } (w, v) \in R_i \text{ and } f(v) = v'$$

We say that F' is a bounded morphic image of F , if there is a surjective bounded morphism from F to F' .

2.15 Proposition

Let ϕ be a formula in $L_{\square, \square_1, \square_2}$, $F = (W, R, R_1, R_2)$ and $F' = (W', R', R'_1, R'_2)$ be two frames and F to F' a surjective bounded morphism. Then the following holds :

$$\text{If } F \Vdash \phi \text{ then } F' \Vdash \phi$$

This can be shown by structural induction on the length of the formula.

2.16 Corollary

If F' is a bounded morphic image of F , then we have $\text{Log}(F) \subseteq \text{Log}(F')$

2.17 Proposition

D is sound and complete w.r.t T_ω .

Proof. Sound is clear. For completeness, we use the well known fact that D has FMP. This means, $D = \text{Log}\{F \mid F \Vdash D\}$ where F is a finite frame. We can pick such a finite frame F and it suffices to find a surjective bounded morphism f from T_ω to F . This would imply $\text{Log}(T_\omega) \subseteq D$.

Now, let $F = (W', R')$ be such a finite rooted frame with root w . (We can pick such because $F, w \Vdash D$ and then generate a subframe by w). We define inductively an assignment of nodes of F to the nodes of T_ω . For the base case, we assign w to the root of T_ω . The induction step looks like the following : Assume a point $x \in T_\omega$ has been assigned to a point $u \in F$ but the successors of x has no assignment. Let s_1, s_2, \dots, s_k be successors of u (k denotes amount of successors and $k \geq 1$ because seriality guarantees us at least one successor). For $n \geq 1, n \in \mathbb{N}$ we assign s_i to the $(n * i)$ th-successor of x . This means we are assigning the successors alternately.

Now we check for f the conditions of bounded morphism. First condition : Let $x, y \in T_\omega$ s.t xRy and $f(x) = s$. But then, y will be assigned to a successor point of s . Hence, $f(x)R'f(y)$. Second condition : Suppose $f(x)R't$ and $f(x) = s$. Since t is a successor of s and $f(x) = s$, then a successor of x , say y , gets the assignment t .

□

3 Topological Space Defintion

3.1 Defintion

A topological space is a pair (X, τ) where τ is a collection of subsets of X (elements of τ are also called open sets) such that :

1. the empty set \emptyset and X are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

The space is called Alexandroff, if we allow the intersection of infinite collection of open sets. A topological model is a structure $M = (X, \tau, v)$ where (X, τ) is a topological space and v is a valuation assigning subsets of X to propositional variables.

3.2 Defintion

Let $M = (X, \tau, v)$ a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows :

$M, x \models \Box\phi$ iff $\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$

$M, x \models \Diamond\phi$ iff $\forall U \in \tau$ s.t $x \in U$ and $\exists u \in U : M, u \models \phi$

3.3 Defintion

Let $A = (X, \chi)$ and $B = (Y, v)$ be topological spaces. The standard product topology τ is the set of subsets of $X \times Y$ such that $X \in \chi$ and $Y \in v$.

Let $N \subseteq X \times Y$. We call N horizontally open if $\forall (x, y) \in N \exists U \in \chi : x \in U$ and $U \times \{y\} \subseteq N$.

We call N vertically open if $\forall (x, y) \in N \exists V \in v : y \in V$ and $\{x\} \times V \subseteq N$

If N is H-open and V-open, then we call it HV-open.

We denote τ_1 is the set of all H-open subsets of $X \times Y$ and τ_2 is the set of all V-open subsets of $X \times Y$

3.4 Defintion

Let X and Y be topological spaces and $f : X \rightarrow Y$ a function. We call f continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X . We say f is open if for each open

set $V \subseteq X$ the set $f[V]$ is open in Y .

3.5 Remark

There is an alternative definition for open sets. Let (X, τ) be a topological space and U a set. U is open iff $\forall x \in U \exists V \subseteq U : V$ is open and $x \in V$. This is true because, the union of open sets is an open set.

Now we define some Kripke frames, which we will use through this chapter.

3.6 Defintion

Let T_2 be the infinite binary tree with reflexive and transitive descendant relation. Formally it is defined as follows : $T_2 = (W, R)$ where $W = \{0,1\}^*$ and sRt iff $\exists u \in W : s * u = t$.

