TECHNISCHE UNIVERSITÄT DRESDEN

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TITLE

CONFERENCE, DATE

Content

- Modal logic and semantics
  - Kripke frames
  - Topological space
  - Neighbourhood frames
- Multimodal logic and product of frames/spaces and logics
  - Notation, Fusion of logics
  - Horizontal and Vertical topology/functions
  - Product of logics and the logic T
- Main result and ideas

Modal logic and Kripke frames and models

- Modal logic extends classical propositional logic. Formally:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi$$

  where $\Box$ is a modal operator and Prop is a set of variable with $p \in \mathrm{Prop}$.

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  where $\Box$ is a modal operator and Prop is a set of variable with $p \in$ Prop.
- A frame $F = (W, R)$ is a pair where
  - W is a non-empty set of worlds
  - $R \subseteq W \times W$ is a binary relation
- A model is a pair $M = (F, R)$ (M is based on F) where
  - F is a frame
  - V is a valuation and is of the form $V : Prop \to 2^W$

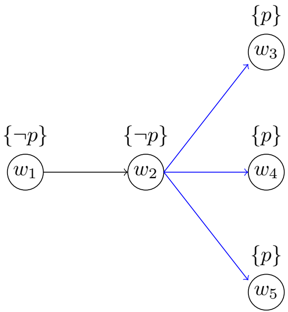
Kripke semantics

- Let $M = (F, V)$ be a model and $w \in W$ a state in M. A formula being true at w is inductively defined as:

$$
\begin{array}{ll}
M, w \Vdash p & \text{iff } w \in V(p) \\
M, w \Vdash \bot & \text{never} \\
M, w \Vdash \neg\phi & \text{iff not } M, w \Vdash \phi \\
M, w \Vdash \phi \vee \psi & \text{iff } M, w \Vdash \phi \vee M, w \Vdash \psi \\
M, w \Vdash \Box\phi & \text{iff } \forall v \in W : wRv \rightarrow M, v \Vdash \phi \\
M, w \Vdash \Diamond\phi & \text{iff } \exists v \in W : wRv \wedge M, v \Vdash \phi
\end{array}
$$

Example

- Let $\phi = \Box p$ and $M = (W, R, V)$ with $W = \{w_1, w_2, w_3, w_4, w_5\}$, $V(p) = \{w_3, w_4, w_5\}$ and $R =$



ϕ is true at w_2

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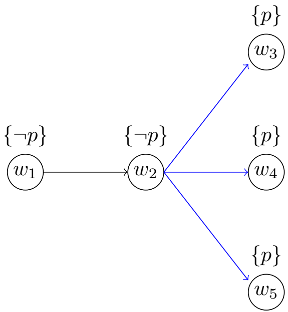

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- Kripke semantics has many applications for example epistemic logic, temporal logic,...

Motivation topological space

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- We want to reason not just about "what is true" but "where is true"
- Topological space deals with open sets, it can describes which points are "nearby"
- applications: spatial reasoning

Topological space

- A topological space is a pair (X, τ), where τ (called topology) is a collection of subsets of X (open sets) such that:
  - $\emptyset$ and X are open
  - the union of arbitrary collection of open sets is open
  - the intersection of finite collection of open sets is open
- A topological model is a structure $M = (X, \tau, \upsilon)$ where (X, τ) is a topological space and υ a valuation of the form $\upsilon : \mathit{Prop} \rightarrow 2^X$

Topological semantics

Let $M = (X, \tau, v)$ be a topological model and $x \in X$ a point in M. A formula being true at x is inductively defined as:

$M, x \vDash p$	iff $x \in v(p)$
$M, x \vDash \bot$	never
$M, x \vDash \neg\phi$	iff $M, x \nvDash \phi$
$M, x \vDash \phi \vee \psi$	iff $M, x \vDash \phi$ or $M, x \vDash \psi$
$M, x \vDash \Box\phi$	iff $\exists U \in \tau$ such that $x \in U$ and $\forall u \in U,\ M, u \vDash \phi$
$M, x \vDash \Diamond\phi$	iff $\forall U \in \tau$: if $x \in U \rightarrow \exists u \in U,\ M, u \vDash \phi$

Example

- Let (X, τ) be a topological space with $X = \{1, 2, 3\}$,
  $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, W\}$ and $V(p) = \{1, 2\}$. Furthermore, let $\phi = \Box p$.

