MTH2004 - Vector Calculus and Applications: Scalar and Vector Fields

January 18, 2022

1 Definitions

Many physical quantities have values at every different point in a particular region of space. For example:

a) The temperature in a room.
b) Gravitational acceleration.
c) Velocity of water flow.

The term **field** is used to mean both the region of space and the value of the physical quantity in that region:

- For a scalar quantity: $\phi(\vec{r}) = \phi(x, y, z)$
- For a vector quantity: $\vec{F}(\vec{r}) = \vec{F}(x, y, z)$

1.1 Level Curves and Level surfaces

The gravitational potential of Earth is a scalar field and near the surface can be approximated as: $\phi(z) = gz$. Where an arbitrary height z = 0 has been chosen as the reference level. The potential field is related to the gravitational potential energy U, between a mass m and the Earth as $U = mgz = m\phi$.

Suppose on a hill we draw a curve corresponding to a constant value of $\phi(\vec{r}) = C$. This curve is called a **level curve** of ϕ.

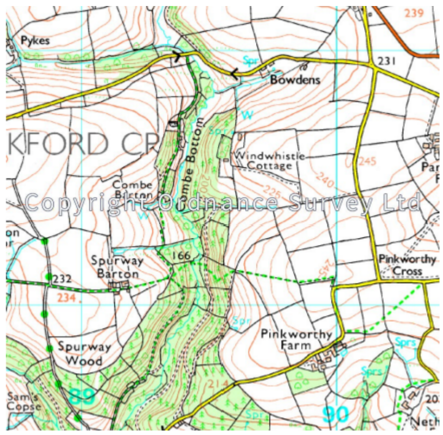

Figure 1: Graph with Contour Lines

These level curves correspond to the contour lines on an ordinance survey map that indicate height. **Level surfaces** of a scalar field are surfaces where all points share the same value of the scalar field, $\phi(\vec{r}) = C$.

Example of Level Curves Consider the function$f(x, y) = x^2 + y^2$, the level curves are $x^2 + y^2 = C$ are centric circles of radius $\sqrt{C}$.

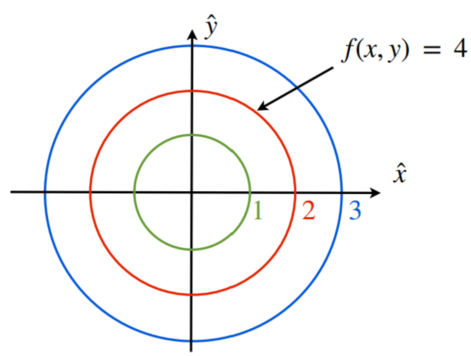

Figure 2: Level Curve example where $C = 16$

Example 2: Level Surfaces For $f(x, y, z) = x^2+y^2+z^2$ the level surfaces are $x^2 + y^2 + z^2 = C$ are are concentric spheres of radius $\sqrt{C}$.

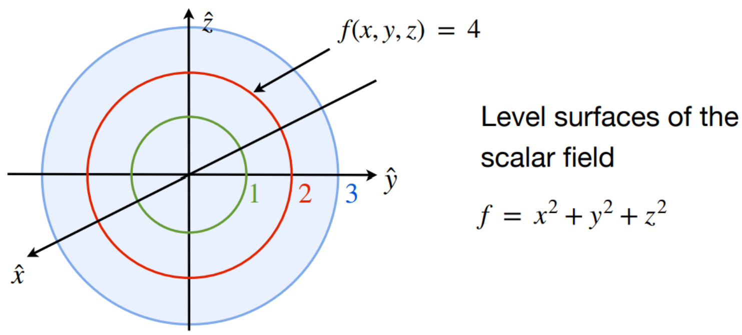

Figure 3: Level Surface example where $C = 16$

1.2 Vector Fields and Field Lines

Vector fields in two dimensions can be visualised by drawing the vector at a sequence of points or on a grid, with **the length and direction of the arrow denoting the magnitude and direction of the vector** respectively.

A **field line** is a curve whose tangent is parallel to the vector field at each point along the curve. With respect to fluid dynamics, field lines are known as **streamlines** and show the direction in which fluid particles travel.

The **density of field liens** is an indication of the magnitude of the vector field.