Fresnel Diffraction Solution Using Sinc Approximation

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April 2022

1 Introduction

To describe the propagation of a light beam through a medium one begins with the general wave equation for a Nonlinear Optical Media derived from the Maxwell equations. In this paper we focus on solving the Helmholtz Paraxial equation, a slowly-varying envelope function for monochromatic waves and short-range approximation to the general wave equation. The solution to the Helmholtz Paraxial equation is known as the Fresnel Diffraction approximation. The purpose of this research is to compare the computational accuracy and times of the Fresnel Diffraction approximation using the spectral method (FFT) and a sinc based method. The paper is organized to 1. Introduce the general wave equation for Nonlinear Optical Media, 2. Introduce the Homogenuous Helmholtz equation (Linear medium), 3. Give an overview of the Derivation for the Helmholtz Paraxial equation, 4. Introduce the solution of the Helmholtz Paraxial equation using the spectral and sinc methods, 5. Give numerical results with the initial conditions of a Gaussian Beam and Circular Aperture, 6. Explain the findings of the research.

In the case of the Fresnel Diffraction Approximation we found the FFT based method introduces artificial periodic boundary conditions and the accuracy of the algorithm depends on propagation distance, wavelength, and observation plane discretization whereas the sinc based method relies on only how well the source field (initial condition) is approximated.

2 Wave equation

[5] To describe the propagation of a light beam through a medium one uses the general wave equation for a Nonlinear Optical Media (Derived from the Maxwell equations) is given by,

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

where the laplacian is denoted as,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For a monochromatic wave (one single frequency), the wave can be modeled in the general form as,

$$u(\boldsymbol{r}, t) = a(\boldsymbol{r}) \cos\left[2\pi\nu t + \phi(\boldsymbol{r})\right]$$

Where,

$$\begin{aligned} a(\boldsymbol{r}) &= amplitude & \phi(\boldsymbol{r}) &= phase \\ \nu &= frequency & 2\pi\nu &= angular\ frequency \end{aligned}$$

The complex representation of the monochromatic wave is thus,

$$U(\boldsymbol{r}, t) = a(\boldsymbol{r}) \exp(i\phi(\boldsymbol{r})) \exp(i2\pi\nu t)$$

3 Helmholtz equation

By substituting $U(\boldsymbol{r}, t) = U(\boldsymbol{r}) \exp(i2\pi\nu t)$ into the wave equation we get the Helmholtz equation or "The wave equation for monochromatic waves". Which becomes,

$$(\nabla^2 + k^2)U(\boldsymbol{r}) = 0$$

where $k = \frac{2\pi}{\lambda}$ is the wave number and λ is the wavelength.

1. Helmholtz Paraxial equation
   Writing the Helmholtz equation as an envelope function where U takes the form,

$$U(\boldsymbol{r}) = \psi(\boldsymbol{r})e^{ikz}$$

the Helmholtz equation becomes,

$$(\partial_z^2 + 2ik\partial_z + \nabla_T^2)\psi(\boldsymbol{r}) = 0$$

Where ∇_T^2 is the transverse laplacian given by $\partial_x^2 + \partial_y^2$. For a close range approximation we proceed with the slowly varying envelope approximation where we assume $\delta\psi \ll \psi$ which allows us to drop the ∂_z^2 term to achieve the Helmholtz Paraxial equation,

$$(2ik\partial_z + \nabla_T^2)\psi(\boldsymbol{r}) = 0$$

in which the solution is known as the Fresnel Diffraction Approximation. For a full derivation of the Helmholtz Paraxial equation refer to [5] where specific conditions are established.

4 Solving the Helmholtz Paraxial equation

Suppose U propagates in the $+z$ direction so it takes on the form,

$$U(x, y, z) = e^{ikz}\psi(x, y, z),$$

and ψ is a slowly-varying envelope function. We note U must sasisfy,

$$-2ik\partial_z = \nabla_T^2\psi$$
$$\psi(x, y, 0) = u(x, y)$$

Where $u(x, y)$ is the initial data and $\nabla_T^2 = \partial_x^2 + \partial_y^2$.
Solving the integral through the Fourier Transform i.e.

