1 Introduction

Modal logic extends classical propositional logic by introducing modal operators, most notably $\Box$ ("necessarily") and $\Diamond$ ("possibly"). In the standard Kripke semantics, these operators are interpreted over a set of possible worlds connected by an accessibility relation. In the following, we give an example for $\Box$.

Example 1.0.1. *Let $\phi = \Box p$ and $M = (W, R, V)$ with $W = \{w_1, w_2, w_3, w_4, w_5\}$, $R = \{(w_1, w_2), (w_2, w_3), (w_2, w_4), (w_2, w_5)\}$ and $V(p) = \{w_3, w_4, w_5\}$. (See figure 1.)*

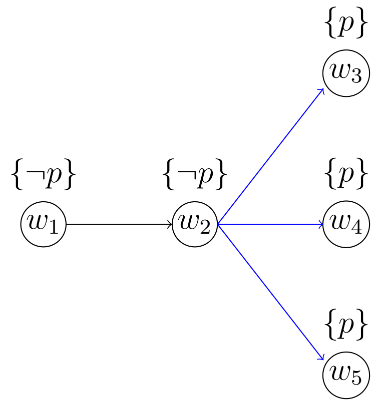

Figure 1: Intuitively, $\Box p$ is interpreted as "In all accessible worlds, p is true". Here we can see $\Box p$ is true at w_2, because all accessible worlds from w_2 (in blue lines), make p true.

However, the term 'modal logic' is often used more broadly for a family of related systems. This include for example temporal or epistemic logic. Due to its relational nature, modal logic is a powerful tool for analyzing structured systems, where the relationships between states or agents are essential. It has wide-ranging applications—for instance, in formal verification. For example in model checking, temporal logic is used to describe the behavior of systems over time. In multi-agent systems, epistemic logic plays a key role in reasoning about knowledge and beliefs.

1.1 Topological semantics

An idea to generalize Kripke semantics was topology. The study of topological semantics in modal logic was initiated by McKinsey and Tarski in 1944 [3]. To assign meaning to modal formulas, we use topological space, which is a pair consisting of a domain X and a collection of open sets, where each set describes which points in X are considered "nearby" to each other. A topological model is a topological space with a valuation V, where we assign subsets of the domain to the variables. $\Box$ and $\Diamond$ are interpreted differently in topological context; they behave more like the interior and closure operators like in topology [16].

Example 1.1.1. *Let $W = \{1, 2, 3\}, \tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, W\}$ (where τ is the collection of open sets) and $V(p) = W$. Furthermore, let $\phi = \Box p$.*

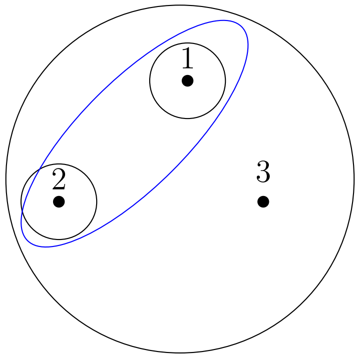

Figure 2: Each ellipse represents an open set. Intuitively, $\Box p$ is satisfied at a point w, if we can find an open set U s.t. $w \in U$ and $U \subseteq V(p)$. This is true for 1 and 2 because we can find $U = \{1, 2\} \subseteq V(p)$ (where $V(p)$ is marked as the blue).

It is well known that the logic for any topological space is $S4$, where $S4$ is the modal logic with the transitivity ($\Box p \to \Box\Box p$) and reflexivity axiom ($\Box p \to p$). This holds because, under topological semantics, the interpretation of $\Box$ naturally satisfies both reflexivity and transitivity. So, if we assume a topological space satisfies $\Box p$ at point w with some valuation $V(p)$, then we can find an open set U around w where $w \in U$. Therefore, $w \in V(p)$ which shows reflexivity. On the other hand, for any point in U, we can again find an open set (U itself) witnessing that $\Box p$ holds there. Hence, transitivity is also satisfied.
Topological semantics provides a rich, and intuitive semantics for $S4$ and related logics. It has a variety of applications e.g. spatial reasoning or program verification. Despite its rich expressiveness, it can not fully capture Kripke semantics.

1.2 Neighborhood frames

That is why Neighbourhood semantics (see [11] for an introduction) as a generalization of Kripke semantics for modal logic was introduced (independently) by Dana Scott [9] and Richard Montague [10]. Neighborhood frames are similar to topological spaces. Instead of collection of open sets, we have a neighborhood function, which assigns to each point a set of subsets of the domain. These subsets are also called neighbourhoods. The Neighborhood semantics only differs slightly in $\Box$ and $\Diamond$. The following example demonstrates it.

Example 1.2.1. *Let* $W = \{1, 2, 3\}$,

$$\tau(x) = \begin{cases} 1 \to \{\{1\}, \{3\}, W\} \\ 2 \to \{\{2\}, \{1, 2\}, W\} \\ 3 \to \{\{3\}, \{1, 3\}\} \end{cases}$$

and $V(p) = \{1, 2\}$. *Assume* $\phi = \Box p$.

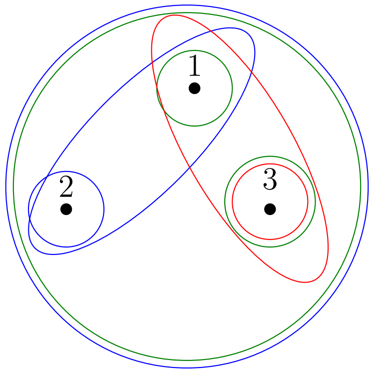

Figure 3: The green ellipses represents neighbourhoods from 1, the blue ellipses those of 2 and red ellipses those of 3. $\Box p$ is satisfied at a point w, if there exists a neighborhood U $\in \tau(w)$ s.t. $U \subseteq V(p)$. This is only true for 2.
Remark : For truth, we don't require that w has to be in U. Therefore, it is not necessary that all neighborhood frames validate reflexivity and hence S4.

Neighborhood semantics is more general than Kripke semantics and in the case of normal reflexive and transitive logics coincides with topological semantics. The original motivation for introducing these general models was to provide a semantics for non-normal modal logics. In this work, we only consider neighborhood functions which are filters. That means, for each element x in W we have : $\emptyset \notin \tau(x)$, if $U \in \tau(x)$ and $U \subseteq V$ then $V \in \tau(x)$ (upward closed) and if $U, V \in \tau(x)$, then $U \cap V \in \tau(x)$ (closed under finite intersection). They impose constraints on neighbourhoods and provide us a new area to explore. Beyond their theoretical motivation, interest in topological semantics and neighborhood frames has grown considerably, partly due to its applications in artificial intelligence.
Oftentimes, it is necessary to combine frames for different modal logics into a complex frame. The natural way of doing that is a product construction. For Kripke frames, the resulting product is the Cartesian product of the two frames with two accessibility relations. For topological semantics, the product of topogical spaces as bi-topological spaces with so-called horizontal and vertical topologies have been considered. In a similar fashion, the product of neighborhood frames was introduced by Sano in [8].

1.3 Our contributions and structure of the thesis

In this work, we study the multimodal logic of products of neighborhood frames. Specifically, the logic consisting of three copies of T along with interaction axiom $\Box p \to \Box_1 p \land \Box_2 p$. By "copies," we mean that we take the union of T with itself, but introduce new modal operators for each component. This is also called "fusion" (notation : $\otimes$). Formally, we define the fusion of T with itself as $T \otimes T \otimes T = K_3 + T_{\Box} + T_{(\Box \to \Box_1)} + T_{(\Box \to \Box_2)}$, where K_3 is the minimal normal modal logic with three modal operators.

We focus on the logic of all product neighbourhood frames $F_1 \times_n^+ F_2$, where both F_1 and F_2 validates T. The product $\times_n^+$ combines three types of functions : horizontal,vertical

and standard. Formally, such logic can be written down as $T \times_n^+ T = Log(\{F_1 \times_n^+ F_2 \mid F_1 \vDash T \text{ and } F_2 \vDash T\})$. We will introduce the horizontal and vertical modalities which correspond to reasoning over the first and second coordinate frames, respectively, while the standard modality reflects interactions in the combined frame.
More specific definitions can be found in Chapter 3.2.
Our main contribution is to investigate whether the logic $T \otimes T \otimes T$ with interaction axiom $\Box p \to \Box_1 p \land \Box_2 p$ corresponds to the logic of such product frames introduced above — and vice versa. Formally, the question is whether the following :

$$T \times_n^+ T = T \otimes T \otimes T + \Box p \to \Box_1 p \land \Box_2 p$$

The question addressed in this work was originally motivated by two earlier papers. In the first [1], the authors explored a similar result in topological semantics, but for the modal logic $S4$. The core idea was to show that the logic is sound and complete w.r.t a specially constructed Kripke frame called "$T_{6,2,2}$", which is a six-branching tree with three accessibility relations. The frame can be interpreted as a topological space which plays a crucial role in constructing a topo-bisimiulation between $T_{6,2,2}$ and a specific product topological space $\mathbb{Q} \times_n^+ \mathbb{Q}$.

In the second work [2], it was shown that for any $L_1, L_2 \in \{D, D4, T, S4\}$ the fusion $L_1 \otimes L_2$ agrees with the logic of product of neighbourhood frames but without using the standard neighborhood product. The main idea is similar to the previous case. The author again consider Kripke tree frames. Let's denote such frame as F. He then constructs two different neighbood interpretation on that frame, called $N(F)$ and $N_\omega(F)$. In $N(F)$, each world is assigned a neighourhood based on the standard successor relation — specifically, the set of all supersets of the image of the accessibility relation. In contrast, $N_\omega(F)$ is built using pseudo-infinite paths to generate neighborhoods that also reflect the successor structure of F. A specific definition can be found in chapter 6.2. By considering the product of these neighbourhood frames, the authors then construct a bounded morphism to establish the desired result.

Our approach is similar. We construct a Kripke tree frame similar to $T_{6,2,2}$ but with infinitely many branches and infinite depth. We show that our logic is sound and complete w.r.t. this frame. Then, we consider its neighbourhood versions and take their product, constructing a bounded morphism to establish the desired result.

We start our journey in **Chapter 2**, where we recall Kripke semantics as well as well-known results such as Filtration Theorem and the Sahlqvist Theorem. Furthermore, we give basic notation for multimodal logic.

Chapter 3 introduces topological semantics and neighbourhood frames. We formally define what topological space is as well as neighbood frames and their products notation. We also provide the corresponding semantics for multimodal logic.

In **Chapter 4**, we begin the first part of our contribution. We formally introduce the infinitely branching and infinite depth tree. Then, using Filtration Theorem, the

Sahlqvist Theorem and an unravelling technique to show that our logic is sound and complete with respect to this tree.

Chapter 5 is dedicated to the author's work presented in [1]. We provide a more detailed proof of his result. To make the main result more accessible, we also reprove a similar result but for modal logic with only one modal operator. We start by formally introducing T_2 and $T_{6,2,2}$ frames where T_2 is the infinite binary tree. Then we show $S4$ is complete w.r.t. $\mathbb{Q}$. The idea to show $S4$ is sound and complete w.r.t. T_2, construct a subset X of $\mathbb{Q}$, which is isomorphic and in topological sense homeomorphic to $\mathbb{Q}$ and then establish a bisimulation between T_2 and X. This method carries over naturally to the multimodal version.

Chapter 6 is split into two parts. First, we present a more detailed proof of [2]. We begin by introducing the crucial trees as Kripke frames. Based on these, we construct their neighbourhood versions in order to build a bounded morphism and demonstrate the desired result. In the second part, we adapt these ideas to show the second part of our result. At the end, we give some completeness results.

2 Preliminaries

This chapter is primarily devoted to the semantics of Kripke. In addition, we define the notions of product frames and multimodal logic. We also present fundamental results in modal logic, such as Filtration Theorem and Sahlqvist Theorem which play a role in our thesis. The content is mainly based on the Modal Logic book [7]. Remark: natural numbers $\mathbb{N}$ are without 0.

Definition 2.0.1. *Let prop be a set of variables. Then a formula ϕ is defined as follows:*

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \Box_i \phi$$

where $p \in Prop$ and $\Box_i$ is a modal operator. Other connectives are expressed through $\neg$ and $\vee$ and dual modal operators $\Diamond_i$ as $\Diamond_i \phi = \neg\Box_i\neg\phi$

Definition 2.0.2. *Let $M = (W, R_1..., R_n, V)$ be a model and $w \in W$ a state in M. The notion of a formula being true at w is inductively defined as follows :*

$$\begin{array}{ll} M, w \Vdash p & \textit{iff } w \in V(p) \\ M, w \Vdash \bot & \textit{never} \\ M, w \Vdash \neg\phi & \textit{iff not } M, w \Vdash \phi \\ M, w \Vdash \phi \vee \psi & \textit{iff } M, w \Vdash \phi \vee M, w \Vdash \psi \\ M, w \Vdash \Box_i\phi & \textit{iff } \forall v \in W : wRv \rightarrow M, v \Vdash \phi \end{array}$$

Definition 2.0.3. *Let $F = (W, R_1, R_2, ...)$ and $F' = (W', R'_1, R'_2, ...)$ be two frames. A bounded morphism from F to F' is a function $f : W \rightarrow W'$ satisfying the following conitions:*

$$\textit{If } (u, v) \in R_i \textit{ then } (f(u), f(v)) \in R'_i$$

$$\textit{If } (f(w), v') \in R'_i \textit{ then } \exists v \in W \textit{ s.t } (w, v) \in R'_i \textit{ and } f(v) = v'$$

We say F' is a bounded morphic image of F, if there is a surjective bounded morphism from F to F'.

