

1 Introduction

The study of topological semantics in modal logic was initiated by McKinsey and Tarski in 1944 [?]. The idea was to generalize Kripke frames using tools from topology. Neighbourhood semantics [?] as a generalization of Kripke semantics for modal logic were invented independently by Dana Scott [?] and Richard Montague [?]. Neighborhood semantics is more general than Kripke semantics and in the case of normal reflexive and transitive logics coincides with topological semantics. The original motivation for introducing was to provide a semantics for non-normal modal logics. But in recent years, interest in topological semantics and neighborhood frames has grown considerably, partly due to its applications in artificial intelligence.

Oftentimes, it is necessary to combine frames for different modal logics into a complex frame. The natural way of doing that is a product construction. For Kripke frames, the resulting product is the Cartesian product of the two frames with two accessibility relations. For topological semantics, the product of topological spaces as bi-topological spaces with so-called horizontal and vertical topologies have been considered. In a similar fashion, the product of neighborhood frames was introduced by Sano in [?].

Now, let L_1 and L_2 be two modal logics. We say $L_1 \otimes L_2$ (called fusion) is the minimal modal logic containing L_1 and L'_2 , where L'_2 is the logic L_2 after renaming all modalities. Furthermore, we say $L_1 \times_n L_2$ is the logic (i.e the set of all valid formulas) of the class products of neighbourhood frames $N_1 \times_n N_2$ such that L_i is valid in N_i for $i = 1, 2$. It was proven in [?] that for any two logics $L_1, L_2 \in \{D4, D, T, S4\} : L_1 \times_n L_2 = L_1 \otimes L_2$. In [?], the authors studied a product of two spaces with three topologies : horizontal, vertical and classic product topology. They proved that the logic of such spaces is $S4 \otimes S4 \otimes S4 + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ where \Box corresponds to the product topology.

The following work will present a detailed proof of the shown results. Additionally, we will show that $T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ (we abbreviate it with TNL) = $T x_n^+ T$. The proof ideas here are inspired by the shown results.

2 Preliminaries

2.1 Kripke

Definition 2.1.1. Let $prop$ be a set of variables. Then a formula ϕ is defined as follows:

$$\phi ::= p \mid \perp \mid \phi \mid \phi \rightarrow \phi \mid \Box_i \phi$$

where $p \in Prop$ and \Box_i is a modal operator. Other connectives are expressed through \perp and \rightarrow and dual modal operators \Diamond_i as $\Diamond_i \phi = \neg \Box_i \neg \phi$

Definition 2.1.2. A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \rightarrow \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms

$$\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$. Let L be a logic and let Γ be a set of formulas. Then $L + \Gamma$ denotes the minimal logic containing L and Γ

Definition 2.1.3. Let L_1 and L_2 be two modal logic with one modality \Box . Then the fusion of these logics are defined as follows :

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \rightarrow \Box_1)} L_{2(\Box \rightarrow \Box_2)}$$

The following logics may be important

$$D = K + \Box p \rightarrow \Diamond p$$

$$T = K + \Box p \rightarrow p$$

$$D4 = D + \Box p \rightarrow \Box \Box p$$

$$S4 = T + \Box p \rightarrow \Box \Box p$$

Now we introduce some special kind of frames, which we will use through this work.

Definition 2.1.4. Let A be a nonempty set.

$$A^* = \{a_1 \dots a_k \mid a_i \in A\}$$

is the set of all finite sequences of elements from A , including the empty sequence Λ . Elements from A^* will be denoted as \vec{a} . The length of a sequence $\vec{a} = a_1 \dots a_k$ is k (also $l(\vec{a}) = k$) and the length of Λ is 0 ($l(\Lambda) = 0$). Concatenation is denoted by " \cdot " : $(a_1 \dots a_k) \cdot (b_1 \dots b_l) = \vec{a} \cdot \vec{b} = a_1 \dots a_k b_1 \dots b_l$.

Definition 2.1.5. Let A be a nonempty set. We define an infinite frame $F_{in}[A] = (A^*, R)$ s.t for $\vec{a}, \vec{b} \in A^*$

$$\vec{a}R\vec{b} \Leftrightarrow \exists x \in A (\vec{b} = \vec{a} \cdot x)$$

Furthermore we define :

$$F_{rn}[A] = (A^*, R^r), \text{ where } R^r = R \cup Id \text{ (reflexive closure)}$$

$$F_{it}[A] = (A^*, R^*), \text{ where } R^* = \bigcup_{i=1}^{\infty} R^i \text{ (transitive closure)}$$

$$F_{rt}[A] = (A^*, R^{r*})$$

where "t" stands for transitive, "n" for non-transitive, "r" for reflexive and "i" for ir-reflexive.

For now, we will use the following notion to generalize : $F_{\xi\eta}$ where $\xi \in \{i, r\}$ and $\eta \in \{t, n\}$

Proposition 2.1.6. Let $F = F_{\xi\eta}[A] = (A^*, R)$ then

$$\vec{a}R(\vec{a} \cdot \vec{c}) \Leftrightarrow \Lambda R\vec{c}$$

Definition 2.1.7. Let $F_1 = F_{\xi_1\eta_1}[A] = (A^*, R_1)$ and $F_2 = F_{\xi_2\eta_2}[B] = (B^*, R_2)$, where $\xi_1, \xi_2 \in \{i, r\}$ and $\eta_1, \eta_2 \in \{t, n\}$. Furthermore, we assume $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ with $A \cap B = \emptyset$. Then we define the frame $F_1 \otimes F_2 = (W, R'_1, R'_2)$ as follows :

$$W = (A \cup B)^*$$

$$\vec{x}R'_1\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in A^* \text{ such that } \Lambda R_1\vec{z}$$

$$\vec{x}R'_2\vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in B^* \text{ such that } \Lambda R_2\vec{z}$$

Proposition 2.1.8 ([?], [?]). Let F_1 and F_2 be as in Definition 2.1.7. Then

$$Log(F_1 \otimes F_2) = Log(F_1) \otimes Log(F_2)$$

Proposition 2.1.9. Let $F_{in} = F_{in}[\mathbb{N}]$, $F_{rn} = F_{rn}[\mathbb{N}]$, $F_{it} = F_{it}[\mathbb{N}]$ and $F_{rt} = F_{rt}[\mathbb{N}]$. Then the following holds:

$$Log(F_{in}) = D$$

$$Log(F_{rn}) = T$$

$$Log(F_{it}) = D4$$

$$Log(F_{rt}) = S4$$

2.2 Topological Space

Definition 2.2.1. A topological space is a pair (X, τ) where τ is a collection of subsets of X (elements of τ are also called open sets) such that :

1. the empty set \emptyset and X are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

The space is called Alexandroff, if we allow the intersection of infinite collection of open sets. A topological model is a structure $M = (X, \tau, v)$ where (X, τ) is a topological space and v is a valuation assigning subsets of X to propositional variables.

Definition 2.2.2. Let $p \in \text{Prop}$. We build formulas as follows :

$$\phi ::= p \mid \perp \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi$$

Definition 2.2.3. Let $M = (X, \tau, v)$ a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows :

$$\begin{aligned} M, x \models p & \quad \text{iff } x \in V(p) \\ M, x \models \perp & \quad \text{never} \\ M, x \models \neg\phi & \quad \text{iff } M, x \not\models \phi \\ M, x \models \phi \vee \psi & \quad \text{iff } M, x \models \phi \text{ or } M, x \models \psi \\ M, x \models \Box\phi & \quad \text{iff there exists } U \in \tau \text{ such that } x \in U \text{ and } \forall u \in U : M, u \models \phi \end{aligned}$$

Definition 2.2.4. Let $A = (X, \chi)$ and $B = (Y, v)$ be topological spaces and $N \subseteq X \times Y$.

N is horizontally open iff $\forall (x, y) \in N \exists U \in \chi : x \in U \text{ and } U \times \{y\} \subseteq N$.

N vertically open iff $\forall (x, y) \in N \exists V \in v : y \in V \text{ and } \{x\} \times V \subseteq N$.

If N is H -open and V -open, then we call it HV -open.

We denote τ_1 is the set of all H -open subsets of $X \times Y$ and τ_2 is the set of all V -open subsets of $X \times Y$. We say τ is the standard product of subsets $X \times Y$ s.t. $X \in \chi$ and $Y \in v$.

Definition 2.2.5. Let $A = (X, \chi)$ and $B = (Y, v)$ be topological spaces and $(x, y) \in X \times Y$. The truth in $M = (X \times Y, \tau_1, \tau_2, \tau, v)$ is similar as in Definition 2.2.3. For \Box , \Box_1 and \Box_2 we interpret this as follows :

$$\begin{aligned} M, (x, y) \models \Box_1\phi & \quad \text{iff } \exists U \in \tau_1 : (x, y) \in U \text{ and } \forall (x', y') \in U : (x', y') \models \phi \\ M, (x, y) \models \Box_2\phi & \quad \text{iff } \exists V \in \tau_2 : (x, y) \in V \text{ and } \forall (x', y') \in V : (x', y') \models \phi \\ M, (x, y) \models \Box\phi & \quad \text{iff } \exists U \in \chi \exists V \in v : (x, y) \in U \times V \text{ and } \forall (x', y') \in U \times V : (x', y') \models \phi \end{aligned}$$

Definition 2.2.6. A topo-bisimulation between two topological models $M = (X, \tau, v)$ and $M' = (X', \tau', v')$ is a nonempty relation $\Leftrightarrow \subseteq X \times X'$ s.t. if $x \Leftrightarrow x'$ then

1. Base: $x \in V(p)$ iff $x' \in v'(p)$ for any propositional variable p
2. Forth: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ s.t. $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \Leftrightarrow y'$
3. Back: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ s.t. $x \in U$ and for every $y \in U$ there is $y' \in U'$ with $y \Leftrightarrow y'$

Proposition 2.2.7. Let $M = (X, \tau, v)$ and $M' = (X', \tau', v')$. Assume $w \in M$ and $w' \in M'$ are topo-bisimilar points. Then for each formula ϕ we have

$$M, w \models \phi \text{ iff } M', w' \models \phi$$

The proof is by induction.