$$\partial_x^2 = (i\xi)^2, \partial_x^2 = (i\eta)^2$$

we get,

$$-2ik\frac{\partial\hat{\psi}}{\partial z} = ((i2\pi\xi)^2 + (i2\pi\eta)^2)\hat{\psi} \implies \hat{\psi} = ce^{\frac{2\pi^2 z}{ik}(\xi^2+\eta^2)}$$

To satisfy the initial condition we set $c = \hat{u}(\xi, \eta)$ to get,

$$\hat{U}(\xi, \eta) = e^{ikz}\hat{u}(\xi, \eta)e^{\frac{-i2\pi^2 z}{k}(\xi^2+\eta^2)}$$

Taking the inverse Fourier Transform to obtain the solution,

$$U(X,Y) = e^{ikz}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(\frac{-i2\pi^2 z}{k}(\xi^2+\eta^2)\right)\hat{u}(\xi,\eta)e^{i2\pi(X\xi+Y\eta)}d\xi d\eta$$

Where we invoke the convolution theorem to achieve the Fresnel Diffraction Approximation,

$$U(X,Y) = \frac{-ike^{ikz}}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{ik}{2z}((X-x)^2+(Y-y)^2)}u(x,y)dxdy$$

5 Computation FFT vs. Sinc

1. **Spectral Method - FFT Convolution theorem**

   We first note we can write the Fresnel Diffraction Approximation in terms of a convolution. Let the Fresnel kernel be denoted as,

   $$h_F(x,y) = \frac{-ike^{ikz}}{2\pi z}e^{\frac{ik}{2z}(x^2+y^2)}$$

   where,

   $$U(X,Y) = (h_F \star u)(X,Y) = \frac{-ike^{ikz}}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{ik}{2z}((X-x)^2+(Y-y)^2)}u(x,y)dxdy$$

   Now recalling the convolution theorem for Fourier Transforms,

   $$\mathcal{F}(h_F \star u)(\xi,\eta) = \hat{h_F}(\xi,\eta)\hat{u}(\xi,\eta)$$

   where,

   $$\hat{f}(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i2\pi(x\xi+y\eta)}dxdy$$

   We can analytically compute,

   $$\hat{h_f}(\xi,\eta) = \frac{-ike^{ikz}}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{ik}{2z}(x^2+y^2)}e^{-i2\pi(x\xi+y\eta)}dxdy$$

   $$\hat{h_f}(\xi,\eta) = e^{ikz}\exp\left(\frac{-i2\pi^2 z}{k}(\xi^2+\eta^2)\right)$$

Depending on the initial source $u(x, y)$ may need to be computed numerically thus we get,

$$U(X, Y) = \mathcal{F}^{-1}[\hat{h_F}\hat{u}]$$

$$= e^{ikz}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(\frac{-i2\pi^2 z}{k}(\xi^2+\eta^2)\right)\hat{u}(\xi,\eta)e^{i2\pi(X\xi+Y\eta)}d\xi d\eta$$

As the spectral method approximates the inverse Fourier Transform by the composite trapezoidal rule, $U(X, Y)$ is computed as,

$$U(X,Y) \approx e^{ikz}\frac{1}{L^2}\sum_{m=-L/2}^{L/2}\sum_{n=-L/2}^{L/2} \exp\left(\frac{-i2\pi^2 z}{k}(\xi_m^2+\eta_n^2)\right)\hat{u}(\xi_m,\eta_n)e^{i2\pi(X\xi_m+Y\eta_n)}d\xi d\eta$$

Where we can see artificial boundary conditions are imposed and waved reaching the boundary are reintroduced rather than dispersed through infinity.

2. **Sinc Method**

Let $f(x)$ be a band limited function (chopping off the initial condition where we want it to be true 0 outside a certain domain). Then f can be represented exactly by,

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \operatorname{sinc}\left(\frac{x-x_n}{\Delta x}\right)$$

The full derivation can be found in Shannon's Sampling Theorem. Thus we can rewrite our initial condition as,

$$u(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} u_{mn}\operatorname{sinc}\left(\frac{x-x_m}{\Delta x}\right)\operatorname{sinc}\left(\frac{y-y_n}{\Delta y}\right)$$

