MTH2004 - Vector Calculus and Applications: Introduction

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0.0.1 Vectors

A vector describes a quantity that has a magnitude and direction in three-dimensional space. A special type of vector is a **position vector** that always starts at the origin of the coordinate system and points to the position in question. A **unit vector** is a position vector of length one that defines the directions of the coordinate system.
Graphically, a vector is represented by an arrow that starts at a position, e.g. centre of mass, origin, and point in the correct direction.
Symbolically, a vector in this module is depicted as $\vec{x}$ and a unit vector as $\hat{x}$.
Mathematically, a vector is defined by its decomposition in the coordinate system. We will always assume orthogonal coordinate systems. In Cartesian coordinates x, y, z then:

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \tag{1}$$

where v_x, v_y, v_z are the x, y, z components of the vector.

Multiplication with a scalar Multiplying a vector with a scalar is multiplying each component. The result is still a vector:

$$\alpha\vec{v} = \alpha v_x\hat{x} + \alpha v_y\hat{y} + \alpha v_z\hat{z} \tag{2}$$

Adding or subtracting vectors Two vectors can be summed or subtracted from each other by adding or subtracting each component. This also results in a vector:

$$\vec{a} \pm \vec{b} = (a_x \pm b_x)\hat{x} + (a_y \pm b_y)\hat{y} + (a_z \pm b_z)\hat{z} \tag{3}$$

Magnitude of a vector For a position vector, it's magnitude equals the distance from the origin to the position. The result is a scalar:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{4}$$

Unit vector in direction of vector We can define a vector of unit length in the direction of a vector $\vec{v}$ as:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \tag{5}$$

NB: The unit vector does not have to start at the origin!

Scalar Product The scalar or dot product between two vectors is defined as the product between the magnitude of one vector and the projection of the other vector onto it:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\alpha \tag{6}$$

Note that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. Further, perpendicular vectors result in $\vec{a} \cdot \vec{b} = 0$.

Cross (or Vector) Product The Cross Product between two vectors produces a third vector that is perpendicular to both according to the right-hand rule.

$$\vec{a} \times \vec{b} = \hat{x}\hat{y}\hat{x}a_x a_y a_z b_x b_y b_z = |\vec{a}||\vec{b}| \sin\alpha \tag{7}$$

Note that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ and also that $|\vec{a} \times \vec{b}|$ is the 'surface area' of the parallelogram made by $\vec{a}$ and $\vec{b}$.
Further note that **if two vectors are parallel then their cross product is zero**.

Scalar Triple Product The volume of the parallelepiped formed by $\vec{c}$, $\vec{a}$ and $\vec{b}$ is the magnitude of the scalar:

$$\vec{c} \cdot \vec{a} \times \vec{b} = c_x c_y c_z a_x a_y a_z b_x b_y b_z = |\,\vec{c}\,||\vec{a} \times \vec{b}| \cos\theta \tag{8}$$

where θ is the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$.

Note that $\vec{c} \cdot \vec{a} \times \vec{b} = \vec{a} \cdot \vec{b} \times \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a}$. Further note that **three vectors are linearly independent if and only if their scalar triple product is non-zero**.

Vector Triple Product The vector $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$. Since the plane defined by $\vec{b}$ and $vecc$ is perpendicular to $\vec{b} \times \vec{c}$, the triple product lies in this plane.

0.0.2 Differentiation Methods

Product Rule For two functions ft and gt:

$$\frac{d}{dt}(fg) = f\frac{dg}{dt} + g\frac{df}{dt} \tag{9}$$

Chain Rule For a function $f(g(t))$:

$$\frac{d}{dt}(f(g(t))) = \frac{df}{dg}\frac{dg}{dt} \tag{10}$$

Integration by Parts For two functions $u(t)$ and $v'(t)$

$$\int_a^b uv'\,dt = [uv]_a^b - \int_a^b u'v\,dt \tag{11}$$

0.0.3 Suffix Notation

Many vector expressions can be simplified and more easily derived if we introduce the suffix notation.