Definition 2.2.8. Let X and Y be topological spaces and $f : X \rightarrow Y$ a function. We call f continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X . We say f is open if for each open set $V \subseteq X$ the set $f[V]$ is open in Y .

Remark 2.2.9. There is an alternative definition for open sets. Let (X, τ) be a topological space and U a set. U is open iff $\forall x \in U \exists V \subseteq U : V$ is open and $x \in V$. This is true because, the union of open sets is an open set.

3 Completeness result for TNL

We introduce a frame called $T_{\omega, \omega, \omega[rn]}$. We will show TNL is sound and complete w.r.t $T_{\omega, \omega, \omega[rn]}$. The idea is to pick a class of frame C s.t $Log(C) = Log(T_{\omega, \omega, \omega[rn]})$ and then show the class has FMP. In the end, we will use an unravelling technique to show completeness.

3.1 Filtration and Sahlqvist for multimodal logic

Definition 3.1.1. Let $M = (W, R_1, \dots, R_n, V)$ be a model and $w \in W$ a state in M . The notion of a formula being true at w is inductively defined as follows :

$$\begin{aligned} M, w &\models p \text{ iff } w \in V(p) \\ M, w &\models \perp \text{ never} \\ M, w &\models \neg \phi \text{ iff not } M, w \models \phi \\ M, w &\models \phi \vee \psi \text{ iff } M, w \models \phi \vee M, w \models \psi \\ M, w &\models \Box_i \phi \text{ iff } \forall v \in W : w R_i v \rightarrow M, v \models \phi \end{aligned}$$

Definition 3.1.2. A set Σ is closed under subformulas, if for all formulas ϕ and ϕ' the following holds :

1. if $\neg\phi \in \Sigma$ then $\phi \in \Sigma$
2. if $\phi \vee \phi' \in \Sigma$ then $\phi, \phi' \in \Sigma$
3. if $\Box_i\phi \in \Sigma$ then $\phi \in \Sigma$

Example : Suppose we have a multimodal logic with \Box, \Box_1 called $\mathcal{L}_{\Box, \Box_1}$ where $\phi = \Box p \rightarrow \Box_1 q$ and $\phi \in \Sigma$. Then $\Sigma = \{\phi, \neg\Box p, \Box_1 q, \Box p, p, q\}$ is closed under subformulas.

Definition 3.1.3. Let $M = (W, R_1, \dots, R_n, V)$ be a model and suppose Σ is a set of formulas. We define a relation \equiv on W as follows :

$$w \equiv v \text{ iff } \forall \phi \in \Sigma : M, w \Vdash \phi \Leftrightarrow M, v \Vdash \phi$$

It is well known that the \equiv -relation is an equivalence relation. We denote the equivalence class of a state $w \in W$ as $[w]_\Sigma = \{v \mid v \equiv w\}$. Furthermore W_Σ is the set of all equivalence classes, i.e $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$.

Definition 3.1.4. Let $M = (W, R_1, \dots, R_n, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes induced by \equiv . A model $M_\Sigma^f = (W^f, R_1^f, \dots, R_n^f, V^f)$ is called filtration of M through Σ if the following holds :

1. $W^f = W_\Sigma$
2. If $(w, v) \in R_i$ then $([w], [v]) \in R_i^f$
3. If $([w], [v]) \in R_i^f$ then for any $\Box_i\phi \in \Sigma$: if $M, w \Vdash \Box_i\phi$ then $M, v \Vdash \phi$
4. $V^f = \{[w] \mid M, w \Vdash p\}$, for all propositional variables $p \in \Sigma$

Theorem 3.1.5. Consider $L_{\Box_1 \dots \Box_n}$. Let $M^f = (W_\Sigma, R_1^f, \dots, R_n^f, V)$ be a filtration of M through a subformula closed set Σ . Then for all formulas $\phi \in \Sigma$, and all nodes $w \in M$, we have

$$M, w \Vdash \phi \text{ iff } M^f, [w] \Vdash \phi$$

Proof. By induction on ϕ . We will only show non-trivial and, for our purposes, necessary cases.

Case $\phi = p$: Left to right follows immediately from filtration definition. Conversely, suppose $M^f, [w] \Vdash p$. This means $[w] \in V^f(p)$. But this means $V(p)$ can not be empty. Pick any $v \in V(p)$. Obviously, $w \equiv v$ and $M, v \Vdash p$. Hence, $M, w \Vdash p$.

Case $\phi = \neg\psi$: Suppose ψ holds. Then we have : $M, w \Vdash \phi$ iff $M, w \not\Vdash \psi$. Applying induction hypothesis, we get : $M^f, [w] \not\Vdash \psi$. But then, we have $M^f, [w] \Vdash \phi$. Right to left is the same.

Case $\phi = \phi_1 \wedge \phi_2$: Suppose ϕ_1, ϕ_2 holds. Let $M, w \Vdash \phi$. That means $M, w \Vdash \phi_1$ and $M, w \Vdash \phi_2$. Applying induction hypothesis, we get $M^f, [w] \Vdash \phi_1$ and $M^f, [w] \Vdash \phi_2$. But then, $M^f, [w] \Vdash \phi_1 \wedge \phi_2 = \phi$. Right to left is similar.

Case $\phi = \Box_i \psi$: Left to right. Suppose ψ holds and $M, w \Vdash \Box_i \psi$. We need to show $M^f, [w] \Vdash \Box_i \psi$, this means $\forall [v] \in W_\Sigma : [w]R_i[v] \rightarrow M^f, [v] \Vdash \Box_i \psi$. Pick any $[v] \in W_\Sigma$ s.t $[w]R_i[v]$. By condition 3, w.r.t to the modal operator, we have $M, v \Vdash \psi$. By induction hypothesis, we get $M^f, [v] \Vdash \psi$. Because $[v]$ was arbitrary it follows that $M^f, [w] \Vdash \Box_i \psi$.

Right to left. Suppose ψ holds and $M^f, [w] \Vdash \Box_i \psi$. Pick $v \in W$ s.t $wR_i v$. By condition 2, w.r.t to the modal operator, we have $[w]R_i^f[v]$. So, $M^f, [v] \Vdash \psi$. By induction hypothesis, we get $M, v \Vdash \psi$. Because v was arbitrary, we have $M, w \Vdash \Box_i \psi$. \square

Definition 3.1.6. Let $M = (W, R_1, \dots, R_n, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes. We define :

$$R^s = \{[w], [v] \mid \exists w' \in [w], \exists v' \in [v] : w'R_i v'\}$$

Lemma 3.1.7. Let $M = (W, R_1, \dots, R_n, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes induced by \equiv and V^f the standard valuation on W_Σ . Then $(W_\Sigma, R_1^s, \dots, R_n^s, V^f)$ is a filtration of M through Σ .

Proof. It suffices to show R_i^s fullfills the condition 2 and 3 w.r.t to the corresponding modal operator \Box_i . But R_i^s already satisfies condition 2. Let's check the other condition. Let $\Box_i \phi \in \Sigma$, $[w]R_i^s[v]$ and $M, w \Vdash \Box_i \phi$. Because of $[w]R_i^s[v]$ we pick a $w' \in [w]$ and $v' \in [v]$. By definition, we have $w'R_i v'$. Because $w' \equiv w$, we get $M, w' \Vdash \Box_i \phi$. Hence, $M, v' \Vdash \phi$ and by $v' \equiv v$, we get $M, v \Vdash \phi$. \square

Proposition 3.1.8. Let Σ be a finite subformula closed set of $L_{\Box_1, \dots, \Box_n}$. For any model M , if M^f is a filtration through Σ , then M^f contains at most 2^n nodes (where n denotes the size of Σ).

Proof. The states of M^f are the equivalence classes in W_Σ . Let $g : W_\Sigma \rightarrow P(\Sigma)$ defined by $g([w]) = \{\phi \in \Sigma \mid M, w \Vdash \phi\}$. g is well defined. Pick any u and v s.t $u \equiv v$. But then by definition of \equiv , they fullfill the same subformulas. This means $g([v]) = g([u])$. g is also injective. Pick any $[u], [v] \in W_\Sigma$ s.t $g([u]) = g([v])$. We show $[u] \subseteq [v]$. The other inclusion is similar. By assumption we have $u \equiv v$. Pick any $u' \in [u]$. Then we have $u' \equiv u \equiv v$. Hence, $u' \in [v]$. At the end, this means M^f contains at most 2^n nodes. \square

Theorem 3.1.9. Let ϕ be a formula of $L_{\Box_1, \dots, \Box_n}$. If ϕ is satisfiable, then it is satisfiable on a finite model containing at most 2^v nodes, where v is the number of subformulas of ϕ .