Proposition 2.0.4. *Let ϕ be a formula in $L_{\Box_1,...,\Box_n}$, $F = (W, R_1, ..., R_n)$ and $F' = (W', R'_1, ..., R'_n)$ be two frames and F to F' a surjective bounded morphism. Then the following holds :*

$$\textit{If } F \Vdash \phi \textit{ then } F' \Vdash \phi$$

Proof. This can be shown by structural induction on the length of the formula. □

Corollary 2.0.5. *If F' is a bounded morphic image of F, then we have $Log(F) \subseteq Log(F')$*

Proof. The proof is by structural induction.

□

In this work, we deal with products of frames and multimodal logics of product of frames. The study of products of Kripke frames and their modal logics was first initiated by Segerberg [4] and Shehtman [5]. In the following, we will define necessary notions for such logics.

Definition 2.0.6. *Let $F = (W, R_1)$ and $G = (W, R_2)$. Then we define the Kripke product on $W \times V$ as follows :*

$$(w, v)R'_1(w', v') \text{ iff } wR_1w' \text{ and } v = v'$$

$$(w, v)R'_2(w', v') \text{ iff } w = w' \text{ and } vR_2v'$$

R'_1 is also called horizontal and R'_2 vertical relation.

Definition 2.0.7. *A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \to \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms*

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$. Let L be a logic and let Γ be a set of formulas. Then $L+\Gamma$ denotes the minimal logic containing L and Γ

Definition 2.0.8. *Let L_1 and L_2 be two modal logic with one modality $\Box$. Then the fusion of these logics are defined as follows :*

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

The following logics may be important

$$D = K + \Box p \to \Diamond p$$

$$T = K + \Box p \to p$$

$$D4 = D + \Box p \to \Box\Box p$$

$$S4 = T + \Box p \to \Box\Box p$$

We now introduce important notation to prove the multimodal version of the Filtration Theorem.

Definition 2.0.9. *A set Σ is closed under subformulas, if for all formulas ϕ and ϕ' the following holds :*

1. *if $\neg\phi \in \Sigma$ then $\phi \in \Sigma$*
2. *if $\phi \vee \phi' \in \Sigma$ then $\phi, \phi' \in \Sigma$*
3. *if $\Box_i \phi \in \Sigma$ then $\phi \in \Sigma$*

Example : Suppose we have a multimodal logic with $\Box, \Box_1$ called $\mathcal{L}_{\Box,\Box_1}$ where $\phi = \Box p \to \Box_1 q$ and $\phi \in \Sigma$. Then $\Sigma = \{\phi, \neg\Box p, \Box_1 q, \Box p, p, q\}$ is closed under subformulas.

Definition 2.0.10. *Let $M = (W, R_1, ..., R_n, V)$ be a model and suppose Σ is a set of formulas. We define a relation $\equiv$ on W as follows :*

$$w \equiv v \textit{ iff } \forall \phi \in \Sigma : M, w \Vdash \phi \Leftrightarrow M, v \Vdash \phi$$

It is well known that the $\equiv$-relation is an equivalence relation. We denote the equivalence class of a state $w \in W$ as $[w]_\Sigma = \{v \mid v \equiv w\}$. Furthermore W_Σ is the set of all equivalence classes, i.e $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$.

In the following, we define what is known as a "filtration." The definition is similar to the one found in the Modal Logic book, but we consider a multimodal version.

Definition 2.0.11. *Let $M = (W, R_1, ..., R_n, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes induced by $\equiv$. A model $M^f_\Sigma = (W^f, R^f_1, ..., R^f_n, V^f)$ is called filtration of M through Σ if the following holds :*

1. *$W^f = W_\Sigma$*
2. *If $(w, v) \in R_i$ then $([w], [v]) \in R^f_i$*
3. *If $([w], [v]) \in R^f_i$ then for any $\Box_i \phi \in \Sigma$: if $M, w \Vdash \Box_i \phi$ then $M, v \Vdash \phi$*
4. *$V^f = \{[w] \mid M, w \Vdash p\}$, for all propositional variables $p \in \Sigma$*

Theorem 2.0.12. *Consider $L_{\Box_1 ... \Box_n}$. Let $M^f = (W_\Sigma, R^f_1, ..., R^f_n, V)$ be a filtration of M through a subformula closed set Σ. Then for all formulas $\phi \in \Sigma$, and all nodes $w \in M$, we have*

$$M, w \Vdash \phi \textit{ iff } M^f, [w] \Vdash \phi$$

Proof. By induction on ϕ.

Case $\phi = p$: Left to right follows immediately from filtration defintion. Conversely, suppose $M^f, [w] \Vdash p$. This means $[w] \in V^f(p)$. But this means V(p) can not be empty. Pick any $v \in V(p)$. Obviously, $w \equiv v$ and $M, v \Vdash p$. Hence, $M, w \Vdash p$.

Case $\phi = \neg\psi$: Suppose ψ holds. Then we have : $M, w \Vdash \phi$ iff $M, w \nVdash \psi$. Applying induction hypothesis, we get : $M^f, [w] \nVdash \psi$. But then, we have $M^f, [w] \Vdash \phi$. Right to left is the same.

Case $\phi = \phi_1 \vee \phi_2$: Suppose ϕ_1, ϕ_2 holds. Let $M, w \Vdash \phi$. That means $M, w \Vdash \phi_1$ or $M, w \Vdash \phi_2$. Applying induction hypothesis, we get $M^f, [w] \Vdash \phi_1$ or $M^f, [w] \Vdash \phi_2$. But then, $M^f, [w] \Vdash \phi_1 \vee \phi_2 = \phi$. Right to left is similar.

Case $\phi = \Box_i \psi$: Left to right. Suppose ψ holds and $M, w \Vdash \Box_i \psi$. We need to show $M^f, [w] \Vdash \Box_i \psi$, this means $\forall [v] \in W_\Sigma : [w] R_i [v] \to M^f, [v] \Vdash \Box_i \psi$. Pick any $[v] \in W_\Sigma$ s.t $[w] R_i [v]$. By condition 3, w.r.t to the modal operator, we have $M, v \Vdash \psi$. By induction hypothesis, we get $M^f, [v] \Vdash \psi$. Because [v] was arbitrary it follows that $M^f, [w] \Vdash \Box_i \psi$.

Right to left. Suppose ψ holds and $M^f, [w] \Vdash \Box_i\psi$. Pick $v \in W$ s.t wR_iv. By condition 2, w.r.t to the modal operator, we have $[w]R_i^f[v]$. So, $M^f, [v] \Vdash \psi$. By induction hypothesis, we get $M, v \Vdash \psi$. Because v was arbitrary, we have $M, w \Vdash \Box_i\psi$.

□

After showing that filtration preserves truth, we now define a special relation known as the "smallest filtration."

Definition 2.0.13. *Let $M = (W, R_1, ..., R_n, V)$ be a model, Σ is closed under subformulas and W_Σ the set of equivalence classes. We define :*

$$R^s = \{[w], [v] \mid \exists w' \in [w], \exists v' \in [v] : w'R_iv'\}$$

Lemma 2.0.14. *Let $M = (W, R_1, ..., R_n, V)$ be a model, Σ is closed under subformlas and W_Σ the set of equivalence classes induced by $\equiv$ and V^f the standard valuation on W_Σ. Then $(W_\Sigma, R_1^s, ..., R_n^s, V^f)$ is a filtration of M through Σ.*

Proof. It suffices to show R_i^s fulfills the condition 2 and 3 w.r.t to the corresponding modal operator $\Box_i$. But R_i^s already satisfies condition 2. Let's check the other condition. Let $\Box_i\phi \in \Sigma$, $[w]R_i^s[v]$ and $M, w \Vdash \Box_i\phi$. Because of $[w]R_i^s[v]$ we pick a $w' \in [w]$ and $v' \in [v]$. By defintion, we have $w'R_iv'$. Because $w' \equiv w$, we get $M, w' \Vdash \Box_i\phi$. Hence, $M, v' \Vdash \phi$ and by $v' \equiv v$, we get $M, v \Vdash \phi$.

□

Proposition 2.0.15. *Let Σ be a finite subformula closed set of $L_{\Box_1,...,\Box_n}$. For any model M, if M^f is a filtration through Σ, then M^f contains at most 2^n nodes (where n denotes the size of Σ).*

Proof. The states of M^f are the equivalence classes in W_Σ. Let $g : W_\Sigma \to P(\Sigma)$ defined by $g([w]) = \{\phi \in \Sigma \mid M, w \Vdash \phi\}$. g is well defined. Pick any u and v s.t $u \equiv v$. But then by defintion of $\equiv$, they fulfill the same subformulas. This means g([v]) = g([u]).
g is also injective. Pick any $[u], [v] \in W_\Sigma$ s.t g([u]) = g([v]). We show $[u] \subseteq [v]$. The other inclusion is similar. By assumption we have $u \equiv v$. Pick any $u' \in [u]$. Then we have $u' \equiv u \equiv v$. Hence, $u' \in [v]$. At the end, this means M^f contains at most 2^n nodes.

□

Theorem 2.0.16. *Let ϕ be a formula of $L_{\Box_1,...,\Box_n}$. If ϕ is satisfiable, then it is satisfiable on a finite model containing at most 2^v nodes, where v is the number of subformulas of ϕ.*

Proof. Assume that ϕ is satisfiable on a model on M. Take any filtration of M through the set of subformulas of ϕ. By Filtration Theorem, we get that ϕ is satisfied in the filtration model M^f. Furthermore, it is bounded by 2^v.

□

The Filtration Theorem helps us to establish the Finite Model Property (FMP), which is essential for our result. We now define a special class of formulas called Sahlqvist formulas.

Definition 2.0.17. *A modal formula ϕ is positive if all variables occurs in the scope of an even number of negations. In the other hand, a formula is negative, if all variables occurs in the scope of an odd number of negations. A boxed atom is a modal formula of the form $\Box^n p$ for some $n \in \mathbb{N}$, where p is a propositional variable and $\Box^n p$ is defined as follows : $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$.*

Sahlqvist formulas can be build by the following inductive definition :

$$\phi ::= \neg\Box^d p \mid \pi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \Box\phi$$

where p is an atom, $d \geq 0$ and π is a positive formula.

Examples for Sahlqvist formulas:

$$\neg\Box p \vee \Box\Box p \ \ \textit{(transitivity)}$$

$$\neg\Box p \vee p \ \ \textit{(reflexivity)}$$

$$\Box\neg\Box\Box\Box\Box p \vee \Box\Diamond\Box\Diamond p$$

$$\Box\Box\Box\Box(\Box\neg\Box p \vee p) \wedge \neg\Box\Box p \vee \Box p$$

Non Sahlqvist Formulas :

$$\neg\Box\Diamond p \vee \Diamond\Box p$$

$$\neg\Diamond\Box p \vee \Box\neg p$$

We can extend this defintion for multimodal logics. We say a boxed atom can be $\Box_i^n p$.

Sahlqvist formulas possess important properties, which are guaranteed by the Sahlqvist Theorem. It says that, when given a normal modal logic K and a set of Sahlqvist formulas, the resulting logic is complete w.r.t to the class of frames, which satisfies the corresponding first-order formula of the Sahlqvist formulas. This also holds for multimodal logic. We will not prove it here, but we will use it later in this work. The main idea is to use Sahlqvist to show that our logic is sound and complete w.r.t. a certain class of frames. Using the smallest filtration and Filtration Theorem, we can gurantee the finite model property while preservering the relevant frame properties.

3 Product of Topological spaces and Neighborhood frames

Topology is a fundamental branch in mathmatics that explores the properties of space that remain invariant under continuous transformations. Unlike geometry, where we rely on precise measurements, topology focuses on the more flexible notions of closeness, boundary and countinuity without relying on specific distances.
One particularly rich and fruitful area of study is the interaction between modal logic and topology. Topological spaces provide an intuitive semantic framework for modal logic by interpreting modal operators in terms of spatial notions like interior, closure, and neighbourhoods. It is also well known, that the modal logic of any topological space is $S4$.
Neighborhood frames generalizes the idea of topology by loosen the conditions that define open sets in topological space. Instead of relying on a global systems of open sets, there exists a neighborhood function, which assigns to each point a collection of sets — its neighbourhoods. This provides us a more flexible framework to work with.
It is often of both theoretical and practical interest to combine topological spaces or neighborhood frames. These product constructions become particularly relevant in the context of multimodal logic, where we aim to reason with multiple modal operators simultaneously. Each modality can be interpreted in its own topological or neighbourhood dimension, allowing us to model systems with distinct but interacting notions of necessity — such as time and space or knowledge and belief.
In this chapter, we introduce the notation for the product of topological spaces, define what is meant by "horizontal" and "vertical" topology and provide the semantics for multimodal logic in topological terms. We will proceed similarly for neighborhood frames.