The solution the Helmholtz Paraxial equation is given by the convolution with the Fresnel kernel h_f,

$$U(X,Y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} u_{mn}\left(h_f \star (\operatorname{sinc}\left(\frac{x-x_m}{\Delta x}\right)\operatorname{sinc}\left(\frac{y-y_n}{\Delta y}\right))\right)(X,Y)$$

To generalize the right hand side (let us assume $\delta = \Delta x, \Delta y$) and let,

$$\Phi(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_f(X-x, Y-y)\,\mathrm{sinc}(\frac{x}{\delta})\,\mathrm{sinc}(\frac{y}{\delta})dxdy$$

Where we use the convolution property of the Fourier Transform and the fact that the Transform,

$$\mathcal{F}\left[\mathrm{sinc}(\frac{x}{\delta})\right](\xi) = \delta\,\mathrm{rect}(\delta\xi)$$

Giving us the inverse Fourier transform vanishes outside of the square $-\frac{1}{2} \le \xi\delta \le \frac{1}{2}$,$-\frac{1}{2} \le \eta\delta \le \frac{1}{2}$ thus let $W = \frac{1}{2\delta}$ then,

$$\Phi(X,Y) = \delta^2 \int_{-W}^{W}\int_{-W}^{W} \hat{h}_f(\xi,\eta)e^{i2\pi(X\xi+Y\eta)}d\xi d\eta$$

Which we can be accurately computed on a chosen domain. We can now rewrite our solution as,

$$U(X,Y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} u_{nm}\Phi(X-x_n, Y-y_m)$$

We can see clearly now the Sinc based method never assumes periodic boundary conditions. Thus the numerical solution will only depend on the fineness of the grid rather than the is propagation distance, wavelength, and observation plane discretization

A numerically efficient method is derived in [1] which for conciseness of the paper will not be derived but is used in the creation of the numerical results. The method derived was writing the solution in terms of matrix multiplications,

$$\boldsymbol{U}_{mn} = e^{ikz}\boldsymbol{w}^x_{mj}\boldsymbol{u}_{jl}(\boldsymbol{w}^y_{nl})^T$$

$$\begin{aligned} \boldsymbol{U}_{mn} &= U(X_m, Y_n) & \qquad \boldsymbol{w}^x_{mj} &= \phi(X_m - x_j) \\ \boldsymbol{u}_{jl} &= u(x_j, y_l) & \qquad \boldsymbol{w}^y_{nl} &= \phi(Y_n - y_l) \end{aligned}$$

Where,

$$\phi(X) = \frac{\delta}{\pi}\sqrt{\frac{k}{2z}}\exp\left(i\frac{X^2k}{2z}\right)\left(C(\mu_2) - C(\mu_1) - iS(\mu_2) - iS(\mu_1)\right))$$

$$C(x) = \int_0^x \cos(\mu^2)d\mu \qquad\qquad S(x) = \int_0^x \sin(\mu^2)d\mu$$

$$\mu_1 = -\pi\sqrt{\frac{2z}{k}}W - \sqrt{\frac{k}{2z}}X \qquad\qquad \mu_1 = \pi\sqrt{\frac{2z}{k}}W - \sqrt{\frac{k}{2z}}X$$

6 Numerical results

1. Gaussian beam initial condition
   Consider the Gaussian beam given by,

$$u(x, y) = \exp\left(-\frac{x^2 + y^2}{w_0^2}\right)$$

with wavelength $\lambda = 1\mu m$ and beam radius $w_0 = 1\text{cm}$. The exact solution given in [1] is

$$U(X, Y) = \frac{1}{\sqrt{1 + \left(\frac{2z}{kw_0^2}\right)^2}}\exp\left[-\frac{1}{w_0^2}\frac{X^2 + Y^2}{1 + \left(\frac{2z}{kw_0^2}\right)^2} + ikz - i\arctan\left(\frac{2z}{kw_0^2}\right) + i\frac{k}{2z}\frac{X^2 + Y^2}{1 + \left(\frac{kw_0^2}{2z}\right)^2}\right]$$

