MTH2004 - Vector Calculus and Applications

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1 Introduction and Preliminaries

1.1 Vectors

A vector describes a quantity that has a magnitude and direction in three-dimensional space. A special type of vector is a **position vector** that always starts at the origin of the coordinate system and points to the position in question. A **unit vector** is a position vector of length one that defines the directions of the coordinate system.
Graphically, a vector is represented by an arrow that starts at a position, e.g. centre of mass, origin, and point in the correct direction.
Symbolically, a vector in this module is depicted as $\vec{x}$ and a unit vector as $\hat{x}$.
Mathematically, a vector is defined by its decomposition in the coordinate system. We will always assume orthogonal coordinate systems. In Cartesian coordinates x, y, z then:

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \tag{1}$$

where v_x, v_y, v_z are the x, y, z components of the vector.

Multiplication with a scalar Multiplying a vector with a scalar is multiplying each component. The result is still a vector:

$$\alpha\vec{v} = \alpha v_x\hat{x} + \alpha v_y\hat{y} + \alpha v_z\hat{z} \tag{2}$$

Adding or subtracting vectors Two vectors can be summed or subtracted from each other by adding or subtracting each component. This also results in a vector:

$$\vec{a} \pm \vec{b} = (a_x \pm b_x)\hat{x} + (a_y \pm b_y)\hat{y} + (a_z \pm b_z)\hat{z} \tag{3}$$

Magnitude of a vector For a position vector, it's magnitude equals the distance from the origin to the position. The result is a scalar:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{4}$$

Unit vector in direction of vector We can define a vector of unit length in the direction of a vector $\vec{v}$ as:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \tag{5}$$

NB: The unit vector does not have to start at the origin!

Scalar Product The scalar or dot product between two vectors is defined as the product between the magnitude of one vector and the projection of the other vector onto it:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\alpha \tag{6}$$

Note that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. Further, perpendicular vectors result in $\vec{a} \cdot \vec{b} = 0$.

Cross (or Vector) Product The Cross Product between two vectors produces a third vector that is perpendicular to both according to the right-hand rule.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{x} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = |\vec{a}||\vec{b}| \sin\alpha \tag{7}$$

Note that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ and also that $|\vec{a} \times \vec{b}|$ is the 'surface area' of the parallelogram made by $\vec{a}$ and $\vec{b}$.
Further note that **if two vectors are parallel then their cross product is zero**.

Scalar Triple Product The volume of the parallelepiped formed by $\vec{c}$, $\vec{a}$ and $\vec{b}$ is the magnitude of the scalar:

$$\vec{c} \cdot \vec{a} \times \vec{b} = \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = |\,\vec{c}\,||\vec{a} \times \vec{b}| \cos\theta \tag{8}$$

where θ is the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$.

Note that $\vec{c} \cdot \vec{a} \times \vec{b} = \vec{a} \cdot \vec{b} \times \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a}$. Further note that **three vectors are linearly independent if and only if their scalar triple product is non-zero**.

Vector Triple Product The vector $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$. Since the plane defined by $\vec{b}$ and *vecc* is perpendicular to $\vec{b} \times \vec{c}$, the triple product lies in this plane.

1.2 Differentiation Methods

Product Rule For two functions ft and gt:

$$\frac{d}{dt}(fg) = f\frac{dg}{dt} + g\frac{df}{dt} \tag{9}$$

Chain Rule For a function $f(g(t))$:

$$\frac{d}{dt}(f(g(t))) = \frac{df}{dg}\frac{dg}{dt} \tag{10}$$

Integration by Parts For two functions $u(t)$ and $v'(t)$

$$\int_a^b uv'\,dt = [uv]_a^b - \int_a^b u'v\,dt \tag{11}$$

1.3 Suffix Notation

Many vector expressions can be simplified and more easily derived if we introduce the suffix notation.

Summation Convention A suffix, that appears twice and no more within a term implies that the term is to be summed from $i = 1$ to 3. This repeated suffix is also referred to as a dummy suffix.
e.g.