3.1 Topological semantics and product of topological spaces

We begin by formally introducing the concept of a topological space.

Definition 3.1.1. *A topological space is a pair (X, τ) where τ (also called topology) is a collection of subsets of X (elements of τ are also called open sets) such that :*

1. *the empty set $\emptyset$ and X are open*
2. *the union of an arbitrary collection of open sets is open*
3. *the intersection of finite collection of open sets is open*

A topological model is a structure $M = (X,\tau,v)$ where (X,τ) is a topological space and v is a valuation assigining subsets of X to propositional variables.

Remark 3.1.2. *There is an equivalent defintion for open sets. Let (X,τ) be a topological space and U a set. U is open iff $\forall x \in U\ \exists V \subseteq U : V$ is open and $x \in V$. This is true because, the union of open sets is an open set.*

Definition 3.1.3. *Let (X, τ) be a topological space. Then the space is called Alexandroff, if arbitrary intersection of open sets are open. An equivalent defintion is that every point has a smallest open set. That means for each $x \in X$ we have $U_x = \bigcap\{U \subseteq X \mid U \text{ is open and } x \in U\} \in \tau$.*

In the following, we present several topological spaces to build intuition. Some of them are Alexandroff spaces, but we also include an example of a non-Alexandroff space.

Example 3.1.4. *Let $X = \{1, 2, 3\}$. We consider different topologies on X:*

- *If $\tau = \{\emptyset, X\}$, then (X, τ) is called the trivial topological space.*
- *If $\tau = \mathcal{P}(X)$, then (X, τ) is called the discrete topological space.*
- *If $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, then (X, τ) is topological space.*

These spaces are Alexandroff. In the following we will see a non-Alexandroff space.

Example 3.1.5. *Let $X = \mathbb{R}$ with the standard topology. That means the basis for open sets is $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} \mid a < b\}$ with $a, b \in \mathbb{R}$. Now, we consider a family of open sets $U_n = \{-\frac{1}{n}, \frac{1}{n}\}$ for $n \in \mathbb{N}$. But, it is the case that $\bigcap_{n=1}^{\infty} U_n = \{0\}$ is not open.*

Definition 3.1.6. *Let $M = (X, \tau, v)$ be a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows:*

$$
\begin{array}{ll}
M, x \vDash p & \textit{iff } x \in v(p) \\
M, x \vDash \bot & \textit{never} \\
M, x \vDash \neg\phi & \textit{iff } M, x \nvDash \phi \\
M, x \vDash \phi \vee \psi & \textit{iff } M, x \vDash \phi \textit{ or } M, x \vDash \psi \\
M, x \vDash \Box\phi & \textit{iff there exists } U \in \tau \textit{ such that } x \in U \textit{ and } \forall u \in U,\ M, u \vDash \phi
\end{array}
$$

In the study of product spaces in topology, we work with horizontal and vertical topologies. Such topologies allows us to reason across the coordinates of the product space. It allows us to define modalities that reflect reasoning in each dimension.

Definition 3.1.7. *Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \upsilon)$ be topological spaces and $N \subseteq X \times Y$.*

We define

Horizontally open: *N is horizontally open iff*

$$\forall (x, y) \in N\ \exists U \in \chi \text{ such that } x \in U \text{ and } U \times \{y\} \subseteq N.$$

Vertically open: *N is vertically open iff*

$$\forall (x, y) \in N\ \exists V \in \upsilon \text{ such that } y \in V \text{ and } \{x\} \times V \subseteq N.$$

(a) vertically (b) horizontally

Figure 4: Each rectangle represents N and the redlines are subsets of N

If N is H-open and V-open, then we call it HV-open.
We denote τ_1 is the set of all H-open subsets of $X \times Y$ and τ_2 is the set of all V-open subsets of $X \times Y$. We say τ is the set of all standard product of subsets $X \times Y$ s.t. $X \in \chi$ and $Y \in \upsilon$.

Example 3.1.8. *Let $\mathcal{X} = (\mathbb{R}, \chi)$ and $Y = (\mathbb{R}, \upsilon)$ with standard topology, that means the basis $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} \mid a < b\}$. Now consider $\mathcal{X} \times \mathcal{Y}$. The horizontal topology is generated by $\mathcal{B}_1 = \{U \times \{y\} \mid U \in \chi$ and $y \in \mathbb{R}\}$ and vertical topology by $\mathcal{B}_2 = \{\{x\} \times V \mid x \in \mathbb{R}$ and $V \in \upsilon\}$.*

Definition 3.1.9. *Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \upsilon)$ be topological spaces and $(x, y) \in X \times Y$. The truth in $M = (X \times Y, \tau_1, \tau_2, \tau, \upsilon)$ is similar as in Definition 2.1.6. For $\Box$, $\Box_1$ and $\Box_2$ we interpret this as follows :*

$$M, (x, y) \vDash \Box_1 \phi \quad \textit{iff } \exists U \in \tau_1 : (x, y) \in U \textit{ and } \forall (x', y') \in U : (x', y') \vDash \phi$$
$$M, (x, y) \vDash \Box_2 \phi \quad \textit{iff } \exists V \in \tau_2 : (x, y) \in V \textit{ and } \forall (x', y') \in V : (x', y') \vDash \phi$$
$$M, (x, y) \vDash \Box \phi \quad \textit{iff } \exists U \in \chi \, \exists V \in \upsilon : (x, y) \in U \times V \textit{ and } \forall (x', y') \in U \times V : (x', y') \vDash \phi$$

Definition 3.1.10. *Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \upsilon)$ be two topological spaces. Then the full product of these spaces is defined as follows :*

$$\mathcal{X} \times_t^+ \mathcal{Y} = (X \times Y, \tau_1, \tau_2, \tau)$$

with horizontal, vertical and standard topologies.

Definition 3.1.11. *For two unimodal logics L_1 and L_2 we define the full t-product of them as follows :*

$$L_1 \times_t^+ L_2 = Log(\{\mathcal{X} \times_t^+ \mathcal{Y} \mid \mathcal{X} \vDash L_1 \textit{ and } \mathcal{Y} \vDash L_2\})$$

where $\mathcal{X}, \mathcal{Y}$ are topological spaces.

Definition 3.1.12. *A topo-bisimulation between two topological models $M = (X, \tau, \upsilon)$ and $M' = (X', \tau', \upsilon')$ is a nonempty relation $\Leftrightarrow \subseteq X \times X'$ s.t. if $x \Leftrightarrow x'$ then*

1. Base: $x \in V(p)$ iff $x' \in \upsilon'(p)$ for any propositional variable p
2. Forth: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ s.t. $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \Leftrightarrow y'$
3. Back: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ s.t. $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \Leftrightarrow y'$

Proposition 3.1.13. *Let $M = (X, \tau, \upsilon)$ and $M' = (X', \tau', \upsilon')$. Assume $w \in M'$ and $w' \in M'$ are topo-bisimilar points. Then for each formula ϕ we have*

$$M, w \vDash \phi \text{ iff } M', w' \vDash \phi$$

Proof. The proof is by structural induction. □

Definition 3.1.14. *Let X and Y be topological spaces and $f : X \to Y$ a function. We call f continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X. We say f is open if for each open set $V \subseteq X$ the set $f[V]$ is open in Y.*

3.2 Product and semantics of neighborhood frames

Neighborhood frames are more flexible than topological space. Its logic is not $S4$ in general. It is well known, that neighborhood frames fully captures the semantics of Kripke. Moreover, neighborhood frames are expressive enough to characterize non-normal modal logics, which makes them a powerful tool. We formally introduce neighbourhood frames.

Definition 3.2.1. *Let X be a non-empty set. A function $\tau : X \to 2^{2^X}$ is called a neighbourhood function. A pair $F = (X,\tau)$ is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X,τ,v), where v assigns a subset of X to a variable.*

Definition 3.2.2. *Let $M = (X,\tau,v)$ be a neighourhood model and $x \in X$. The truth of a formula is defined inductively as follows :*

$M, x \vDash p$	iff $x \in V(p)$
$M, x \vDash \bot$	never
$M, x \vDash \neg\phi$	iff $M, x \nvDash \phi$
$M, x \vDash \phi \vee \psi$	iff $M, x \vDash \phi \vee M, x \vDash \psi$
$M, x \vDash \Box\phi$	iff $\exists V \in N(x) \forall y \in V : M, y \models \phi$

A formula is valid in a n-model M if it is valid at all points of M ($M \vDash \phi$). Formula is valid in a n-frame F if it is valid in all models based on F (notation $F \vDash \phi$). We write $F \vDash L$ if for any $\phi \in L, F \vDash \phi$. Logic of a class of n-frames $\mathcal{C}$ as $Log(\mathcal{C}) = \{\phi \mid F \vDash \phi \text{ for all } F \in \mathcal{C}\}$. We define $nV(L) = \{F \mid F \text{ is an n-frame and } F \models \phi\}$.

Example 3.2.3. *Suppose $W = \{1, 2, 3\}$, and define a neighborhood function $\tau : W \to 2^{2^W}$ as follows :*

- $N(1) = \{\{2\}, \{1, 2\}, \{2, 3\}, W\}$
- $N(2) = \{W\}$
- $N(3) = \{\{3\}, \{1, 3\}, \{2, 3\}, W\}$

We define a valuation $V : \{p, q\} \to 2^W$ by $V(p) = \{2, 3\}$ and $V(q) = W$. Let $\phi = \Box p \to p$ and $\psi = \Box p \wedge \Box\Box q$. Then the following holds :

- $(W, \tau), 1 \models \psi$
- $(W, \tau), 3 \models \psi$
- $(W, \tau), 3 \models \phi$
- $(W, \tau), 2 \not\models \psi$
- $(W, \tau), 1 \not\models \phi$

The last example shows that the reflexive axiom is not valid, and hence S4 is not valid in general.

Definition 3.2.4. *Let X be a non-empty set and τ neighborhood function. We call τ is a filter if for each $x \in X$ the collection $\tau(x)$ satisfies the following conditions :*

1. $\emptyset \notin \tau(x)$
2. If $U \in \tau(x)$ and $U \subseteq V$ then $V \in \tau(x)$ (upward closed)
3. If $U, V \in \tau(x)$, then $U \cap V \in \tau(x)$ (closed under finite intersection)

Definition 3.2.5. *Let $X = (X, \tau_1, ...)$ and $Y = (Y, \sigma_1, ...)$ be n-frames. Then the function $f\colon X \to Y$ is called bounded morphism if*

1. f is surjective
2. $\forall x \in X\, \forall U \in \tau_i(x) : f(U) \in \sigma_i(f(x))$
3. $\forall x \in X\, \forall V \in \sigma_i(f(x))\, \exists U \in \tau_i(x) : f(U) \subseteq V$

The neighborhood frame in Example 3.2.3 is a filter.

Corollary 3.2.6. *Let $X = (X, \tau_1, ...)$ and $Y = (Y, \sigma_1, ...)$ be n-frames and $f : X \to Y$ a bounded morphism. Then*

$$Log(X) \subseteq Log(Y)$$

Proof. The proof is by structural induction. □

In topology, we defined horizontal and vertical topologies for products of topological spaces. In the following, we will define a similar construction for neighborhood frames.

Definition 3.2.7. *Let $\mathcal{X} = (X, \tau_1)$ and $\mathcal{Y} = (Y, \tau_2)$ be two n-frames. Then the product of these two frames is a n-2-frame and is defined as follows :*

$$\mathcal{X} \times_n \mathcal{Y} = (X \times Y, \tau_1', \tau_2')$$
$$\tau_1'(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$
$$\tau_2'(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

Additionally, we say the full product of n-frames $\mathcal{X} \times_n^+ \mathcal{Y}$ is :

$$\mathcal{X} \times_n^+ \mathcal{Y} = (X \times Y, \tau_1', \tau_2', \tau) \text{ where}$$
$$\tau(x, y) = \{U \subseteq X \times Y \mid \exists W \in \tau_1(x)\, \exists V \in \tau_2(y) : W \times V \subseteq U\}$$

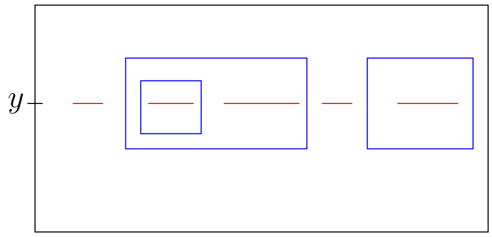

Figure 5: This is an illustration for τ_1' with a fix point (x, y). The black box is $X \times Y$. The red lines represents the combinations of any neighbourhood of x with y. The blue rectangles are the subsets of $X \times Y$ s.t. the condition is fulfilled.

