

# 1 Definition

## 1.1 Defintion

Let  $Prop$  be a set of variables. Then a formula  $\phi$  is defined as follows :

$$\phi ::= p \mid \perp \mid \phi \mid \phi \rightarrow \phi \mid \Box_i \phi$$

where  $p \in Prop$  and  $\Box_i$  is a modal operator. Other connectives are expressed through  $\perp$  and  $\rightarrow$  and dual modal operators  $\Diamond_i$  as  $\Diamond_i \phi = \neg \Box_i \neg \phi$

## 1.2 Defintion

A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ( $\frac{\phi(p_i)}{\phi(\psi)}$ ), Modus Ponens ( $\frac{\phi, \phi \rightarrow \psi}{\psi}$ ), Generalization rules ( $\frac{\phi}{\Box_i \phi}$ ) and the following axioms

$$\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$$

$K_n$  denotes the minimal normal modal logic with  $n$  modalities and  $K = K_1$  Let  $L$  be a logic and let  $\Gamma$  be a set of formulas. Then  $L+\Gamma$  denotes the minimal logic containing  $L$  and  $\Gamma$

## 1.3 Definition

Let  $L1$  and  $L2$  be two modal logic with one modality  $\Box$ . Then the fusion of these logics are defined as follows :

$$L1 \otimes L2 = K2 + L1(\Box \rightarrow \Box_1)L2(\Box \rightarrow \Box_2)$$

The follow logics may be important

$$D = K + \Box p \rightarrow \Diamond p$$

$$T = K + \Box p \rightarrow p$$

$$D4 = D + \Box p \rightarrow \Box \Box p$$

$$S4 = T + \Box p \rightarrow \Box \Box p$$

## 2 Topological Space Defintion

### 2.1 Defintion

A topological space is a pair  $(X, \tau)$  where  $\tau$  is a collection of subsets of  $X$  (elements of  $\tau$  are also called open sets) such that :

1. the empty set  $\emptyset$  and  $X$  are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

A topological model is a structure  $M = (X, \tau, v)$  where  $(X, \tau)$  is a topological space and  $v$  is a valuation assigning subsets of  $X$  to propositional variables.

### 2.2 Defintion

Let  $M = (X, \tau, v)$  a topological model and  $x \in X$ . The satisfaction of a formula at the point  $x$  in  $M$  is defined inductively as follows :  $M, x \models \Box\phi$  iff ,  $\exists U \in \tau$  s.t  $x \in U$  and  $\forall u \in U : M, u \models \phi$

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$M, x \models \Diamond\phi$  iff ,  $\forall U \in \tau$  s.t  $x \in U$  and  $\exists u \in U : M, u \models \phi$

### 2.3 Defintion

Let  $A = (X, \chi)$  and  $B = (Y, v)$  be topological spaces. The standard product topology  $\tau$  is the set of subsets of  $X \times Y$  such that  $X \in \chi$  and  $Y \in v$ .

Let  $N \subseteq X \times Y$ . We call  $N$  horizontally open if  $\forall (x, y) \in N \exists U \in \chi : x \in U$  and  $U \times \{y\} \subseteq N$ .

We call  $N$  vertically open if  $\forall (x, y) \in N \exists V \in v : y \in V$  and  $\{x\} \times V \subseteq N$

If  $N$  is H-open and V-open, then we call it HV-open.

We denote  $\tau_1$  is the set of all H-open subsets of  $X \times Y$  and  $\tau_2$  is the set of all V-open subsets of  $X \times Y$

## 3 Neighbourhood

### 3.1 Defintion

Let  $X$  be a non-empty set. A function  $N : X \rightarrow 2^{2^X}$  is called a neighbourhood function. A pair  $F = (X, N)$  is called a neighbourhood frame (or n-frame). A model based on  $F$  is a tuple  $(X, N, v)$ , where  $v$  assigns a subset of  $X$  to a variable

### 3.2 Defintion

Let  $M = (X, N, v)$  be a neighbourhood model and  $x \in X$ . The truth of a formula is defined inductively as follows :

$$M, x \models \Box\phi \text{ iff } \exists V \in N(x) \forall y \in V : M, y \models \phi$$

A formula is valid in a n-model  $M$  if it is vali at all points of  $M$  ( $M \models \phi$ ). Formula is valid in a n-frame  $F$  if it is valid in all models based on  $F$  (notation  $F \models \phi$ )