Proof. Assume that ϕ is satisfiable on a model on M . Take any filtration of M through the set of subformulas of ϕ . By Filtration Theorem, we get that ϕ is satisfied in the filtration model M^f . Furthermore, it is bounded by 2^v . \square

Now we define Sahlqvist formulas for our purposes.

Definition 3.1.10. A modal formula ϕ is positive if all variables occurs in the scope of an even number of negations. In the other hand, a formula is negative, if all variables occurs in the scope of an odd number of negations. A boxed atom is a modal formula of the form $\Box^n p$ for some $n \in \mathbb{N}$, where p is a propositional variable and $\Box^n p$ is defined as follows : $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$.

Furthermore, a Sahlqvist antecedent is built from \perp, \top , negative formulas and boxed atoms by applying \Diamond and \wedge . A Sahlqvist implication is a modal formula of the form $\phi \rightarrow \psi$, where ϕ is a Sahlqvist antecedent and ψ a positive formula.

Now, a Sahlqvist formula is built from Sahlqvist implications by applying \Box and \vee .

Examples for Sahlqvist formulas:

$$\begin{aligned} & \Box\Box p \rightarrow \Box p \\ & \Diamond \neg p \rightarrow p \\ & \Diamond \Box \Box \Box \Box p \rightarrow \Box \Diamond \Box \Diamond p \\ & \Box \Box \Box \Box (\Diamond \Box p \rightarrow p) \vee \Box \Box p \rightarrow \Box p \end{aligned}$$

Non Sahlqvist Formulas :

$$\begin{aligned} & \Box \Diamond p \rightarrow \Diamond \Box p \\ & \Diamond \Box p \rightarrow \Box \neg p \end{aligned}$$

We can extend this definition for multimodal logics. We say a boxed atom can be $\Box_i^n p$. A Sahlqvist antecedent can also be build by applying \Diamond_i . A Sahlqvist formula can be build by Sahlqvist implications by applying additionally \Box_i .

Sahlqvist formulas possess important properties, which are guaranteed by the Sahlqvist Theorem. It says that, when given a normal modal logic K and a set of Sahlqvist formulas, the resulting logic is complete w.r.t to the class of frames, which satisfies the corresponding first-order formula of the Sahlqvist formulas. This also holds for multimodal logic. We will not prove it here, but we will use this for our logic to show completeness w.r.t to a suitable class of frames.

Definition 3.1.11. We say for a modal logic Λ has the finite model property (FMP) if for every formula ϕ that is not provable in Λ , is falsifiable in a finite model.

Proposition 3.1.12. The logic $T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ has FMP.

Proof. The idea is to pick a class of frame \mathcal{F} , where TNL is sound and complete with respect to and then show by filtration the FMP.

Let $\mathcal{C} = \{F \mid F \models TNL\}$. Obviously, TNL is sound w.r.t \mathcal{C} . Completeness can be shown by using the Sahlqvist Theorem. We remember $T = K + \Box_i p \rightarrow p$ for $i \in \{1, 2, \epsilon\}$. These axioms and $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$ are Sahlqvist formulas. By Sahlqvist, we have TNL is complete w.r.t $\{F \mid F \models \forall x R_i(x, x) \text{ and } F \models \forall x \forall y (R_1(x, y) \vee R_2(x, y)) \rightarrow R(x, y)\}$.

Now assume a formula ϕ is not derivable from TNL . By completeness we get ϕ is falsifiable in a model M and a world w . Hence, $M, w \models \neg\phi$. Now we build the set $\text{subf}(\neg\phi)$ which denotes the closed subformulas set of $\neg\phi$. Let $M^s = (W_\Sigma, R^s, R_1^s, R_2^s, V^s)$ be the smallest filtration of M through $\text{subf}(\neg\phi)$ and V^s is the standard valuation on W_Σ . By filtration Theorem, they preserve truth. It remains to show $F^s = (W_\Sigma, R^s, R_1^s, R_2^s) \in \mathcal{C}$. For that, every point must have an edge to itself in every relation and it must hold that $R_1^s, R_2^s \subseteq R^s$. For the first one, we show this for R^s because the rest is similar. Pick $w' \in [w] \in W_\Sigma$. Because M is based on a frame $F \in \mathcal{C}$, we have $w'Rw'$. By the definition of smallest filtration, we have that $[w']R^s[w']$. For the second one, we show only for R_1^s because it is the same for R_2^s . Pick $[w], [v]$ s.t $[w]R_1^s[v]$. By definition of smallest filtration there are points $w' \in [w]$ and $v' \in [v]$ s.t $w'R_1v'$. Furthermore, it holds $R_1 \subseteq R$, so $w'Rv'$. By smallest filtration, we get $[w']R^s[v']$. Because $w' \equiv w$ and $v' \equiv v$, we have $[w'] = [w]$ and $[v'] = [v]$. It follows $[w]R^s[v]$. □

3.2 Completeness w.r.t $T_{\omega, \omega, \omega[rn]}$

Definition 3.2.1. Let $T_{\omega[rn]}$ ($r = \text{reflexive}$, $n = \text{non-transitive}$) denote the infinite branching and infinite depth tree, which is reflexive and non-transitive. Formally the tree can be defined as : $T_{\omega[rn]} = (W, R)$ where $W = \mathbb{N}^*$ and sRt iff $\exists u \in \mathbb{N} \cup \{\epsilon\} : s \cdot u = t$ (" \cdot " is the concatenation operator)

The $T_{\omega, \omega, \omega[rn]}$ tree has three relations with infinite branching and infinite depth and we have $R_1, R_2 \subseteq R$. Before characterizing it, we say \mathbb{N}_1^* is the set of finite number combinations which has a subscript "1" to denote that these numbers relate to R_1 (examples : $2_1, 4231123_1, 32_1\epsilon 45_1\epsilon 9_1 = 32459_1$).

Now let $T_{\omega, \omega, \omega[rn]} = (W, R, R_1, R_2)$ where $W = (\mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2)^*$,

$$sRt \text{ iff } \exists u \in \mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_1t \text{ iff } \exists u \in \mathbb{N}_1 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_2t \text{ iff } \exists u \in \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

where $s, t \in W$. Again, the " \cdot " operator acts here as a concatenation operator.

Definition 3.2.2. Let $F = (W, R_1, R_2, \dots)$ and $F' = (W', R'_1, R'_2, \dots)$ be two frames. A bounded morphism from F to F' is a function $f : W \rightarrow W'$ satisfying the following conditions:

$$\text{If } (u, v) \in R_i \text{ then } (f(u), f(v)) \in R'_i$$

$$\text{If } (f(w), v') \in R'_i \text{ then } \exists v \in W \text{ s.t. } (w, v) \in R_i \text{ and } f(v) = v'$$

We say F' is a bounded morphic image of F , if there is a surjective bounded morphism from F to F' .

Proposition 3.2.3. Let ϕ be a formula in $L_{\Box_1, \dots, \Box_n}$, $F = (W, R_1, \dots, R_n)$ and $F' = (W', R'_1, \dots, R'_n)$ be two frames and F to F' a surjective bounded morphism. Then the following holds :

$$\text{If } F \models \phi \text{ then } F' \models \phi$$

This can be shown by structural induction on the length of the formula.

Corollary 3.2.4. If F' is a bounded morphic image of F , then we have $\text{Log}(F) \subseteq \text{Log}(F')$

Proposition 3.2.5. T is sound and complete w.r.t $T_{\omega[rn]}$.

Proof. Sound is clear. For completeness, we use the well known fact that T has FMP. This means, $T = \text{Log}\{F \mid F \models T\}$ where F is a finite frame. We can pick such a finite frame F and it suffices to find a surjective bounded morphism f from T_w to F . This would imply $\text{Log}(T_{\omega[rn]}) \subseteq T$.

Now, let $F = (W', R')$ be such a finite rooted frame with root w . We define inductively an assignment of nodes of F to the nodes of T_w . For the base case, we assign w to the root of T_w . The induction step looks like the following : Assume a point $x \in T_w$ has been assigned to a point $u \in F$ but the successors of x has no assignment. Let s_1, s_2, \dots, s_k be successors of u (k denotes amount of successors and $k \geq 1$ because reflexivity guarantees us at least one successor). For $n \geq 1, n \in \mathbb{N}, i \in \{1, \dots, k\}$ we assign s_i to the $(n * i)$ th-successor of x . This means we are assigning the successors alternately.

Now we check for f the conditions of bounded morphism. First condition : Let $x, y \in T_w$ s.t xRy and $f(x) = s$. But then, y will be assigned to a successor point of s . Hence, $f(x)R'f(y)$. Second condition : Suppose $f(x)R't$ and $f(x) = s$. Since t is a successor of s and $f(x) = s$, then a successor of x , say y , gets the assignment t .

□

Proposition 3.2.6. TNL is sound and complete w.r.t $T_{\omega, \omega, \omega[rn]}$.

Proof. For soundness, we have that $T_{\omega, \omega, \omega[rn]} \models \Box p \rightarrow \Box_1 p \wedge \Box_2 p$, because by definition we have $R_1, R_2 \subseteq R$. The rest is clear. For completeness, we use the fact TNL has FMP. Let $F = (W', R', R'_1, R'_2)$ be a finite rooted frame with root w and $F \models TNL$. We define inductively an assignment similar to 2.17. We assign w to the root of $T_{\omega, \omega, \omega[rn]}$. For induction step we start by only assigning points from R_1 to R'_1 and R_2 to R'_2 . After that, the remaining points will be assigned to the points in R' . The procedure works

similar as described previously.