Summation Convention A suffix, that appears twice and no more within a term implies that the term is to be summed from $i = 1$ to 3. This repeated suffix is also referred to as a dummy suffix.
e.g.

$$\vec{a}\cdot\vec{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i \tag{12}$$

$$(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d}) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i c_j d_j = a_i b_i c_j d_j = a_i c_j b_i d_j \tag{13}$$

Kronecker Delta The Kronecker delta is defined as:

$$\delta_{ij} = \{\, 1\; if$$

i = j
$0 if$i $\neq j$ (14)

The Kronecker delta also can represents the 3x3 unit matrix:

$$\delta_{ij} = 100010001 \tag{15}$$

and this is symmetric, i.e. $\delta_{ij} = \delta_{ji}$

Kronecker delta and summation notation Consider $\delta_{ij}a_j = \sum_{j=1}^{3}\delta_{ij}a_j$. For $i = 1$ only $\delta_{i1} \neq 0$, hence $\delta_{1j}a_j = a_1$. Also, $\delta_{2j}a_j = a_2$ and $\delta_{3j}a_j = a_3$. Thus:

$$\delta_{ij}a_j = a_i \tag{16}$$

Alternating Tensor The alternating tensor (or permutation tensor) ϵ_{ijk} is defined as:

$$\epsilon_{ijk} = \{\, 1\; if$$

(i,j,k) = (1,2,3),(2,3,1)or(3,1,2)
$-\ 1 if$(i,j,k) = (1,3,2), (2,1,3)or(3,2,1)
$0 if any of$i,jork$are equal$ (17)

The alternating tensor also represents a 3x3x3 object with 27 elements of which ony 6 are non zero:

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1,\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \tag{18}$$

Importantly, ϵ_{ijk} remains unchanged if the suffixes are reordered by shifting to the right and putting the last suffix first, or by shifting to the left and putting the first suffix last:

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} \tag{19}$$

Further, the sign of the tensor changes if two suffixes are interchanged:

$$\epsilon_{ijk} = -\epsilon_{ikj} \tag{20}$$

Also, the alternating tensor is useful for expressing cross products:

$$(\vec{a} \times \vec{b})_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}a_jb_k = \epsilon_{ijk}a_jb_k \tag{21}$$

Proof:
To verify this, consider the case $i = 1$:

$$(\vec{a} \times \vec{b})_1 = \epsilon_{1jk}a_jb_k = \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{1jk}a_jb_k \tag{22}$$

The only non-zero contributions to ϵ_{1jk} are for values $j = 2,\ k = 3$ and

$j = 3,\ k = 2$. Hence:

$$(\vec{a} \times \vec{b})_1 = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2 \tag{23}$$

Similarly for $i = 2, 3$ A useful equivalence is the scalar triple product in suffix notation, which can be written as:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_i (\vec{b} \times \vec{c})_i = \epsilon_{ijk} a_i b_j c_k \tag{24}$$

0.0.4 Relationship between δ_{ij} and ϵ_{ijk}

Consider the following relationship:

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \tag{25}$$

Note that k is a dummy suffix as it appears twice, and there are four free suffixes: i, j, k, l and m. Therefore, this expression represents $3^4 = 81$ different equations.
Proof:
We only need to consider one case, e.g $i = 1$, since the three coordinate axes are equivalent. Consider the possible values for j.
• If $j = 1$, $\epsilon_{ijk} = \epsilon_{11k} = 0$, and so the L.H.S is zero the R.H.S is $\delta_{1l}\delta_{1m} - \delta_{1l}$ which is also zero.
• If $j = 2$ $\epsilon_{ijk} = \epsilon_{12k} = 0$ unless $k = 3$ then $\epsilon_{ijk} = \epsilon_{123} = 1$. Therefore only the $k = 3$ terms contribute to the sum. When $k = 3$, the term $\epsilon_{klm} = 0$ unless l and m are 1 and 2. Therefore, the L.H.S takes the value +1 if $l = 1$ and $m = 2$, -1 if $l = 2$ and $m = 1$, and zero otherwise. The R.H.S is $\delta_{1l}\delta_{2m} - \delta_{1m}\delta_{2l}$ which also has the same equality for the given values of m and l.
• If $j = 3$, similar arguments as for $j = 2$ apply.
NB: This relation will be useful whn considering terms involving two vector cross products.