Definition 3.2.8. *For two unimodal logics L_1 and L_2 we define the n-product of them as follows :*

$$L_1 \times_n L_2 = Log(\{\mathcal{X} \times_n \mathcal{Y} \mid \mathcal{X} \in nV(L_1) \text{ and } \mathcal{Y} \in nV(L_2)\})$$

In similar way, we can define $L_1 \times_n^+ L_2$ with three topologies.

Proposition 3.2.9 ([8])**.** *For two unimodal logics L_1 and L_2 it holds :*

$$L_1 \otimes L_2 \subseteq L_1 \times_n L_2$$

Definition 3.2.10. *Let $\mathcal{L}_{\Box,\Box_1,\Box_2}$ be a modal language with three modal operators $\Box, \Box_1$ and $\Box_2$. We define the T-neighborhood product logic TNL as the least set of formulas in $L_{\Box,\Box_1,\Box_2}$ containing all axioms $T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$, and closed under modus ponens, substitution and $\Box-, \Box_1-$ and $\Box_2-$ necessitation.*

After introducing the necessary notation, we can now formally state our main research question as follows: Does the following equivalence hold?

$$TNL = T \times_n^+ T$$

The main idea is to introduce an infinite depth and branching tree called $T_{\omega,\omega,\omega[rn]}$ and show $Log(T_{\omega,\omega,\omega[rn]}) = TNL$. Then we use the idea in [2] to use the pseudo-infinite paths construction to build an n-frame called $N_\omega(T_{\omega[rn]})$ and construct a bounded morphism-like map from $N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]})$ to $T_{\omega,\omega,\omega[rn]}$. We will introduce the notation later.

4 Completeness result for TNL

We introduce a frame called $T_{\omega,\omega,\omega[rn]}$. We will show TNL is sound and complete w.r.t $T_{\omega,\omega,\omega[rn]}$. The idea is to pick a class of frame $\mathcal{C}$ s.t $Log(\mathcal{C}) = TNL$ and then show TNL has the finite model property. In the end, we will use an unravelling technique to show completeness.

Definition 4.0.1. *We say for a modal logic Λ has the finite model property (FMP) if for every formula ϕ that is not provable in Λ, is falsifiable in a finite model.*

Proposition 4.0.2. *The logic $T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$ has FMP.*

Proof. The idea is to pick a class of frame $\mathcal{C}$, where TNL is sound and complete with respect to and then show by filtration the FMP.

Let $\mathcal{C} = \{F \mid F \Vdash TNL\}$. Obviously, TNL is sound w.r.t $\mathcal{C}$. Completeness can be shown by using the Sahlqvist Theorem. We recall T = K + $\Box_i p \to p$ for $i \in \{1, 2, \epsilon\}$. These axioms and $\Box p \to \Box_1 p \wedge \Box_2 p$ are Sahlqvist formulas. By Sahlqvist, we have TNL is complete w.r.t $\{F \mid F \Vdash \forall x\, R_i(x, x)$ and $F \Vdash \forall x \forall y(R_1(x, y) \vee R_2(x, y)) \to R(x, y)\}$.

Now assume a formula ϕ is not derivable from TNL. By completeness we get ϕ is falsifiable in a model M and a world w. Hence, $M, w \Vdash \neg\phi$. Now we build the set subf($\neg\phi$) which denotes the closed subformulas set of $\neg\phi$. Let $M^s = (W_\Sigma, R^s, R^s_1, R^s_2, V^s)$ be the smallest filtration of M through subf($\neg\phi$) and V^s is the standard valuation on W_Σ. By Filtration theorem, they preserve truth. It remains to show $F^s = (W_\Sigma, R^s, R^s_1, R^s_2) \in \mathcal{C}$. For that, every point must have an edge to itself in every relation and it must hold that $R^s_1, R^s_2 \subseteq R^s$. For the first one, we show this for R^s because the rest is similar. Pick $w' \in [w] \in W_\Sigma$. Because M is based on a frame $F \in \mathcal{C}$, we have $w'Rw'$. By the defintion of smallest filtration, we have that $[w']R^s[w']$. For the second one, we show only for R^s_1 because it is the same for R^s_2. Pick [w],[v] s.t $[w]R^s_1[v]$. By defintion of smallest filtration there are points $w' \in [w]$ and $v' \in [v]$ s.t $w'R_1v'$. Furthermore, it holds $R_1 \subseteq R$, so $w'Rv'$. By smallest filtration, we get $[w']R^s[v']$. Because $w' \equiv w$ and $v' \equiv v$, we have $[w'] = [w]$ and $[v'] = [v]$. It follows $[w]R^s[v]$.

$\square$

Now we define $T_{\omega,\omega,\omega[rn]}$ which is the infinitely branching and infinite-depth tree with three reflexive and non-transitive relations. We also introduce a simpler version of it called $T_{\omega[rn]}$, which is similar to the above but has only one relation. We introduce this simpler structure for two main reasons: We will use it later in our proof; and second, we will show that the logic T is sound and complete w.r.t. $T_{\omega[rn]}$. The technique we use there can be generalized and will be used to prove the main result of this chapter.

Definition 4.0.3. *Let $T_{\omega[rn]}$ (r = reflexive, n = non-transitive) denote the infinite branching and infinite depth tree, which is reflexive and non-transitive. Formally the tree can be defined as : $T_{\omega[rn]} = (W, R)$ where $W = \mathbb{N}^*$,*

$$sRt \text{ iff } \exists u \in \mathbb{N} \cup \{\epsilon\} : s \cdot u = t$$

The $T_{\omega,\omega,\omega[rn]}$ tree has three relations with infinite branching, infinite depth and we have $R_1, R_2 \subseteq R$. Before characterizing it, we say $\mathbb{N}_1^*$ is the set of finite number combinations which has subscript "1" to denote these numbers relate to R_1.
(examples : $2_1, 4231123_1$, $32_1\epsilon 45_1\epsilon 9_1 = 32459_1$). The similar holds for $\mathbb{N}_2^*$.

Now let $T_{\omega,\omega,\omega[rn]} = (W, R, R_1, R_2)$ where $W = (\mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2)^*$,

$$sRt \text{ iff } \exists u \in \mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_1t \text{ iff } \exists u \in \mathbb{N}_1 \cup \{\epsilon\} : s \cdot u = t$$

$$sR_2t \text{ iff } \exists u \in \mathbb{N}_2 \cup \{\epsilon\} : s \cdot u = t$$

where $s, t \in W$. The "$\cdot$" operator acts here as a concatenation operator.

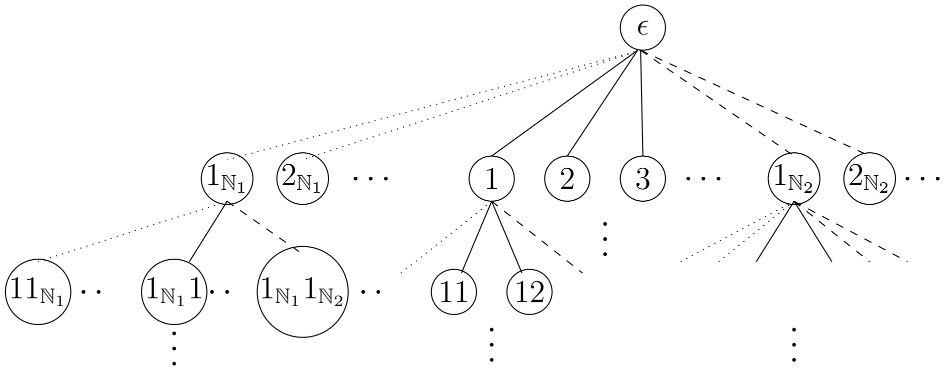

Figure 6: This is $T_{\omega,\omega,\omega[rn]}$. The dotted lines on each node are from R_1 and the dashed lines represents R_2. The whole tree is R.

Proposition 4.0.4. *T is sound and complete w.r.t $T_{\omega[rn]}$.*

Proof. Sound is clear. For completeness, we use the well known fact that T has FMP. This means, $T = Log\{F \mid F \Vdash T\}$ where F is a finite frame. We can pick such a finite frame F and it suffices to find a surjective bounded morphism f from T_w to F. This would imply $Log(T_{w[rn]}) \subseteq T$.
Now, let $F = (W', R')$ be such a finite rooted frame with root w. We define inductively an assignment of nodes of F to the nodes of T_ω. For the base case, we assign w to the root of T_ω. The induction step looks like the following : Assume a point $x \in T_\omega$ has been assigned to a point $u \in F$ but the successors of x has no assignment. Let $s_1, s_2, ..., s_k$ be successors of u (k denotes amount of successors and $k \geq 1$ because reflexivity guarantees us at least one successor). For $n \geq 1, n \in \mathbb{N}, i \in \{1, ..., k\}$ we assign s_i to the $(n * i)$th-successor of x. This means we are assigining the successors alternatingly.
Now we check for f the conditions of bounded morphism. First condition : Let $x, y \in T_\omega$ s.t xRy and $f(x) = s$. But then, y will be assigned to a successor point of s. Hence, $f(x)R'f(y)$. Second condition : Suppose $f(x)R't$ and $f(x) = s$. Since t is a successor of s and $f(x) = s$, then a successor of x, say y, gets the assignment t.

□

Proposition 4.0.5. *TNL is sound and complete w.r.t* $T_{\omega,\omega,\omega[rn]}$.

Proof. For soundness, we have that $T_{\omega,\omega,\omega[rn]} \Vdash \Box p \to \Box_1 p \wedge \Box_2 p$, because by defintion we have $R_1, R_2 \subseteq R$. The rest is clear. For completeness, we use the fact *TNL* has FMP. Let $F = (W', R', R'_1, R'_2)$ be a finite rooted frame with root w and $F \Vdash TNL$. We define inductively an assignment similar to Proposition 4.0.4. We assign w to the root of $T_{\omega,\omega,\omega[rn]}$. For induction step we start by only assigining points from R_1 to R'_1 and R_2 to R'_2. After that, the remaining points will be assigned to the points in R'. The procedure works similar as described previously.
We will only check the conditions for R and R'. For R_1, R'_1 and R_2, R'_2, we can argue as before. We pick any xRy with $f(x) = s$. If we also have xR_1y or xR_2y, then it follows $f(x)R'f(y)$ because $R_1, R_2 \subseteq R$ and $R'_1, R'_2 \subseteq R'$. Else, the successor of s was assigned to y, so $f(x)R'f(y)$.
Let $f(x)R't$ and $f(x) = s$. If we have $f(x)R'_1t$ or $f(x)R'_2t$, then the second condition follows because $R_1, R_2 \subseteq R$. Else, we have that t is a successor of $f(x) = s$, and a successor of x was assigned to t. □

5 Logic of product of topological spaces with three topologies

In this chapter, we will reprove the logic of the product of two topological spaces equipped horizontal, vertical and standard product topologies is $S4 \otimes S4 \otimes S4 + \Box p \to \Box_1 p \wedge \Box_2 p$. To begin, we first present the single-modality version, as it is easier to understand and the core idea of the proof is based on this simpler case. We will define the binary tree T_2 and view it as an Alexandroff space. We use completeness of $S4$ w.r.t. T_2, it suffices to find a bisimulation between T_2 and the topological space $\mathbb{Q}$. A similar approach will then be applied to the multimodal case.

Definition 5.0.1. *Let T_2 be the infinite binary tree with reflexive and transitive descendant relation.*
Formally it is defined as follows : $T_2 = (W, R)$ where $W = \{0,1\}^*$,

$$sRt \text{ iff } \exists u \in W^* : s \cdot u = t$$

The $T_{6,2,2}$ tree (similar to $T_{\omega,\omega,\omega[rn]}$) is the infinite six branching tree, where all nodes of $T_{6,2,2}$ is R-related, the first two R_1-related and the last two R_2-related. Formally we can define this tree as follows : $T_{6,2,2} = (W, R, R_1, R_2)$, where $W = \{0,1,2,3,4,5\}^*$,

$$sRt \text{ iff } \exists u \in \{0,1,2,3,4,5\}^* : s \cdot t = u$$

$$sR_1t \text{ iff } \exists u \in \{0,1\}^* : s \cdot t = u$$

$$sR_2t \text{ iff } \exists u \in \{5,6\}^* : s \cdot t = u$$

where $s, t \in W$ and "$\cdot$" is the concatenation-operator.