Figure 1

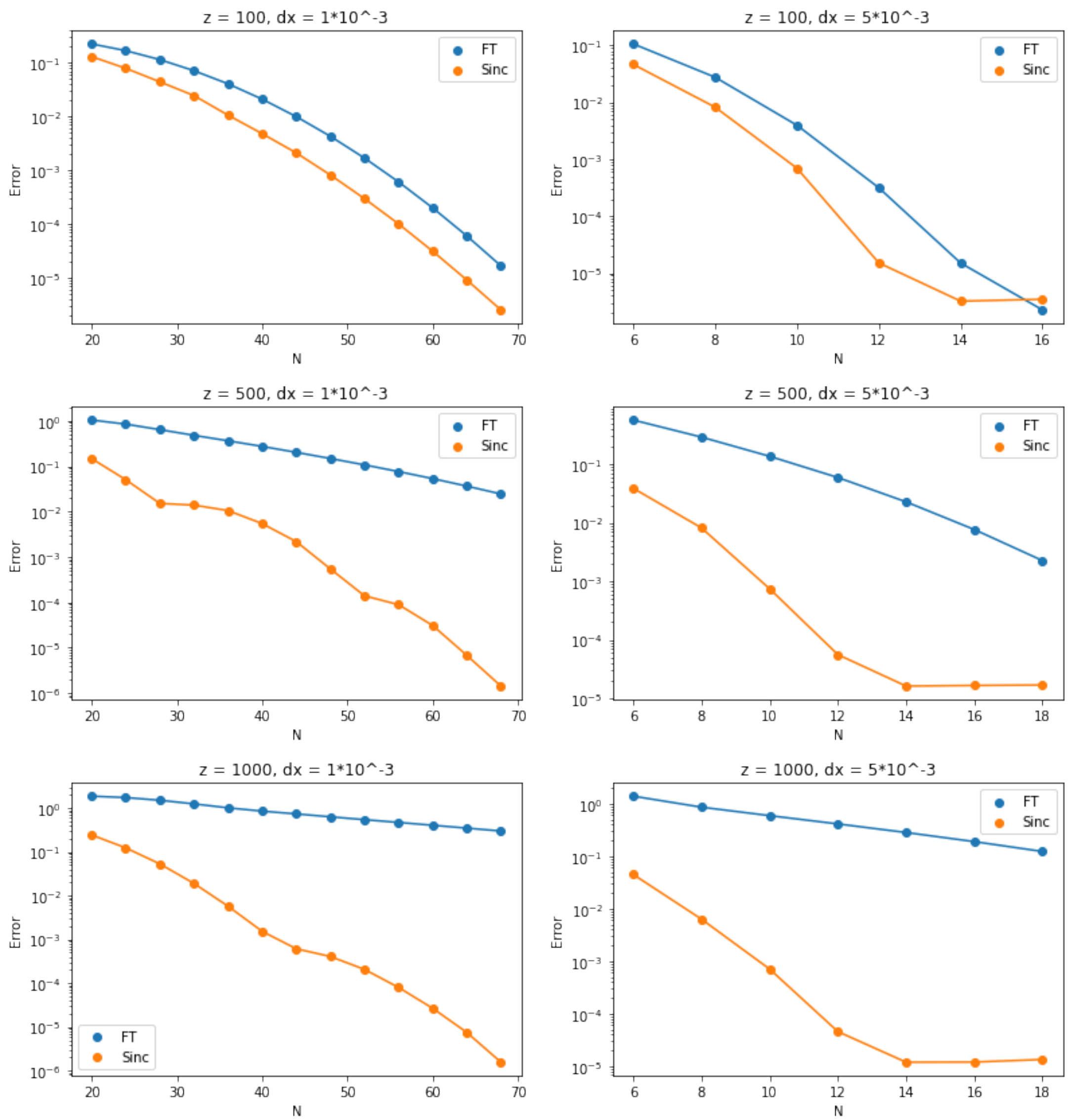


Figure 1: Figure shows the comparison of relative error versus number of points on $N \times N$ grid for three propagation distances $z = 100, 500, 1000$m and two sample discretization lengths (left) $\Delta x = 10^{-3}$, and (right) $\Delta x = 5 \cdot 10^{-3}$. We find here (with minor numerical noise) the accuracy of the Sinc method is independent of propagation distance(z) whereas FFT based method's accuracy reduces with the propagation of distance.