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i \tag{12a}$$

$$(\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d}) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_i c_j d_j = a_i b_i c_j d_j = a_i c_j b_i d_j \tag{12b}$$

Kronecker Delta The Kronecker delta is defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{13}$$

The Kronecker delta also can represents the 3x3 unit matrix:

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

and this is symmetric, i.e. $\delta_{ij} = \delta_{ji}$

Kronecker delta and summation notation Consider $\delta_{ij} a_j = \sum_{j=1}^{3} \delta_{ij} a_j$. For $i = 1$ only $\delta_{i1} \neq 0$, hence $\delta_{1j} a_j = a_1$. Also, $\delta_{2j} a_j = a_2$ and $\delta_{3j} a_j = a_3$. Thus:

$$\delta_{ij} a_j = a_i \tag{15}$$

Alternating Tensor The alternating tensor (or permutation tensor) ϵ_{ijk} is defined as:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3), (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (1,3,2), (2,1,3) \text{ or } (3,2,1) \\ 0 & \text{if any of } i, j \text{ or } k \text{ are equal} \end{cases} \tag{16}$$

The alternating tensor also represents a 3x3x3 object with 27 elements of which ony 6 are non zero:

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \; \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \tag{17}$$

Importantly, ϵ_{ijk} remains unchanged if the suffixes are reordered by shifting to the right and putting the last suffix first, or by shifting to the left and putting the first suffix last:

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} \tag{18}$$

Further, the sign of the tensor changes if two suffixes are interchanged:

$$\epsilon_{ijk} = -\epsilon_{ikj} \tag{19}$$

Also, the alternating tensor is useful for expressing cross products:

$$(\vec{a} \times \vec{b})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a_j b_k = \epsilon_{ijk} a_j b_k \tag{20}$$

Proof:
To verify this, consider the case $i = 1$:

$$(\vec{a} \times \vec{b})_1 = \epsilon_{1jk} a_j b_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{1jk} a_j b_k \tag{21a}$$

The only non-zero contributions to ϵ_{1jk} are for values $j = 2,\ k = 3$ and $j = 3,\ k = 2$. Hence:

$$(\vec{a} \times \vec{b})_1 = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2 \tag{21b}$$

Similarly for $i = 2, 3$ A useful equivalence is the scalar triple product in suffix notation, which can be written as:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_i (\vec{b} \times \vec{c})_i = \epsilon_{ijk} a_i b_j c_k \tag{22}$$

1.4 Relationship between δ_{ij} and ϵ_{ijk}

Consider the following relationship:

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \tag{23}$$

Note that k is a dummy suffix as it appears twice, and there are four free suffixes: i, j, k, l and m. Therefore, this expression represents $3^4 = 81$ different equations.
Proof:
We only need to consider one case, e.g $i = 1$, since the three coordinate axes are equivalent. Consider the possible values for j.
• If $j = 1$, $\epsilon_{ijk} = \epsilon_{11k} = 0$, and so the L.H.S is zero the R.H.S is $\delta_{1l} \delta_{1m} - \delta_{1l}$ which is also zero.
• If $j = 2$ $\epsilon_{ijk} = \epsilon_{12k} = 0$ unless $k = 3$ then $\epsilon_{ijk} = \epsilon_{123} = 1$. Therefore only the $k = 3$ terms contribute to the sum. When $k = 3$, the term $\epsilon_{klm} = 0$ unless l and m are 1 and 2. Therefore, the L.H.S takes the value +1 if $l = 1$ and $m = 2$, -1 if $l = 2$ and $m = 1$, and zero otherwise. The R.H.S is $\delta_{1l} \delta_{2m} - \delta_{1m} \delta_{2l}$ which also has the same equality for the given values of m and l.

- If $j = 3$, similar arguments as for $j = 2$ apply.

NB: This relation will be useful whn considering terms involving two vector cross products.