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# Modal Logic of Products of Topological Spaces and Neighbourhood Frames

Dresden, September 29, 2025

# Content

- Syntax for the basic modal language and semantics
  - Kripke frames
  - Topological space
  - Neighbourhood frames
- Multimodal logic and product of frames/spaces and logics
  - Multimodal logic and product of frames
  - Horizontal and Vertical topology/functions
  - Product of logics
- Main result and ideas

# Basic modal language, Kripke frames and models

- Basic modal language extends classical propositional logic. Formally:

## Definition

Let  $\text{Prop}$  be a set of variable. Then a formula  $\phi$  is defined as follows:

$$\phi ::= p \mid \perp \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi$$

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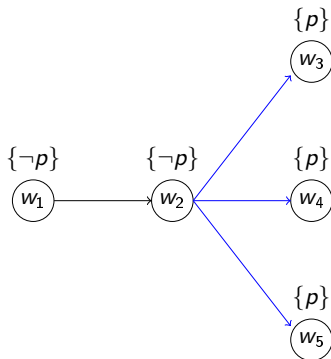
A frame  $F = (W, R)$  is a pair where

- $W$  is a non-empty set of worlds
- $R \subseteq W \times W$  is a binary relation

A model is a pair  $M = (F, V)$  where  $V$  is a valuation and is of the form  $V : \text{Prop} \rightarrow 2^W$

## Example

- Let  $\phi = \Box p$  and  $M = (W, R, V)$  with  $W = \{w_1, w_2, w_3, w_4, w_5\}$ ,  $V(p) = \{w_3, w_4, w_5\}$  and  $R =$



$M, w_2 \Vdash \phi$

# Kripke semantics

## Definition

Let  $M = (F, V)$  be a model and  $w \in W$  a state in  $M$ . A formula being true at  $w$  is inductively defined as:

$M, w \Vdash p$	iff $w \in V(p)$
$M, w \Vdash \perp$	never
$M, w \Vdash \neg\phi$	iff not $M, w \Vdash \phi$
$M, w \Vdash \phi \vee \psi$	iff $M, w \Vdash \phi \vee M, w \Vdash \psi$
$M, w \Vdash \Box\phi$	iff $\forall v \in W : wRv \rightarrow M, v \Vdash \phi$
$M, w \Vdash \Diamond\phi$	iff $\exists v \in W : wRv \wedge M, v \Vdash \phi$

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- deals with open sets, topology

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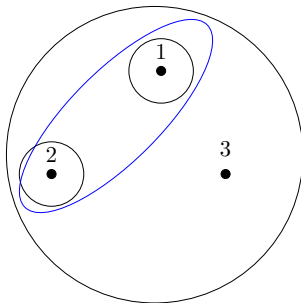
A topological model is a structure  $M = (X, \tau, v)$  where  $(X, \tau)$  is a topological space and  $v$  a valuation of the form  $v : Prop \rightarrow 2^X$

## Example

- Let  $(X, \tau, \nu)$  be a topological model with  $X = \{1, 2, 3\}$ ,  
 $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ ,  $V(p) = \{1, 2\}$  and  $\phi = \Box p$

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$M, 1 \Vdash \phi$

$M, 3 \nVdash \phi$

# Topological semantics

## Definition

Let  $M = (X, \tau, v)$  be a topological model and  $x \in X$  a point in  $M$ . A formula being true at  $x$  is inductively defined as:

$$M, x \models p \quad \text{iff } x \in v(p)$$

$$M, x \models \perp \quad \text{never}$$

$$M, x \models \neg\phi \quad \text{iff } M, x \not\models \phi$$

$$M, x \models \phi \vee \psi \quad \text{iff } M, x \models \phi \text{ or } M, x \models \psi$$

$$M, x \models \Box\phi \quad \text{iff } \exists U \in \tau \text{ such that } x \in U \text{ and } \forall u \in U, M, u \models \phi$$

$$M, x \models \Diamond\phi \quad \text{iff } \forall U \in \tau : \text{if } x \in U \rightarrow \exists u \in U, M, u \models \phi$$

# Neighbourhood frames

- Generalize Kripke semantics
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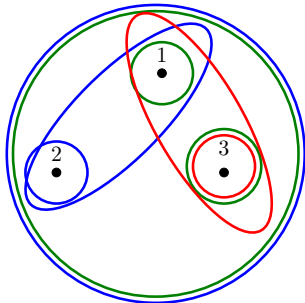
## Definition