Example

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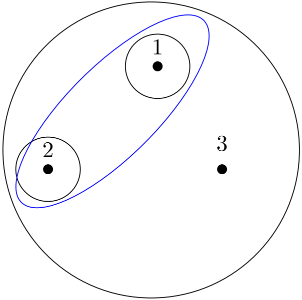

$M, 1 \Vdash \phi$

$M, 3 \nVdash \phi$

Neighbourhood frames

- generalize Kripke semantics
- captures non-normal modal logics
- application in Epistemic logic

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- generalize Kripke semantics
- captures non-normal modal logics
- application in Epistemic logic
- A neighbourhood frame is a pair (X, τ) where τ is a function $\tau : X \to 2^{2^X}$.
- A neighbourhood model is a structure $M = (X, \tau, \upsilon)$, where υ is a valuation of the form $\upsilon : \mathit{Prop} \to 2^X$

Neighbourhood semantics

Let $M = (X, \tau, \upsilon)$ be a neighbourhood model and $x \in X$ a point in M. A formula being true at x is inductively defined as:

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M, x \Vdash p & \text{iff } x \in V(p) \\
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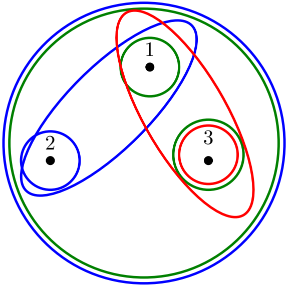
M, x \Vdash \phi \vee \psi & \text{iff } M, x \vDash \phi \vee M, x \vDash \psi \\
M, x \Vdash \Box\phi & \text{iff } \exists V \in \tau(x) \forall y \in V : M, y \models \phi
\end{array}$$

Example

Assume $\phi = \Box p$. Let $W = \{1, 2, 3\}$, $V(p) = \{1, 2\}$ and

$$\tau(x) = \begin{cases} 1 \to \{\{1\}, \{3\}, W\} \\ 2 \to \{\{2\}, \{1, 2\}, W\} \\ 3 \to \{\{3\}, \{1, 3\}\} \end{cases}$$



True at all points

Multimodal logic and product of frames

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- Mutlimodal logic allows us to reason about knowledge,time etc. simultaneously
- for example we can combine temporal and epistemic logic to reason about "when does the agent knows something" (product of frames)
- horizontal and vertical relation allows us to reason in one dimension
- Let $F = (W, R_1)$ and $G = (V, R_2)$. We define the Kripke product on $W \times V$ as :

$$(w, v)R_1'(w', v') \text{ iff } wR_1w' \text{ and } v = v' \text{ (horizontal)}$$

$$(w, v)R_2'(w', v') \text{ iff } w = w' \text{ and } vR_2v' \text{(vertical)}$$

Fusion

- we can also combine logics with fusion
- Let L_1 and L_2 be modal logics with one modality $\Box$. Then the fusion is defined as:

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \to \Box_1)} L_{2(\Box \to \Box_2)}$$

Horizontal and Vertical topology

- in topological space, horizontal and vertical can be also defined

Let $\mathcal{X} = (X, \chi)$ and $\mathcal{Y} = (Y, \upsilon)$ be topological spaces and $N \subseteq X \times Y$

Horizontally open: N is horizontally open iff

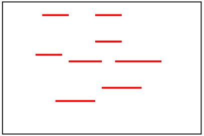
$$\forall (x, y) \in N \; \exists U \in \chi \text{ such that } x \in U \text{ and } U \times \{y\} \subseteq N.$$

Vertically open: N is vertically open iff

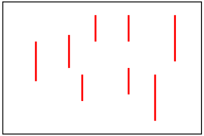
$$\forall (x, y) \in N \; \exists V \in \upsilon \text{ such that } y \in V \text{ and } \{x\} \times V \subseteq N.$$

τ_1 (horizontal topology) is the set of all horizontally open sets and τ_2 (vertical topology) the set of all vertically open sets.

