

1 Definition

1.1 Defintion

Let $Prop$ be a set of variables. Then a formula ϕ is defined as follows :

$$\phi ::= p \mid \perp \mid \phi \mid \phi \rightarrow \phi \mid \Box_i \phi$$

where $p \in Prop$ and \Box_i is a modal operator. Other connectives are expressed through \perp and \rightarrow and dual modal operators \Diamond_i as $\Diamond_i \phi = \neg \Box_i \neg \phi$

1.2 Defintion

A normal modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \rightarrow \psi}{\psi}$), Generalization rules ($\frac{\phi}{\Box_i \phi}$) and the following axioms

$$\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$$

K_n denotes the minimal normal modal logic with n modalities and $K = K_1$ Let L be a logic and let Γ be a set of formulas. Then $L+\Gamma$ denotes the minimal logic containing L and Γ

1.3 Definition

Let $L1$ and $L2$ be two modal logic with one modality \Box . Then the fusion of these logics are defined as follows :

$$L1 \otimes L2 = K2 + L1(\Box \rightarrow \Box_1)L2(\Box \rightarrow \Box_2)$$

The follow logics may be important

$$D = K + \Box p \rightarrow \Diamond p$$

$$T = K + \Box p \rightarrow p$$

$$D4 = D + \Box p \rightarrow \Box \Box p$$

$$S4 = T + \Box p \rightarrow \Box \Box p$$

2 Topological Space Defintion

2.1 Defintion

A topological space is a pair (X, τ) where τ is a collection of subsets of X (elements of τ are also called open sets) such that :

1. the empty set \emptyset and X are open
2. the union of an arbitrary collection of open sets is open
3. the intersection of finite collection of open sets is open

A topological model is a structure $M = (X, \tau, v)$ where (X, τ) is a topological space and v is a valuation assigning subsets of X to propositional variables.

2.2 Defintion

Let $M = (X, \tau, v)$ a topological model and $x \in X$. The satisfaction of a formula at the point x in M is defined inductively as follows : $M, x \models \Box\phi$ iff , $\exists U \in \tau$ s.t $x \in U$ and $\forall u \in U : M, u \models \phi$

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$M, x \models \Diamond\phi$ iff , $\forall U \in \tau$ s.t $x \in U$ and $\exists u \in U : M, u \models \phi$

2.3 Defintion

Let $A = (X, \chi)$ and $B = (Y, v)$ be topological spaces. The standard product topology τ is the set of subsets of $X \times Y$ such that $X \in \chi$ and $Y \in v$.

Let $N \subseteq X \times Y$. We call N horizontally open if $\forall (x, y) \in N \exists U \in \chi : x \in U$ and $U \times \{y\} \subseteq N$.

We call N vertically open if $\forall (x, y) \in N \exists V \in v : y \in V$ and $\{x\} \times V \subseteq N$

If N is H-open and V-open, then we call it HV-open.

We denote τ_1 is the set of all H-open subsets of $X \times Y$ and τ_2 is the set of all V-open subsets of $X \times Y$

3 Neighbourhood

3.1 Defintion

Let X be a non-empty set. A function $N : X \rightarrow 2^{2^X}$ is called a neighbourhood function. A pair $F = (X, N)$ is called a neighbourhood frame (or n-frame). A model based on F is a tuple (X, N, v) , where v assigns a subset of X to a variable

3.2 Defintion

Let $M = (X, N, v)$ be a neighbourhood model and $x \in X$. The truth of a formula is defined inductively as follows :

$$M, x \models \Box\phi \text{ iff } \exists V \in N(x) \forall y \in V : M, y \models \phi$$

A formula is valid in a n-model M if it is valid at all points of M ($M \models \phi$). Formula is valid in a n-frame F if it is valid in all models based on F (notation $F \models \phi$). For Logic L we write $F \models L$, if for any $\phi \in L, F \models \phi$. We define $nV(L) = \{N \mid N \text{ is an n-frame and } N \models \phi\}$.

3.3 Defintion

Let $F = (W, R)$ be a Kripke frame.