A neighbourhood frame is a pair  $(X, \tau)$  where  $\tau$  is a function  $\tau : X \rightarrow 2^{2^X}$ .  
A neighbourhood model is a structure  $M = (X, \tau, v)$ , where  $v$  is a valuation of the form  $v : Prop \rightarrow 2^X$

## Example

Assume  $\phi = \Box p$ . Let  $X = \{1, 2, 3\}$ ,  $V(p) = \{1, 2\}$  and

$$\tau(x) = \begin{cases} 1 \rightarrow \{\{1\}, \{3\}, W\} \\ 2 \rightarrow \{\{2\}, \{1, 2\}, W\} \\ 3 \rightarrow \{\{3\}, \{1, 3\}\} \end{cases}$$



True at 1,2



# Neighbourhood semantics

## Definition

Let  $M = (X, \tau, v)$  be a neighbourhood model and  $x \in X$  a point in  $M$ . A formula being true at  $x$  is inductively defined as:

$M, x \Vdash p$	iff $x \in V(p)$
$M, x \Vdash \perp$	never
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$M, x \Vdash \phi \vee \psi$	iff $M, x \Vdash \phi \vee M, x \Vdash \psi$
$M, x \Vdash \Box\phi$	iff $\exists V \in \tau(x) \forall y \in V : M, y \Vdash \phi$

# Modal logic and the logic $T$

## Definition

A modal logic is a set of modal formulas containing all propositional tautologies, closed under Substitution ( $\frac{\phi(p_i)}{\phi(\psi)}$ ), Modus Ponens ( $\frac{\phi, \phi \rightarrow \psi}{\psi}$ ).

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A modal logic is normal, if it contains  $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$  (K) and is closed under Generalization ( $\frac{\phi}{\Box \phi}$ )

## Definition

$$T = K + \Box p \rightarrow p$$

# Multimodal logic and Fusion

- Simultaneously reasoning about knowledge,time etc.

## Definition

Let  $L_1$  and  $L_2$  be modal logics with one modality  $\Box$ . The fusion is defined as:

$$L_1 \otimes L_2 = K_2 + L_1(\Box \rightarrow \Box_1) + L_2(\Box \rightarrow \Box_2)$$

# Product of frames

- Reasoning in one dimension via horizontal and vertical relations

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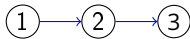
Let  $F = (W, R_1)$  and  $G = (V, R_2)$ . We define the Kripke product on  $W \times V$  as :

$(w, v)R'_1(w', v')$  iff  $wR_1w'$  and  $v = v'$  (horizontal)

$(w, v)R'_2(w', v')$  iff  $w = w'$  and  $vR_2v'$  (vertical)

# Example

$R_1$

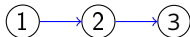


$R_2$

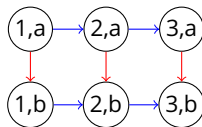


# Example

$R_1$



$R_2$





# Horizontal and Vertical topology

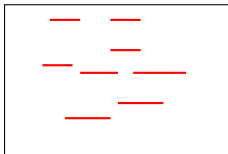
## Definition

Let  $\mathcal{X} = (X, \chi)$  and  $\mathcal{Y} = (Y, \nu)$  be topological spaces and  $N \subseteq X \times Y$

**Horizontally open:**  $N$  is horizontally open iff

$$\forall (x, y) \in N \exists U \in \chi \text{ such that } x \in U \text{ and } U \times \{y\} \subseteq N.$$

$\tau_1$  (horizontal topology) is the set of all horizontally open sets



# Horizontal and Vertical topology

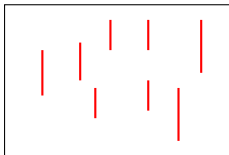
## Definition

Let  $\mathcal{X} = (X, \chi)$  and  $\mathcal{Y} = (Y, v)$  be topological spaces and  $N \subseteq X \times Y$

**Vertically open:**  $N$  is vertically open iff

$$\forall (x, y) \in N \exists V \in v \text{ such that } y \in V \text{ and } \{x\} \times V \subseteq N.$$