Illustration, standard product



horizontally



vertically

- we can also reason in both directions at once with the standard topology
- we denote the standard toplogy as τ where

$$\tau = \{N \subseteq X \times Y \mid \exists U \in \chi \exists V \in \upsilon : N = U \times V\}$$

horizontal, vertical and standard functions

Let $\mathcal{X} = (X, \tau_1)$ and $\mathcal{Y} = (Y, \tau_2)$ be two n-frames. We define the full product as

$$\mathcal{X} \times_n^+ \mathcal{Y} = (X \times Y, \tau_1', \tau_2', \tau) \text{ where}$$

$$\tau_1'(x, y) = \{U \subseteq \mathrm{X} \times \mathrm{Y} \mid \exists V \in \tau_1(x) : V \times \{y\} \subseteq U\}$$

$$\tau_2'(x, y) = \{U \subseteq \mathrm{X} \times \mathrm{Y} \mid \exists V \in \tau_2(y) : \{x\} \times V \subseteq U\}$$

$$\tau(x, y) = \{U \subseteq \mathrm{X} \times \mathrm{Y} \mid \exists W \in \tau_1(x) \exists V \in \tau_2(y) : W \times V \subseteq U\}$$

Product of logics and the logic T

Let L_1 and L_2 be two unimodal logic. We define the full n-product of them as:

$$L_1 \times_n^+ L_2 = Log(\{\mathcal{X} \times_n^+ \mathcal{Y} \mid \mathcal{X} \Vdash L_1 \text{ and } \mathcal{Y} \Vdash L_2\})$$

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- this thesis deals with the logic T
- $T = K + \Box p \to p$

Main Research Question

- Does the following equality holds?

$$T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p = T \times_n^+ T$$

where $T \otimes T \otimes T = K_3 + T_\Box + T_{(\Box \to \Box_1)} + T_{(\Box \to \Box_2)}$ is the logic with three modalities and $\Box p \to \Box_1 p \wedge \Box_2 p$ the interaction axiom

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- main motivation comes from epistemic logic
- we will only sketch the right to left inclusion
- proof is splitted into two parts (ideas are from [1], [2])

Sketch of the main ideas

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- Show $Log(\mathcal{C}) = T \otimes T \otimes T + \Box p \to \Box_1 p \land \Box_2 p$ via Sahlqvist Theorem
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- We show finite model property of the logic with Filtration Theorem
- Construct an infinite branching and infinite depth tree with three reflexive relations ($T_{\omega,\omega,\omega[rn]}$)
- Show $Log(T_{\omega,\omega,\omega[rn]}) = T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$
- By FMP pick a finite frame $F \in \mathcal{C}$ and construct a bounded morphism from $T_{\omega,\omega,\omega[rn]}$ to F

Sketch continue

$$Log(T_{\omega,\omega,\omega[rn]}) = T \otimes T \otimes T + \Box p \to \Box_1 p \wedge \Box_2 p$$

- we introduce a simpler version of $T_{\omega,\omega,\omega[rn]}$ which is $T_{\omega[rn]}$

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- we introduce a simpler version of $T_{\omega,\omega,\omega[rn]}$ which is $T_{\omega[rn]}$
- construct a neighbourhood version of $T_{\omega[rn]}$ called $N_\omega(T_{\omega[rn]})$
- we can show $T = Log(N_\omega(T_{\omega[rn]}))$