Theorem 5.0.2. *S4 is complete with respect to T_2.*

Proof. The idea is to use the fact S4 has finite model property. By that we can pick a finite rooted S4-frame and then show that this frame is the bounded morphic image of T_2. For details see [12]. □

Theorem 5.0.3. *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to $\mathbb{Q}$.*

Proof. For a proof see e.g [13, Page 217, Theorem 2]. □

Our strategy is as follows. We use completeness of $S4$ w.r.t. T_2, view T_2 as an Alexandroff space, define a dense subset X of $\mathbb{Q}$ without endpoints, and then establish a topo-bisimulation between X and T_2. This will allow us to transfer counterexamples from T_2 to X, which by Cantor's theorem is order-isomorphic and hence homeomorphic to $\mathbb{Q}$.
Now let us define X as $X = \bigcup_{n \in \mathbb{N}} X_n$ where $X_0 = \{0\}$ and

$$X_{n+1} = X_n \cup \{x - \frac{1}{3^n}, x + \frac{1}{3^n} \mid x \in X_n\}$$

Lemma 5.0.4. *For $n > 0$ and $x, y \in X_n, x \neq y$ we have : $|x - y| \geq \frac{1}{3^{n-1}}$*

Proof. The proof is by induction on n. For the base case, if $n = 1$, then $X = 0, 1, -1$. For induction step, assume $u, v \in X_{n-1}$ with $u \neq v$. By induction hypothesis, it holds $|u - v| \geq \frac{1}{3^{n-2}}$. Suppose we pick $x = u + \frac{1}{3^{n-1}}$ and $y = v - \frac{1}{3^{n-1}}$. Hence, $x, y \in X_n$. But then, $|x - y| = \left|(u + \frac{1}{3^{n-1}}) - (v - \frac{1}{3^{n-1}})\right| = \left|u - v + \frac{2}{3^{n-1}}\right| \geq \frac{1}{3^{n-1}}$. If $u - v$ is positive, then the inequality follows immediately. Now assume $u - v$ is negative. By indcution hypothesis we get $u - v \leq -\frac{1}{3^{n-2}}$. So the worst case is, when $u - v$ gets $-\frac{1}{3^{n-2}}$. But then, $\left|-\frac{1}{3^{n-2}} + \frac{2}{3^{n-1}}\right| = \frac{1}{3^{n-1}}$. □

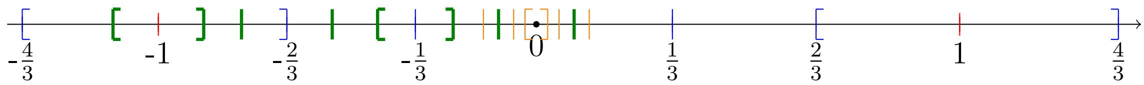

Figure 7: This is an excerpt of X. Some points are visualized as brackets to distinguish them from nearby points at the same level.

In Figure 7, each colour represents a different level. The points in red are from X_1, blue points from X_2, green points from a part of X_3 and orange from a part of X_4. We can observe that, the distance between a point and its source is getting exponentially smaller.

It follows from Lemma 5.0.4 that $(X, <)$ is a countable, dense, and linear ordered without endpoints. By Cantor's Theorem, X is homeomorphic to $\mathbb{Q}$. It also follows that for each $x \in X$ with $x \neq 0$ there exists an index n_x with $x \in X_{n_x}$ but $x \notin X_{n_x - 1}$. But also, we can find a unique $y \in X_{n_x-1}$ with $x = y - \frac{1}{3^{n_x-1}}$ or $x = y + \frac{1}{3^{n_x-1}}$. Furthermore, the basis for the open intervals for any $x \in X$ will be $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ where n_x is the index where x gets produced.

Now we define f from X onto T_2 by recursion. If $x = 0$, then we let $f(0)$ be the root r of T_2. If $x \neq 0$, then $x \in X_{n_x} - X_{n_x-1}$ and we let

$$f(x) = \begin{cases} \text{the left successor of f(y)} & \text{if } x = y - \frac{1}{3^{n_x-1}} \\ \text{the right successor of f(y)} & \text{if } x = y + \frac{1}{3^{n_x-1}} \end{cases}$$

Proposition 5.0.5. *We claim, f is open and continuous.*

Proof. We define the basis for the Alexandroff topology on T_2 as $\{B_t\}_{t \in T_2}$ where $B_t = \{s \in T_2 \mid tRs\}$. To show f is open, we pick an X-interval $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ and show $f((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})) = B_{f(x)}$.
$\subseteq$: If we pick $x \in ((x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}))$, then obviously $f(x)Rf(x)$. Assume we pick a point $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ but $y \neq x$. Then we have $n_y > n_x$ and hence $f(x)Rf(y)$.
$\supseteq$: Assume $f(x)Rt$. We will show by induction on the length k between $f(x)$ and t s.t there exists $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ s.t $f(y) = t$. Base case : If $k = 0$, then we have $t = f(x)$ and we can pick $y = x$. For induction step, assume it holds for $f(x)Rt'$ with length n, $f(y) = t'$ and $t'Rt$. For y, there must exists an index $n_y > n_x$ s.t. $y \in X_{n_y} - X_{n_y-1}$. Now

let $y' = y - \frac{1}{3^{n_y}}$ or $y' = y + \frac{1}{3^{n_y}}$. It holds that, $n'_y > n_y > n_x$, so $y' \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$. By defintion of f, $f(y')$ is either left or right successor of $f(y)$ and by $f(y) = t'Rt$, we get $f(y') = t$. It follows f is open.
To show f is continuous, it suffices to show that for each $t \in T_2$, the f-inverse image of B_t is open. Let $x \in f^{-1}(B_t)$. Then $tRf(x)$. As argued before, we can find an open interval in X s.t. $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)}$. By $tRf(x)$, we get $B_{f(x)} \subseteq B_t$. Hence, we found an open interval $I = (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$, s.t. $I = f^{-1}(B_{f(x)}) \subseteq f^{-1}(B_t)$. This holds because we can pick $y \in I$. By applying f, we get $f(y) \in B_{f(x)} \subseteq B_t$. But then $y \in f^{-1}(B_t)$. By that fact, we can follow with Remark 3.1.2, that $f^{-1}(B_t)$ is open and hence f is continuous.

□

To complete the proof, if $S4 \nvdash \phi$, then by Theorem 5.0.2, there is a valuation v on T_2 s.t. $(T_2, v), r \not\models \phi$. We define a valuation ξ on X by $\xi(p) = f^{-1}(v(p))$. Since f is continuous and open, $f(0) = r$ and by the choice of the valuation, we have that 0 and r are topo-bisimilar. By Proposition 3.1.13 it follows $(X, \xi), 0 \not\models \phi$. Because X is homeomorphic to $\mathbb{Q}$, we obtain ϕ refutable on $\mathbb{Q}$.

Definition 5.0.6. *Let $\mathcal{L}_{\Box,\Box_1,\Box_2}$ be a modal language with three modal operators $\Box, \Box_1$ and $\Box_2$. We define the topological product logic TPL as the least set of formulas in $\mathcal{L}_{\Box,\Box_1,\Box_2}$ containing all axioms $S4 \otimes S4 \otimes S4 + \Box p \to \Box_1 p \wedge \Box_2 p$, and closed under modus ponens, substitution and $\Box$-, $\Box_1$- and $\Box_2$-necessitation.*

Theorem 5.0.7. *TPL is complete w.r.t. $T_{6,2,2}$.*

Proof. A generalization of Theorem 5.0.2 can be seen on [6]. □

Now we are ready to show TPL is complete w.r.t. $\mathbb{Q} \times \mathbb{Q}$ with three topologies. The idea is similar as previous shown. We view $T_{6,2,2}$ as equipped with three Alexandroff topologies defined from R, R_1 and R_2. It suffices to show that there exists a 3-topo-bisimulation between $T_{6,2,2}$ and $X \times X = X'$ where each X is as defined in Theorem 5.0.3.

Theorem 5.0.8 ([1])**.** *TPL is complete w.r.t. $\mathbb{Q} \times \mathbb{Q}$.*

Proof. We define h from $X \times X$ onto $T_{6,2,2}$ by recursion following the inducitve definition of X. If $(x, y) = (0, 0)$, then $h(0, 0)$ is the root r of $T_{6,2,2}$. If $(x, y) \neq (0, 0)$, then as argued in Lemma 5.0.4, there exists $n_{(x,y)}$ with $(x, y) \in X'_{n_{(x,y)}} - X'_{n_{(x,y)}-1}$ and that there exists a unique $(u, v) \in X'_{n_{(x,y)}-1}$ such that $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v)$ or $(x, y) = (u, v \pm \frac{1}{3^{n_{(x,y)}-1}})$ or $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v \pm \frac{1}{3^{n_{(x,y)}-1}})$.
As visiualized in Figure 8, we can see as an example X'_0, X'_1 and an excerpt of X'_2. For clarity, we illustrated some points of X'_1 as a line. We clearly can see, that any point has his own "square", where the points within it can be only build from its center point. This is because, the "square size" gets exponentially smaller with each level, due to its construction. That means, in each iteration they can not intersect.

Now we define h as

$$h(x,y) = \begin{cases} \text{The left } R_1\text{-successor of } h(u,v) & \text{if } (x,y) = (u - \frac{1}{3^{n(x,y)-1}}, v) \\ \text{The right } R_1\text{-successor of } h(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n(x,y)-1}}, v) \\ \text{The left } R_2\text{-successor of } h(u,v) & \text{if } (x,y) = (u, v - \frac{1}{3^{n(x,y)-1}}) \\ \text{The right } R_2\text{-successor of } h(u,v) & \text{if } (x,y) = (u, v + \frac{1}{3^{n(x,y)-1}}) \\ \text{The first remaining successor of } h(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n(x,y)-1}}, v + \frac{1}{3^{n(x,y)-1}}) \\ & \text{or } (x,y) = (u - \frac{1}{3^{(x,y)-1}}, v - \frac{1}{3^{(x,y)-1}}) \\ \text{The last remaining successor of } h(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n(x,y)-1}}, v - \frac{1}{3^{n(x,y)-1}}) \\ & \text{or } (x,y) = (u - \frac{1}{3^{(x,y)-1}}, v + \frac{1}{3^{(x,y)-1}}) \end{cases}$$

It is still left to prove, that h is open and continuous w.r.t. all three topologies. We

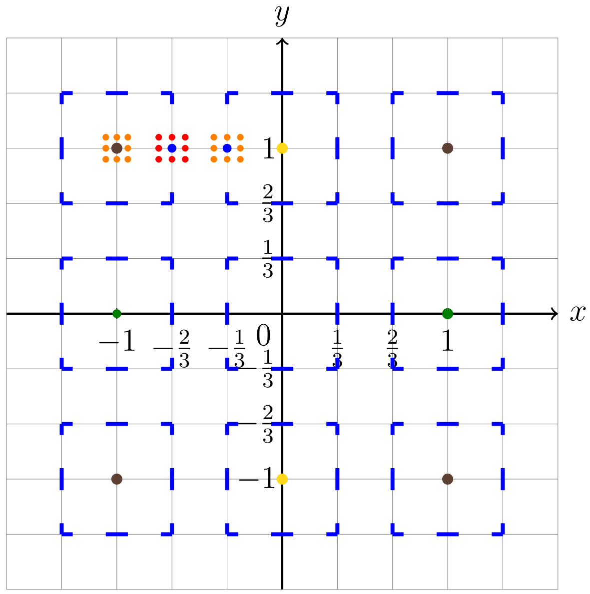

Figure 8: This is an excerpt of $X \times X$.

will show it for τ_1 and τ. For τ_2 it is similar as for τ_1.

First we observe the basis for τ_1 is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \mid (x,y) \in X \times X\}$$

and for τ is

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}}) \mid (x,y) \in X \times X\}$$

We also observe that a basis for the topology on $T_{6,2,2}$ from R_1 is $\{B^1_t\}_{t \in T_{6,2,2}}$ where $B^1_t = \{s \in T_{6,2,2} \mid tR_1s\}$ and from R is $\{B_t\}_{t \in T_{6,2,2}}$ where $B_t = \{s \in T_{6,2,2} \mid tRs\}$. The

following arguments are carried over from Proposition 5.0.5.
Now, to see that g is open w.r.t. τ_1, we pick an open set $(x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times \{y\} \in \tau_1$. In similar way, we can show $g((x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)}$. Thus g is open. To show g is continuous, it suffices to show that for each $t \in T_{6,2,2}$, the g-inverse image of B^1_t belongs to τ_1. Let $(x,y) \in g^{-1}(B^1_t)$. Then $tR_1g(x,y)$. Hence, it holds for (x,y) that $g((x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)}$. By $tR_1g(x,y)$, we get $B^1_{g(x,y)} \subseteq B^1_t$. Thus, we found a neighborhood U of (x,y) s.t. $U = ((x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times \{y\}) \subseteq g^{-1}(B^1_t)$, implying g is continuous.
To see g is open w.r.t. τ, we pick pick an open set $(x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times (y-\frac{1}{3^{n(x,y)}}, y+\frac{1}{3^{n(x,y)}})$. A similar argument as in Proposition 5.0.5, yields us $g((x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times (y-\frac{1}{3^{n(x,y)}}, y+\frac{1}{3^{n(x,y)}})) = B_{g(x,y)}$.
To show g is continuous, it suffices to show for each $t \in T_{6,2,2}$ that $g^{-1}(B_t) \in \tau$. Let $(x,y) \in g^{-1}(B_t)$. Then $tRg(x,y)$. But then, it holds that : $g((x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times (y-\frac{1}{3^{n(x,y)}}, y+\frac{1}{3^{n(x,y)}})) = B_{g(x,y)} \subseteq B_t$. Thus, we found a $U \in \tau$ s.t. $U = (x-\frac{1}{3^{n(x,y)}}, x+\frac{1}{3^{n(x,y)}}) \times (y-\frac{1}{3^{n(x,y)}}, y+\frac{1}{3^{n(x,y)}}) \subseteq g^{-1}(B_t)$.