[11pt]article MTH2004 - Vector Calculus and Applications: Scalar and Vector Fields

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0.1 Definitions

Many physical quantities have values at every different point in a particular region of space. For example:

[scale = 1]survey$_map$

Figure 1: Graph with Contour Lines

a) The temperature in a room.
b) Gravitational acceleration.
c) Velocity of water flow.

The term **field** is used to mean both the region of space and the value of the physical quantity in that region:

- For a scalar quantity: a **scalar field** $\phi(\vec{r}) = \phi(x, y, z)$
- For a vector quantity: a **vector field** $\vec{F}(\vec{r}) = \vec{F}(x, y, z)$

0.1.1 Level Curves and Level surfaces

The gravitational potential of Earth is a scalar field and near the surface can be approximated as: $\phi(z) = gz$. Where an arbitrary height z = 0 has been chosen as the reference level. The potential field is related to the gravitational potential energy U, between a mass m and the Earth as $U = mgz = m\phi$.

Suppose on a hill we draw a curve corresponding to a constant value of $\phi(\vec{r}) = C$. This curve is called a **level curve** of ϕ.

These level curves correspond to the contour lines on an ordinance survey map that indicate height. **Level surfaces** of a scalar field are surfaces where all points share the same value of the scalar field, $\phi(\vec{r}) = C$.

Example of Level Curves Consider the function$f(x, y) = x^2 + y^2$, the level curves are $x^2 + y^2 = C$ are centric circles of radius $\sqrt{C}$.

[scale = .70]level$_curve_eg$

Figure 2: Level Curve example where $C = 16$

Example 2: Level Surfaces For $f(x, y, z) = x^2+y^2+z^2$ the level surfaces are $x^2 + y^2 + z^2 = C$ are are concentric spheres of radius $\sqrt{C}$.

[scale $=$ .70]level$_surface_eg$

Figure 3: Level Surface example where $C = 16$

0.1.2 Vector Fields and Field Lines

Vector fields in two dimensions can be visualised by drawing the vector at a sequence of points or on a grid, with **the length and direction of the arrow denoting the magnitude and direction of the vector** respectively.
A **field line** is a curve whose tangent is parallel to the vector field at each point along the curve. With respect to fluid dynamics, field lines are known as **streamlines** and show the direction in which fluid particles travel.
The **density of field liens** is an indication of the magnitude of the vector field.

0.2 Differentiating Scalar Fields

We define the **gradient** of a scalar field $f(x, y, z)$ as the vector
:

$$grad\ f = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \tag{26}$$

where ∇ is a differential operator:

$$grad = \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\ ) = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{27}$$

NB: despite f being a scalar field, ∇f is a vector field!

∇f with respect to level surfaces The gradient ∇f is orientated perpendicular to the level surfaces $f(\vec{r}) = C$, in the direction of the steepest ascent, with magnitude equal to the rate of change in that direction. Mathematically:

$$|\nabla f| = \frac{df}{ds} \tag{28}$$

Where s is the distance measured along the surface normal vector (parallel to ∇f) $\hat{n}$.

