TECHNISCHE UNIVERSITÄT DRESDEN

hello

IP = PSPACE

CONFERENCE, DATE

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- QBF $\leq_m^P$ QBF-Truth$_{NNF}$ can be easily done by the verifier
- it suffices to show QBF' $\in$ IP because we can reduce any problem in PSPACE to QBF in polynomial time

  How do we find an algorithm for QBF' s.t it satisfies the IP conditions?

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x	y	$x \wedge y = x * y$	$x \wedge y = x + y - x * y$	$\neg(x \wedge y) = 1 - (x * y)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

$$\phi(x_1, x_2, ..., x_n) = 1 \Leftrightarrow \phi_{arith}(x_1, x_2, ..., x_n) = 1$$

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- If ϕ is true then $\phi_{arith} > 0$
- If ϕ is false then $\phi_{arith} = 0$
- This can be shown by structural induction

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- Suppose ϕ_1 and ϕ_2 are true, that means $\phi_{1arith} > 0$ and $\phi_{2arith} > 0$ (it's similar for the other case)
- $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$ also holds
- For $\forall \phi_1$, we have by induction that $\phi_1[x := 0], \phi_1[x := 1]$ is true, so the multiplication of two positive value is positive (same for $\exists \phi_1$)

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$\phi_{arith} = \prod_{x_1 \in \{0,1\}} \cdots \prod_{x_m \in \{0,1\}} \phi'_{arith} = 3^{2^m}$

- It holds that for formula ϕ with string length n : $\phi_{arith} \leq 2^{2^n}$
- We solve this problem by using modulo with a suitable value

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- the calculation mod k must preserve ">0" for valid and "=0" for invalid formulas
- Number Theory Theorem : for any $a \leq 2^{2^n}$, $a > 0$, there exist a prime number $k \in [2^n, 2^{3n}]$ s.t $a \not\equiv 0$ (mod k)

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