

## § Linear Algebra §

### Problem 1: Basic Vector Operations

(1)  $\|\mathbf{a}\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ ,  $\|\mathbf{b}\|_2 = \sqrt{(-8)^2 + 1^2 + 2^2} = \sqrt{69}$ .

(2)  $\|\mathbf{a} - \mathbf{b}\|_2 = \sqrt{9^2 + 1^2 + 1^2} = \sqrt{83}$ .

(3)  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal.

*Proof.* The inner product of vectors  $\mathbf{a}$  and  $\mathbf{b}$  is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b} = 1 \times (-8) + 2 \times 1 + 3 \times 2 = 0, \quad (1.1)$$

therefore  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal.  $\square$

### Problem 2: Basic Matrix Operations

According to the consensus, the matrix notation should be the bold upper-case letter like  $\mathbf{A}$  or  $\mathbf{A}$ , not  $A$ .

(1)

$$\begin{aligned} [\mathbf{A}, \mathbf{I}_3] &= \begin{bmatrix} 1 & -3 & 3 & : & 1 & 0 & 0 \\ 3 & -5 & 3 & : & 0 & 1 & 0 \\ 6 & -6 & 4 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & : & 1 & 0 & 0 \\ 0 & 4 & -6 & : & -3 & 1 & 0 \\ 0 & 12 & -14 & : & -6 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & : & 1 & 0 & 0 \\ 0 & 4 & -6 & : & -3 & 1 & 0 \\ 0 & 0 & 4 & : & 3 & -3 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -3 & 0 & : & -\frac{5}{4} & \frac{9}{4} & \frac{3}{4} \\ 0 & 4 & 0 & : & \frac{3}{2} & -\frac{7}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & : & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & : & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \\ 0 & 1 & 0 & : & \frac{3}{8} & -\frac{7}{8} & \frac{3}{8} \\ 0 & 0 & 1 & : & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}, \end{aligned} \quad (2.1)$$

where  $\mathbf{I}_3$  is the  $3 \times 3$  identity matrix. Therefore we have

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{7}{8} & \frac{3}{8} \\ \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}. \quad (2.2)$$

The determinant of matrix  $\mathbf{A}$  can be calculated as

$$\det(\mathbf{A}) = 1 \times \begin{vmatrix} -5 & 3 \\ -6 & 4 \end{vmatrix} - (-3) \times \begin{vmatrix} 3 & 3 \\ 6 & 4 \end{vmatrix} + 3 \times \begin{vmatrix} 3 & -5 \\ 6 & -6 \end{vmatrix} = 1 \times (-2) + 3 \times (-6) + 3 \times 12 = 16, \quad (2.3)$$

where  $|\cdot|$  denotes the determinant.

(2) The rank of matrix  $\mathbf{A}$  is 3 because as is shown in Eq. (2.2) the matrix  $\mathbf{A}$  is invertible.

(3) The trace of matrix  $\mathbf{A}$  is

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^3 a_{ii} = 1 + (-5) + 4 = 0. \quad (2.4)$$

$$\mathbf{A} + \mathbf{A}^T = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 9 \\ 0 & -10 & -3 \\ 9 & -3 & 8 \end{bmatrix}. \quad (2.5)$$

(4)

$$\mathbf{A} + \mathbf{A}^T = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 9 \\ 0 & -10 & -3 \\ 9 & -3 & 8 \end{bmatrix}. \quad (2.6)$$

(5)  $\mathbf{A}$  is not an orthogonal matrix.

*Proof.* Assume  $\mathbf{A}$  is an orthogonal matrix, therefore

$$\mathbf{A}\mathbf{A}^T = \mathbf{I}_3, \quad (2.7)$$

Take the determinant at both side, it can be derived that

$$|\det(\mathbf{A})| = \sqrt{|\mathbf{A}||\mathbf{A}^T|} = |\det(\mathbf{I}_3)| = 1, \quad (2.8)$$

which contradicts with Eq. (2.3). Therefore, the assumption is false.  $\square$

(6) Let  $f(\lambda)$  be the characteristic function of matrix  $\mathbf{A}$  and

$$f(\lambda) = \begin{vmatrix} \lambda - 1 & 3 & -3 \\ -3 & \lambda + 5 & -3 \\ -6 & 6 & \lambda - 4 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2, \quad (2.9)$$

therefore the eigenvalues are  $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ . Let the corresponding eigenvectors be  $\boldsymbol{\alpha}_i, i = 1, 2, 3$ .

