Previous Questions

2025 年 6 月 27 日

1 环境配置

见 1.py

2 向量与矩阵

2.1 旋转矩阵

代码实现部分见 2-1.py 相当于将向量 v 顺时针旋转 θ

2.2 矩阵指数

代码实现部分见 2-2.py

考虑泰勒展开 $e^{i\theta\hat{P}}$，易计算验证泡利矩阵的平方都为 I，即 $\hat{P}^2 = I$。故

$$\begin{aligned} e^{i\theta\hat{P}} &= \sum_{k=0}^{+\infty} \frac{i^k\theta^k\hat{P}^k}{k!} = \sum_{k=0}^{+\infty}(-1)^k\frac{\theta^{2k}}{(2k)!}I + \sum_{k=0}^{+\infty}(-1)^k i\frac{\theta^{2k+1}}{(2k+1)!}\hat{P} \\ &= \cos\theta I + i\sin\theta\hat{P} \end{aligned} \tag{1}$$

只需 $\hat{P}^2 = I$ 即可。

2.3 矩阵关于向量的期望

代码实现部分见 2-3.py

事实上泡利矩阵还满足厄米性质，即 $\hat{P}^\dagger = \hat{P}$，故

$$\begin{aligned}
v^\dagger \hat{Q} v &= (1,0)(\cos\theta/2 I + i\sin\theta/2\hat{P})^\dagger \hat{Q}(\cos\theta/2 I + i\sin\theta/2\hat{P})(1,0)^T \\
&= (1,0)(\cos\theta/2\hat{Q} - i\sin\theta/2\hat{P}\hat{Q})(\cos\theta/2 I + i\sin\theta/2\hat{P})(1,0)^T \\
&= (1,0)(\cos^2\theta/2\hat{Q} + \sin^2\theta/2\hat{P}\hat{Q}\hat{P} + i\sin\theta/2\cos\theta/2(\hat{Q}\hat{P} - \hat{P}\hat{Q}))(1,0)^T
\end{aligned} \tag{2}$$

有两种情况：

1. $\hat{P} = \hat{Q}$，则 $\hat{P}\hat{Q}\hat{P} = \hat{Q}, \hat{P}\hat{Q} = \hat{Q}\hat{P}$。
2. $\hat{P}\hat{Q} = \pm i\hat{R}$，则 $\hat{P}\hat{Q}\hat{P} = \pm i\hat{R}\hat{P} = -\hat{Q}, \hat{P}\hat{Q} - \hat{Q}\hat{P} = \pm 2i\hat{R}$（$\hat{R}$ 也为泡利矩阵）

故

$$v^\dagger \hat{Q} v = \begin{cases} \hat{Q}_{1,1} & (\hat{P} = \hat{Q}) \\ \cos\theta\hat{Q}_{1,1} \pm \sin\theta\hat{R}_{1,1} & (\hat{P} \neq \hat{Q}) \end{cases} \tag{3}$$

总结一下可得

$$v^\dagger \hat{Q} v = \begin{cases} 1 & (\hat{P} = \hat{Q} = \sigma_z) \\ \cos\theta & (\hat{P} = \sigma_{x,y}, \hat{Q} = \sigma_z) \\ \sin\theta & (\hat{P} = \sigma_x, \hat{Q} = \sigma_y) \\ -\sin\theta & (\hat{P} = \sigma_y, \hat{Q} = \sigma_x) \\ 0 & (otherwise.) \end{cases} \tag{4}$$