□

To complete the proof, if $TPL \nvdash \phi$, then by Theorem 5.0.8, there is a valuation υ on $T_{6,2,2}$ such that $(T_{6,2,2}, \upsilon), r \nvDash \phi$. Now we define a valuation ξ on $X \times X$ by $\xi(p) = h^{-1}(\upsilon(p))$. Since h is continuous and open w.r.t. all three topologies and $h(0,0) = r$, we have that $(0,0)$ and r are 3-topo-bisimilar. Therefore, $(X \times X, \xi), (0,0) \nvDash \phi$. Similar as in Theorem 5.0.3, we argue that $X \times X$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ w.r.t. all three topologies, it follows that ϕ is refutable on $\mathbb{Q} \times \mathbb{Q}$.

Corollary 5.0.9. *In the language $L_{\Box,\Box_1,\Box_2}$, TPL is the logic of products of arbitrary topologies.*

Proof. Let X and Y be arbitrary topological spaces. By the interpretation of $\Box$, it follows $Log(X), Log(Y) \supseteq S4$. Let's consider the full product of the spaces $X \times^+_t Y$. Hence, with theorem 5.0.8, it is easy to see that $Log(X \times^+_t Y) \supseteq S4 \times^+_t S4 = TPL$.

□

6 Products of neighbourhood frames, completeness results

We now turn our focus to neighbourhood semantics, and in particular, we revisit the main theorem from the paper "Modal Logic of Products of Neighbourhood Frames" [2]. This work established that, for any $L_1, L_2 \in \{D, D4, T, S4\}$ we have $L_1 \otimes L_2 = L_1 \times_n L_2$. Our goal is to reprove this theorem within our framework. We then adapt the key ideas to prove the second part of our result. We recall, that in Chapter 4 we showed: $TNL = Log(T_{\omega,\omega,\omega[rn]})$. Using the techniques in this chapter, we can establish the following : $TNL = T \times_n^+ T$ which is the desired outcome of our contribution.
We start by introducing the necessary trees as Kripke frame. We begin by introducing the necessary trees as Kripke frames. Then, we define the constructions $N(F)$ and $N_\omega(F)$ which form the foundation of our proofs. Using these, we construct bounded morphisms to derive the main results. The chapter ends with the corresponding completeness theorems.

6.1 Kripke tree frames, natural neighborhood version and their logic

Definition 6.1.1. *Let A be a nonempty set.*

$$A^* = \{a_1...a_k \mid a_i \in A\}$$

is the set of all finite sequences of elements from A, including the empty sequence Λ. Elements from A^* will be denoted as $\vec{a}$. The length of a sequence $\vec{a} = a_1...a_k$ is k (also $l(\vec{a}) = k$) and the length of Λ is 0 ($l(\Lambda) = 0$). Concatenation is denoted by " $\cdot$ " : $(a_1...a_k) \cdot (b_1...b_l) = \vec{a} \cdot \vec{b} = a_1...a_k b_1...b_l$.

Definition 6.1.2. *Let A be a nonempty set. We define an infinite frame $F_{in}[A] = (A^*, R)$ s.t for $\vec{a}, \vec{b} \in A^*$*

$$\vec{a} R \vec{b} \Leftrightarrow \exists x \in A \, (\vec{b} = \vec{a} \cdot x)$$

Furthermore we define :

$$F_{rn}[A] = (A^*, R^r), \text{ where } R^r = R \cup Id \text{ (reflexive closure)}$$

$$F_{it}[A] = (A^*, R^*), \text{ where } R^* = \bigcup_{i=1}^{\infty} R^i \text{ (transitive closure)}$$

$$F_{rt}[A] = (A^*, R^{r*})$$

where "t" stands for transtive, "n" for non-transitive, "r" for reflexive and "i" for irreflexive.
For now, we will use the following notion to generalize : $F_{\xi\eta}$ where $\xi \in \{i, r\}$ and $\eta \in \{t, n\}$

Definition 6.1.3. *Let* $F_1 = F_{\xi_1\eta_1}[A] = (A^*, R_1)$ *and* $F_2 = F_{\xi_2\eta_2}[B] = (B^*, R_2)$*, where* $\xi_1, \xi_2 \in \{i, r\}$ *and* $\eta_1, \eta_2 \in \{t, n\}$*. Furthermore, we assume* $A = \{a_1, a_2, ...\}$ *and* $B = \{b_1, b_2, ...\}$ *with* $A \cap B = \emptyset$*. Then we define the frame* $F_1 \otimes F_2 = (W, R_1', R_2')$ *as follows :*

$$W = (A \cup B)^*$$

$$\vec{x} R_1' \vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in A^* \text{such that } \Lambda R_1 \vec{z}$$

$$\vec{x} R_2' \vec{y} \Leftrightarrow \vec{y} = \vec{x} \cdot \vec{z} \text{ for some } \vec{z} \in B^* \text{such that } \Lambda R_2 \vec{z}$$

Proposition 6.1.4 ([14], [15])**.** *Let* F_1 *and* F_2 *be as in Definition 6.1.3 . Then*

$$Log(F_1 \otimes F_2) = Log(F_1) \otimes Log(F_2)$$

Proposition 6.1.5 ([17])**.** *Let* $F_{in} = F_{in}[\mathbb{N}], F_{rn} = F_{rn}[\mathbb{N}], F_{it} = F_{it}[\mathbb{N}]$ *and* $F_{rt} = F_{rt}[\mathbb{N}]$*. Then the following holds:*

$$Log(F_{in}) = D$$

$$Log(F_{rn}) = T$$

$$Log(F_{it}) = D4$$

$$Log(F_{rt}) = S4$$

Definition 6.1.6. *Let* $F = (W, R)$ *be a Kripke frame. We define an n-frame* $N(F) = (W, \tau)$ *as follows. For any* $w \in W$ *we have :*

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}$$

Lemma 6.1.7. *Let* $F = (W, R)$ *be a Kripke frame. Then*

$$Log(F) = Log(N(F))$$

Proof. The proof is by structural induction. □

6.2 Main Construction

In the following, we construct a useful neighborhood frame called $N_\omega[F]$ based on a tree frame F. We will show, that the construction has the same logic as the orignal frame. At the end, we construct a bounded morphism from the product of $N_\omega[F]$ frames to the fusion of frames F.

Definition 6.2.1. *Let* $F = (A^*, R) = F_{\xi\eta}[A]$ *and* $0 \notin A$*. We define "pseudo-infinite" sequences*

$$X = \{a_1 a_2 a_3 ... \mid a_i \in A \cup \{0\} \text{ and } \exists N \forall k \geq N : a_k = 0\}$$

Furthermore, we define $f_F : X \to A^*$ *to be the function, that deletes all zeros.*

Example : Say $12034002340^\omega \in X$ *(*0^ω *denotes infinitely many zeros). Then* $f_F(12034002340^\omega) = 1234234$

Definition 6.2.2. *Let $F = (A^*, R) = F_{\xi\eta}[A]$ and $0 \notin A$. Assume the function f_F and the set X as defined before. For $\alpha \in X$ such that $\alpha = a_1 a_2 ...$ we define*

$$\begin{aligned} st(a) &= min\{N \mid \forall k \geq N : a_k = 0\} \\ a\mid_k &= a_1 a_2 ... a_k \\ U_k(\alpha) &= \{\beta \mid f_F(\alpha) R f_F(\beta) \text{ and } \alpha\mid_m = \beta\mid_m, \text{ where } m = max(k, st(\alpha))\} \end{aligned}$$

Remark : Let $\alpha \in X$ with $st(\alpha) = n$. Then we have that $U_n(\alpha) = U_j(\alpha)$ for any $j \leq n$

Lemma 6.2.3. *$U_k(\alpha) \subseteq U_m(\alpha)$, whenever $k \geq m$.*

Proof. Let $\beta \in U_k(\alpha)$. Since $\alpha\mid_k = \beta\mid_k$ and $k \geq m$, we have $\alpha\mid_m = \beta\mid_m$. It follows, $\beta \in U_m(\alpha)$. □

Definition 6.2.4. *Due to Lemma 6.2.3 the sets $U_n(\alpha)$ forms a filter base. So we can define :*

$$\tau(\alpha) \text{ is a filter with base } \{U_n(\alpha) \mid n \in \mathbb{N}\}$$

$$N_\omega = (X, \tau) \text{ is the n-frame based on } F$$

In the following, we will show $f_F : N\omega(F) \to N(F)$ is a bounded morphism. We begin by proving a small lemma, which helps us to show it.

Lemma 6.2.5. *Let $F = (A^*, R) = F_{\xi\eta}[A]$. Based on that, let $N_\omega(F) = (X, \tau)$, $N(F) = (A^*, \sigma)$ and $f_F : N\omega(F) \to N(F)$. Then for any $m \in \mathbb{N}$ and $x \in X$ with $x = a_1 a_2 ...$ we have*

$$f_F(U_m(x)) = R(f_F(x))$$

Proof. $\subseteq$: Let $f_F(\alpha) \in f_F(U_m(x))$ with $\alpha \in U_m(x)$. By defintion of $U_m(x)$, we get $f_F(\alpha) \in R(f_F(x))$.

$\supseteq$: We pick $\vec{a} \in R(f_F(x))$. We have to find $\beta \in U_m(x)$ s.t $f_F(\beta) = \vec{a}$. We assume R is irreflexive and non-transitive. The other cases are similar.
Because $\vec{a} \in R(f_F(x))$, there must exists $c \in A$ such that $\vec{a} = f_F(x) \cdot c$. We construct $\beta = x\mid_m \cdot c\, 0^\omega$. Hence, $f(\beta) = f_F(x\mid_m) \cdot f_F(c)$ and because $0 \notin A$ we get $f_F(x\mid_m) \cdot c = \vec{a}$. □

Lemma 6.2.6. *Let $F = (A^*, R) = F_{\xi\eta}[A]$. Then $f_F : N_\omega(F) \to N(F)$ is a bounded morphism.*

Proof. From now on this proof we will omit the subindex in f_F. Let $N_\omega(F) = (X, \tau)$ and $N(F) = (A^*, \sigma)$.

For surjectivity, we pick any $\vec{x} \in A^*$. But then, $\vec{x}\, 0^\omega \in X$. Hence, $f(\vec{x}\, 0^\omega) = \vec{x}$.
For the next condition, assume that $x \in X$ and $U \in \tau(x)$. We need to prove that

$f(U) \in \sigma(f(x))$. That means $R(f(x)) \subseteq f(U)$. Because $U \in \tau(x)$, there is a m such that $U_m(x) \subseteq U$. By Lemma 6.2.5 we have $f(U_m(x)) = R(f(x))$. It follows,

$$R(f(x)) = f(U_m(x)) \subseteq f(U)$$

Assume $x \in X$ and V is a neighborhood of x, i.e $R(f(x)) \subseteq V$. We need to prove that there exists $U \in \tau(x)$, such that $f(U) \subseteq V$. As U we pick $U_m(x)$ for any $m \in \mathbb{N}$. By Lemma 6.2.5 we get $f(U_m(x)) = R(f(x))$. Hence,

$$f(U_m(x)) = R(f(x)) \subseteq V$$

$\square$

Corollary 6.2.7. *For frame $F = F_{\xi\eta}[A]$ we have $Log(N_\omega(F)) \subseteq Log(F)$.*

Proof. It follows from Corollary 3.2.5, Lemma 6.1.7 and Lemma 6.2.6

$$Log(N_\omega(F)) \subseteq Log(N(F)) = Log(F)$$

$\square$

Proposition 6.2.8. *Let $F_{in} = F_{in}[\mathbb{N}], F_{rn} = F_{rn}[\mathbb{N}], F_{it} = F_{it}[\mathbb{N}]$ and $F_{rt} = F_{rt}[\mathbb{N}]$. Then*

$$Log(N_\omega(F_{in})) = D$$
$$Log(N_\omega(F_{rn})) = T$$
$$Log(N_\omega(F_{it})) = D4$$
$$Log(N_\omega(F_{rt})) = S4$$

Proof. In all these cases, the inclusion from left to right follows from Proposition 6.1.5 and Corollary 6.2.7. Now the converse direction. Assume $X = (X, \tau)$.
It is easy to check that $X \vDash D$ iff for each $x \in X : \emptyset \notin \tau(x)$. For $N_\omega(F_{in})$ and $N_\omega(F_{it})$ this holds.
It is easy to check that $X \vDash T$ iff we have $x \in U \in \tau(x)$ for any x and U. For $N_\omega(F_{rn})$ and $N_\omega(F_{rt})$ this holds.
Now we check $X \vDash 4$ iff for each $U \in \tau(x) : \{y \mid U \in \tau(y)\} \in \tau(x)$.
$\supseteq$: Let $x \in X$ and assume $X, x \vDash \Box p$. That means there exists $U \in \tau(x)$ s.t $U \subseteq V(p)$. By assumption we have $S = \{y \mid U \in \tau(y)\} \in \tau(x)$. But then $X, x \vDash \Box\Box p$ because we can just pick the set S.
$\subseteq$: By contradiction, assume there exists a U s.t $\{y \mid U \in \tau(y)\} \notin \tau(x)$. Let $X, x \vDash \Box p$ where $V(p) = U$. If $X, x \vDash \Box\Box p$ then it must be the case that
$S = \{y \in X \mid X, y \vDash \Box p\} \in \tau(x)$. $X, y \vDash \Box p$ means $U \in \tau(y)$. That means $S = \{y \in X \mid U \in \tau(y)\}$. But by assumption $S \notin \tau(x)$. Hence, $X, x \nvDash \Box\Box p$. But thats a contradiction. This also holds for $N_\omega(F_{it})$ and $N_\omega(F_{rt})$ because we have for any $y \in U_m(x)$ and $k \geq m : U_k(y) \subseteq U_m(x)$.

