TECHNISCHE UNIVERSITÄT DRESDEN

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IP = PSPACE

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Content

- What is IP?
- Arithmetization
- Introducing the protocol
- Problems within the protocol and solutions
- Some protocol examples
- Correctness

What is IP?

- A prover tries to convince the Verifier of membership
- Verifier scpetically checks the Prover's arguemnts before making a decision
- The interaction might involve several rounds of communication
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- A language L is in IP if there is a polynomial verifier V such that, for every word w :

$$\text{if w } \in \text{L then there is a Prover P with } Pr[V \leftrightarrow Paccepts] \geq \frac{2}{3}$$

$$\text{if w } \notin \text{L then for all Prover P with } Pr[V \leftrightarrow Paccepts] \leq \frac{1}{3}$$

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- QBF-Truth$_{NNF}$ (abbrev. with QBF') is QBF-Truth but negations are only applied on variables
- $\exists x \exists y \neg (x \wedge y)$ is not in NNF but $\exists x \exists y (\neg x \vee \neg y)$

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- QBF-Truth$_{NNF}$ (abbrev. with QBF') is QBF-Truth but negations are only applied on variables
- $\exists x \exists y \neg(x \wedge y)$ is not in NNF but $\exists x \exists y(\neg x \vee \neg y)$
- QBF $\leq_m^P$ QBF' can be easily done by the verifier
- it suffices to show QBF' $\in$ IP because we can reduce any problem in PSPACE to QBF in polynomial time

How do we find an algorithm for QBF' s.t it satisfies the IP conditions?

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- $\neg x$ becomes 1-x

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x	y	$x \wedge y = x * y$	$x \wedge y = x + y$	$\neg(x \wedge y) = 1 - (x * y)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	2	0

$$\phi(x_1, x_2, ..., x_n) = 1 \Leftrightarrow \phi_{arith}(x_1, x_2, ..., x_n) > 0$$

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$\prod_{x \in \{0,1\}} \sum_{y \in \{0,1\}} 1 - (x * y) = \phi_{arith}$

- $\phi_{arith} = 2$

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- $\phi_{arith} = 2$
- If ϕ is true then $\phi_{arith} > 0$
- If ϕ is false then $\phi_{arith} = 0$
- This can be shown by structural induction

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- $\phi = x$. If ϕ is true then $\phi_{arith} = 1$ and if ϕ is false then $\phi_{arith} = 0$
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- Suppose ϕ_1 and ϕ_2 are true, that means $\phi_{1arith} > 0$ and $\phi_{2arith} > 0$ (it's similar for the other case)
- $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$ also holds
- For $\forall \phi_1$, we have by induction that $\phi_1[x := 0], \phi_1[x := 1]$ is true, so the multiplication of two positive value is positive (same for $\exists \phi_1$)

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$\phi_{arith} = \prod_{x_1 \in \{0,1\}} \cdots \prod_{x_m \in \{0,1\}} \phi'_{arith} = 4^{2^m}$

- It holds that for formula ϕ with string length n : $\phi_{arith} \leq 2^{2^n}$
- We solve this problem by using modulo with a suitable value

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- k must be presentable in linear many bits
- the calculation mod k must preserve ">0" for valid and "=0" for invalid formulas
- It holds that: for any $a \leq 2^{2^n}, a > 0$, there exist a prime number $k \in [2^n, 2^{3n}]$ s.t $a \not\equiv 0$ (mod k)

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- If $\phi = \phi_1 \wedge \phi_2$, then ask prover to send a_1 and a_2 and check $c = a_1 * a_2$. If it's true then ask the prover to prove that the of ϕ_1 is a_1 and ϕ_2 is a_2
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- For $\phi = \phi_1 \vee \phi_2$, we ask for a_1 and a_2 s.t c = $a_1 + a_2$
- In case $\phi = \forall x \phi_1$ we asked for a polynomial p(x) that represents the arithmetic presentation of ϕ_1 where x is free and we check c = p(0) * p(1)
- If it is true, the verifier sends randomly a number d between $\{0, ..., k-1\} = GF(K)$ and caluclate p(d). Now the verifier expects the prover to prove the value of $\phi_1[x := d]$ is p(d)
- The same process happens when we have $\phi = \exists \phi_1$, but we check $c = p(0) + p(1)$

Protocol continue

- When every variable got a number in GF(K), say $y_1, ..., y_n$ the verifier calculates $\phi_{arith}(y_1, ..., y_n)$ and accept if its equal to $q(y_1, ..., y_n)$ (last polynomial sent by prover) else reject