The properties of the gradient For any scalar functions of position $f(x, y, z)$ and $g(x, y, z)$ and any constant c: $\nabla(f + g) = \nabla f + \nabla g$
$\nabla(cf) = c\nabla f$
$\nabla(fg) = f\nabla g + g\nabla f \; [productrule]$
$\nabla f(g) = f'(g)\nabla g \; [chainrule]$

0.2.1 The directional derivative

Previously we considered the rate of change of a scalar field in the direction normal to its level surfaces. But we may calculate the rate of change in any arbitrary direction.
Consider a direction, defined as the unit vector $\hat{a}$, and a displacement vector $d\vec{r} = \hat{a}ds$, where ds is the distance along $\vec{a}$. Then:

$$df = \nabla f \cdot \hat{a}\; ds \Rightarrow \frac{df}{ds} = \nabla f \cdot \hat{a} = (\hat{a} \cdot \nabla)f \tag{29}$$

This is known as the **directional derivative** of f in the direction of $\hat{a}$. We can also write it as:

$$\frac{df}{ds} = |\nabla f| \cos(\theta) \tag{30}$$

where θ is the angle between ∇f and $\hat{a}$.
NB: Since $-1 \leq \cos\theta \leq 1$:
• f increases most rapidly in the direction of ∇f
• f decreases most rapidly in the direction of $-\nabla f$

0.2.2 Vector functions

Parameterised curves

A curve C in three-dimensional space is an inherently on-dimensional object. The position vector of any point along the curve may be characterised using one parameter:

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \tag{31}$$

Particle moving in time:

$$\vec{r'}(t) = \vec{v}[velocity]$$
$$—\vec{r}(t)| = v[speed]$$
$$\vec{r''}(t) = \vec{a}(t)[acceleration]$$
$$\hat{v} = \frac{\vec{v}(t)}{|\vec{v}(t)}$$

Tangent vector and tangent lines

The derivative of the position vector $\vec{r}(t)$ w.r.t the parameter t is the **tangent vector**:

$$\vec{r'}(t) = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

(32)

The **tangent line** to the curve at point P with position vector $\vec{r}(t_0)$ is the line through the point P and parallel with $\vec{r'}(t_0)$:

$$\vec{q}(w) = \vec{t_0} + w\vec{r'}(t_0) \tag{33}$$

Length of a curve

Planar curve The length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$ is:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{34}$$

Suppose that C can be described by the parametric $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y}$, $t \in [\alpha, \beta]$, where $\frac{dx}{dt} = x'(t) > 0$. This means that C is traversed once, from left to right as t increases from α to β and $x(\beta) = b$.
If $\frac{dx}{dt} = x'(t) \neq 0$, the slope $\frac{dy}{dx}$ of the tangent to the curve can be found via the chain rule:

$$\text{dy}\overline{dt = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{y'(t)}{x'(t)}}$$

If we think of the curve as being traced by a moving particle, then $x'(t)$ and $y'(t)$ are the horizontal and vertical velocities of the particle, respectively, and the expression above says that the slope of the tangent is the ratio of these velocities.
Using the substitution rule, we obtain:

$$\text{L} = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \frac{dx}{dt}\, dt$$

since $\frac{dx}{dt} = x'(t) > 0$, we have:

$$\text{L} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Generalisation The length of a smooth curve $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$ for $a \leq t \leq b$, is:

$$\ell = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b |\vec{r'}(t)|\, dt = \int_a^b \sqrt{\vec{r'}(t) \cdot \vec{r'}(t)}\, dt \tag{35}$$

If we replace the fixed value b by the parameter t, we obtain the arc length as a function of t:

$$s(t) = \int_{t_0}^t |\vec{r'}(\tau)| d\tau \tag{36}$$

which is the directed distance between points $P(\vec{r}(t_0))$ to $P(\vec{r}(t))$ along the curve.
Further, if we differentiate and square the arc length we obtain:

$$(ds)^2 = d\vec{r} \cdot d\vec{r} = (dx)^2 + (dy)^2 + (dz)^2 \tag{37}$$

Where ds is called the **line element** of the curve C .