$\tau_2$  (vertical topology) is the set of all vertically open sets



# Standard topology

- Basis of the standard topology is  $\tau$  where

$$\tau = \{U \times V \mid U \in \chi V \in v\}$$

# Horizontal, Vertical and Standard functions

## Definition

Let  $\mathcal{X} = (X, \tau_1)$  and  $\mathcal{Y} = (Y, \tau_2)$  be two  $n$ -frames. We define the full product as

$$\mathcal{X} \times_n^+ \mathcal{Y} = (X \times Y, \tau'_1, \tau'_2, \tau) \text{ where}$$

$$\tau'_1(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$

$$\tau'_2(x, y) = \{U \subseteq X \times Y \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

$$\tau(x, y) = \{U \mid \exists W \in \tau_1(x) \exists V \in \tau_2(y) : W \times V \subseteq U\}$$

# Product of logics

## Definition

Let  $L_1$  and  $L_2$  be two unimodal logics. We define the full  $n$ -product of them as:

$$L_1 \times_n^+ L_2 = \text{Log}(\{\mathcal{X} \times_n^+ \mathcal{Y} \mid \mathcal{X} \Vdash L_1 \text{ and } \mathcal{Y} \Vdash L_2\})$$

where  $\text{Log}(\mathcal{C}) = \{\phi \mid F \Vdash \phi \text{ for } F \in \mathcal{C}\}$

# Main Research Question

## Theorem

$$T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p = T \times_n^+ T$$

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- $T \otimes T \otimes T = K_3 + T_{\Box} + T_{(\Box \rightarrow \Box_1)} + T_{(\Box \rightarrow \Box_2)}$  is the logic with three modalities
- $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$  is the interaction axiom

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- $\Box p \rightarrow \Box_1 p \wedge \Box_2 p$  is the interaction axiom
- Sketch the inclusion from right to left
- Proof is splitted into two parts (ideas are from [1], [2])



# Sketch of the main ideas

- Pick  $\mathcal{C} = \{F \mid F \Vdash T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p\}$

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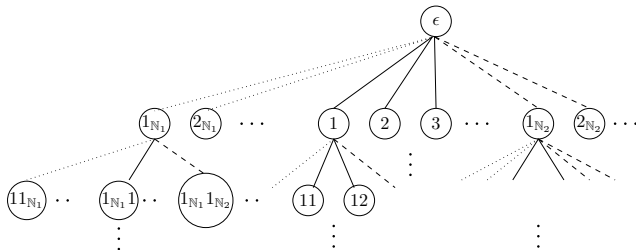
- Use **Filtration Theorem**

## Sketch continue

- Construct an infinite branching and infinite depth tree with three reflexive relations ( $T_{\omega, \omega, \omega[rn]}$ )

# Sketch continue

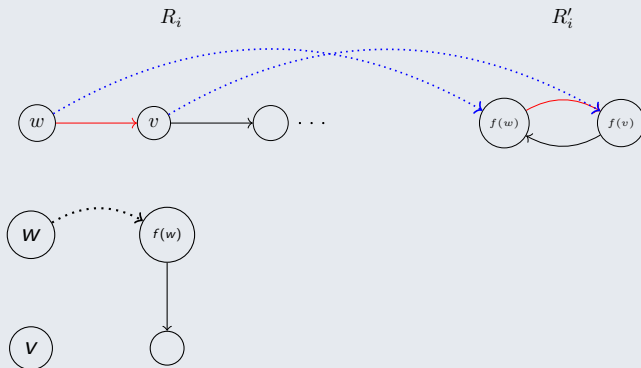
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# Sketch continue

## Definition

Let  $F = (W, R_1, R_2, \dots)$  and  $F' = (W', R'_1, R'_2, \dots)$  be two frames. A bounded morphism  $f : W \rightarrow W'$  can be illustrated as:

