

hello

IP = PSPACE

CONFERENCE, DATE

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- $\exists x \exists y \neg (x \wedge y)$ is not in NNF but $\exists x \exists y (\neg x \vee \neg y)$
- $\text{QBF} \leq_m^P \text{QBF-Truth}_{\text{NNF}}$ can be easily done by the verifier
- it suffices to show $\text{QBF}' \in \text{IP}$ because we can reduce any problem in PSPACE to QBF in polynomial time

How do we find an algorithm for QBF' s.t it satisfies the IP conditions?

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x	y	$x \wedge y = x * y$	$x \vee y = x + y$	$\neg(x \wedge y) = 1 - (x * y)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	2	0

$$\phi(x_1, x_2, \dots, x_n) = 1 \Leftrightarrow \phi_{arith}(x_1, x_2, \dots, x_n) > 0$$

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Arithmetize : $\neg(x \wedge y) \xrightarrow{\text{arith.}} 1 - (x * y) \xrightarrow{\exists \text{arith.}} \sum_{y \in \{0,1\}} 1 - (x * y) \xrightarrow{\forall \text{arith.}}$

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- $\phi_{\text{arith}} = 2$

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- If ϕ is false then $\phi_{\text{arith}} = 0$

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- This can be shown by structural induction

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- Suppose ϕ_1 and ϕ_2 are true, that means $\phi_{1arith} > 0$ and $\phi_{2arith} > 0$ (it's similar for the other case)
- $\phi_1 \wedge \phi_2, \phi_1 \vee \phi_2$ also holds
- For $\forall \phi_1$, we have by induction that $\phi_1[x := 0], \phi_1[x := 1]$ is true, so the multiplication of two positive value is positive (same for $\exists \phi_1$)

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$\phi_{arith} = \prod_{x_1 \in \{0,1\}} \dots \prod_{x_m \in \{0,1\}} \phi'_{arith} = 4^{2^m}$

- It holds that for formula ϕ with string length n : $\phi_{arith} \leq 2^{2^n}$
- We solve this problem by using modulo with a suitable value

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- k must be presentable in linear many bits
- the calculation mod k must preserve ">0" for valid and "=0" for invalid formulas
- Number Theory Theorem : for any $a \leq 2^{2^n}$, $a > 0$, there exist a prime number $k \in [2^n, 2^{3n}]$ s.t $a \not\equiv 0 \pmod{k}$

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- If $\phi = \phi_1 \wedge \phi_2$, then ask prover to send a_1 and a_2 and check $c = a_1 * a_2$. If it's true then ask the prover to prove that the of ϕ_1 is a_1 and ϕ_2 is a_2
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- For $\phi = \phi_1 \vee \phi_2$, we ask for a_1 and a_2 s.t $c = a_1 + a_2$
- In case $\phi = \forall x \phi_1$ we asked for a polynomial $p(x)$ that represents the arithmetic presentation of ϕ_1 where x is free and we check $c = p(0) * p(1)$
- If it is true, the verifier sends randomly a number d between $\{0, \dots, k-1\} = GF(K)$ and calculate $p(d)$. Now the verifier expects the prover to prove the value of $\phi_1[x := d]$ is $p(d)$
- The same process happens when we have $\phi = \exists \phi_1$, but we check $c = p(0) + p(1)$

Protocol continue

- When every variable got a number in $\text{GF}(K)$, say y_1, \dots, y_n the verifier calculates $\phi_{arith}(y_1, \dots, y_n)$ and accept if its equal to $q(y_1, \dots, y_n)$ (last polynomial sent by prover) else reject

Example

$$\phi = \forall x \exists y (\neg x \vee y) \wedge \exists z \exists w (z \vee w)$$

$$\phi_{arith} = \underbrace{\left(\prod_x \prod_y (1 - x) + y \right)}_{\phi_{1arith}} * \underbrace{\left(\sum_z \sum_w z + w \right)}_{\phi_{2arith}} \quad x, y, z, w \in \{0, 1\}$$

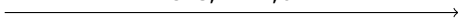
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Prover

$c=8, k=11, b$



Verifier

check $c > 0, 8, b$

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asks for $a_1 = \phi_1$ and $a_2 = \phi_2$

sends $a_1 = 2, a_2 = 4$

check $c = a_1 * a_2$

prove $a_1 = \phi_1, a_2 = \phi_2$

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send me $p_1(x)$ for ϕ_1 and $p_2(x)$ for ϕ_2

Example continue

Prover

calculate

$\phi_{1arith}(x),$

$\phi_{2arith}(z)$

sends $p_1(x) = \phi_{1arith}(x), \phi_{2arith}(z)$

$$\frac{\text{sends } p_1(x) = \phi_{1arith}(x), \phi_{2arith}(z)}{p_1(x) = \prod_y (1 - x) + y = x^2 - 3x + 2} \rightarrow$$

