Report of CATAM 1.2 Ordinary Differential Equations

January 24, 2022

1 Q1 Instability of Leapfrog Method

Starting from $x_0 = 0, Y_0 = 0$, the Leapfrog method was used from $x = 0$ to $x = 10$ with step size h specified below. The numerical solution Y_n is compared against the analytical solution $y(x_n)$ and the error E_n is defined as $E_n = Y_n - y(x_n)$.

Clearly, the numerical solution is unstable and becomes very large within a few iterations, whereas the analytical solution decays to 0 quickly. When h is decreased, the error greatly increases. By plotting $\log |E_n|$ against x_n and using linear regression to obtain the slope, we can estimate the exponential growth rate γ, where $|E_n| \sim e^{\gamma x}$.

We observe that as h decreases, the magnitude of E_n at small x decreases, but at large x increases. Furthermore the length of the interval where $|E_n|$ is smaller than that for a larger value of h also increases. The rate of increase of γ rapidly decreases, and it appears $\gamma \to 4$ asymptotically, which we will prove in the next section. Because of the overcorrections made by the algorithm, a smaller step size makes the result worse.

		h=0.4		h=0.2	
x_n	$y(x_n)$	Y_n	E_n	Y_n	E_n
0.0	0.00	0.	0.	0.	0.
0.4	0.47	1.20	7.32×10^{-1}	2.25×10^{-2}	-4.46×10^{-1}
0.8	0.41	-2.23	-2.64	-1.51	-1.92
1.2	0.29	9.42	9.13	-7.98	-8.27
1.6	0.20	-3.16×10^{1}	-3.18×10^{1}	-3.56×10^{1}	-3.58×10^{1}
2.0	0.13	1.11×10^{2}	1.11×10^{2}	-1.55×10^{2}	-1.55×10^{2}
2.4	0.09	-3.87×10^{2}	-3.87×10^{2}	-6.71×10^{2}	-6.71×10^{2}
2.8	0.06	1.35×10^{3}	1.35×10^{3}	-2.90×10^{3}	-2.90×10^{3}
3.2	0.04	-4.71×10^{3}	-4.71×10^{3}	-1.26×10^{4}	-1.26×10^{4}
3.6	0.03	1.64×10^{4}	1.64×10^{4}	-5.44×10^{4}	-5.44×10^{4}
4.0	0.02	-5.72×10^{4}	-5.72×10^{4}	-2.36×10^{5}	-2.36×10^{5}
4.4	0.01	2.00×10^{5}	2.00×10^{5}	-1.02×10^{6}	-1.02×10^{6}
4.8	0.01	-6.96×10^{5}	-6.96×10^{5}	-4.42×10^{6}	-4.42×10^{6}
5.2	0.01	2.43×10^{6}	2.43×10^{6}	-1.91×10^{7}	-1.91×10^{7}
5.6	0.00	-8.46×10^{6}	-8.46×10^{6}	-8.28×10^{7}	-8.28×10^{7}
6.0	0.00	2.95×10^{7}	2.95×10^{7}	-3.58×10^{8}	-3.58×10^{8}
6.4	0.00	-1.03×10^{8}	-1.03×10^{8}	-1.55×10^{9}	-1.55×10^{9}
6.8	0.00	3.59×10^{8}	3.59×10^{8}	-6.71×10^{9}	-6.71×10^{9}
7.2	0.00	-1.25×10^{9}	-1.25×10^{9}	-2.91×10^{10}	-2.91×10^{10}
7.6	0.00	4.36×10^{9}	4.36×10^{9}	-1.26×10^{11}	-1.26×10^{11}
8.0	0.00	-1.52×10^{10}	-1.52×10^{10}	-5.45×10^{11}	-5.45×10^{11}
8.4	0.00	5.30×10^{10}	5.30×10^{10}	-2.36×10^{12}	-2.36×10^{12}
8.8	0.00	-1.85×10^{11}	-1.85×10^{11}	-1.02×10^{13}	-1.02×10^{13}
9.2	0.00	6.44×10^{11}	6.44×10^{11}	-4.42×10^{13}	-4.42×10^{13}
9.6	0.00	-2.25×10^{12}	-2.25×10^{12}	-1.91×10^{14}	-1.91×10^{14}
10.0	0.00	7.83×10^{12}	7.83×10^{12}	-8.28×10^{14}	-8.28×10^{14}