Figure 2

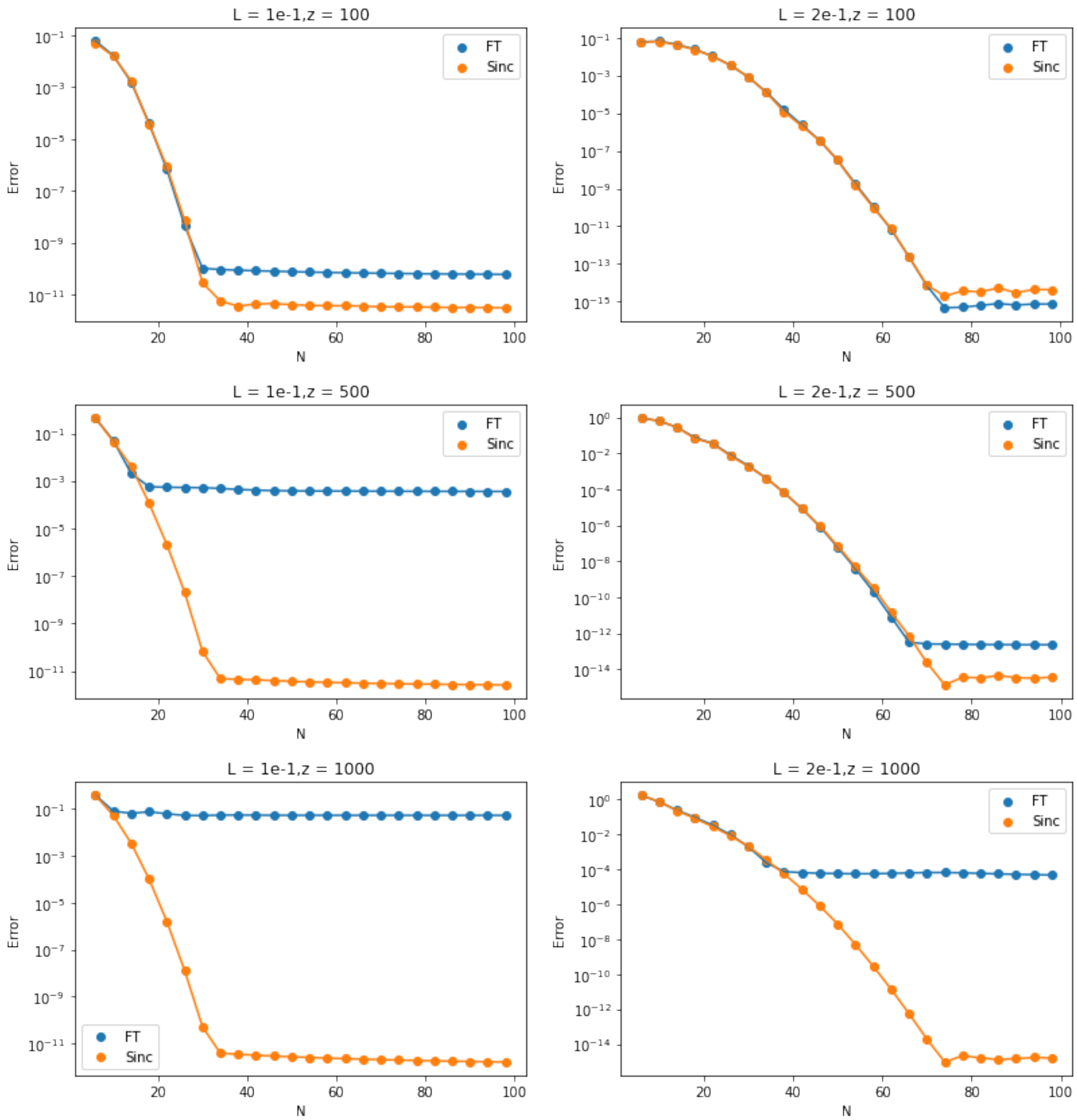


Figure 2: Figure shows the comparison of relative error versus number of points on $N \times N$ grid for three propagation distances $z = 100, 500, 1000$m and two sample lengths (left) $L = 10^{-1}$, and (right) $L = 2 \cdot 10^{-1}$. We find here (Similar to figure 1) the accuracy of the Sinc method is independent of propagation distance(z) whereas FFT based method's accuracy reduces with the propagation of distance.