The $T_{6,2,2}$ tree is the infinite six branching tree, where all nodes of $T_{6,2,2}$ is R -related, the first two R_1 -related and the last two R_2 -related. Formally we can define this tree as follows : $T_{6,2,2} = (W, R, R_1, R_2)$, where $W = \{0,1,2,3,4,5\}^*$,

$$sRt \text{ iff } \exists u \in \{0,1,2,3,4,5\}^* : s * t = u$$

$$sR_1t \text{ iff } \exists u \in \{0,1\}^* : s * t = u$$

$$sR_2t \text{ iff } \exists u \in \{5,6\}^* : s * t = u$$

where s and t are elements of the set where the element u can come from, w.r.t to the relation. For example in the case sRt , s and t are elements of $\{0,1,2,3,4,5\}^*$.

4 Neighbourhood

4.1 Defintion

Let X be a non-empty set. A function $\tau : X \rightarrow 2^{2^X}$ is called a neighbourhood function. A pair $F = (X, \tau)$ is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X, τ, v) , where v assigns a subset of X to a variable

4.2 Defintion

Let $M = (X, \tau, v)$ be a neighbourhood model and $x \in X$. The truth of a formula is defined inductively as follows :

$$M, x \models \Box\phi \text{ iff } \exists V \in N(x) \forall y \in V : M, y \models \phi$$

A formula is valid in a n-model M if it is valid at all points of M ($M \models \phi$). Formula is valid in a n-frame F if it is valid in all models based on F (notation $F \models \phi$). For Logic L we write $F \models L$, if for any $\phi \in L$, $F \models \phi$. We define $nV(L) = \{F \mid F \text{ is an n-frame and } F \models \phi\}$.

4.3 Defintion

Let $F = (W, R)$ be a Kripke frame. We define an n-frame $N(F) = (W, \tau)$ as follows. For any $w \in W$ we have :

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

4.4 Defintion

Let $X = (X, \tau_1, \dots)$ and $Y = (Y, \sigma_1, \dots)$ be n-frames. Then the function $f: X \rightarrow Y$ is called bounded morphism if

1. f is surjective
2. $\forall x \in X \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X \forall V \in \sigma_i(f(x)) \exists U \in \tau_i(x) : f(U) \subseteq V$

4.5 Defintion

Let $X = (X, \tau_1)$ and $Y = (Y, \tau_2)$ be two n-frames. Then the product of these two frames is an n-2-frame and is defined as follows :

$$\begin{aligned} X \times Y &= (X \times Y, \tau'_1, \tau'_2) \\ \tau'_1(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\} \\ \tau'_2(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\} \end{aligned}$$

4.6 Defintion

For two unimodal logics L_1 and L_2 we define the n-product of them as follows :

$$L_1 \times_n L_2 = \text{Log}(\{X \times Y \mid X \in nV(L_1) \text{ and } Y \in nV(L_2)\})$$

Now we define some Kripke frames we need for this chapter.

4.7 Defintion

Let $T_{\omega[in]}$ (i = irreflexiv, n = non-transitiv) denote the infinite branching and infinite depth tree, which is irreflexiv and non-transitive. Formally the tree can be defined as : $T_{\omega[in]} = (W, R)$ where $W = \mathbb{N}^*$ and sRt iff $\exists u \in \mathbb{N} : s * u = t$ (the '*' is the concatenation operator)

The $T_{\omega, \omega, \omega[in]}$ tree is similary defined as the $T_{6,2,2}$ tree but with infinite branching and infinite depth. Before characterizing it, we say $\mathbb{N}_{R_1}^*$ is the set of finite number combinations which has a subscript R_1 to denote that these numbers relate to R_1 (examples : $0_{R_1}, 0123_{R_1}$). $\mathbb{N}_{R_1}^+$ is the set $\mathbb{N}_{R_1}^* - \{\epsilon\}$. $\mathbb{N}_R^+, \mathbb{N}_{R_2}^+$ are defined similar.

Now let $T_{\omega, \omega, \omega[in]} = (W, R, R_1, R_2)$ where $W = \mathbb{N}_R^+ \cup \mathbb{N}_{R_1}^+ \cup \mathbb{N}_{R_2}^+ \cup \{\epsilon\}$,

$$sRt \text{ iff } \exists u \in \mathbb{N}_R \cup \mathbb{N}_{R_1} \cup \mathbb{N}_{R_2} : s * u = t$$

$$sR_1t \text{ iff } \exists u \in \mathbb{N}_{R_1} : s * u = t$$

$$sR_2t \text{ iff } \exists u \in \mathbb{N}_{R_2} : s * u = t$$

where s,t are elements of the positive closure set where the element u can come from (additionally s can be ϵ), w.r.t to the relation. For example if we consider sR_1t , then $s, t \in \mathbb{N}_{R_1}^+$ but also $s = \epsilon$. Of course the '*' operator acts here again as a concatenation operator.