Table 1: Results and Errors for Leapfrog Method at $h = 0.4$, $h = 0.2$

		h=0.1		h=0.05	
x_n	$y(x_n)$	Y_n	E_n	Y_n	E_n
0	0.00	0.00	0.00	0.00	0.00
2	0.13	-81.60	-81.70	-25.60	-25.70
4	0.02	-199570.40	-199570.40	-72802.20	-72802.20
6	0.00	-4.87×10^{8}	-4.87×10^{8}	-2.06×10^{8}	-2.06×10^{8}
8	0.00	-1.19×10^{12}	-1.19×10^{12}	-5.83×10^{11}	-5.83×10^{11}
10	0.00	-2.91×10^{15}	-2.91×10^{15}	-1.65×10^{15}	-1.65×10^{15}

Table 2: Results and Errors for Leapfrog Method at $h = 0.1$, $h = 0.05$

h	0.4	0.2	0.1	0.05
γ	3.12246	3.66335	3.90038	3.97371
$\frac{\log\left(4h+\sqrt{16h^2+1}\right)}{h}$	3.12246	3.66334	3.90035	3.97380

Table 3: Error growth rate γ at different values of step size h

2 Q2 Difference Equation

Analytical Solution

The complementary solution is for the homogeneous difference equation

$$Y_{n+1} = Y_{n-1} - 8hY_n \tag{1}$$

Whose characteristic equation is $s^2 + 8hs - 1 = 0$ with solutions $s_1 = -4h + \sqrt{16h^2+1}, s_2 = -4h - \sqrt{16h^2+1}$. Then the complementary solution is:

$$Y_n = As_1^n + Bs_2^n = A\left(-4h + \sqrt{16h^2+1}\right)^n + B\left(-4h - \sqrt{16h^2+1}\right)^n \tag{2}$$

The particular solution can be found by setting $Y_n = Ce^{-nh}$, with the result that $C = \frac{3}{4-\sinh h/h}$. Now A and B can be determined by setting $n = 0, n = 1$ and using the initial conditions $Y_0 = 0, Y_1 = 3h$. The final solution is

$$\begin{aligned} Y_n = \frac{3}{2\sqrt{16h^2+1}} &\left[\left(h - \frac{e^{-h} + 4h + \sqrt{16h^2+1}}{4 - \sinh h/h}\right)\left(-4h + \sqrt{16h^2+1}\right)^n \right. \\ &\left. + \left(-h + \frac{e^{-h} + 4h - \sqrt{16h^2+1}}{4 - \sinh h/h}\right)\left(-4h - \sqrt{16h^2+1}\right)^n\right] + \frac{3}{4 - \sinh h/h}e^{-nh} \end{aligned} \tag{3}$$

The instability arises from the term $\left(-4h - \sqrt{16h^2+1}\right)^n$. As $|-4h - \sqrt{16h^2+1}| > 1$, this term oscillates and diverges as n becomes large, becoming the dominating contribution to the error E_n. For a fixed $x = nh$, $\left|\left(-4h - \sqrt{16h^2+1}\right)^n\right| = e^{\gamma x}$ where $\gamma = \frac{\log\left(4h+\sqrt{16h^2+1}\right)}{h}$. Indeed, this formula accurately agrees with the fitted values previously, as seen in table 3.

As $h \to 0$, $\frac{\sinh h}{h} \to \frac{\cosh h}{1} = 1$. With $h \to 0$ and $x = nh$ fixed, $Y_n \to e^{-x} - e^{-4x}$ because the coefficient on the diverging term $-h + \frac{e^{-h}+4h-\sqrt{16h^2+1}}{4-\sinh h/h} \to 0$, thus removing the instability and gives the same analytical solution in the previous part. Therefore for any fixed x, the instability can be suppressed at a small value of h. However any non-zero value of h will eventually create instability after this point x, as the diverging term $\sim he^{4x} \to \infty$ as $x \to \infty$. So there is no sufficiently small value of h that can suppress all instability.

3 Q3 Euler and RK4 Method

With the same initial condition, the ODE was numerically integrated using the Euler and RK4 Method. The numerical solutions are plotted, along with analytical solution, in figure 1.

	Euler	RK4
x_n	Y_n	Y_n
0.0	0.000	0.000
0.4	1.200	0.406
0.8	0.084	0.382
1.2	0.489	0.286
1.6	0.068	0.199
2.0	0.201	0.136
2.4	0.042	0.092
2.8	0.084	0.062
3.2	0.023	0.041
3.6	0.035	0.028
4.0	0.012	0.019

Table 4: Results computed by Euler and RK4 method

We observe that the Euler solution is oscillatory around the analytical solution, while the RK4 solution gives an underestimate.