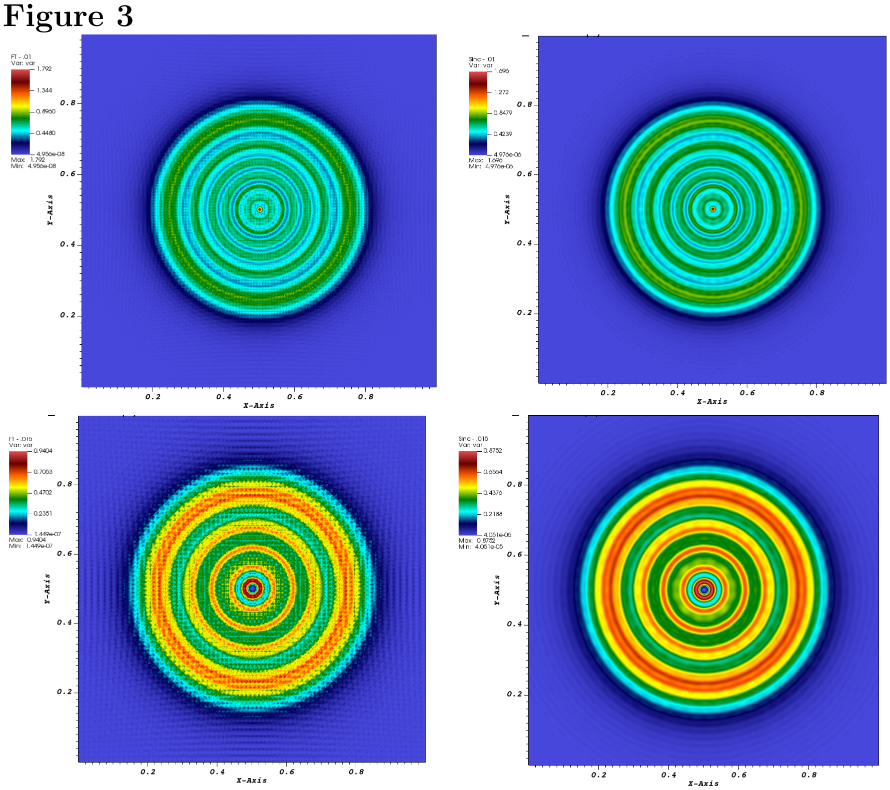
2. Circular aperture initial condition

$$u(x,y,z) = A circ\left(\frac{r}{r_0}\right)\frac{e^{ik\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

$$circ(r) = \begin{cases} 1, & \text{if}\, r \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

where the wavelength is $\lambda = 10\mu m$, $r = \sqrt{x^2+y^2}, r_0 = 10^{-3}$ is the circular aperature radius, $A = 3 \cdot 10^{-2}$ is the source amplitude, and $(x_0, y_0, z_0) = (0, 0, -3 \cdot 10^{-2})$ so the source is placed slightly behind $z = 0$. The figures are evaluated over an $N \times N$ grid with $N = 400$ and in the domain $(x, y) \in [-2 \cdot 10^{-3}, 2 \cdot 10^{-3}]^2$

Figure 3



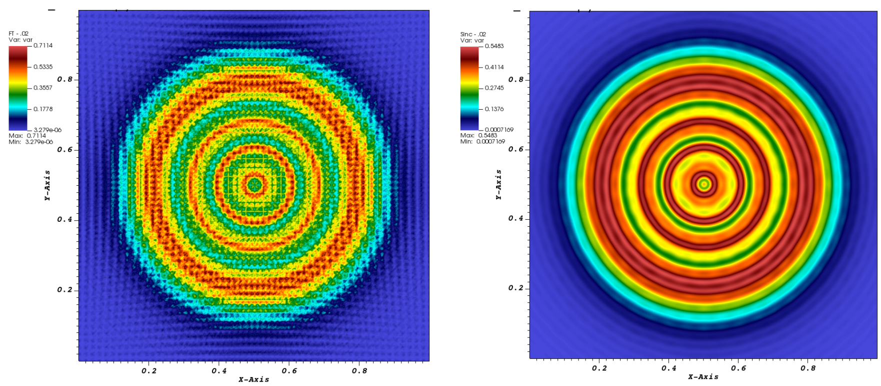


Figure 3: Figure show the Irradiance plots ($|U|^2$) for three propagation distances $z = .01, .015, .02$m for the (left) FFT based method and (right) Sinc based method. We see clearly here the FFT method introduces artificial periodic boundary conditions as waves hitting the boundary are reintroduced and the Sinc method is true zero outside of the computational domain.