Example

$$\phi = \forall x \exists y(\neg x \vee y) \wedge \exists z \exists w\,(z \vee w)$$

$$\phi_{arith} = \underbrace{(\prod_x \sum_y (1-x)+y)}_{\phi_{\mathbf{1}arith}} * \underbrace{(\sum_z \sum_w z+w)}_{\phi_{\mathbf{2}arith}} \quad x, y, z, w \in \{0, 1\}$$

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Prover $\xrightarrow{\text{c=12,k=11,b}}$ Verifier

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Prover		Verifier
	c=12,k=11,b $\longrightarrow$	check c>0, 8, b
	asks for $a_1 = \phi_1$ and $a_2 = \phi_2$ $\longleftarrow$	
	sends $a_1 = 3$, $a_2 = 4$ $\longrightarrow$	check $c = a_1 * a_2$
	prove $a_1 = \phi_1, a_2 = \phi_2$ $\longleftarrow$	

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Prover | Verifier

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Verifier: check c>0, 8, b

← asks for $a_1 = \phi_1$ and $a_2 = \phi_2$

sends $a_1 = 3$, $a_2 = 4$ →

Verifier: check $c = a_1 * a_2$

← prove $a_1 = \phi_1, a_2 = \phi_2$
send me $p_1(x)$ for ϕ_1 and $p_2(x)$ for ϕ_2

Example continue

Prover

calculate $\phi_{1arith}(x)$, $\phi_{2arith}(z)$

$$\xrightarrow[p_1(x) = \sum_y (1-x) + y = -2x + 3]{\text{sends } p_1(x) = \phi_{1arith}(x),\ \phi_{2arith}(z)}$$

Verifier

check $a_{\mathbf{1}} = p_{\mathbf{1}}(0) * p_{\mathbf{1}}(1)$

check $a_{\mathbf{2}} = p_{\mathbf{2}}(0) + p_{\mathbf{2}}(1)$

Example continue

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	sends $p_1(d) = 10, p_2(d) = 5$ $\longleftarrow$ ask for $p_1'(d, y), p_2'(d, w)$	Choose randomly $d \in GF(k)$,say d=2

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Choose randomly $d \in GF(k)$,say d=2

$$\xrightarrow[\text{sends } p_2'(d,w) = \phi_{2arith}(d,w)]{\text{sends } p_1'(d,y) = \phi_{1arith}(d,y)}$$

$p_1'(d,0) * p_1'(d,1) = p_1(d)$
$p_2'(d,0) * p_2'(d,1) = p_2(d)$
Choose c $\in$ GF(k)

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	sends $p_1(d) = 10, p_2(d) = 5$ $\longleftarrow$ ask for $p_1'(d,y), p_2'(d,w)$	Choose randomly $d \in GF(k)$,say d=2
	sends $p_1'(d,y) = \phi_{1arith}(d,y)$ $\longrightarrow$ sends $p_2'(d,w) = \phi_{2arith}(d,w)$	$p_{\mathbf{1}}'(d,0) * p_{\mathbf{1}}'(d,1) = p_{\mathbf{1}}(d)$ $p_{\mathbf{2}}'(d,0) * p_{\mathbf{2}}'(d,1) = p_{\mathbf{2}}(d)$ Choose c $\in$ GF(k)

The verifier check $\phi_{arith}(d,c,d,c) = p_1'(d,c) * p_2'(d,c)$. It accepts.

Next Example

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Prover Verifier

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$$\text{Prover} \quad \xrightarrow{\text{c= 3, p = 131, b}} \quad \text{Verifier}$$

check c>0, 131, b

$$\xleftarrow{\text{asks for } p_1(x)}$$

$$\xrightarrow[\text{prover has to lie}]{\text{sends } p_1(x) = 2x + x}$$

check $p_1(0) * p_1(1) = c$

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Prover

c= 3, p = 131, b $\longrightarrow$

Verifier

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asks for $p_1(x)$ $\longleftarrow$

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prover has to lie

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choose $d \in GF(p)$

sends $p_1(d)$ $\longleftarrow$

If $\phi_{arith}(d) = p_1(d = 3) = 7$ then we lost. Chances should be low

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	← asks for $p_1(x)$
	sends $p_1(x) = 2x + x$ → prover has to lie	check $p_1(0) * p_1(1) = c$ choose $d \in GF(p)$
	← sends $p_1(d)$

If $\phi_{arith}(d) = p_1(d = 3) = 7$ then we lost. Chances should be low

	sends $p_1'(d, y) = 7 * y$ →	check $p_1'(d, 0) + p_1'(d, 1) = p_1(d)$. Chooses g = 4

The verifier check $\phi_{arith}(d, g) = p_1'(d, g)$. $12 \neq 28$. Verfier rejects.