$$(\mathbf{A} - \lambda_i \mathbf{I}_3)\boldsymbol{\alpha}_i = \mathbf{0}, \quad i = 1, 2, 3, \quad (2.10)$$

and the corresponding eigenvectors are

$$\boldsymbol{\alpha}_1 = [1 \quad 1 \quad 2]^T, \quad \boldsymbol{\alpha}_{2,3} = [1 \quad 1 + c_{2,3} \quad c_{2,3}]^T, \quad (2.11)$$

where  $c_{2,3} \in \mathbb{R}$ . Without loss of generality, we take  $c_2 = 0$  and  $c_3 = -1$ , and we have  $\boldsymbol{\alpha}_2 = [1 \quad 1 \quad 0]^T$  and  $\boldsymbol{\alpha}_3 = [1 \quad 0 \quad -1]^T$ .

(7) Use the result from Eq. (2.9), the matrix  $\mathbf{A}$  can be diagonalized as

$$\mathbf{\Lambda} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \quad (2.12)$$

(8) The  $\ell_{2,1}$  norm of  $\mathbf{A}$  is

$$\|\mathbf{A}\|_{2,1} = \sum_{i=1}^3 \sqrt{\sum_{j=1}^3 a_{ij}^2} = \sqrt{46} + \sqrt{70} + \sqrt{34} \approx 20.98, \quad (2.13)$$

and the Frobenius norm of  $\mathbf{A}$  is

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i,j=1,2,3} a_{ij}^2} = \sqrt{150} = 5\sqrt{6} \approx 12.247. \quad (2.14)$$

(9) The nuclear norm of  $\mathbf{A}$  is

$$\|\mathbf{A}\|_* = \text{tr}(\sqrt{\mathbf{A}\mathbf{A}^*}) = \sum_{i=1}^3 \sigma_i(\mathbf{A}) \approx 14.728, \quad (2.15)$$

and the spectral norm of  $\mathbf{A}$  is

$$\|\mathbf{A}\|_2 = \max \sigma_i(\mathbf{A}) \approx 12.065. \quad (2.16)$$

MATLAB Code for Check

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1 A = [1, -3, 3; 3, -5, 3; 6, -6, 4]; % define the matrix A
2 inv(A) % calculate and print the inverse of A
3 det(A) % the determinant of A
4 rank(A) % the rank of A
5 trace(A) % the trace of A
6 A + A.' % the sum of A and the transpose of A
7 sum(sum(A * A.' ~ eye(3))) % check if A is orthogonal
8 [X, D] = eig(A) % the eigenvectors and the corresponding eigenvalues of A
9 sum(sqrt(sum(A.^2))) % 1-2,1 norm of A
10 norm(A, 'fro') % Frobenius norm of A
11 sum(svd(A)) % nuclear norm of A
12 max(svd(A)) % spectral norm of A
    
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### Problem 3: Linear Equations

(1) It is evident to solve the linear equation

$$\begin{cases} x_1 = -1, \\ x_2 = 0, \\ x_3 = 1. \end{cases} \quad (3.1)$$

(2) Let

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad (3.2)$$

and we have  $\mathbf{Ax} = \mathbf{b}$  as

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}. \quad (3.3)$$

(3) Since there is a unique solution shown in Eq. (3.1), we know

$$\text{rank}(\mathbf{A}) = 3. \quad (3.4)$$

(4)