Figure 4

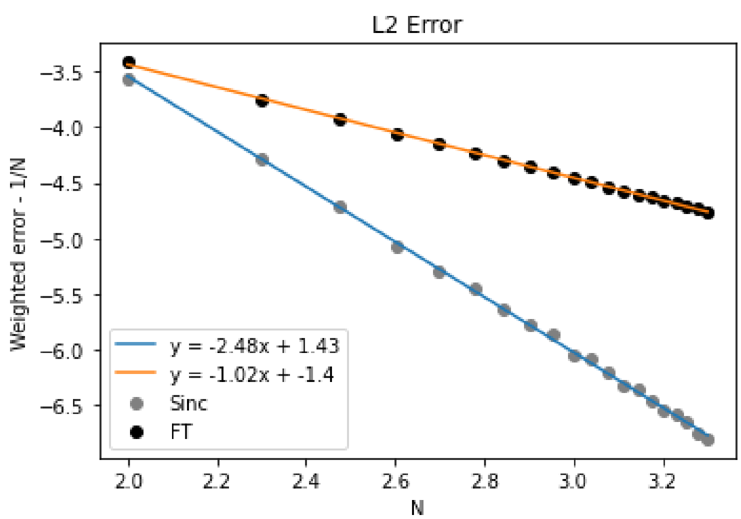


Figure 4: Figure shows the weighted error for the convergence rates of the circular aperture for the FFT and Sinc based methods. We find the FFT based method is approximately $O(N^{-1})$ and the sinc method is approximately $O(N^{-2.5})$.

Figure 5

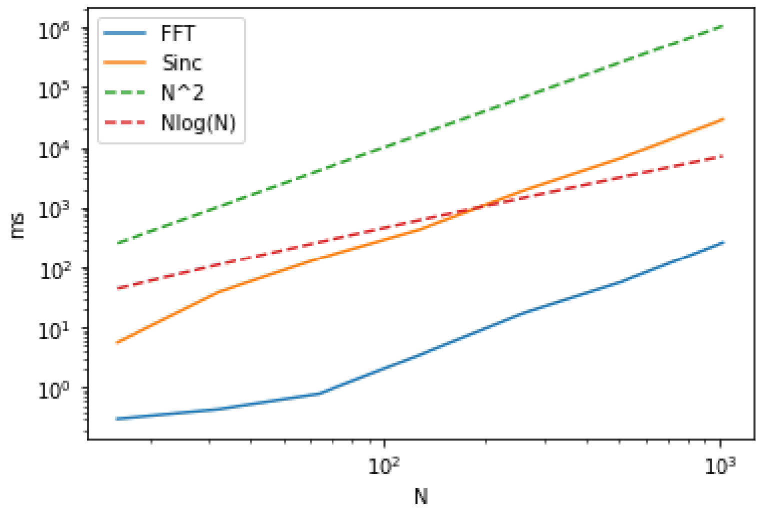


Figure 5: Figure shows computation time in ms for the FFT and Sinc based methods. Both are found to be $O(N^2)$ for the solution to the Fresnel Diffraction Approximation.

7 Conclusion

1. The results found in [1] were replicated and showed functions with discontinuities or functions with discontinuous derivatives are solved more efficiently with the Sinc Method.

2. We found the computational complexity of the solution in the Fresnel Diffraction approximation to be of $O(N^2)$ for both FFT and sinc methods. The FFT method is commonly regarded as an $N \log(N)$ algorithm but the solution in this case involves more than FFT alone and rather needs the creation of grids and matrix multiplications. The findings in [1] indicate when the Fresnel number is large (i.e. when $D^2/(\lambda z)$) is large, where D is the spatial extent of the optical field), the sinc algorithm can be reduced to $O(N)$ through the creation of banded matrices. This analysis was not attempted in the study and for further clarification refer to [1].

3. We found the order of accuracy for the circular aperture to be $O(N^{-1})$ for FFT as expected due to the discontinuous derivative, and $O(N^{-2.5})$

for the sinc approximation.

8 References

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