Verifier

check $a_1 = p_1(0) * p_1(1)$

check $a_2 = p_2(0) + p_2(1)$

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sends $p_1(d) = 0, p_2(g) = 7$

$$\xleftarrow{\text{ask for } p'_1(d, y), p'_2(g, w)}$$

Choose randomly $d, g \in$

$GF(k)$, say $d=2, g=3$

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Prover sends $p'_1(d, y) = \phi_{1arith}(d, y)$ to Verifier

Prover sends $p'_2(g, w) = \phi_{2arith}(g, w)$ to Verifier

Verifier $p'_1(d, 0) * p'_1(d, 1) = p_1(d)$
 $p'_2(g, 0) * p'_2(g, 1) = p_2(g)$
Choose $c, k \in GF(k)$

Example continue

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sends $p_1(x) = \phi_{1arith}(x), \phi_{2arith}(z)$ → check $a_1 = p_1(0) * p_1(1)$
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 $p'_2(g, 0) * p'_2(g, 1) = p_2(g)$
Choose $c, k \in GF(k)$

The verifier check $\phi_{arith}(d, c, g, k) = p'_1(d, c) * p'_2(g, k)$. It accepts.

Next Example

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Prover can not tell the truth because the verifier would reject instantly.

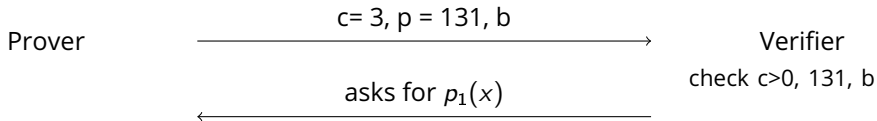
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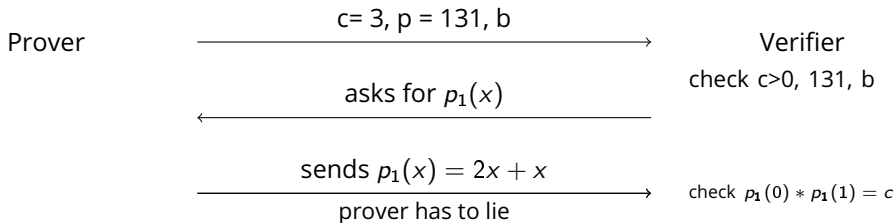
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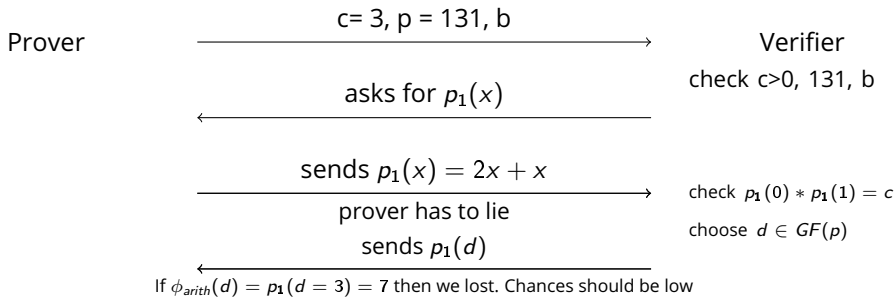
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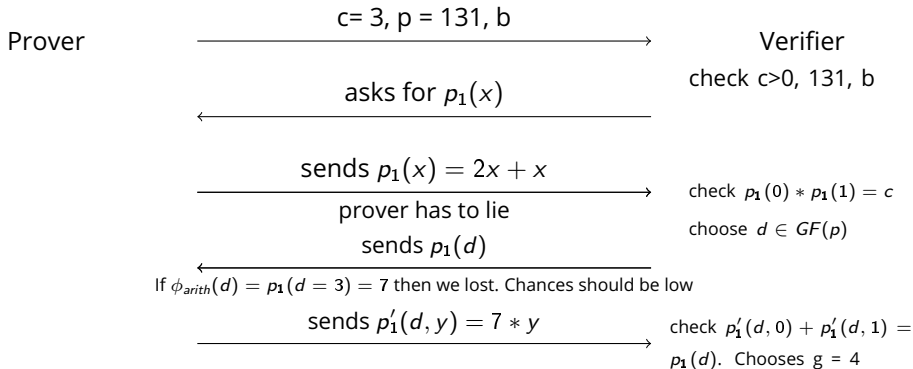


The verifier check $\phi_{\text{arith}}(d, g) = p'_1(d, g)$. $12 \neq 28$. Verifier rejects.

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Simple QBF

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$$\Pr[p(a_1, \dots, a_m) = 0] \leq \frac{d}{|S|}$$

- Prover sends wrong polynomial $p \neq h = \phi_{arith}$. Verifier chooses randomly $c \in GF(p)$ where $p \geq 2^n$. Furthermore we have $\deg(g - h) \leq 2n$.

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-