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Modal Logic of Products of Topological Spaces and Neighbourhood Frames

Dresden, September 29, 2025

Content

- Syntax for the basic modal language and semantics
 - Kripke frames
 - Topological space
 - Neighbourhood frames
- Multimodal logic and product of frames/spaces and logics
 - Multimodal logic and product of frames
 - Horizontal and Vertical topology/functions
 - Product of logics
- Main result and ideas

Basic modal language, Kripke frames and models

- Basic modal language extends classical propositional logic. Formally:

Definition

Let Prop be a set of variable. Then a formula ϕ is defined as follows:

$$\phi ::= p \mid \perp \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi$$

where \Box is a modal operator and $p \in \text{Prop}$

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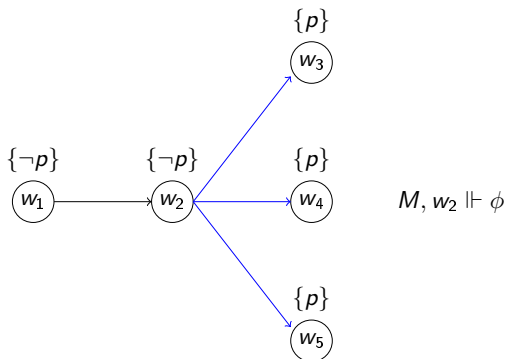
A frame $F = (W, R)$ is a pair where

- W is a non-empty set of worlds
- $R \subseteq W \times W$ is a binary relation

A model is a pair $M = (F, V)$ where V is a valuation and is of the form $V : \text{Prop} \rightarrow 2^W$

Example

- Let $\phi = \Box p$ and $M = (W, R, V)$ with $W = \{w_1, w_2, w_3, w_4, w_5\}$, $V(p) = \{w_3, w_4, w_5\}$ and $R =$



Kripke semantics

Definition

Let $M = (F, V)$ be a model and $w \in W$ a state in M . A formula being true at w is inductively defined as:

$M, w \Vdash p$	iff $w \in V(p)$
$M, w \Vdash \perp$	never
$M, w \Vdash \neg\phi$	iff not $M, w \Vdash \phi$
$M, w \Vdash \phi \vee \psi$	iff $M, w \Vdash \phi \vee M, w \Vdash \psi$
$M, w \Vdash \Box\phi$	iff $\forall v \in W : wRv \rightarrow M, v \Vdash \phi$
$M, w \Vdash \Diamond\phi$	iff $\exists v \in W : wRv \wedge M, v \Vdash \phi$

Topological space

- deals with open sets, topology

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- The intersection of finite collection of open sets is open

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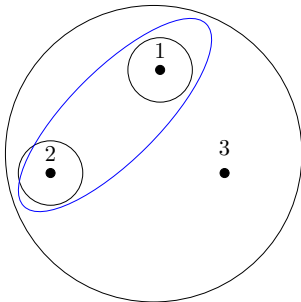
A topological model is a structure $M = (X, \tau, v)$ where (X, τ) is a topological space and v a valuation of the form $v : Prop \rightarrow 2^X$

Example

- Let (X, τ, ν) be a topological model with $X = \{1, 2, 3\}$,
 $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, $V(p) = \{1, 2\}$ and $\phi = \Box p$

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$M, 1 \Vdash \phi$

$M, 3 \nVdash \phi$

Topological semantics

Definition

Let $M = (X, \tau, v)$ be a topological model and $x \in X$ a point in M . A formula being true at x is inductively defined as:

$$M, x \models p \quad \text{iff } x \in v(p)$$

$$M, x \models \perp \quad \text{never}$$

$$M, x \models \neg\phi \quad \text{iff } M, x \not\models \phi$$

$$M, x \models \phi \vee \psi \quad \text{iff } M, x \models \phi \text{ or } M, x \models \psi$$

$$M, x \models \Box\phi \quad \text{iff } \exists U \in \tau \text{ such that } x \in U \text{ and } \forall u \in U, M, u \models \phi$$

$$M, x \models \Diamond\phi \quad \text{iff } \forall U \in \tau : \text{if } x \in U \rightarrow \exists u \in U, M, u \models \phi$$