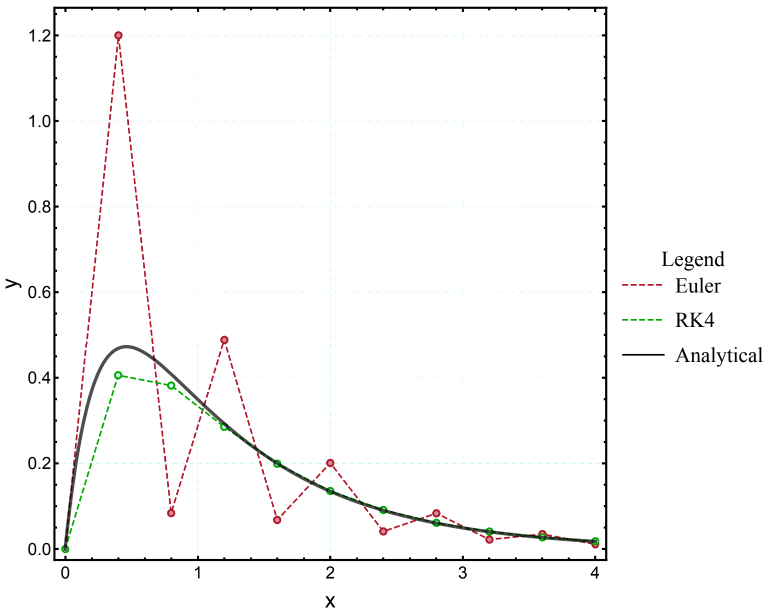

Figure 1: Euler and RK4 Method

4 Q4 Errors and Order of Accuracy

The three methods were used with $h = \frac{0.4}{n}, n = 2^k, k = 0, 1, ..., 15$ and their errors E_n are compared at $x_n = 0.4$. We observe that the errors greatly decrease as k increases, by 5 order of magnitudes for Euler, 8 for LF and 12 for RK4.

	Euler	LF	RK4
k	E_n	E_n	E_n
0	7.32×10^{-1}	7.32×10^{-1}	-6.26×10^{-2}
1	1.43×10^{-1}	-4.46×10^{-1}	-1.92×10^{-3}
2	6.37×10^{-2}	-1.60×10^{-1}	-8.42×10^{-5}
3	3.00×10^{-2}	-4.48×10^{-2}	-4.41×10^{-6}
4	1.46×10^{-2}	-1.15×10^{-2}	-2.53×10^{-7}
5	7.20×10^{-3}	-2.91×10^{-3}	-1.51×10^{-8}
6	3.57×10^{-3}	-7.28×10^{-4}	-9.24×10^{-10}
7	1.78×10^{-3}	-1.82×10^{-4}	-5.71×10^{-11}
8	8.89×10^{-4}	-4.55×10^{-5}	-3.55×10^{-12}
9	4.44×10^{-4}	-1.14×10^{-5}	-2.22×10^{-13}
10	2.22×10^{-4}	-2.85×10^{-6}	-1.37×10^{-14}
11	1.11×10^{-4}	-7.12×10^{-7}	2.11×10^{-15}
12	5.55×10^{-5}	-1.78×10^{-7}	-8.88×10^{-16}
13	2.77×10^{-5}	-4.45×10^{-8}	-1.23×10^{-14}
14	1.39×10^{-5}	-1.11×10^{-8}	5.05×10^{-15}
15	6.93×10^{-6}	-2.78×10^{-9}	4.30×10^{-14}

Table 5: Comparison of Errors of Euler, LF, RK4 Method

Suppose the error is $|E_n| = Ch^k$, then $\log|E_n| = k \log h + \log C$. Using linear regression on the linear part of the graph, the slope gives the order of accuracy of the algorithms, as shown in the table 6 below. This indeed matches the theoretical description of the methods given in the preamble, which gives the order as 1, 2, 4, respectively.

