Maik Thanh Nguyen

TITLE

CONFERENCE, DATE

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- Modal logic and semantics
  - Kripke frames
  - Topological space
  - Neighbourhood frames
- Multimodal logic and product of frames/spaces and logics
  - Notation, Fusion of logics
  - Horizontal and Vertical topology/functions and semantics
  - Product of logics
- Main result and ideas

Modal logic and Kripke frames and models

- Modal logic extends classical propositional logic. Formally:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi$$

where $\Box$ is a modal operator and Prop is a set of variable with $p \in \mathrm{Prop}$.

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- Modal logic extends classical propositional logic. Formally:

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  where $\Box$ is a modal operator and Prop is a set of variable with $p \in$ Prop.
- A frame $F = (W, R)$ is a pair where
  - W is a non-empty set of worlds
  - $R \subseteq W \times W$ is a binary relation
- A model is a pair $M = (F, R)$ (M is based on F) where
  - F is a frame
  - V is a valuation and is of the form $V : Prop \rightarrow 2^W$

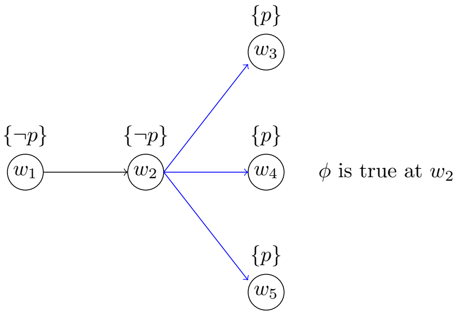
Kripke semantics

- Let $M = (F, V)$ be a model and $w \in W$ a state in M. A formula being true at w is inductively defined as:

$$
\begin{array}{ll}
M, w \Vdash p & \text{iff } w \in V(p) \\
M, w \Vdash \bot & \text{never} \\
M, w \Vdash \neg\phi & \text{iff not } M, w \Vdash \phi \\
M, w \Vdash \phi \vee \psi & \text{iff } M, w \Vdash \phi \vee M, w \Vdash \psi \\
M, w \Vdash \Box\phi & \text{iff } \forall v \in W : wRv \rightarrow M, v \Vdash \phi \\
M, w \Vdash \Diamond\phi & \text{iff } \exists v \in W : wRv \wedge M, v \Vdash \phi
\end{array}
$$

Example

- Let $\phi = \Box p$ and $M = (W, R, V)$ with $W = \{w_1, w_2, w_3, w_4, w_5\}$, $V(p) = \{w_3, w_4, w_5\}$ and $R =$



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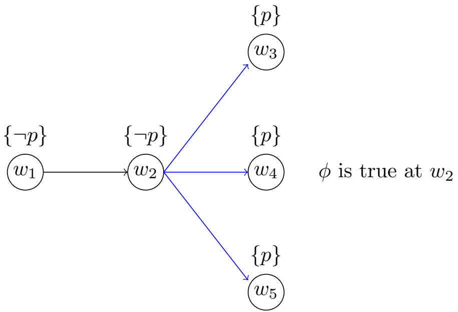

- Kripke semantics has many applications for example epistemic logic, temporal logic,...

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Topological space

- We want to reason not just about "what is true" but "where is true"
- Topological space deals with open sets, it can describes which points are "nearby"
- applications: spatial reasoning for example reasoning about neighboring regions or reachability (AI, robotics)
- A topological space is a pair (X, τ), where τ (called topology) is a collection of subsets of X (open sets) such that:
  - $\emptyset$ and X are open
  - the union of arbitrary collection of open sets is open
  - the intersection of finite collection of open sets is open
- A topological model is a structure $M = (X, \tau, \upsilon)$ where (X, τ) is a topological space and υ a valuation of the form $\upsilon : Prop \to 2^X$

Topological semantics

Let $M = (X, \tau, v)$ be a topological model and $x \in X$ a point in M. A formula being true at x is inductively defined as:

$$
\begin{array}{ll}
M, x \vDash p & \text{iff } x \in v(p) \\
M, x \vDash \bot & \text{never} \\
M, x \vDash \neg\phi & \text{iff } M, x \nvDash \phi \\
M, x \vDash \phi \vee \psi & \text{iff } M, x \vDash \phi \text{ or } M, x \vDash \psi \\
M, x \vDash \Box\phi & \text{iff } \exists U \in \tau \text{ such that } x \in U \text{ and } \forall u \in U,\ M, u \vDash \phi \\
M, x \vDash \Diamond\phi & \text{iff } \forall U \in \tau : \text{if } x \in U \to \exists u \in U,\ M, u \vDash \phi
\end{array}
$$

Example

- Let (X, τ) be a topological space with $X = \{1, 2, 3\}$, $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, W\}$ and $V(p) = \{1, 2\}$. Furthermore, let $\phi = \Box p$.