We will only check the conditions for R and R' . For R_1, R'_1 and R_2, R'_2 , we can argue as before. We pick any xRy with $f(x) = s$. If we also have xR_1y or xR_2y , then it follows $f(x)R'f(y)$ because $R_1, R_2 \subseteq R$ and $R'_1, R'_2 \subseteq R'$. Else, the successor of s was assigned to y , so $f(x)R'f(y)$.

Let $f(x)R't$ and $f(x) = s$. If we have $f(x)R'_1t$ or $f(x)R'_2t$, then the second condition follows because $R_1, R_2 \subseteq R$. Else, we have that t is a successor of $f(x) = s$, and a successor of x was assigned to t . \square

4 Logic of product of topological spaces with three topologies

In this section, we will prove the logic of the product of two topological spaces with horizontal, vertical and classic product topology is $S4 \otimes S4 \otimes S4 + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ (we abbreviate the logic $S4 \otimes S4 \otimes \dots$ as TPL). First we will show a version for one modality because it is easier to understand and the proof idea is based on this version.

Definition 4.0.1. *Let T_2 be the infinite binary tree with reflexive and transitive descendant relation.*

Formally it is defined as follows : $T_2 = (W, R)$ where $W = \{0, 1\}^$ and sRt iff $\exists u \in W^* : s \cdot u = t$.*

The $T_{6,2,2}$ tree is the infinite six branching tree, where all nodes of $T_{6,2,2}$ is R -related, the first two R_1 -related and the last two R_2 -related. Formally we can define this tree as follows : $T_{6,2,2} = (W, R, R_1, R_2)$, where $W = \{0, 1, 2, 3, 4, 5\}^$,*

$$sRt \text{ iff } \exists u \in \{0, 1, 2, 3, 4, 5\}^* : s \cdot t = u$$

$$sR_1t \text{ iff } \exists u \in \{0, 1\}^* : s \cdot t = u$$

$$sR_2t \text{ iff } \exists u \in \{5, 6\}^* : s \cdot t = u$$

where $s, t \in W$ and \cdot is the concatenation-operator.

Theorem 4.0.2. *$S4$ is complete with respect to T_2 .*

Proof. The idea is to use the fact $S4$ has finite model property. By that we can pick a finite rooted $S4$ -frame and then show that this frame is the bounded morphic image of T_2 . For details see [Goldblatt]. \square

Theorem 4.0.3. *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to \mathbb{Q} .*

Proof. For a proof see e.g [?, Page 217, Theorem 2]. \square

Our strategy is as follows. We use completeness of $S4$ w.r.t. T_2 , view T_2 as an Alexandroff space, define a dense subset X of \mathbb{Q} without endpoints, and then establish a topo-bisimulation between X and T_2 . This will allow us to transfer counterexamples from T_2 to X , which by Cantor's theorem is order-isomorphic and hence homeomorphic to \mathbb{Q} .

Now let us define X as $X = \bigcup_{n \in \mathbb{N}} X_n$ where $X_0 = \{0\}$ and

$$X_{n+1} = X_n \cup \left\{ x - \frac{1}{3^n}, x + \frac{1}{3^n} \mid x \in X_n \right\}$$

Lemma 4.0.4. *For $n > 0$ and $x, y \in X_n, x \neq y$ we have : $|x - y| \geq \frac{1}{3^{n-1}}$*

Proof. The proof is by induction on n . For the base case, if $n = 1$, then $X = 0, 1, -1$. For induction step, assume $u, v \in X_{n-1}$ with $u \neq v$. By induction hypothesis, it holds $|u - v| \geq \frac{1}{3^{n-2}}$. Suppose we pick $x = u + \frac{1}{3^{n-1}}$ and $y = v - \frac{1}{3^{n-1}}$. Hence, $x, y \in X_n$. But then, $|x - y| = |(u + \frac{1}{3^{n-1}}) - (v - \frac{1}{3^{n-1}})| = |u - v + \frac{2}{3^{n-1}}| \geq \frac{1}{3^{n-1}}$. If $u - v$ is positive, then the inequality follows immediately. Now assume $u - v$ is negative. By induction hypothesis we get $u - v \leq -\frac{1}{3^{n-2}}$. So the worst case is, when $u - v$ gets $-\frac{1}{3^{n-2}}$. But then, $|\frac{1}{3^{n-2}} - \frac{2}{3^{n-1}}| = \frac{1}{3^{n-1}}$. \square

It follows from Lemma 4.0.4 that $(X, <)$ is a countable, dense, and linear ordered without endpoints. By Cantor's Theorem, X is homeomorphic to \mathbb{Q} . It also follows that for each $x \in X$ with $x \neq 0$ there exists an index n_x with $x \in X_{n_x}$ but $x \notin X_{n_x-1}$. But also, we can find a unique $y \in X_{n_x-1}$ with $x = y - \frac{1}{3^{n_x-1}}$ or $x = y + \frac{1}{3^{n_x-1}}$. Furthermore, the basis for the open intervals for any $x \in X$ will be $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ where n_x is the index where x gets produced.

Now we define f from X onto T_2 by recursion. If $x = 0$, then we let $f(0)$ be the root r of T_2 . If $x \neq 0$, then $x \in X_{n_x} - X_{n_x-1}$ and we let

$$f(x) = \begin{cases} \text{the left successor of } f(y) & \text{if } x = y - \frac{1}{3^{n_x-1}} \\ \text{the right successor of } f(y) & \text{if } x = y + \frac{1}{3^{n_x-1}} \end{cases}$$

Proposition 4.0.5. *We claim, f is open and continuous.*

Proof. We define the basis for the Alexandroff topology on T_2 as $\{B_t\}_{t \in T_2}$ where $B_t = \{s \in T_2 \mid tRs\}$. To show f is open, we pick an X -interval $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ and show $f((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})) = B_{f(x)}$.

\subseteq : If we pick $x \in ((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}))$, then obviously $f(x)Rf(x)$. Assume we pick a point $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ but $y \neq x$. Then we have $n_y > n_x$ and hence $f(x)Rf(y)$.

\supseteq : Assume $f(x)Rt$. We will show by induction on the length k between $f(x)$ and t s.t. there exists $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ s.t. $f(y) = t$. Base case : If $k = 0$, then we have $t = f(x)$ and we can pick $y = x$. For induction step, assume it holds for $f(x)Rt'$ with length n , $f(y) = t'$ and $t'Rt$. For y , there must exist an index $n_y > n_x$ s.t. $y \in X_{n_y} - X_{n_y-1}$. Now let $y' = y - \frac{1}{3^{n_y}}$ or $y' = y + \frac{1}{3^{n_y}}$. It holds that, $n'_y > n_y > n_x$, so $y' \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$. By definition of f , $f(y')$ is either left or right successor of $f(y)$ and by $f(y) = t'Rt$, we get $f(y') = t$. It follows f is open.

To show f is continuous, it suffices to show that for each $t \in T_2$, the f -inverse image of B_t is open. Let $x \in f^{-1}(B_t)$. Then $tRf(x)$. As argued before, we can find an open interval in X s.t. $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)}$. By $tRf(x)$, we get $B_{f(x)} \subseteq B_t$. Hence, we found an open interval $I = (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$, s.t. $I = f^{-1}(B_{f(x)}) \subseteq f^{-1}(B_t)$. This holds because we can pick $y \in I$. By applying f , we get $f(y) \in B_{f(x)} \subseteq B_t$. But then $y \in f^{-1}(B_t)$. By that fact, we can follow with Remark 2.2.8, that $f^{-1}(B_t)$ is open and hence f is continuous. \square

To complete the proof, if $S4 \not\models \phi$, then by Theorem 4.0.2, there is a valuation v on T_2 s.t. $(T_2, v), r \not\models \phi$. We define a valuation ξ on X by $\xi(p) = f^{-1}(v(p))$. Since f is continuous and open, $f(0) = r$ and by the choice of the valuation, we have that 0 and r are topo-bisimilar. By Proposition 2.2.7 it follows $(X, \xi), 0 \not\models \phi$. Because X is homeomorphic to \mathbb{Q} , we obtain ϕ refutable on \mathbb{Q} .

Definition 4.0.6. Let $\mathcal{L}_{\Box, \Box_1, \Box_2}$ be a modal language with three modal operators \Box, \Box_1 and \Box_2 . We define the topological product logic TPL as the least set of formulas in $\mathcal{L}_{\Box, \Box_1, \Box_2}$ containing all axioms $S4 \otimes S4 \otimes S4 + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$, and closed under modus ponens, substitution and \Box -, \Box_1 - and \Box_2 -necessitation.

Theorem 4.0.7. TPL is complete w.r.t. $T_{6,2,2}$.

Proof. A generalization of Theorem 4.0.2 can be seen on [?]. \square

Now we are ready to show TPL is complete w.r.t. $\mathbb{Q} \times \mathbb{Q}$ with three topologies. The idea is similar as previous shown. We view $T_{6,2,2}$ as equipped with three Alexandroff topologies defined from R, R_1 and R_2 . It suffices to show that there exists a 3-topo-bisimulation between $T_{6,2,2}$ and $X \times X = X'$ where each X is as defined in Theorem 4.0.3.

