Due: 23:55 Sunday 17 March 2019 .

Financial Engineering - Assignment 1

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Submission as a group is allowed. Please submit your solutions together with codes on time.

1) Suppose you are given the integral

$$\Theta = \int_2^4 5x^4 dx.$$

   a) Compute the integral analytically.
   b) Write an algorithm in R to find the Monte Carlo estimate $\hat{\Theta}_N$ of Θ where you use N number of generated $U(0,1)$ random variables.
   c) Now by using parts (a) and (b) you can come up with the corresponding Monte Carlo error. To understand the behavior of the error, generate $M = 1000$ realizations (i.e. run your algorithm in (b) 1000 times) for the estimates $\hat{\Theta}_{100}$, $\hat{\Theta}_{1000}$ and $\hat{\Theta}_{10000}$. Plot the histogram for each of the cases $(N, M) = (100, 1000)$, $(N, M) = (1000, 1000)$, $(N, M) = (10000, 1000)$. What type of behavior do you observe as N increases?

2) Suppose that we have a discrete random variable with possible values $c_1, c_2, \ldots, c_5$. The distribution of the random variable is defined by probabilities

$$p_1 = 0.1,\ p_2 = 0.1,\ p_3 = 0.3,\ p_4 = 0.1,\ p_5 = 0.4.$$

In order to use the *inverse transform method* to generate from X, first we find the cumulative probabilities

$$P_1 = 0.1,\ P_2 = 0.2,\ P_3 = 0.5,\ P_4 = 0.6,\ P_5 = 1.0,$$

and then we draw a uniform random number, say U = 0.55. For each cumulative probability P, we check if $U > P$, which yields the vector

$$[1, 1, 1, 0, 0],$$

where 1 corresponds to "TRUE" and 0 to "FALSE". To select the correct value to return as realization of X, we must sum the ones in this vector and add 1. This gives us the index, 4, hence the value to return is c_4.

   a) Use above idea and write an algorithm in R (or in another program) for generating M random values from a given random variable X with n possible values $c_1, \ldots, c_n$ and corresponding probabilities $p_1, \ldots, p_n$.
   b) Now use your algorithm to generate 10000 values for the random variable Y with possible values $c_1, c_2, \ldots, c_5$ with probabilities

$$p_1 = 0.1,\ p_2 = 0.3,\ p_3 = 0.4,\ p_4 = 0.15,\ p_5 = 0.05.$$

   Visualize your results by plotting a histogram.

3) Write an algorithm in which you use the *composition method* to generate M random values from a random variable with $Hyperexp(\lambda_1, \alpha_1, \lambda_2, \alpha_2)$ distribution, i.e, X with the density

$$f(x) = \sum_{i=1}^{2} \alpha_i \lambda_i e^{-\lambda_i x}.$$

Run your algorithm for $M = 10000$, $\lambda_1 = 0.5$, $\lambda_2 = 2$, $\alpha_1 = 0.7$, $\alpha_2 = 0.3$. Visualize your results by plotting, e.g., a histogram.

4) Let G be a distribution function with density g and suppose, for constants $a < b$, we want to generate a random variable from the distribution function

$$F(x) = \frac{G(x) - G(a)}{G(b) - G(a)}, \quad a \leq x \leq b.$$

a) If X has distribution G, then F is the conditional distribution of X given what information?

b) Show that the *acceptance-rejection method* reduces in this case to generating a random variable X having distribution G and then accepting it if it lies between a and b.

5) Suppose you want to simulate from $X \sim Beta(4, 3)$, that is X has the density:

$$f(x) = 60x^3(1 - x^2), \quad 0 \leq x \leq 1.$$

Simulate 1000 values by using the *acceptance-rejection method* where $Y \sim U(0, 1)$. *Hint: In order to choose a reasonable constant c you can check the maximum value of* $Beta(4, 3)$ *over* $(0, 1)$ *(In R, this amounts to maximize the function* $dbeta(x, 4, 3)$ *over* $(0, 1)$*).*

6) Suppose you are given the random vector $(U, -\log(U))$ where $U \sim U(0, 1)$.

a) Write an algorithm in which you use Monte Carlo simulation to approximate the covariance $\Theta = Cov(U, -\log(U))$ by using N uniform random variates.

b) Actually, it is known that for $U \sim U(0, 1)$, the new random variable, say Y, defined by $Y = -\log(U)$ is exponentially distributed with parameter 1 ($exp(1)$). Use this and try to compute the covariance analytically.

c) Now combine part (a) and (b): Run your algorithm for

$$N = 10,\ 100,\ 200,\ 500,\ 1000,\ 2000,\ 5000,\ 7000,\ 10000.$$

Plot the Monte-Carlo error and also provide a table of errors as a function of the given N values.