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- Example : $\phi = \forall x_1, ..., \forall x_m (x_1 \vee ... \vee x_m) \xrightarrow{arith\phi(x_1)} \phi_{arith}(x) = \prod_{x_2} ... \prod_{x_m} (x_1 + x_2 + ... + x_m) \rightarrow deg(\phi_{arith}(x)) \leq 2^{m-1}$
- A QBF ϕ is called simple, if any occurence of a variable is seperated by at most one universal quantifier from its point of quantification.
- Example : $\forall x_1 \forall x_2 \exists x_3 \; [(x_1 \vee x_2) \wedge \forall x_4 (x_2 \vee x_3 \vee x_4)]$
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Simple QBF

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- We can reduce any QBF formula into a Simple QBF in polynomial time
- Let $\phi = ...Qx_i ... \forall x_j \psi(x_i)$ where $Q \in \{\forall, \exists\}$ and $\forall x_j$ is the first universal quantifier after Q_{xi}. We transform ϕ as follows :

$$\phi' = ...Qx_i ... \forall x_j \exists x_i' (x_i \leftrightarrow x_i') \wedge \psi(x_i')$$

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- Example : $\exists x(\forall y \forall z(x \vee (y \vee z))) \wedge (\forall u(u \vee x))$
- Reduced : $\exists x(\forall y \exists x'(x \leftrightarrow x') \wedge \forall z(x' \vee (y \vee z))) \wedge (\forall u(u \vee x))$
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- Prover sends wrong polynomial $p \neq h = \phi_{arith}$ in the i-th round. Verifier chooses randomly $c \in GF(p)$ where $p \geq 2^n$. Furthermore we have $deg(g - h) \leq 2n$. Then $Pr[p(c) = h(c)] = \Pr[\text{Error i-th round}] \leq \frac{2n}{2^n}$.

Correctness continue

- That means Pr[No Error in i-th round]$\geq 1 - \frac{2n}{2^n}$
- Because random number are chosen independently, and after $m \leq n$ rounds, we have :

$$Pr[Error] = 1 - Pr[\text{No Error}] = 1 - \prod_{i=1}^{m} Pr[\text{No Error in i-th round}]$$

$$\leq (1 - (1 - \frac{2n}{2^n}))^n$$

- The last approximation is true because :
$\prod_{i=1}^{m} Pr[\text{no error in i-th round}] \geq (1 - \frac{2n}{2^n})^m \geq (1 - \frac{2n}{2^n})^n$

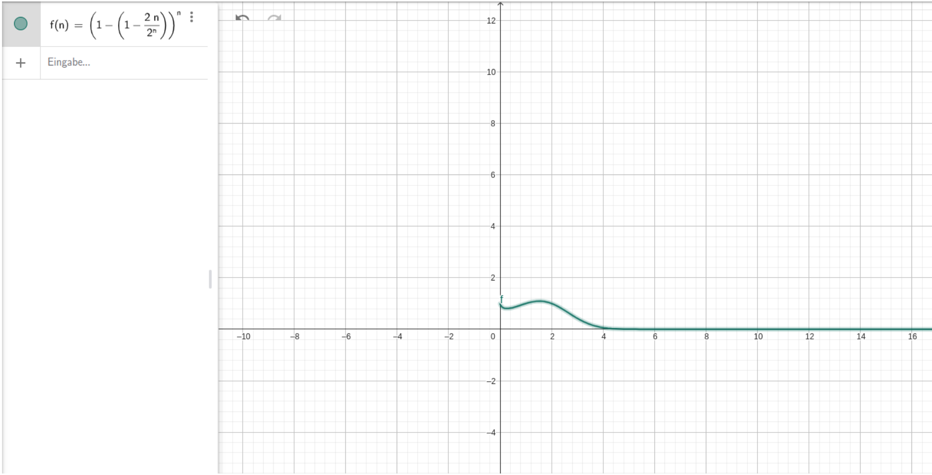

Figure: $n \to \infty$