Theorem 4.0.8. TPL is complete w.r.t. $\mathbb{Q} \times \mathbb{Q}$.

Proof. We define h from $X \times X$ onto $T_{6,2,2}$ by recursion following the inductive definition of X . If $(x, y) = (0, 0)$, then $h(0, 0)$ is the root r of $T_{6,2,2}$. If $(x, y) \neq (0, 0)$, then as argued in Lemma 4.0.4, there exists $n_{(x,y)}$ with $(x, y) \in X'_{n_{(x,y)}} - X'_{n_{(x,y)}-1}$ and that there exists a unique $(u, v) \in X'_{n_{(x,y)}-1}$ such that $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v)$ or $(x, y) = (u, v \pm \frac{1}{3^{n_{(x,y)}-1}})$ or $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v \pm \frac{1}{3^{n_{(x,y)}-1}})$.

As visualized in the picture, we can see as an example X'_0, X'_1 and an excerpt of X'_2 . For clarity, we illustrated some points of X'_1 as a line. We clearly can see, that any point has his own "square", where the points within it can be only build from his center point. This is because, the square size gets exponentially smaller with each iteration, due to its construction. That means, in each iteration they can not intersect.

Now we define h as

$$h(x, y) = \begin{cases} \text{The left } R_1\text{-successor of } h(u, v) & \text{if } (x, y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{The right } R_1\text{-successor of } h(u, v) & \text{if } (x, y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{The left } R_2\text{-successor of } h(u, v) & \text{if } (x, y) = (u, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The right } R_2\text{-successor of } h(u, v) & \text{if } (x, y) = (u, v + \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The first remaining successor of } h(u, v) & \text{if } (x, y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v + \frac{1}{3^{n_{(x,y)}-1}}) \\ & \text{or } (x, y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{The last remaining successor of } h(u, v) & \text{if } (x, y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ & \text{or } (x, y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v + \frac{1}{3^{n_{(x,y)}-1}}) \end{cases}$$

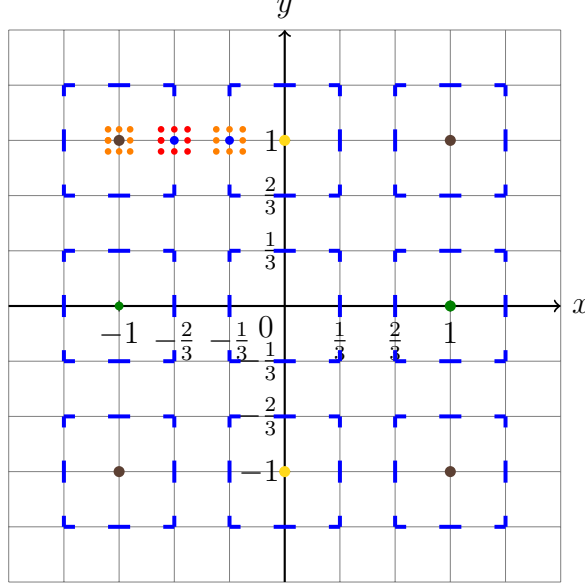


Figure 1:

It is still left to prove, that h is open and continuous w.r.t. all three topologies. We will show it for τ_1 and τ . For τ_2 it is similar as for τ_1 .

First we observe the basis for τ_1 is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \mid (x, y) \in X \times X\}$$

and for τ is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}}) \mid (x, y) \in X \times X\}$$

We also observe that a basis for the topology on $T_{6,2,2}$ from R_1 is $\{B_t^1\}_{t \in T_{6,2,2}}$ where $B_t^1 = \{s \in T_{6,2,2} \mid tR_1s\}$ and from R is $\{B_t\}_{t \in T_{6,2,2}}$ where $B_t = \{s \in T_{6,2,2} \mid tRs\}$. The following arguments are carried over from Proposition 4.0.5.

Now, to see that g is open w.r.t. τ_1 , we pick an open set $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \in \tau_1$. In similar way, we can show $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B_{g(x,y)}^1$. Thus g is open. To show g is continuous, it suffices to show that for each $t \in T_{6,2,2}$, the g -inverse image of B_t^1 belongs to τ_1 . Let $(x, y) \in g^{-1}(B_t^1)$. Then $tR_1g(x, y)$. Hence, it holds for (x, y) that $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B_{g(x,y)}^1$. By $tR_1g(x, y)$, we get $B_{g(x,y)}^1 \subseteq B_t^1$. Thus, we found a neighborhood U of (x, y) s.t. $U = ((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) \subseteq g^{-1}(B_t^1)$, implying g is continuous. □

5 Neighborhood

5.1 Neighborhood frames

Definition 5.1.1. Let X be a non-empty set. A function $\tau : X \rightarrow 2^{2^X}$ is called a neighbourhood function. A pair $F = (X, \tau)$ is called a neighbourhood frame (or n -frame). A model based on F is a tuple (X, τ, v) , where v assigns a subset of X to a variable

Definition 5.1.2. Let $M = (X, \tau, v)$ be a neighbourhood model and $x \in X$. The truth of a formula is defined inductively as follows :

$$\begin{aligned} M, x \Vdash p & \text{ iff } x \in V(p) \\ M, x \Vdash \neg\phi & \text{ iff } M, x \nVdash \phi \\ M, x \Vdash \phi \vee \psi & \text{ iff } M, x \Vdash \phi \vee M, x \Vdash \psi \\ M, x \Vdash \Box\phi & \text{ iff } \exists V \in N(x) \forall y \in V : M, y \Vdash \phi \end{aligned}$$

A formula is valid in a n -model M if it is valid at all points of M ($M \models \phi$). Formula is valid in a n -frame F if it is valid in all models based on F (notation $F \models \phi$). We write $F \models L$ if for any $\phi \in L$, $F \models \phi$. Logic of a class of n -frames C as $\text{Log}(C) = \{\phi \mid F \models \phi \text{ for some } F \in C\}$. We define $nV(L) = \{F \mid F \text{ is an } n\text{-frame and } F \models \phi\}$.

Definition 5.1.3. Let X be a non-empty set and τ neighborhood function. We call τ is a filter if for each $x \in X$ the collection $\tau(x)$ satisfies the following conditions :

1. $\emptyset \notin \tau(x)$
2. If $U \in \tau(x)$ and $U \subseteq V$ then $V \in \tau(x)$ (upward closed)
3. If $U, V \in \tau(x)$, then $U \cap V \in \tau(x)$

Definition 5.1.4. Let $F = (W, R)$ be a Kripke frame. We define an n -frame $N(F) = (W, \tau)$ as follows. For any $w \in W$ we have :

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

Lemma 5.1.5. Let $F = (W, R)$ be a Kripke frame. Then

$$\text{Log}(F) = \text{Log}(N(F))$$

The proof is by structural induction.

Definition 5.1.6. Let $X = (X, \tau_1, \dots)$ and $Y = (Y, \sigma_1, \dots)$ be n -frames. Then the function $f : X \rightarrow Y$ is called bounded morphism if

1. f is surjective
2. $\forall x \in X \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X \forall V \in \sigma_i(f(x)) \exists U \in \tau_i(x) : f(U) \subseteq V$

Corollary 5.1.7. *Let $X = (X, \tau_1, \dots)$ and $Y = (Y, \sigma_1, \dots)$ be n -frames and $f : X \rightarrow Y$ a bounded morphism. Then*

$$\text{Log}(X) \subseteq \text{Log}(Y)$$

The proof is by structural induction.

Definition 5.1.8. *Let $X = (X, \tau_1)$ and $Y = (Y, \tau_2)$ be two n -frames. Then the product of these two frames is an n -2-frame and is defined as follows :*

$$\begin{aligned} X \times_n Y &= (X \times Y, \tau'_1, \tau'_2) \\ \tau'_1(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\} \\ \tau'_2(x, y) &= \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\} \end{aligned}$$

Additionally, we say the full product of n -frames $X \times_n^+ Y$ is :

$$\begin{aligned} X \times_n^+ Y &= (X \times Y, \tau'_1, \tau'_2, \tau) \text{ where} \\ \tau(x, y) &= \{U \subseteq X \times Y \mid \exists W \in \tau_1(x) \exists V \in \tau_2(y) : W \times V \subseteq U\} \end{aligned}$$

Definition 5.1.9. *For two unimodal logics L_1 and L_2 we define the n -product of them as follows :*

$$L_1 \times_n L_2 = \text{Log}(\{X \times Y \mid X \in nV(L_1) \text{ and } Y \in nV(L_2)\})$$

Proposition 5.1.10. *([?]). For two unimodal logics L_1 and L_2 it holds :*

$$L_1 \otimes L_2 \subseteq L_1 \times_n L_2$$

5.2 Main Construction

In the following, we will construct a useful neighborhood frame called $N_\omega[F]$ based on a frame F . We will use it later, to show $L_1 \times_n L_2 = L_1 \otimes L_2$ where $L_1, L_2 \in \{D, S4, D4, T\}$

Definition 5.2.1. *Let $F = (A^*, R) = F_{\xi\eta}[A]$ and $0 \notin A$. We define "pseudo-infinite" sequences*

$$X = \{a_1 a_2 a_3 \dots \mid a_i \in A \cup \{0\} \text{ and } \exists N \forall k \geq N : a_k = 0\}$$

Furthermore, we define $f_F : X \rightarrow A^$ to be the function, that deletes all zeros.*

Example : Say $12034002340^\omega \in X$ (0^ω denotes infinitely many zeros). Then $f_F(12034002340^\omega) = 1234234$

Definition 5.2.2. Let $F = (A^*, R) = F_{\xi\eta}[A]$ and $0 \notin A$. Assume the function f_F and the set X as defined before. For $\alpha \in X$ such that $\alpha = a_1 a_2 \dots$ we define

$$\begin{aligned} st(\alpha) &= \min\{N \mid \forall k \geq N : a_k = 0\} \\ \alpha \upharpoonright_k &= a_1 a_2 \dots a_k \\ U_k(\alpha) &= \{\beta \mid f_F(\alpha) R f_F(\beta) \text{ and } \alpha \upharpoonright_m = \beta \upharpoonright_m, \text{ where } m = \max((k, st(\alpha)))\} \end{aligned}$$

Remark : Let $\alpha \in X$ with $st(\alpha) = n$. Then we have that $U_n(\alpha) = U_j(\alpha)$ for any $j \leq n$

Lemma 5.2.3. $U_k(\alpha) \subseteq U_m(\alpha)$, whenever $k \geq m$.