Example

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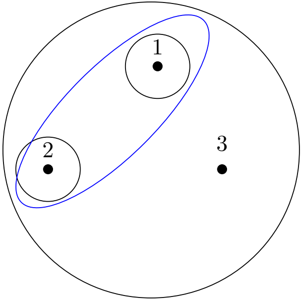

$M, 1 \Vdash \phi$

$M, 3 \nVdash \phi$

Neighbourhood frames

- generalize Kripke semantics
- captures non-normal modal logics
- application in Epistemic logic

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- captures non-normal modal logics
- application in Epistemic logic
- A neighbourhood frame is a pair (X, τ) where τ is a function $\tau : X \to 2^{2^X}$.
- A neighbourhood model is a structure $M = (X, \tau, \upsilon)$, where υ is a valuation of the form $\upsilon : Prop \to 2^X$

Neighbourhood semantics

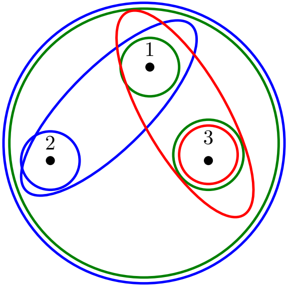
Let $M = (X, \tau, v)$ be a neighbourhood model and $x \in X$ a point in M. A formula being true at x is inductively defined as:

$$
\begin{aligned}
&M, x \Vdash p && \text{iff } x \in V(p) \\
&M, x \Vdash \bot && \text{never} \\
&M, x \Vdash \neg\phi && \text{iff } M, x \nVdash \phi \\
&M, x \Vdash \phi \vee \psi && \text{iff } M, x \vDash \phi \vee M, x \vDash \psi \\
&M, x \Vdash \Box\phi && \text{iff } \exists V \in \tau(x) \forall y \in V : M, y \models \phi
\end{aligned}
$$

Example

Assume $\phi = \Box p$. Let $W = \{1, 2, 3\}$, $V(p) = \{1, 2\}$ and

$$\tau(x) = \begin{cases} 1 \to \{\{1\}, \{3\}, W\} \\ 2 \to \{\{2\}, \{1, 2\}, W\} \\ 3 \to \{\{3\}, \{1, 3\}\} \end{cases}$$



True at all points

Multimodal logic and product of frames

- Mutlimodal logic allows us to reason about knowledge,time etc. simultaneously
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Multimodal logic and product of frames

- Mutlimodal logic allows us to reason about knowledge,time etc. simultaneously
- for example we can combine temporal and epistemic logic to reason about "when does the agent knows something" (product of frames)
- horizontal and vertical relation allows us to reason in one dimension, for example we a fix time point and ask what does the agent knows at that time?
- Let $F = (W, R_1)$ and $G = (V, R_2)$. We define the Kripke product on $W \times V$as :

$$(w, v)R_1'(w', v') \text{ iff } wR_1 w' \text{ and } v = v' \text{ (horizontal)}$$

$$(w, v)R_2'(w', v') \text{ iff } w = w' \text{ and } vR_2 v' \text{(vertical)}$$

Fusion

- we can also combine logics with fusion
- Let L_1 and L_2 be modal logics with one modality $\Box$. Then the fusion is defined as:

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

Horizontal and Vertical topology

- in topological space, horizontal and vertical can be also defined

Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \upsilon)$ be topological spaces and $N \subseteq X \times Y$

Horizontally open: N is horizontally open iff

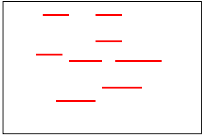
$$\forall (x, y) \in N \; \exists U \in \chi \text{ such that } x \in U \text{ and } U \times \{y\} \subseteq N.$$

Vertically open: N is vertically open iff

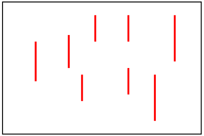
$$\forall (x, y) \in N \; \exists V \in \upsilon \text{ such that } y \in V \text{ and } \{x\} \times V \subseteq N.$$

τ_1 (horizontal topology) is the set of all horizontally open sets and τ_2 (vertical topology) the set of all vertically open sets.

Illustration



horizontally



vertically