Integration of Vectors

March 2, 2022

1 Evaluation of Line Integrals

In a line integral, we integrate a given function $f(x, y, z)$ along a curve C in space from point a at location $r(\vec{a})$ to point b at location $r(\vec{b})$.
In order to achieve this, we describe the curve C by its parametric representation in Cartesian coordinates: $r(\vec{t}) = (x(t), y(t), z(t))$. The curve C is called the path of integration. $P = r(\vec{a})$ is its start point $Q = r(\vec{b})$ is its end point. The curve C is oriented positively in the direction from P to Q and is denoted by an arrow. If the points P and Q coincide the path is **closed**.

1.1 Line integral of vector field

The line integral of a vector field $\vec{F}$ over a curve C with parametric representation $r(\vec{t})$ is:

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(r(\vec{t})) \cdot \frac{dr(\vec{t})}{dt} dt \tag{1}$$

where $d\vec{r}$ is the curve's displacement vector or line element. The Line Integral can also be represented as:

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C F_x\, dx + F_y\, dy + F_z\, dz \tag{2}$$

1.2 Other forms of line integrals

The most common type of line integral is $\int_C \vec{F} \cdot d\vec{r}$. This line integral occurs in areas of physics to calculate the work done by a constant force field $\vec{F}$ in moving an object along the curve C.
If we introduce the **unit tangent vector** $\vec{T} \equiv \frac{d\vec{r}}{ds}$ for arc length s along the

curve C, this i also often expressed as the line integral with respect to arc length of the tangential component of the vector field:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds \tag{3a}$$

Even though integrals with respect to arc length are unchanged when orientation is reversed, it is still true that:

$$\int_C \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r} \tag{3b}$$

this is because the unit tangent vector $\vec{T}$ is replaced by its negative when C is replaced by $-C$

However, line integrals also come in other forms where ϕ is a scalar field, and $ds = |d\vec{r}|$. All are solved in the same way via parameterization of the curve.

1. $\int_C \phi\, ds$ **The result is a scalar as** ϕds **is a scalar**
2. $\int_C \phi\, d\vec{r}$ **The result is a vector because** $d\vec{r}$ **is a vector**
3. $\int_C \vec{F} \times d\vec{r}$ **The result is a vector as** $\vec{F} \times d\vec{r}$ **is a vector**

1.3 Conservative vector fields

In general, a line integral depends on the vector field, the start and end points, and the shape of the path the curve takes. There are special cases when the line integral does not depend on the shape of the path taken.

A vector field $\vec{F}$ is said to be **conservative** if its line integral around any closed curve C is zero, **i.e**

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \tag{4a}$$

An equivalent definition is that $\vec{F}$ is conservative if the line integral along a curve only depends on the star and end point but not on the path taken, **i.e**:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \tag{4b}$$

where C_1 and C_2 are any two curves that share the same start and end points

1.3.1 Potential function of a vector field

If a vector field $\vec{F}$ is related to a scalar field ϕ by $\vec{F} = \nabla\phi$ with $\nabla\phi$ existing everywhere within some region D, then $\vec{F}$ is conservative within D.
Conversely, if $\vec{F}$ is conservative, then $\vec{F}$ can be written as the gradient of a scalar field, $\vec{F} = \nabla\phi$

Notes

1. The potential ϕ is not unique since an arbitrary constant can be added to ϕ without it affecting $\vec{F} = \nabla\phi$.
2. The curl of a conservative field is zero: $\nabla \times \vec{F} = \nabla \times \nabla\phi = \vec{0}$. Thus, a vector field with zero curl mut be conservative.
3. The line integral of a conservative vector field depends only on the start and end points of the curve, and not on the path taken

2 Surface integrals

Similarly to integrating fields over a curve yielding a line integral, we can also integrate fields over surfaces. To achieve this, we must first see how to represent a surface.

2.1 Representation of surfaces

Surfaces are two-dimensional objects and require two parameters to be defined. We choose the name those two parameters u and v, and define the surface as the structure traced out by the position vector:

$$\vec{r}(u, v) = (f(u, v), g(u, v), h(u, v)) \tag{5}$$

Where u and v vary in some region R of the uv-plane.