$\square$

Now we construct the product neighborhood frames $N_\omega(F_1) \times_n N_\omega(F_2)$ and construct a bounded morphism to $N(F_1 \otimes F_2)$. The idea is inspired by the monomodal version.

Assume $F_1 = (A^*, R_1) = F_{\eta_1\xi_1}[A]$ and $F_2 = (B^*, R_2) = F_{\eta_2\xi_2}[B]$ with $A \cap B = \emptyset$, $A = \{a_1, a_2, a_3, ...\}$ and $B = \{b_1, b_2, b_3, ...\}$. Consider the product of n-frames $F_1' = (X_1, \tau_1) = N_\omega(F_1)$ and $F_2' = (X_2, \tau_2) = N_\omega(F_2)$ is

$$X = (X_1 \times X_2, \tau_1', \tau_2') = N_\omega(F_1) \times_n N_\omega(F_2)$$

Furthermore, we have $F_1 \otimes F_2 = ((A \cup B)^*, R_1', R_2')$ as defined in Definition 6.1.3. We consider the neighborhood version

$$N(F_1 \otimes F_2) = ((A \cup B)^*, \sigma_1', \sigma_2')$$

Now we define $g : X_1 \times X_2 \to (A \cup B)^*$ as follows. For $(\alpha, \beta) \in X_1 \times X_2$ with $\alpha = a_1a_2...$ and $\beta = b_1b_2...$ we define $g(\alpha, \beta)$ to be the finite sequence which we get after eliminating all zeros from the infinite sequence $a_1b_1a_2b_2...$

Example : Let $\alpha = 012340^\omega$ and $\beta = 0ab00e0^\omega$. Then $g(\alpha, \beta) = 1a2b34e$.

In order to show g is bounded morphism, we show a small lemma, which helps us to prove it.

Lemma 6.2.9. *Let X and $N(F_1 \otimes F_2)$ be as defined before and $(\alpha, \beta) \in X_1 \times X_2$. Then for any $m > max\{st(\alpha), st(\beta)\}$ we have*

$$R_1'(g(\alpha, \beta)) = g(U_m(\alpha) \times \{\beta\})$$

.

Proof. We will show both direction by assuming $F_1 = F_{in}[A]$. For $F_{it}[A], F_{rt}[A]$ and $F_{rn}[A]$ it is similar.
$\subseteq$: Let $\vec{w} \in R_1'(g(\alpha, \beta))$. By definition we get, there exists $\vec{c} \in A^*$ where $\vec{w} = g(\alpha, \beta) \cdot \vec{c}$ and $\Lambda R_1 \vec{c}$. Because R_1 is irreflexive and non-transitive, we get $\vec{c} \in A$. We construct (ζ, β) where $\zeta \in U_m(\alpha)$ and $g(\zeta, \beta) = \vec{w}$. For that, we can take $\zeta = \alpha \mid_m \cdot \vec{c}0^\omega$. Obviously, $\zeta \in U_m(\alpha)$. Because $m \in max\{st(\alpha), st(\beta)\}$, we have $g(\alpha \mid_m, \beta \mid_m) = g(\alpha, \beta)$. Hence, $g(\zeta, \beta) = g(\alpha \mid_m, \beta \mid_m) \cdot g(\vec{c}0^\omega, 0^\omega) = g(\alpha \mid_m, \beta \mid_m) \cdot \vec{c} = \vec{w}$.

$\supseteq$: Assume $\zeta \in U_m(\alpha)$. We have to show $g(\zeta, \beta) \in R_1'(g(\alpha, \beta))$. Because R_1 is irreflexive and non-transitive, it suffices to find a $\vec{d} \in A$ s.t. $g(\alpha, \beta) \cdot \vec{d} = g(\zeta, \beta)$. By choosing m is maximal and $\zeta \mid_m = \alpha \mid_m$, we have $g(\zeta_m, \beta_m) = g(\alpha, \beta)$. We also know $f_F(\alpha) R_1 f_F(\zeta)$, that means there exists a $\vec{d} \in A$ s.t. $f_F(\alpha) \cdot \vec{d} = f_F(\zeta)$. This $\vec{d}$ must appear at a point after ζ_m. We can follow $g(\zeta, \beta) = g(\alpha, \beta) \cdot \vec{d}$. Hence, $g(\zeta, \beta) \in R_1'(g(\alpha, \beta))$.

Remark : If $m \leq max\{st(\alpha), st(\beta)\}$, then we cannot gurantee the equality. Assume R_1 as before and $\alpha = 123000^\omega$ and $\beta = d0b0a0^\omega$. Lets pick $m = 3$ and $\zeta = 123100^\omega$.

Then we have $g(\alpha, \beta) = 1d23ba$ and $g(\zeta, \beta) = 1d23b1a$. Obviously, we don't have $g(\zeta, \beta) \in R_1'(g(\alpha, \beta))$.
Furthermore, we can show $R_2'(g(\alpha, \beta)) = g(\alpha \times U_m(\beta))$ similar as above.

□

Lemma 6.2.10. *Function $g : X \to N(F_1 \otimes F_2)$ is a bounded morphism.*

Proof. Let $\vec{z} = z_1 z_2 ... z_n \in (A \cup B)^*$. Define for $i \leq n$:

$$x_i = \begin{cases} z_i, & \text{if } z_i \in A; \\ 0, & \text{if } z_i \notin A. \end{cases} \qquad y_i = \begin{cases} z_i, & \text{if } z_i \in B; \\ 0, & \text{if } z_i \notin B. \end{cases}$$

Let $\alpha = x_1 x_2 ... x_n \, 0^\omega$ and $\beta = y_1 y_2 ... y_n \, 0^\omega$. Then $g(\alpha, \beta) = \vec{z}$. Hence, g is surjective.

For the next conditions we check only for τ_1' and σ_1. The other case is similar. Assume $(\alpha, \beta) \in X_1 \times X_2$ and $U \in \tau_1'(\alpha, \beta)$. We have to show $g(U) \in \sigma_1(g(\alpha, \beta))$. That means $R_1'(g(\alpha, \beta)) \subseteq g(U)$. Pick a $m > max\{st(\alpha), st(\beta)\}$ s.t. $U_m(\alpha) \times \{\beta\} \subseteq U$. We can pick such m, because $U \in \tau_1'(\alpha, \beta)$. So there exists $U_k(\alpha) \in \tau(\alpha)$ s.t. $U_k(\alpha) \times \{\beta\} = U$. If $k > max\{st(\alpha), st(\beta)\}$ then we are done. Else, by Lemma 6.2.3, we have for any $n \geq k : U_n(\alpha) \subseteq U_k(\alpha)$. So we can lift the k til we reach m. Then we use Lemma 6.2.9 to get the following :

$$R_1'(g(\alpha, \beta)) = g(U_m(\alpha) \times \{\beta\}) \subseteq g(U)$$

For the last condition we assume $(\alpha, \beta) \in X_1 \times X_2$ and $V \in \sigma_1(g(\alpha, \beta))$ (or rather $R_1'(g(\alpha, \beta)) \subseteq V$). We need to prove there exists $U \in \tau_1'(\alpha, \beta)$, such that $g(U) \subseteq V$. As U we take $U_m(\alpha) \times \{\beta\}$ for some $m > max\{st(\alpha), st(\beta)\}$. Hence, by Lemma 6.2.9 we get :

$$g(U_m(\alpha) \times \{\beta\}) = R_1'(g(\alpha, \beta)) \subseteq V$$

□

6.3 Adapting the key ideas

Now we will show $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \to N(T_{\omega,\omega,\omega[rn]})$ is a bounded morphism. Let $(T_{\omega[rn]})_1 = (\mathbb{N}_1, R_1)$ and $(T_{\omega[rn]})_2 = (\mathbb{N}_2, R_2)$ as defined in Definition 4.0.3. Then we take the n-frames $N_\omega(T_{\omega[rn]})_1 = (X_1, \tau_1)$ and $N_\omega(T_{\omega[rn]})_2 = (X_2, \tau_2)$. For the proof, we will omit the subscripts after the frame.

Let's consider the full product of the n-frames :

$$N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) = (X_1 \times X_2, \tau_1', \tau_2', \tau)$$

We say $N(T_{\omega,\omega,\omega[rn]}) = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*, \sigma_1, \sigma_2, \sigma)$ where the tree $T_{\omega,\omega,\omega[rn]} = ((\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^* R_1', R_2', R)$ is as defined in Definition 4.0.3.

In order to define the bounded morphism, we have to fix a bijection first. Let $h : \mathbb{N}_1 \times \mathbb{N}_2 \to \mathbb{N}$ be a bijection.
Next, we define function $k : (\mathbb{N}_1 \cup \{0\}) \times (\mathbb{N}_2 \cup \{0\}) \to \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{0\}$ as follows :

$$f(a,b) = \begin{cases} a, & \text{if } b = 0; \\ b, & \text{if } a = 0; \\ h(a,b), & \text{otherwise.} \end{cases}$$

Let $(\alpha, \beta) \in X_1 \times X_2$ with $\alpha = a_1a_2...$ and $\beta = b_1b_2....$ We define

$$g'(\alpha, \beta) = f(a_1, b_1)f(a_2, b_2)...$$

At last, we define $g(\alpha, \beta)$ as $g'(\alpha, \beta)$ but removing all zeros.

Example 6.3.1. *Assume h is the Cantor pairing function $h(m,n) = \frac{(m+n)(m+n+1)}{2} + n$ and $(\alpha, \beta) = (12_{\mathbb{N}_1}021_{\mathbb{N}_1}0^\omega, 1_{\mathbb{N}_2}047_{\mathbb{N}_2}0^\omega) \in X_1 \times X_2$.*

Then $g'(\alpha, \beta) = f(1,1)f(2,0)f(0,4)f(2,7)f(1,0)f(0,0)... = h(1,1)24\,h(2,7)10^\omega = 424\,52\,10^\omega$. Hence, $g(\alpha, \beta) = 424\,52\,1$.

Lemma 6.3.2. *Function $g : N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \to N(T_{\omega,\omega,\omega[rn]})$ is a bounded morphism.*

Proof. Let $\vec{z} = z_1z_2...z_n \in (\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N})^*$. We define for $i \leq n$

$$x_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_1; \\ a_1, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases} \qquad y_i = \begin{cases} z_i, & \text{if } z_i \in \mathbb{N}_2; \\ b_2, & \text{if } z_i = h(a_1, b_2); \\ 0, & \text{other} \end{cases}$$

Let $\alpha = x_1x_2...x_n\,0^\omega$ and $\alpha = y_1y_2...y_n\,0^\omega$. But then $g(\alpha, \beta) = \vec{z}$. Hence, g is surjective.
For the next conditions we show it for τ and σ. For τ_1', σ_1 and τ_2', σ_2 we can argue similar as in Lemma 5.2.9 and 5.2.10.
First we need to show, $R(g(\alpha, \beta)) = g(U_m(\alpha) \times U_m(\beta))$ for $m > max\{st(\alpha), st(\beta)\}$.
$\supseteq$: Pick $\zeta \in U_m(\alpha), \upsilon \in U_m(\beta)$. Show $g(\zeta, \upsilon) \in R(g(\alpha, \beta))$. By definition, we need to find a $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$ s.t. $g(\alpha, \beta) \cdot \vec{c} = g(\zeta, \upsilon)$. By choosing $m > max\{st(\alpha), st(\beta)\}$, $\alpha \mid_m = \zeta \mid_m$ and $\beta \mid_m = \upsilon \mid_m$, we get $g(\alpha, \beta) = g(\zeta_m, \upsilon_m)$. Furthermore, we have $f_F(\alpha)R_1f_F(\zeta)$ and $f_F(\beta)R_2f_F(\upsilon)$, that means $\exists \vec{d} \in \mathbb{N}_1 \cup \{\epsilon\}$ and $\exists \vec{e} \in \mathbb{N}_2 \cup \{\epsilon\}$ s.t. $f_F(\alpha) \cdot \vec{d} = f_F(\zeta)$ and $f_F(\beta) \cdot \vec{e} = f_F(\upsilon)$. So, $\vec{d}$ must appear at a point after $\zeta \mid_m$ and $\vec{e}$ in $\upsilon \mid_m$. But then we have $g(\zeta, \upsilon) = g(\alpha, \beta) \cdot g(\vec{d}, \vec{e})$.
Now there are four cases :

$$\begin{cases} g(\vec{d}, \vec{e}) \in \mathbb{N}_1, & \text{if } \vec{d} \in \mathbb{N}_1 \;\&\; \vec{e} = \epsilon \\ g(\vec{d}, \vec{e}) \in \mathbb{N}_2, & \text{if } \vec{d} = \epsilon \;\&\; \vec{e} \in \mathbb{N}_2 \\ g(\vec{d}, \vec{e}) \in \{\epsilon\}, & \text{if } \vec{d} = \vec{e} = \epsilon \\ g(\vec{d}, \vec{e}) \in \mathbb{N}, & \text{other} \end{cases}$$

Hence, $g(\zeta, \upsilon) \in R(g(\alpha, \beta))$.