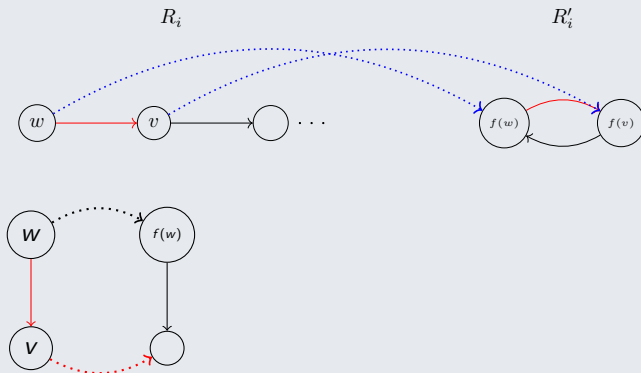




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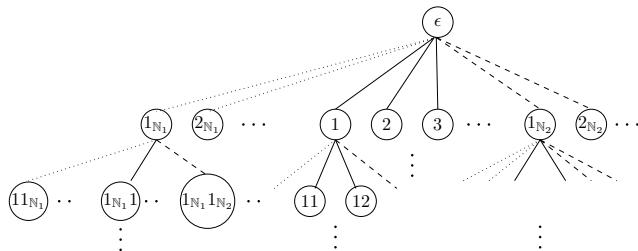


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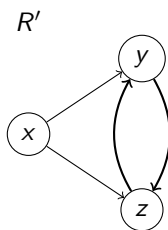
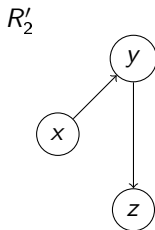
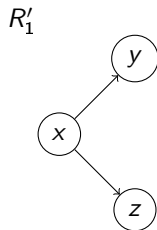
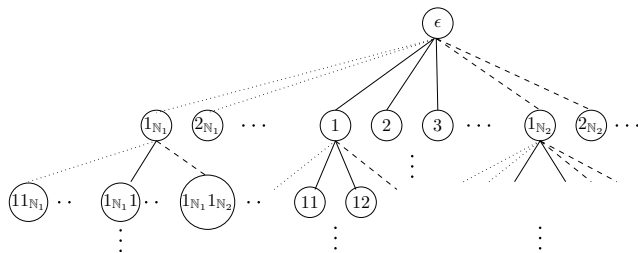
## Claim

$$\text{Log}(T_{\omega, \omega, \omega[rn]}) = T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

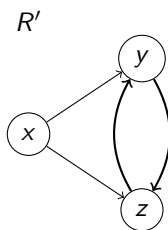
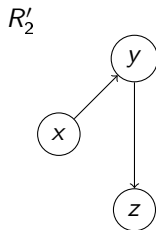
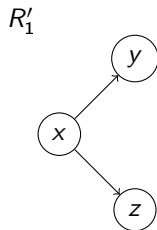
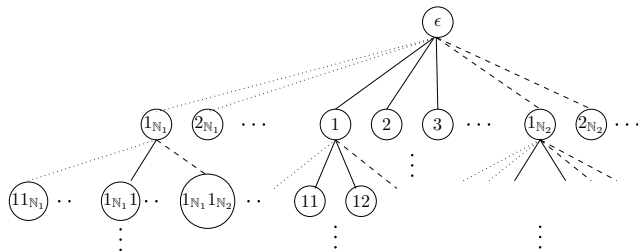
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# Required components

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- Show  $T = \text{Log}(N_{\omega}(T_{\omega[rn]}))$

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- Show  $T = \text{Log}(N_{\omega}(T_{\omega[rn]}))$
- Construct a bounded morphism

$$N_{\omega}(T_{\omega[rn]}) \times_n^+ N_{\omega}(T_{\omega[rn]}) \rightarrow T_{\omega,\omega,\omega[rn]}$$

- With some further steps we can conclude

$$T \times_n^+ T \subseteq T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$



## Future works

- Many ways to continue the research
- Discover how it works for logic  $K$
- Combine different logics for example  $D \times_n^+ K, T \times_n^+ K, \dots$
- For logic  $\Lambda$  with  $T \subseteq \Lambda \subseteq S4$ , does the following hold:

$$\Lambda \otimes \Lambda \otimes \Lambda + \Box p \rightarrow \Box_1 p \wedge \Box_2 p = \Lambda \times_n^+ \Lambda$$

# Conclusion

# References

- [1] Johan van Benthem, Guram Bezhanishvili, Balder ten Cate, and Darko Sarenac.  
Multimodal logics of products of topologies. *Studia Logica*, 84:369–392, 2006.
- [2] Andrei Kudinov. Modal logic of some products of neighbourhood frames.  
In *Advances in Modal Logic, Volume 9*, pages 286–294, London, 2012. College Publications.