Proof. Let $\beta \in U_k(\alpha)$. Since $\alpha \upharpoonright_k = \beta \upharpoonright_k$ and $k \geq m$, we have $\alpha \upharpoonright_m = \beta \upharpoonright_m$. It follows, $\beta \in U_m(\alpha)$. \square

Definition 5.2.4. Due to Lemma 5.2.3 the sets $U_n(\alpha)$ forms a filter base. So we can define :

$\tau(\alpha)$ is a filter with base $\{U_n(\alpha) \mid n \in \mathbb{N}\}$

$N_\omega = (X, \tau)$ is the n -frame based on F

Lemma 5.2.5. Let $F = (A^*, R) = F_{\xi\eta}[A]$. Based on that, let $N_\omega(F) = (X, \tau)$, $N(F) = (A^*, \sigma)$ and $f_F : N_\omega(F) \rightarrow N(F)$. Then for any $m \in \mathbb{N}$ and $x \in X$ with $x = a_1 a_2 \dots$ we have

$$f_F(U_m(x)) = R(f_F(x))$$

Proof. \subseteq : Let $f_F(\alpha) \in f_F(U_m(x))$ with $\alpha \in U_m(x)$. By definition of $U_m(x)$, we get $f_F(\alpha) \in R(f_F(x))$.

For the other direction, we pick $\vec{a} \in R(f_F(x))$. We have to find $\beta \in U_m(x)$ s.t $f_F(\beta) = \vec{a}$. We assume R is irreflexive and non-transitive. The other cases are similar. Because $\vec{a} \in R(f_F(x))$, there must exists $c \in A$ such that $\vec{a} = f_F(x) * c$. We construct $\beta = x \upharpoonright_m \cdot c 0^\omega$. Hence, $f(\beta) = f_F(x \upharpoonright_m) * f_F(c)$ and because $0 \notin A$ we get $f_F(x \upharpoonright_m) * c = \vec{a}$. \square

Lemma 5.2.6. Let $F = (A^*, R) = F_{\xi\eta}[A]$. Then $f_F : N_\omega(F) \rightarrow N(F)$ is a bounded morphism.

Proof. From now on this proof we will omit the subindex in f_F . Let $N_\omega(F) = (X, \tau)$ and $N(F) = (A^*, \sigma)$.

For surjectivity, we pick any $\vec{x} \in A^*$. But then, $\vec{x} 0^\omega \in X$. Hence, $f(\vec{x} 0^\omega) = \vec{x}$.

For the next condition, assume that $x \in X$ and $U \in \tau(x)$. We need to prove that $f(U) \in \sigma(f(x))$. That means $R(f(x)) \subseteq f(U)$. Because $U \in \tau(x)$, there is a m such that $U_m(x) \subseteq U$. By Lemma 5.2.5 we have $f(U_m(x)) = R(f(x))$. It follows,

$$R(f(x)) = f(U_m(x)) \subseteq f(U)$$

Assume $x \in X$ and V is a neighborhood of x , i.e $R(f(x)) \subseteq V$. We need to prove that there exists $U \in \tau(x)$, such that $f(U) \subseteq V$. As $U \in U_m(x)$ for any $m \in \mathbb{N}$. By Lemma 5.2.5 we get $f(U_m(x)) = R(f(x))$. Hence,

$$f(U_m(x)) = R(f(x)) \subseteq V$$

□

Corollary 5.2.7. *For frame $F = F_{\xi\eta}[A]$ we have $\text{Log}(N_\omega(F)) \subseteq \text{Log}(F)$.*

Proof. It follows from Lemma 5.1.5, Corollary 5.1.7 and Lemma 5.2.6

$$\text{Log}(N_\omega(F)) \subseteq \text{Log}(N(F)) = \text{Log}(F)$$

□

Proposition 5.2.8. *Let $F_{in} = F_{in}[\mathbb{N}]$, $F_{rn} = F_{rn}[\mathbb{N}]$, $F_{it} = F_{it}[\mathbb{N}]$ and $F_{rt} = F_{rt}[\mathbb{N}]$. Then*

$$\text{Log}(N_\omega(F_{in})) = D$$

$$\text{Log}(N_\omega(F_{rn})) = T$$

$$\text{Log}(N_\omega(F_{it})) = D4$$

$$\text{Log}(N_\omega(F_{rt})) = S4$$

Proof. In all these cases, the inclusion from left to right follows from Proposition 2.1.9 and Corollary 5.2.7. Now the converse direction. Assume $X = (X, \tau)$.

It is easy to check that $X \models D$ iff for each $x \in X : \emptyset \notin \tau(x)$. For $N_\omega(F_{in})$ and $N_\omega(F_{it})$ this holds.

It is easy to check that $X \models T$ iff we have $x \in U \in \tau(x)$ for any x and U . For $N_\omega(F_{rn})$ and $N_\omega(F_{rt})$ this holds.

Now we check $X \models 4$ iff for each $U \in \tau(x) : \{y \mid U \in \tau(y)\} \in \tau(x)$.

\supseteq : Let $x \in X$ and assume $X, x \models \Box\Box p$. That means there exists $U \in \tau(x)$ s.t $U \subseteq V(p)$. By assumption we have $S = \{y \mid U \in \tau(y)\} \in \tau(x)$. But then $X, x \models \Box\Box p$ because we can just pick the set S .

\subseteq : By contradiction, assume there exists a U s.t $\{y \mid U \in \tau(y)\} \notin \tau(x)$. Let $X, x \models \Box p$ where $V(p) = U$. If $X, x \models \Box\Box p$ then it must be the case that

$S = \{y \in X \mid X, y \models \Box p\} \in \tau(x)$. $X, y \models \Box p$ means $U \in \tau(y)$. That means $S = \{y \in X \mid U \in \tau(y)\}$. But by assumption $S \notin \tau(x)$. Hence, $X, x \not\models \Box\Box p$. But that's a contradiction. This also holds for $N_\omega(F_{it})$ and $N_\omega(F_{rt})$ because we have for any $y \in U_m(x)$ and $k \geq m : U_k(y) \subseteq U_m(x)$.

□

Lemma 5.2.9. *Let $T_{\omega, \omega, \omega[rn]}$ be as in Definition 3.2.1. Then*

$$\text{Log}(T_{\omega, \omega, \omega[rn]}) = \text{Log}(N(T_{\omega, \omega, \omega[rn]}))$$

The proof is by structural induction.

Assume $F_1 = (A^*, R_1) = F_{\eta_1 \xi_1}[A]$ and $F_2 = (B^*, R_2) = F_{\eta_2 \xi_2}[B]$ with $A \cap B = \emptyset$, $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$. Consider the product of n-frames $F'_1 = (X_1, \tau_1) = N_\omega(F_1)$ and $F'_2 = (X_2, \tau_2) = N_\omega(F_2)$ is

$$X = (X_1 \times X_2, \tau'_1, \tau'_2) = N_\omega(F_1) \times_n N_\omega(F_2)$$

Furthermore, we have $F_1 \otimes F_2 = ((A \cup B)^*, R'_1, R'_2)$ as defined in Defintion 2.0.7. We consider the neighborhood version

$$N(F_1 \otimes F_2) = ((A \cup B)^*, \sigma'_1, \sigma'_2)$$

Now we define $g : X_1 \times X_2 \rightarrow (A \cup B)^*$ as follows. For $(\alpha, \beta) \in X_1 \times X_2$ with $\alpha = a_1 a_2 \dots$ and $\beta = b_1 b_2 \dots$ we define $g(\alpha, \beta)$ to be the finite sequence which we get after eliminating all zeros from the infinite sequence $a_1 b_1 a_2 b_2 \dots$.

Example : Let $\alpha = 012340^\omega$ and $\beta = 0ab00e0^\omega$. Then $g(\alpha, \beta) = 1a2b34e$.

Lemma 5.2.10. *Let X and $N(F_1 \otimes F_2)$ be as defined before and $(\alpha, \beta) \in X_1 \times X_2$. Then for any $m > \max\{st(\alpha), st(\beta)\}$ we have*

$$R'_1(g(\alpha, \beta)) = g(U_m(\alpha) \times \{\beta\})$$

.

Proof. We will show both direction by assuming $F_1 = F_{in}[A]$. For $F_{it}[A], F_{rt}[A]$ and $F_{rn}[A]$ is it similar.