Neighbourhood frames

- Generalize Kripke semantics
- Captures non-normal modal logics

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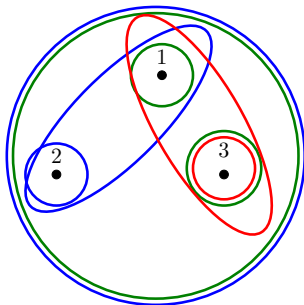
Definition

A neighbourhood frame is a pair (X, τ) where τ is a function $\tau : X \rightarrow 2^{2^X}$.
A neighbourhood model is a structure $M = (X, \tau, v)$, where v is a valuation of the form $v : Prop \rightarrow 2^X$

Example

Assume $\phi = \Box p$. Let $X = \{1, 2, 3\}$, $V(p) = \{1, 2\}$ and

$$\tau(x) = \begin{cases} 1 \rightarrow \{\{1\}, \{3\}, W\} \\ 2 \rightarrow \{\{2\}, \{1, 2\}, W\} \\ 3 \rightarrow \{\{3\}, \{1, 3\}\} \end{cases}$$



True at 1,2

Neighbourhood semantics

Definition

Let $M = (X, \tau, v)$ be a neighbourhood model and $x \in X$ a point in M . A formula being true at x is inductively defined as:

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$M, x \Vdash \neg\phi$	iff $M, x \nVdash \phi$
$M, x \Vdash \phi \vee \psi$	iff $M, x \models \phi \vee M, x \models \psi$
$M, x \Vdash \Box\phi$	iff $\exists V \in \tau(x) \forall y \in V : M, y \models \phi$

Modal logic and the logic T

Definition

A modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ($\frac{\phi(p_i)}{\phi(\psi)}$), Modus Ponens ($\frac{\phi, \phi \rightarrow \psi}{\psi}$).

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A modal logic is normal, if it contains $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ (K) and is closed under Generalization ($\frac{\phi}{\Box \phi}$)

Definition

$$T = K + \Box p \rightarrow p$$

Multimodal logic and Fusion

- Simultaneously reasoning about knowledge,time etc.

Definition

Let L_1 and L_2 be modal logics with one modality \Box . The fusion is defined as:

$$L_1 \otimes L_2 = K_2 + L_1(\Box \rightarrow \Box_1) + L_2(\Box \rightarrow \Box_2)$$

Product of frames

Definition

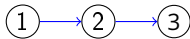
Let $F = (W, R_1)$ and $G = (V, R_2)$. We define the Kripke product on $W \times V$ as :

$(w, v)R'_1(w', v')$ iff wR_1w' and $v = v'$ (horizontal)

$(w, v)R'_2(w', v')$ iff $w = w'$ and vR_2v' (vertical)

Example

R_1

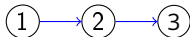


R_2

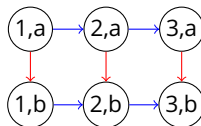


Example

R_1



R_2



Horizontal and Vertical topology

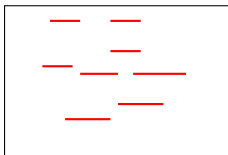
Definition

Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \nu)$ be topological spaces and $N \subseteq X \times Y$

Horizontally open: N is horizontally open iff

$$\forall (x, y) \in N \exists U \in \chi \text{ such that } x \in U \text{ and } U \times \{y\} \subseteq N.$$

τ_1 (horizontal topology) is the set of all horizontally open sets



Horizontal and Vertical topology

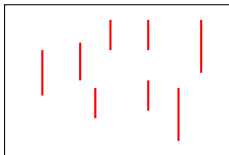
Definition

Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, v)$ be topological spaces and $N \subseteq X \times Y$

Vertically open: N is vertically open iff

$$\forall (x, y) \in N \exists V \in v \text{ such that } y \in V \text{ and } \{x\} \times V \subseteq N.$$

τ_2 (vertical topology) is the set of all vertically open sets



Standard topology

- Basis of the standard topology is τ where

$$\tau = \{U \times V \mid U \in \chi V \in v\}$$

Horizontal, Vertical and Standard functions

Definition

Let $\mathcal{X} = (X, \tau_1)$ and $\mathcal{Y} = (Y, \tau_2)$ be two n -frames. We define the full product as

$$\mathcal{X} \times_n^+ \mathcal{Y} = (X \times Y, \tau'_1, \tau'_2, \tau) \text{ where}$$

$$\tau'_1(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$

$$\tau'_2(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

$$\tau(x, y) = \{U \mid \exists W \in \tau_1(x) \exists V \in \tau_2(y) : W \times V \subseteq U\}$$