2.2 Tangent plane and surface normal

If in the surface $\vec{r}(u, v)$ we fi one parameter, say $u = u_0$, and vary the other, then we obtain the equation of a curve on the surface. **e.g**

On a spherical shell of radius a, we can fix $\theta = \theta_0 = \pi/4$, to obtain:

$$\vec{r}(\theta_0, \varphi) = (a\sin(\theta_0)\cos(\varphi), a\sin(\theta_0)\sin(\varphi), a\cos(\theta_0)) \tag{6a}$$

$$= \frac{a}{\sqrt{2}}(\cos(\varphi), \sin(\varphi), 1) \tag{6b}$$

This is a circle of radius $a/\sqrt{2}$ on the plane $z = a/\sqrt{2}$

More generally, for $\vec{r}(u, v)$ we have two curves at a point $P = (u_0, v_0)$ and we can define the tangent vectors to those curves:

$$\frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u}\right) \tag{7a}$$

$$\frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right) \tag{7b}$$

The tangent vectors define the tangent plane to the surface at P. The cross product of $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$ gives the normal vector $\vec{n}$ of the tangent plane at P **i.e**

$$\vec{n} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}, \;\; (\neq 0) \tag{7c}$$

By convention we define the **surface normal** to be the unit vector locally perpendicular to the surface and pointing outwards for closed surfaces, **i.e**

$$\hat{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}|} \tag{7d}$$

2.3 Surface Area

We already know from geometry that the area of the parallelogram spanned by two vectors $\vec{a}$ and $\vec{b}$ with an angle θ between them is $|\vec{a} \times \vec{b} = ab|\sin(\theta)|$. We apply this parameterised surface to define the area of the parallelogram spanned by the two surface tangent vectors.
This allows us to find the area of the parameterised surface:

$$\text{Area} = \iint_S dA = \iint_R \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\, dv \tag{8}$$

where R is the region of the uv-surface.

2.4 Explicitly defined surface by graph

Sometimes a surface is defined explicitly tby the graph of a function. In this case, we parameterise the surface using $u = x$ and $v = y$:

$$\vec{r}(u, v) = \vec{r}(x, y) = (x, y, f(x, y)) \tag{9a}$$

and the tangent vectors:

$$\frac{\partial \vec{r}}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x}\right) \tag{9b}$$

$$\frac{\partial \vec{r}}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y}\right) \tag{9c}$$

Hence,

$$\vec{n} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) \tag{9d}$$

$$\left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \tag{9e}$$

So that the surface element is:

$$dA = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx\ dy \tag{9f}$$

Then the area of the surface $z = f(x, y)$ becomes

$$\iint_S dA = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx\ dy \tag{9g}$$

2.5 Surface integral of a scalar field

We define the integral of a scalar field φ over the surface S as:

$$\iint_S \varphi dA = \iint_R \varphi(\vec{r}(u, v)) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\ dv \tag{10}$$

Note when $\varphi = 1$ we calculate the area of the parameterised surface.

2.6 Surface integral of a vector field and flux

For a surface S parameterised as $\vec{r}(u, v)$ and a vector field $\vec{F}(\vec{r})$, we define the integral of that vector field along the sruface as:

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_R \vec{F} \cdot \hat{n} \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\ dv \tag{11}$$

The scalar quantity $\iint_S \vec{F} \cdot \hat{n} dA$ is called the **flux** of the vector field $\vec{F}$ through the surface S.
NB: for closed surfaces we use the notation:

$$\oiint \vec{F} \cdot d\vec{S} \tag{12}$$

2.6.1 Physical interpretation

Suppose a fluid flows with velocity $\vec{u}$ through a pipe with cross-sectional area A. The **flux** through the pipe is the total volume of fluid passing through the pipe per unit time, or the flux across the surface S with are A.

1. Suppose the pipe's cross-sectional area is constant and equal to A, and that **the velocity is uniform and directed parallel** to the walls of the pipe with speed $u_0 = |\vec{u}|$. Then the fluid moves along the pipe as if it were a solid block. In a time interval Δt the fluid passes a distance $u_0 \Delta t$ and so a block of fluid of volume $u_0 \Delta t A$ emerges from the end of the pipe. The flow rate or flux Q of fluid through the pipe is therefore this volume per unit time, **i.e** $Q = u_0 A$.

2. Suppose **the speed** of the flow is still parallel to the walls but **depends on the position within the pipe i.e** $|\vec{u}| = u_0(x, y)$. Assume the cross-section of the pipe is a square $0 \leq x \leq 1$, $0 \leq y \leq 1$. A small surface element with area dA is a small rectangle with sides of lengths dx and dy , located a point (x, y) on S, **i.e** $dA = dx\ dy$
   The flux dQ of fluid across the surface element dA is: $dQ = u_0(x, y)dA = u_0(x, y)\ dx\ dy$. The total flux follows from adding up all small contributions dQ:

$$Q = \iint_S u_0(x, y)\ dx\ dy = \int_0^1 \int_0^1 u_0(x, y)\ dx\ dy \tag{13}$$

3. Consider the **general case** where both **velocity field $\vec{u}$ and surface S are arbitrary i.e** $\vec{u}$ may not be parallel to the walls and S may be a non-planar surface. Again, we consider the flux of $\vec{u}$ across the surface element dA. However, $\vec{u}$ is not necessarily perpendicular to the surface dA. Therefore, we need to extract the component of $\vec{u}$ perpendicular to the surface dA. We make use of the unit normal vector $\hat{n}$ that is perpendicular to the surface dA. Hence, the required component is $\vec{u} \cdot \hat{n}$. The flux across surface element dA is $dQ = \vec{u} \cdot \hat{n}\, dA$. The total flux Q is:

$$Q = \iint_S \vec{u}(x, y) \cdot \hat{n}\, dA \tag{14}$$

   Since $\hat{n}$ points outwards from the pipe, a positive (negative) value of Q implied fluid flowing out (into) the pipe.

3 Volume Integrals

Consider an object of volume V and density ρ. If ρ is uniform, then the mass of the object is $M = \rho V$. But hat if the density is a function of position, some $\rho(\vec{r})$?
We divide the volume into N small pieces with volume δV_i, $i = 1, 2, \dots, N$, called the **volume elements**. Then the mass of the volume element δV_i at position $\vec{r_i}$ is simply $\delta M_i = \rho(\vec{r_i})\delta V_i$. The total mass is the sum over all positions:

$$M = \sum_{i=1}^{N} \delta M_i = \sum_{i=1}^{N} \rho(\vec{r_i})\delta V_i \tag{15a}$$

In the limit of infinitesimally small volume elements and $N \to \infty$, we obtain a volume integral:

$$\iiint_V \rho dV = \lim_{N\to\infty} \sum_{i=1}^{N} \rho(\vec{r_i})\delta V_i \tag{15b}$$

If we set $\rho = 1$, we actually calculate the total volume itself

Definition Consider a closed surface in space enclosing a volume V, then the following **volume integrals, space integrals** or **triple integrals**:

$$\iiint_V \vec{A}\, dV \text{ and } \iiint_V \phi\, dV \tag{16}$$

where $\vec{A}$ is a vector field, ϕ is a scalar field, and V is a 3D volume.

3.1 Volume elements in different coordinate systems

1. **Cartesian coordinates** (x, y, z): $dV = dxdydz$
2. **Cylindrical coordinates** (R, φ, z): $dV = RdRd\varphi dz$
3. **Spherical coordinates** (r, θ, ϕ): $dV = r^2 \sin(\theta) dr d\theta d\varphi$

3.2 The Jacobian

We already know from geometry that the volume of the parallelepiped spanned by three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $|(\vec{a} \times \vec{b}) \cdot \vec{c}$. Applying this to the parameterised volume to define the volume of the parallelepiped spanned by three surface tangent vectors we get the **Jacobian**:

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) \cdot \frac{\partial \vec{r}}{\partial w}\Bigg| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{17a}$$

This allows us to write a volume, or triple, integral originally written in terms of Cartesian coordinates

$$\iiint_V f(x, y, z)\ dxdydz$$

instead in terms of another coordinate system (u, v, w):

$$\iiint_V f(u, v, w) J\ dudvdw$$

where f, J and the integration limits must be in terms of the new variables u, v and w

4 Integral Theorems

There are two important theorems that link line, surface and volume integrals with the definitions of the divergence and curl operators. These theorems have significance in deriving mathematical equations representing physical laws.

4.1 Divergence Theorem

The transformation between **triple** and **surface** integrals, which involves the **divergence** of a vector function $\vec{F}$.

Theorem Consider a continuously differentiable vector field $\vec{F}$ defined on a volume V that is enclosed by a surface S. Then:

$$\iiint_V \nabla \cdot \vec{F}\ dV = \oiint_S \vec{F} \cdot d\vec{S} \tag{18}$$

where $d\vec{S} = \hat{n}dS$.
This theorem says that the total expansion of a vector field over a volume is equal to the flux of that vector field through the bounding surface.

4.2 Stokes' Theorem

This describes the transformation between **surface** and **line** integrals, which involves the **curl** of a vector function $\vec{F}$

Theorem Consider a continuously differentiable vector field $\vec{F}$ defined within an open surface S. Then,

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} \tag{19}$$

where $d\vec{S} = \hat{n}dS$ and the direction of the line integral around C and the unit normal $\hat{n}$ are oriented in a right-hand sense **This theorem says that the total flux of curl of a vector field through a surface is equal to the closed line integral of that vector field around the surface boundary**