\subseteq : Let $\vec{w} \in R'_1(g(\alpha, \beta))$. By definition we get, there exists $\vec{c} \in A^*$ where $\vec{w} = g(\alpha, \beta) \cdot \vec{c}$ and $\Lambda R_1 \vec{c}$. Because R_1 is irreflexive and non-transitive, we get $\vec{c} \in A$. We construct (ζ, β) where $\zeta \in U_m(\alpha)$ and $g(\zeta, \beta) = \vec{w}$. For that, we can take $\zeta = \alpha \upharpoonright_m \cdot \vec{c} 0^\omega$. Obviously, $\zeta \in U_m(\alpha)$. Because $m \in \max\{st(\alpha), st(\beta)\}$, we have $g(\alpha \upharpoonright_m, \beta \upharpoonright_m) = g(\alpha, \beta)$. Hence, $g(\zeta, \beta) = g(\alpha \upharpoonright_m, \beta \upharpoonright_m) \cdot g(\vec{c} 0^\omega, 0^\omega) = g(\alpha \upharpoonright_m, \beta \upharpoonright_m) \cdot \vec{c} = \vec{w}$.

\supseteq : Assume $\zeta \in U_m(\alpha)$. We have to show $g(\zeta, \beta) \in R'_1(g(\alpha, \beta))$. Because R_1 is irreflexive and non-transitive, it suffices to find a $\vec{c} \in A$ s.t. $g(\alpha, \beta) \cdot \vec{c} = g(\zeta, \beta)$. By choosing m is maximal and $\zeta \upharpoonright_m = \alpha \upharpoonright_m$, we have $g(\zeta_m, \beta_m) = g(\alpha, \beta)$. We also know $f_F(\alpha) R_1 f_F(\zeta)$, that means there exists a $\vec{d} \in A$ s.t. $f_F(\alpha) \cdot \vec{c} = f_F(\zeta)$. This \vec{d} must appear at a point after ζ_m . We can follow $g(\zeta, \beta) = g(\alpha, \beta) \cdot \vec{d}$. Hence, $g(\zeta, \beta) \in R'_1(g(\alpha, \beta))$.

Remark : If $m \leq \max\{st(\alpha), st(\beta)\}$, then we cannot gurantee the equality. Assume R_1 as before and $\alpha = 123000^\omega$ and $\beta = d0b0a0^\omega$. Lets pick $m = 3$ and $\zeta = 123100^\omega$. Then we have $g(\alpha, \beta) = 1d23ba$ and $g(\zeta, \beta) = 1d23b1a$. Obviously, we don't have $g(\zeta, \beta) \in R'_1(g(\alpha, \beta))$.

Furthermore, we can show $R'_2(g(\alpha, \beta)) = g(\alpha \times U_m(\beta))$ similar as above.

□

Lemma 5.2.11. *Function $g : X \rightarrow N(F_1 \otimes F_2)$ is a bounded morphism.*

Proof. Let $\vec{z} = z_1 z_2 \dots z_n \in (A \cup B)^*$. Define for $i \leq n$:

$$x_i = \begin{cases} z_i, & \text{if } z_i \in A; \\ 0, & \text{if } z_i \notin A. \end{cases} \quad y_i = \begin{cases} z_i, & \text{if } z_i \in B; \\ 0, & \text{if } z_i \notin B. \end{cases}$$

Let $\alpha = x_1 x_2 \dots x_n 0^\omega$ and $\beta = y_1 y_2 \dots y_n 0^\omega$. Then $g(\alpha, \beta) = \vec{z}$. Hence, g is surjective.

For the next conditions we check only for τ'_1 and σ_1 . The other case is similar. Assume $(\alpha, \beta) \in X_1 \times X_2$ and $U \in \tau'_1(\alpha, \beta)$. We have to show $g(U) \in \sigma_1(g(\alpha, \beta))$. That means $R'_1(g(\alpha, \beta)) \subseteq g(U)$. Pick a $m > \max\{st(\alpha), st(\beta)\}$ s.t. $U_m(\alpha) \times \{\beta\} \subseteq U$. We can pick such m , because $U \in \tau'_1(\alpha, \beta)$. So there exists $U_k(\alpha) \in \tau(\alpha)$ s.t. $U_k(\alpha) \times \{\beta\} = U$. If $k > \max\{st(\alpha), st(\beta)\}$ then we are done. Else, by Lemma 5.2.3, we have for any $n \geq k : U_n(\alpha) \subseteq U_k(\alpha)$. So we can lift the k til we reach m . Then we use Lemma 5.2.11 to get the following :

$$R'_1(g(\alpha, \beta)) = g(U_m(\alpha) \times \{\beta\}) \subseteq g(U)$$

For the last condition we assume $(\alpha, \beta) \in X_1 \times X_2$ and $V \in \sigma_1(g(\alpha, \beta))$ (or rather $R'_1(g(\alpha, \beta)) \subseteq V$). We need to prove there exists $U \in \tau'_1(\alpha, \beta)$, such that $g(U) \subseteq V$. As U we take $U_m(\alpha) \times \{\beta\}$ for some $m > \max\{st(\alpha), st(\beta)\}$. Hence, by Lemma 5.2.11 we get :

$$g(U_m(\alpha) \times \{\beta\}) = R'_1(g(\alpha, \beta)) \subseteq V$$

□

Now we will show $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \rightarrow N(T_{\omega, \omega, \omega[rn]})$ is a bounded morphism. Let $(T_{\omega[rn]})_1 = (\mathbb{N}_1, R_1)$ and $(T_{\omega[rn]})_2 = (\mathbb{N}_2, R_2)$ as defined in Defintion 3.2.1. Then we take the n -frames $N_\omega(T_{\omega[rn]})_1 = (X_1, \tau_1)$ and $N_\omega(T_{\omega[rn]})_2 = (X_2, \tau_2)$. For the proof, we will omit the subscripts after the frame.

Let's consider the full product of the n -frames :

$$N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) = (X_1 \times X_2, \tau'_1, \tau'_2, \tau)$$

We say $N(T_{\omega, \omega, \omega[rn]}) = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*, \sigma_1, \sigma_2, \sigma)$ where the tree $T_{\omega, \omega, \omega[rn]} = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^* R'_1, R'_2, R)$ is as defined in Def.3.2.1

In order to define the bounded morphism, we have to fix a bijection first. Let $h : \mathbb{N}_1 \times \mathbb{N}_2 \rightarrow \mathbb{N}$ be a bijection.

Next, we define function $k : (\mathbb{N}_1 \cup \{0\}) \times (\mathbb{N}_2 \cup \{0\}) \rightarrow \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{0\}$ as follows :

$$f(a, b) = \begin{cases} a, & \text{if } b = 0; \\ b, & \text{if } a = 0; \\ h(a, b), & \text{otherwise.} \end{cases}$$

Let $(\alpha, \beta) \in X_1 \times X_2$ with $\alpha = a_1 a_2 \dots$ and $\beta = b_1 b_2 \dots$. We define

$$g'(\alpha, \beta) = f(a_1, b_1) f(a_2, b_2) \dots$$

At last, we define $g(\alpha, \beta)$ as $g'(\alpha, \beta)$ but removing all zeros.

Lemma 5.2.12. *Function $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \rightarrow N(T_{\omega, \omega, \omega[rn]})$ is a bounded morphism.*

Proof. Let $\vec{z} = z_1 z_2 \dots z_n \in (\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*$. We define for $i \leq n$

$$x_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_1; \\ a_1, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases} \quad y_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_2; \\ b_2, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases}$$

Let $\alpha = x_1 x_2 \dots x_n 0^\omega$ and $\beta = y_1 y_2 \dots y_n 0^\omega$. But then $g(\alpha, \beta) = \vec{z}$. Hence, g is surjective. For the next conditions we show it for τ and σ . For τ'_1, σ_1 and τ'_2, σ_2 we can argue similar as in Lemma 5.2.11 and 5.2.12.

First we need to show, $R(g(\alpha, \beta)) = g(U_m(\alpha) \times U_m(\beta))$ for $m > \max\{st(\alpha), st(\beta)\}$. The proof is similar as in Lemma 5.2.1.

\supseteq : Pick $\zeta \in U_m(\alpha), v \in U_m(\beta)$. Show $g(\zeta, v) \in R(g(\alpha, \beta))$. By definition, we need to find a $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$ s.t. $g(\alpha, \beta) \cdot \vec{c} = g(\zeta, v)$. By choosing $m > \max\{st(\alpha), st(\beta)\}$, $\alpha \upharpoonright_m = \zeta \upharpoonright_m$ and $\beta \upharpoonright_m = v \upharpoonright_m$, we get $g(\alpha, \beta) = g(\zeta_m, v_m)$. Furthermore, we have $f_F(\alpha) R_1 f_F(\zeta)$ and $f_F(\beta) R_2 f_F(v)$, that means $\exists \vec{d} \in \mathbb{N}_1 \cup \{\epsilon\}$ and $\exists \vec{e} \in \mathbb{N}_2 \cup \{\epsilon\}$ s.t. $f_F(\alpha) \cdot \vec{d} = f_F(\zeta)$ and $f_F(\beta) \cdot \vec{e} = f_F(v)$. So, \vec{d} must appear at a point after $\zeta \upharpoonright_m$. The same holds for \vec{e} in v . But then we have $g(\zeta, v) = g(\alpha, \beta) * g(\vec{d}, \vec{e})$. Now there are four cases :

$$\begin{cases} g(\vec{d}, \vec{e}) \in \mathbb{N}_1, & \text{if } \vec{d} \in \mathbb{N}_1 \text{ \& } \vec{e} = \epsilon \\ g(\vec{d}, \vec{e}) \in \mathbb{N}_2, & \text{if } \vec{d} = \epsilon \text{ \& } \vec{e} \in \mathbb{N}_2 \\ g(\vec{d}, \vec{e}) \in \{\epsilon\}, & \text{if } \vec{d} = \vec{e} = \epsilon \\ g(\vec{d}, \vec{e}) \in \mathbb{N}, & \text{other} \end{cases}$$

Hence, $g(\zeta, v) \in R(g(\alpha, \beta))$.