Sketch continue

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- we can show $T = Log(N_\omega(T_{\omega[rn]}))$
- construct a bounded morphism

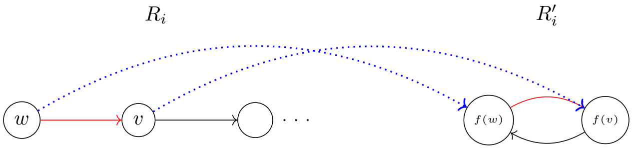
$$N_\omega(T_{\omega[rn]}) \times_n^+ N_\omega(T_{\omega[rn]}) \rightarrow T_{\omega,\omega,\omega[rn]}$$

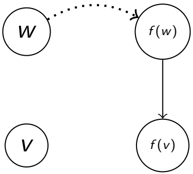
- with some further steps we can conclude

$$T \times_n^+ T \subseteq T \otimes T \otimes T + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$$

$T_{\omega,\omega,\omega[rn]}$ and a more detailed sketch

Let $F = (W, R_1, R_2, \ldots)$ and $F' = (W', R'_1, R'_2, \ldots)$ be two frames. A bounded morphism $f : W \to W'$ can be illustrated as:





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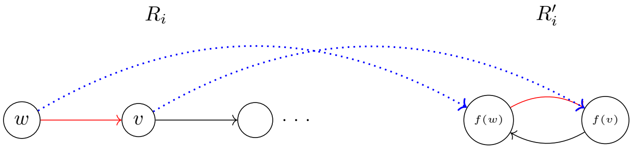

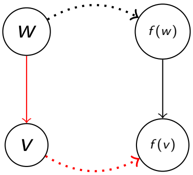

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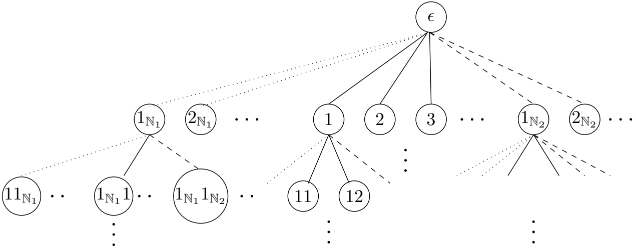

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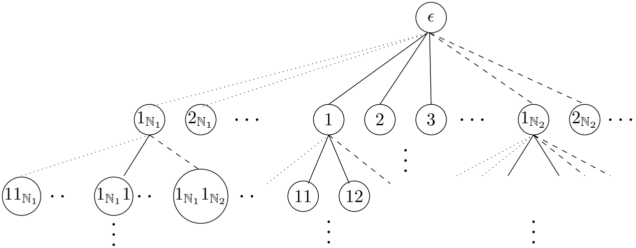

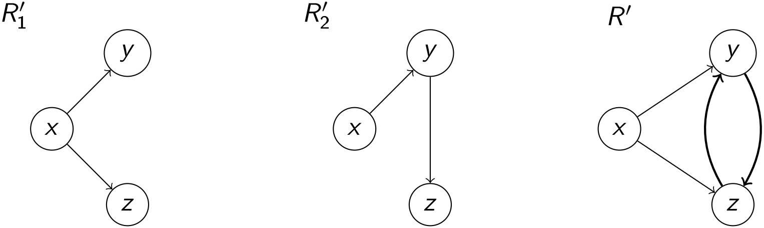

Future works

- Many ways to continue the Research
- Discover how it works for logic K
- Combine different logics for example $D \times_n^+ K$, $T \times_n^+ K, \ldots$
- For logic Λ with $T \subseteq \Lambda \subseteq S4$, does the following hold:

$$\Lambda \otimes \Lambda \otimes \Lambda + \Box p \to \Box_1 p \wedge \Box_2 p = \Lambda \times_n^+ \Lambda$$

Conclusion

References