Product of logics

Definition

Let L_1 and L_2 be two unimodal logics. We define the full n -product of them as:

$$L_1 \times_n^+ L_2 = \text{Log}(\{\mathcal{X} \times_n^+ \mathcal{Y} \mid \mathcal{X} \Vdash L_1 \text{ and } \mathcal{Y} \Vdash L_2\})$$

where $\text{Log}(\mathcal{C}) = \{\phi \mid F \Vdash \phi \text{ for } F \in \mathcal{C}\}$

Main Research Question

Theorem

$$T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p = T \times_n^+ T$$

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- $T \otimes T \otimes T = K_3 + T_{\Box} + T_{(\Box \rightarrow \Box_1)} + T_{(\Box \rightarrow \Box_2)}$ is the logic with three modalities
- $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$ is the interaction axiom

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- $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$ is the interaction axiom
- Sketch the inclusion from right to left
- Proof is splitted into two parts (ideas are from [1], [2])

Sketch of the main ideas

- Pick $\mathcal{C} = \{F \mid F \Vdash T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p\}$

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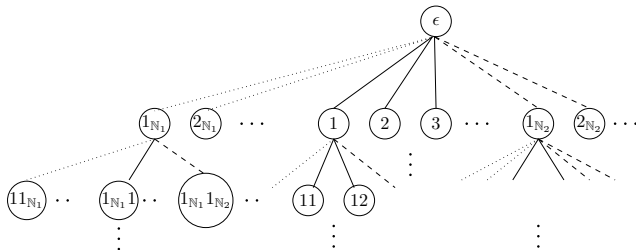
- Use **Filtration Theorem**

Sketch continue

- Construct an infinite branching and infinite depth tree with three reflexive relations ($T_{\omega, \omega, \omega[rn]}$)

Sketch continue

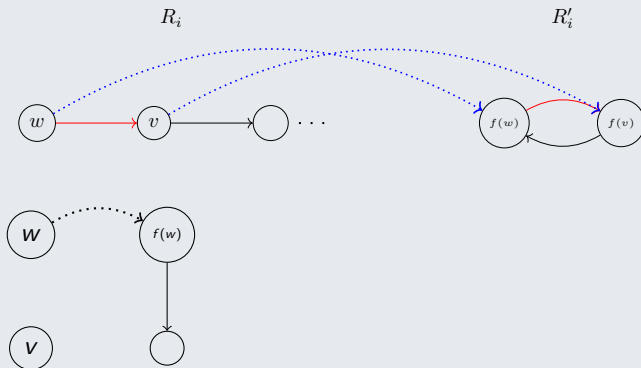
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Sketch continue

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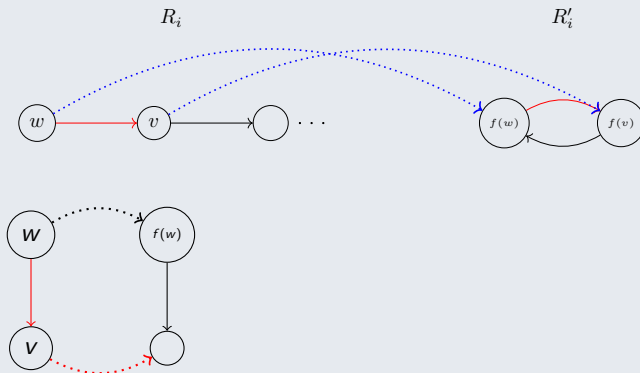
Let $F = (W, R_1, R_2, \dots)$ and $F' = (W', R'_1, R'_2, \dots)$ be two frames. A bounded morphism $f : W \rightarrow W'$ can be illustrated as:



Sketch continue

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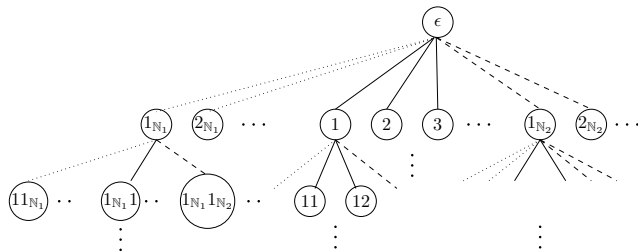