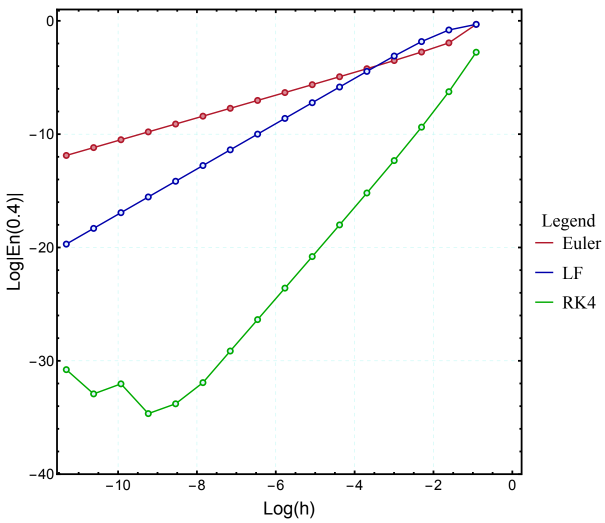

Figure 2: Log-Log Error Plot

	Euler	LF	RK4
k	1.02	1.98	4.06

Table 6: Order of accuracy k for the different methods

5 Q5 Analytic Solution to Linear, Lightly Damped Oscillator, $\delta = 0$

The complementary solution is for the homogeneous equation

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \Omega^2 y = 0 \tag{4}$$

Whose characteristic equation is $s^2 + \gamma s + \Omega^2 = 0$, with solution $s = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\Omega^2}}{2}$. Let $k = \sqrt{4\Omega^2 - \gamma^2}$, then can write the complementary solution

$$y_c = e^{-\frac{\gamma}{2}t}\left(C\sin\frac{kt}{2} + D\cos\frac{kt}{2}\right) \tag{5}$$

The particular solution can be found by substituting $y_p = A\sin\omega t + B\cos\omega t$ into the original equation. We obtain:

$$A = \frac{(\Omega^2 - \omega^2)a}{(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad B = \frac{-\gamma\omega a}{(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{6}$$

The full analytical solution is then:

$$\begin{aligned} y = y_p + y_c = & \frac{(\Omega^2 - \omega^2)a}{(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2}\sin\omega t + \frac{-\gamma\omega a}{(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2}\cos\omega t \\ & + e^{-\frac{\gamma}{2}t}\left(C\sin\frac{kt}{2} + D\cos\frac{kt}{2}\right) \end{aligned} \tag{7}$$

As $t \to \infty$, the complementary solution decays, and we are left with the particular solution y_p, which can be written as $y = A_s\sin(\omega t - \phi_s)$ (assuming $\Omega > \omega$):

$$y_p = \frac{a}{\sqrt{(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2}}\sin\left(\omega t - \tan^{-1}\frac{\gamma\omega}{\Omega^2 - \omega^2}\right) \tag{8}$$

For $\Omega < \omega$, it is necessary to consider that both A and B are negative, so have:

$$y_p = \frac{a}{\sqrt{(\omega^2 - \Omega^2)^2 + \gamma^2\omega^2}}\sin\left(\omega t - \left(\pi - \tan^{-1}\frac{\gamma\omega}{\omega^2 - \Omega^2}\right)\right) \tag{9}$$

For $\Omega = \omega$:

$$y_p = \frac{a}{\gamma\omega}\sin\left(\omega t - \frac{\pi}{2}\right) \tag{10}$$

In summary:

$$A_s = \frac{a}{\sqrt{(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \phi_s = \begin{cases} \tan^{-1}\frac{\gamma\omega}{\Omega^2 - \omega^2} & \Omega > \omega \\ \frac{\pi}{2} & \Omega = \omega \\ \pi - \tan^{-1}\frac{\gamma\omega}{\omega^2 - \Omega^2} & \Omega < \omega \end{cases} \tag{11}$$

6 Q6

Substituting in with initial conditions $y = \frac{dy}{dt} = 0$, we obtain $D = -B$ and $C = \frac{\gamma D - 2\omega A}{k}$.