$\subseteq$: Pick $\vec{w} \in R(g(\alpha, \beta))$. We will construct $\zeta \in U_m(\alpha)$ and $\upsilon \in U_m(\beta)$ s.t. $g(\zeta, \upsilon) = \vec{w}$. By definition there exists a $\vec{c} \in \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N} \cup \{\epsilon\}$ s.t. $g(\alpha, \beta) \cdot \vec{c} = \vec{w}$. Depending on the $\vec{c}$, we build $\zeta = \alpha \mid_m \cdot \vec{d} 0^\omega$ and $\upsilon = \beta \mid_m \cdot \vec{e} 0^\omega$ where

$$\begin{cases} \vec{d} = \vec{c}, \vec{e} = 0, & \text{if } \vec{c} \in \mathbb{N}_1; \\ \vec{d} = 0, \vec{e} = \vec{c}, & \text{if } \vec{c} \in \mathbb{N}_2; \\ \vec{d} = \vec{e} = 0 & \text{if } \vec{c} \in \{\epsilon\} \\ \vec{d} = \vec{\alpha}, \vec{e} = \vec{b} \text{ where } h(\vec{\alpha}, \vec{\beta}) = \vec{c} & \text{if } \vec{c} \in \mathbb{N}\,(\vec{\alpha} \in \mathbb{N}_1, \vec{\beta} \in \mathbb{N}_2); \end{cases}$$

We have $\zeta \in U_m(\alpha)$ and $\upsilon \in U_m(\beta)$. Furthermore, because of $g(\zeta \mid_m, \upsilon \mid_m) = g(\alpha, \beta)$, it holds that $g(\zeta, \upsilon) = g(\alpha, \beta) \cdot g(\vec{d}, \vec{e}) = g(\alpha, \beta) \cdot \vec{c} = \vec{w}$.

Now to check the conditions, we assume $x \in X_1$ and $y \in X_2$ and $U \in \tau(x, y)$. We need to prove $g(U) \in \sigma(g(x, y))$ In other words, show $R(g(x, y)) \subseteq g(U)$. We pick a $m > max\{st(x), st(y)\}$ s.t. $U_m(x) \times U_m(y) \subseteq U$. But then we get

$$R(g(x, y)) = g(U_m(x) \times U_m(y)) \subseteq g(U)$$

Assume $x \in X_1$ and $x \in X_2$ and $R(g(x, y)) \subseteq V$. We need to find a $U \in \tau(x, y)$, s.t. $f(U) \subseteq V$. For U we pick $U_m(x) \times U_m(y)$ for some $m > \{st(x), st(y)\}$. But then

$$g(U_m(x) \times U_m(y)) = R(g(x, y)) \subseteq V$$

□

6.4 Completeness Results

Lemma 6.4.1. *Let $T_{\omega,\omega,\omega[rn]}$ be as in Definition 4.0.3. Then*

$$Log(T_{\omega,\omega,\omega[rn]}) = Log(N(T_{\omega,\omega,\omega[rn]}))$$

Proof. The proof is by structural induction. □

Corollary 6.4.2. *Let $F_1 = (A^*, R_1) = F_{\xi_1,\eta_1}[A]$ and $F_2 = (B^*, R_2) = F_{\xi_2,\eta_2}[B]$. Then*

$$Log(N_\omega(F_1) \times_n N_\omega(F_2)) \subseteq Log(F_1) \otimes Log(F_2)$$

. It follows from Corollary 3.2.5, Proposition 6.1.4 and Lemma 6.2.10.

Corollary 6.4.3. *Let $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$. Then*

$$Log(N_\omega(F_1) \times_n N_\omega(F_2)) = Log(F_1) \otimes Log(F_2)$$

Proof. The left to right inclusion follows from Corollary 6.4.2

To prove right to left, due to Proposition 6.1.5 and Proposition 6.2.8 we have $Log(N_\omega(F_i)) = Log(F_i)$ $(i \in \{1,2\})$. Due to Proposition 3.2.8 we get $Log(N_\omega(F_1)) \otimes Log(N_\omega(F_2)) \subseteq Log(N_\omega(F_1)) \times_n Log(N_\omega(F_2))$. Because $N_\omega(F_1) \times_n N_\omega(F_2)$ is a frame in the n-product, we get $Log(N_\omega(F_1)) \times_n Log(N_\omega(F_2)) \subseteq Log(N_\omega(F_1) \times_n N_\omega(F_2))$.

□

Corollary 6.4.4. *Let $N_\omega(T_{\omega[rn]})_1 = F_1$ and $N_\omega(T_{\omega[rn]})_2 = F_2$ as defined before. Then*

$$T \times_n^+ T \subseteq Log(T_{\omega,\omega,\omega[rn]})$$

It follows from Corollary 3.2.5, Proposition 6.2.8, Lemma 6.3.2, Lemma 6.4.1.

Proof. First, it is the case that $Log(F_1) \times_n^+ Log(F_2) \subseteq Log(F_1 \times_n^+ F_2)$. We can argue as before. By Lemma 6.3.2 and Lemma 6.4.1 it holds $Log(F_1 \times_n^+ F_2) \subseteq Log(N(T_{\omega,\omega,\omega[rn]})) = Log(T_{\omega,\omega,\omega[rn]})$. But by Proposition 6.2.8 we have $Log(F_1) = Log(F_2) = T$. Hence, $Log(F_1) \times_n^+ Log(F_2) = T \times_n^+ T \subseteq Log(T_{\omega,\omega,\omega[rn]})$. □

Theorem 6.4.5 ([2])**.** *Let $L_1, L_2 \in \{S4, D4, D, T\}$. Then*

$$L_1 \times_n L_2 = L_1 \otimes L_2$$

Proof. Right to left is by Proposition 3.2.8.
For left to right, assume $L_1 = Log(F_1)$ and $L_2 = Log(F_2)$ for some $F_1, F_2 \in \{F_{in}, F_{rn}, F_{it}, F_{rt}\}$.

It follows from Corollary 3.2.5 Lemma 6.2.10 :
$L_1 \times_n L_2 = Log(N_\omega(F_1)) \times_n^+ Log(N_\omega(F_2)) \subseteq Log(N_\omega(F_1) \times_n^+ N_\omega(F_2)) \subseteq Log(F_1 \otimes F_2) = Log(F_1) \otimes Log(F_2) = L_1 \otimes L_2$ □

Theorem 6.4.6. *Let $N_\omega(T_{\omega[rn]})_1 = F_1'$ and $N_\omega(T_{\omega[rn]})_2 = F_2'$ as defined before. Then*

$$T \times_n^+ T = T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$$

Proof. For left to right it is easy to check the axioms. We pick any frame F_1 and F_2 s.t. $F_1 \vDash T$ and $F_2 \vDash T$. We consider $F_1 \times_n^+ F_2$ the full product and let $(x,y) \in F_1 \times_n^+ F_2$. Assume $F_1 \times_n^+ F_2, (x,y) \vDash \Box_1 p$. By definition, there exists a $U \in \tau_1'(x,y)$ s.t $U \subseteq V(p)$. But we have $U \supseteq V \times \{y\}$ for some $V \in \tau(x)$. By assumption, $F_1 \vDash T$, so we can follow $x \in V$. Hence, $(x,y) \in V(p)$. For $\Box_2 p \to p$ it's done similar.
Let's check $\Box p \to p$. Assume $F_1 \times_n^+ F_2, (x,y) \vDash \Box p$. By definition there exists a $U \in \tau(x,y)$ s.t. $U \subseteq V(p)$. But $U \supseteq V \times W$ for some $V \in \tau_1(x)$ and $W \in \tau_2(y)$. Because of the assumption, we must have $x \in V$ and $y \in W$. Hence, $(x,y) \in V(p)$. It follows $F_1 \times_n^+ F_2, (x,y) \vDash p$.
Now we check the extra axiom. Assume $F_1 \times_n^+ F_2, (x,y) \Vdash \Box p$. By definition, there exists a $U \subseteq V(p)$, where $U \supseteq V \times W$ with $V \in \tau_1(x)$ and $W \in \tau_2(y)$. By assumption, we get $y \in W$. That means $U \in \tau_1'(x,y)$ because $U \supseteq V \times \{y\}$. The same argument we can apply to $\tau_2'(x,y)$. Hence, $F_1 \times_n^+ F_2, (x,y) \vDash \Box_1 p \wedge \Box_2 p$. The rest is clear.

Now for the other direction, we apply Corollary 6.4.4 to get $T \times_n^+ T \subseteq Log(T_{\omega,\omega,\omega[rn]})$. By Proposition 4.0.5 we have $Log(T_{\omega,\omega,\omega[rn]}) = TNL$. Hence,

$$T \times_n^+ T \subseteq T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$$

.

$\Box$

Remark 6.4.7. *Let $L \in \{D, D4\}$. Then it is not guaranteed that*

$$L \times_n^+ L = L \otimes L \otimes L + \Box p \to \Box_1 p \wedge \Box_2 p$$

This can be shown by a counterexample. We will only do it for D because for $D4$ it's done similar. Assume $F_1 = N_\omega(T_{\omega[in]})_1 = (X, \tau_1)$ and $F_2 = N_\omega(T_{\omega[in]})_2 = (Y, \tau_2)$. By Proposition 6.2.8, it follows $Log(F_1) = Log(F_2) = D$. Let's consider the full product $X = F_1 \times_n^+ F_2 = (W \times V, \tau_1', \tau_2', \tau)$. Then we get $Log(X) \supseteq D \times_n^+ D$. Now let $(x, y) \in X$ $X, (x, y) \models \Box p$ and $V(p) = U \times V$ where $U \in \tau_1(x)$ and $V \in \tau_2(y)$. By construction of $N_\omega(T_{\omega[rn]})$, $x \notin U$ and $y \notin V$, because $f_F(x)$ and $f_F(y)$ don't have a relation to itself in $T_{\omega[rn]}$. In order to satisfy $\Box_1 p$ (similar for $\Box_2 p$), we have to find $U' \in \tau_1'(x, y)$ s.t. $U' \subseteq V(p)$. But this is not possible because any U' contains only elements, where the second coordinate is y. It follows $X, (x, y) \not\models \Box p \to \Box_1 p \wedge \Box_2 p$. Hence, the extra axiom is not in $Log(X) \supseteq D \times_n^+ D$.

7 Conclusion

In this work, we reproved the theorems from "Multimodal Logics of Products of Topologies" and "Modal Logic of products of neighborhood frames". We showed the logic of the product of two arbitrary topological spaces is $S4 \otimes S4 \otimes S4 + \Box p \to \Box_1 p \wedge \Box_2 p$. The idea was to find a 3-topo-bisimulation between $T_{6,2,2}$ and $X \times X$ which is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$. Additionally, by switching to neighborhood frames, it holds that for any $L_1, L_2 \in \{D, D4, T, S4\}$: $L_1 \otimes L_2 = L_1 \times_n L_2$. We constructed from a Kripke frame F a special neighborhood frame called $N_\omega(F)$ and then found a bounded morphim from the product of these constructed frame to the fusion of two frames. Inspired by these ideas, we first introduced a Kripke frame called $T_{\omega,\omega,\omega[rn]}$ where we proved FMP and used unravelling to show $Log(T_{\omega,\omega,\omega[rn]}) = TNL$. Afterwoods, we constructed a bounded morphism from $N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]})$ onto $N(T_{\omega,\omega,\omega[rn]})$ to finish the proof.

There are tons of ways to continue the research. We may consider the logic K and ask what is the logic of $K \times_n^+ K$? What's about different combinations like : $D \times_n^+ T$, $T \times_n^+ K$, $S4 \times_n^+ D4$ ...? We can also try to find out whether for logic Λ with $T \subseteq \Lambda \subseteq S4$ the following holds :

$$\Lambda \otimes \Lambda \otimes \Lambda + \Box p \to \Box_1 p \wedge \Box_2 p = \Lambda \times_n^+ \Lambda$$

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