Sketch continue

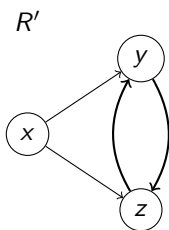
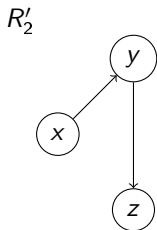
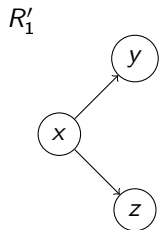
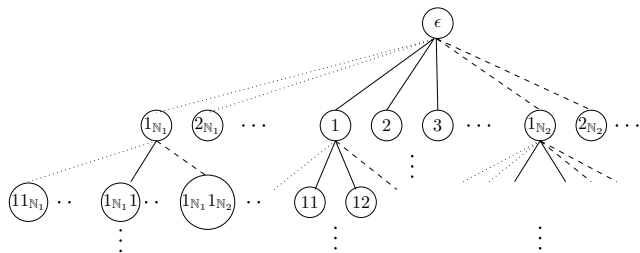
Claim

$$\text{Log}(T_{\omega, \omega, \omega[rn]}) = T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

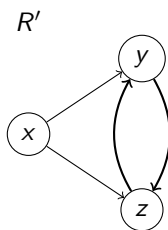
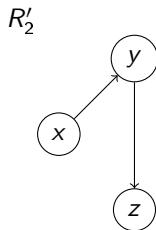
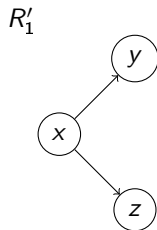
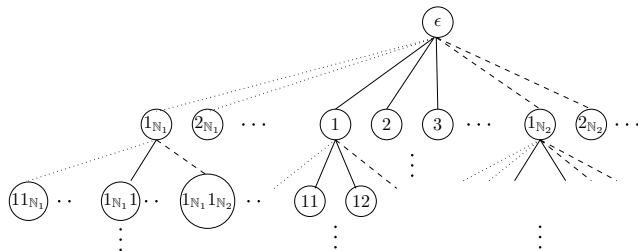
Detailed sketch



Detailed sketch



Detailed sketch



Required components

$$\text{Log}(T_{\omega,\omega,\omega[rn]}) = T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

- Introduce $T_{\omega[rn]}$ a simpler version of $T_{\omega,\omega,\omega[rn]}$

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- Introduce $T_{\omega[rn]}$ a simpler version of $T_{\omega,\omega,\omega[rn]}$
- Construct $N_{\omega}(T_{\omega[rn]})$ the neighbourhood version of $T_{\omega[rn]}$
- Show $T = \text{Log}(N_{\omega}(T_{\omega[rn]}))$

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- Construct $N_{\omega}(T_{\omega[rn]})$ the neighbourhood version of $T_{\omega[rn]}$
- Show $T = \text{Log}(N_{\omega}(T_{\omega[rn]}))$
- Construct a bounded morphism

$$N_{\omega}(T_{\omega[rn]}) \times_n^+ N_{\omega}(T_{\omega[rn]}) \rightarrow T_{\omega,\omega,\omega[rn]}$$

- With some further steps we can conclude

$$T \times_n^+ T \subseteq T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

Future works

- Many ways to continue the research
- Discover how it works for logic K
- Combine different logics for example $D \times_n^+ K, T \times_n^+ K, \dots$
- For logic Λ with $T \subseteq \Lambda \subseteq S4$, does the following hold:

$$\Lambda \otimes \Lambda \otimes \Lambda + \Box p \rightarrow \Box_1 p \wedge \Box_2 p = \Lambda \times_n^+ \Lambda$$

Conclusion

References

- [1] Johan van Benthem, Guram Bezhanishvili, Balder ten Cate, and Darko Sarenac.
Multimodal logics of products of topologies. *Studia Logica*, 84:369–392, 2006.
- [2] Andrei Kudinov. Modal logic of some products of neighbourhood frames.
In *Advances in Modal Logic, Volume 9*, pages 286–294, London, 2012. College Publications.