\subseteq : Pick $\vec{w} \in R(g(\alpha, \beta))$. We will construct $\zeta \in U_m(\alpha)$ and $v \in U_m(\beta)$ s.t. $g(\zeta, v) = \vec{w}$. By definition there exists a $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$ s.t. $g(\alpha, \beta) \cdot \vec{c} = \vec{w}$. Depending on the \vec{c} , we build $\zeta = \alpha \upharpoonright_m \cdot \vec{d} 0^\omega$ and $v = \beta \upharpoonright_m \cdot \vec{e} 0^\omega$ where

$$\begin{cases} \vec{d} = \vec{c}, \vec{e} = 0, & \text{if } \vec{c} \in \mathbb{N}_1; \\ \vec{d} = 0, \vec{e} = \vec{c}, & \text{if } \vec{c} \in \mathbb{N}_2; \\ \vec{d} = \vec{e} = 0 & \text{if } \vec{c} \in \{\epsilon\} \\ \vec{d} = \vec{\alpha}, \vec{e} = \vec{\beta} \text{ where } h(\vec{\alpha}, \vec{\beta}) = \vec{c} & \text{if } \vec{c} \in \mathbb{N} (\vec{\alpha} \in \mathbb{N}_1, \vec{\beta} \in \mathbb{N}_2); \end{cases}$$

We have $\zeta \in U_m(\alpha)$ and $v \in U_m(\beta)$. Furthermore, because of $g(\zeta \upharpoonright_m, v \upharpoonright_m) = g(\alpha, \beta)$ it

holds that $g(\zeta, v) = g(\alpha, \beta) \cdot g(\vec{d}, \vec{e}) = g(\alpha, \beta) \cdot \vec{c} = \vec{w}$.

Now to check the conditions, we assume $x \in X_1$ and $y \in X_2$ and $U \in \tau(x, y)$. We need to prove $g(U) \in \sigma(g(x, y))$. In other words, show $R(g(x, y)) \subseteq g(U)$. We pick a $m > \max\{st(x), st(y)\}$ s.t. $U_m(x) \times U_m(y) \subseteq U$. But then we get

$$R(g(x, y)) = g(U_m(x) \times U_m(y)) \subseteq g(U)$$

Assume $x \in X_1$ and $x \in X_2$ and $R(g(x, y)) \subseteq V$. We need to find a $U \in \tau(x, y)$, s.t. $f(U) \subseteq V$. For U we pick $U_m(x) \times U_m(y)$ for some $m > \{st(x), st(y)\}$. But then

$$g(U_m(x) \times U_m(y)) = R(g(x, y)) \subseteq V$$

□

5.3 Completeness Results

Corollary 5.3.1. *Let $F_1 = (A^*, R_1) = F_{\xi_1, \eta_1}[A]$ and $F_2 = (B^*, R_2) = F_{\xi_2, \eta_2}[B]$. Then*

$$\text{Log}(N_\omega(F_1) \times_n N_\omega(F_2)) \subseteq \text{Log}(F_1) \otimes \text{Log}(F_2)$$

. It follows from Proposition 2.1.8, Lemma 5.1.5, Corollary 5.1.7 and Lemma 5.1.12.

Corollary 5.3.2. *Let $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$. Then*

$$\text{Log}(N_\omega(F_1) \times_n N_\omega(F_2)) = \text{Log}(F_1) \otimes \text{Log}(F_2)$$

Proof. The left to right inclusion follows from Corollary 5.3.1

To prove right to left, we know due to Proposition 2.1.9 and Proposition 5.2.8 we have $\text{Log}(N_\omega(F_i)) = \text{Log}(F_i)$ ($i \in \{1, 2\}$). Due to Proposition 5.1.10 we get $\text{Log}(N_\omega(F_1)) \otimes \text{Log}(N_\omega(F_2)) \subseteq \text{Log}(N_\omega(F_1) \times_n N_\omega(F_2))$. Because $N_\omega(F_1) \times_n N_\omega(F_2)$ is a frame in the n-product, we get $\text{Log}(N_\omega(F_1) \times_n N_\omega(F_2)) \subseteq \text{Log}(N_\omega(F_1) \times_n N_\omega(F_2))$.

□

Corollary 5.3.3. *Let $N_\omega(T_{\omega[rn]})_1 = F_1$ and $N_\omega(T_{\omega[rn]})_2 = F_2$ as defined before. Then*

$$T \times_n^+ T \subseteq \text{Log}(T_{\omega, \omega, \omega[rn]})$$

It follows from Corollary 5.1.7, Proposition 5.2.8, Lemma 5.2.9, Lemma 5.2.12

Proof. First, it is the case that $\text{Log}(F_1) \times_n^+ \text{Log}(F_2) \subseteq \text{Log}(F_1 \times_n^+ F_2)$. We can argue as before. By Lemma 5.2.12 it holds $\text{Log}(F_1) \times_n^+ \text{Log}(F_2) \subseteq \text{Log}(N(T_{\omega, \omega, \omega[rn]})) = \text{Log}(T_{\omega, \omega, \omega[rn]})$. But by Proposition 5.2.8 we have $\text{Log}(F_1) = \text{Log}(F_2) = T$. Hence, $\text{Log}(F_1) \times_n^+ \text{Log}(F_2) = T \times_n^+ T \subseteq \text{Log}(T_{\omega, \omega, \omega[rn]})$.

□

Theorem 5.3.4. *Let $L_1, L_2 \in \{S4, D4, D, T\}$. Then*

$$L_1 \times_n L_2 = L_1 \otimes L_2$$

Proof. Right to left is by Proposition 5.1.10.

For left to right, assume $L_1 = \text{Log}(F_1)$ and $L_2 = \text{Log}(F_2)$ for some $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$.

It follows from Corollary 5.3.2

$$L_1 \times_n L_2 = \text{Log}(N_\omega(F_1)) \times_n^+ \text{Log}(N_\omega(F_2)) \subseteq \text{Log}(N_\omega(F_1) \times_n^+ N_\omega(F_2)) = \text{Log}(F_1 \otimes F_2) = \text{Log}(F_1) \otimes \text{Log}(F_2) = L_1 \otimes L_2 \quad \square$$

Theorem 5.3.5. *Let $N_\omega(T_{\omega[rn]})_1 = F'_1$ and $N_\omega(T_{\omega[rn]})_2 = F'_2$ as defined before. Then*

$$T \times_n^+ T = T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

Proof. For left to right it is easy to check the axioms. We pick any frame F_1 and F_2 s.t. $F_1 \models T$ and $F_2 \models T$. We consider $F_1 \times_n^+ F_2$ the full product and let $(x, y) \in F_1 \times_n^+ F_2$.

Assume $F_1 \times_n^+ F_2, (x, y) \models \Box_1 p$. By definition, there exists a $U \in \tau'_1(x, y)$ s.t. $U \subseteq V(p)$. But we have $U \supseteq V \times \{y\}$ for some $V \in \tau(x)$. By assumption, $F_1 \models T$, so we can follow $x \in V$. Hence, $(x, y) \in V(p)$. For $\Box_2 p \rightarrow p$ it's done similar.

Let's check $\Box p \rightarrow p$. Assume $F_1 \times_n^+ F_2, (x, y) \models \Box p$. By definition there exists a $U \in \tau(x, y)$ s.t. $U \subseteq V(p)$. But $U \supseteq V \times W$ for some $V \in \tau_1(x)$ and $W \in \tau_2(y)$. Because of the assumption, we must have $x \in V$ and $y \in W$. Hence, $(x, y) \in V(p)$. It follows $F_1 \times_n^+ F_2, (x, y) \models p$.

Now we check the extra axiom. Assume $F_1 \times_n^+ F_2, (x, y) \Vdash \Box p$. By definition, there exists a $U \subseteq V(p)$, where $U \supseteq V \times W$ with $V \in \tau_1(x)$ and $W \in \tau_2(y)$. By assumption, we get $y \in W$. That means $U \in \tau'_1(x, y)$ because $U \supseteq V \times \{y\}$. The same argument we can apply to $\tau'_2(x, y)$. Hence, $F_1 \times_n^+ F_2, (x, y) \models \Box_1 p \wedge \Box_2 p$. The rest is clear.

Now for the other direction, we apply Corollary 5.3.3 to get $T \times_n^+ T \subseteq \text{Log}(T_{\omega, \omega, \omega[rn]})$. By Proposition 3.2.6 we have $\text{Log}(T_{\omega, \omega, \omega[rn]}) = TNL$. Hence,

$$T \times_n^+ T \subseteq T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

.

□