$$\begin{aligned}
 [\mathbf{A}, \mathbf{I}_3] &= \begin{bmatrix} 2 & 2 & 3 & : & 1 & 0 & 0 \\ 1 & -1 & 0 & : & 0 & 1 & 0 \\ -1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 3 & : & 1 & 0 & 0 \\ 1 & -1 & 0 & : & 0 & 1 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 1 & -1 & 0 & : & 0 & 1 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 1 \end{bmatrix} \\
 &\sim \begin{bmatrix} 1 & 1 & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & -2 & -\frac{3}{2} & : & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -\frac{3}{4} & : & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & : & -\frac{1}{4} & \frac{3}{2} & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & : & 2 & -9 & -6 \\ 0 & -1 & 0 & : & -1 & 5 & 3 \\ 0 & 0 & 1 & : & -1 & 6 & 4 \end{bmatrix} \\
 &\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 & -4 & -3 \\ 0 & 1 & 0 & : & 1 & -5 & -3 \\ 0 & 0 & 1 & : & -1 & 6 & 4 \end{bmatrix},
 \end{aligned} \quad (3.5)$$

therefore the inverse of  $\mathbf{A}$  is

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}. \quad (3.6)$$

The determinant of  $\mathbf{A}$  can be calculated as

$$\det(\mathbf{A}) = 2 \times \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 2 \times (-1) - 2 \times 1 + 3 \times 1 = -1. \quad (3.7)$$

(5) As is shown in Eq. (3.4),  $\mathbf{A}$  is invertible and with the result in Eq. (3.6)

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad (3.8)$$

and it is exactly the same result with Eq. (3.1).

(6) The inner product

$$\langle \mathbf{x}, \mathbf{b} \rangle = \mathbf{x}^T \mathbf{b} = 1 \times 1 + 0 \times (-1) + 1 \times 2 = 1, \quad (3.9)$$

and the outer product is

$$\mathbf{x} \otimes \mathbf{b} = \mathbf{x} \mathbf{b}^T = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}. \quad (3.10)$$

(7)  $\|\mathbf{b}\|_1 = |1| + |-1| + |2| = 4$ ,  $\|\mathbf{b}\|_2 = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$ ,  $\|\mathbf{b}\|_\infty = \max\{|1|, |-1|, |2|\} = 2$ .

(8) Let  $\mathbf{y} = [y_1 \ y_2 \ y_3]^T$ , we have

$$\mathbf{y}^T \mathbf{A} \mathbf{y} = [y_1 \ y_2 \ y_3] \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 2y_1^2 - y_2^2 + y_3^2 + 3y_1y_2 + 2y_2y_3 + 2y_1y_3, \quad (3.11)$$

and

$$\nabla_{\mathbf{y}} \mathbf{y}^T \mathbf{A} \mathbf{y} = \begin{bmatrix} \frac{\partial}{\partial y_1} \mathbf{y}^T \mathbf{A} \mathbf{y} \\ \frac{\partial}{\partial y_2} \mathbf{y}^T \mathbf{A} \mathbf{y} \\ \frac{\partial}{\partial y_3} \mathbf{y}^T \mathbf{A} \mathbf{y} \end{bmatrix} = \begin{bmatrix} 4y_1 + 3y_2 + 2y_3 \\ 3y_1 - 2y_2 + 2y_3 \\ 2y_1 + 2y_2 + 2y_3 \end{bmatrix}. \quad (3.12)$$

(9) The equation  $\mathbf{A}_1 \mathbf{x} = \mathbf{b}_1$  can be represented as

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}. \quad (3.13)$$

(10)  $\text{rank}(\mathbf{A}_1) = 3$ .

*Proof.* On one hand,  $\text{rank}(\mathbf{A}_1) \geq \text{rank}(\mathbf{A}) = 3$  which is shown in Eq. (3.4). On the other hand,  $\text{rank}(\mathbf{A}_1) \leq \min\{3, 4\} = 3$ . Therefore,  $\text{rank}(\mathbf{A}_1) = 3$ . We can also find the first three equations are linearly independent while the last equation is actually the same with the third equation which makes it meaningless.  $\square$

(11) Yes.

*Proof.* Since  $\text{rank}(\mathbf{A}_1) = \|\mathbf{x}\|_0$ , i.e. rank of  $\mathbf{A}_1$  is equal to the dimension of  $\mathbf{x}$ , the formula can be solved with a unique solution the same as Eq. (3.1).  $\square$