The full analytical solution is then:

$$y = \frac{a}{(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2}\Big[(\Omega^2 - \omega^2)\sin\omega t - \gamma\omega\cos\omega t + e^{-\frac{\gamma}{2}t}\left(\frac{\gamma^2\omega - 2\omega(\Omega^2 - \omega^2)}{k}\sin\frac{kt}{2} + \gamma\omega\cos\frac{kt}{2}\right)\Big] \tag{12}$$

		h=0.4		h=0.2		h=0.1	
t_n	$y(t_n)$	Y_n	E_n	Y_n	E_n	Y_n	E_n
0.0	0.00	0.00	0.	0.00	0.	0.00	0.
0.4	0.02	0.02	6.93×10^{-5}	0.02	2.94×10^{-6}	0.02	1.50×10^{-7}
0.8	0.11	0.11	4.41×10^{-5}	0.11	1.58×10^{-6}	0.11	7.95×10^{-8}
1.2	0.28	0.28	-6.19×10^{-6}	0.28	6.75×10^{-8}	0.28	3.69×10^{-8}
1.6	0.46	0.46	-1.75×10^{-5}	0.46	1.39×10^{-6}	0.46	1.74×10^{-7}
2.0	0.57	0.57	3.62×10^{-5}	0.57	5.91×10^{-6}	0.57	4.78×10^{-7}
2.4	0.53	0.53	1.32×10^{-4}	0.53	1.14×10^{-5}	0.53	7.90×10^{-7}
2.8	0.32	0.32	2.16×10^{-4}	0.32	1.44×10^{-5}	0.32	9.06×10^{-7}
3.2	0.02	0.02	2.29×10^{-4}	0.02	1.24×10^{-5}	0.02	6.92×10^{-7}
3.6	-0.27	-0.27	1.46×10^{-4}	-0.27	4.90×10^{-6}	-0.27	1.67×10^{-7}
4.0	-0.43	-0.43	-1.34×10^{-5}	-0.43	-5.64×10^{-6}	-0.43	-4.91×10^{-7}
4.4	-0.41	-0.41	-1.89×10^{-4}	-0.41	-1.51×10^{-5}	-0.41	-1.02×10^{-6}
4.8	-0.21	-0.21	-3.07×10^{-4}	-0.21	-1.94×10^{-5}	-0.21	-1.19×10^{-6}
5.2	0.05	0.05	-3.13×10^{-4}	0.05	-1.66×10^{-5}	0.05	-9.28×10^{-7}
5.6	0.27	0.27	-2.02×10^{-4}	0.27	-7.70×10^{-6}	0.27	-3.27×10^{-7}
6.0	0.35	0.35	-1.88×10^{-5}	0.35	3.65×10^{-6}	0.35	3.61×10^{-7}
6.4	0.25	0.25	1.61×10^{-4}	0.25	1.27×10^{-5}	0.25	8.48×10^{-7}
6.8	0.03	0.03	2.61×10^{-4}	0.03	1.57×10^{-5}	0.03	9.38×10^{-7}
7.2	-0.21	-0.21	2.43×10^{-4}	-0.21	1.17×10^{-5}	-0.21	6.10×10^{-7}
7.6	-0.36	-0.36	1.18×10^{-4}	-0.36	2.71×10^{-6}	-0.36	3.02×10^{-8}
8.0	-0.33	-0.33	-5.15×10^{-5}	-0.33	-6.94×10^{-6}	-0.33	-5.27×10^{-7}
8.4	-0.15	-0.15	-1.88×10^{-4}	-0.15	-1.28×10^{-5}	-0.15	-8.05×10^{-7}
8.8	0.10	0.10	-2.29×10^{-4}	0.10	-1.22×10^{-5}	0.10	-6.80×10^{-7}
9.2	0.31	0.31	-1.57×10^{-4}	0.31	-5.60×10^{-6}	0.31	-2.18×10^{-7}
9.6	0.38	0.38	-6.89×10^{-6}	0.38	3.87×10^{-6}	0.38	3.61×10^{-7}
10.0	0.28	0.28	1.49×10^{-4}	0.28	1.17×10^{-5}	0.28	7.80×10^{-7}

Table 7: Results and Errors for RK4 Method at $h = 0.4$, $h = 0.2$, $h = 0.1$

The errors are very small, indicating that the numerical method indeed agrees with the analytical solution. The amount of error decreases when step size h is decreased. The RK4 method is very accurate around turning points, the errors occur with the largest magnitude when the graph is crossing the x-axis. This is likely due to the relatively sparse vertical sampling with the graph has large gradient (especially when $h = 0.4$) which leads to errors. In short, the error is sinusoidal in nature and $\pi/2$ out of phase, indicating that it is related to the gradient of the function.

7 Q7

Using RK4 method, the graphs of numerical solutions for $\gamma = 0.25, 0.5, 1.0, 1.9$ are plotted below, with $\omega = 1$ (dashed line) and $\omega = 2$ (solid line).

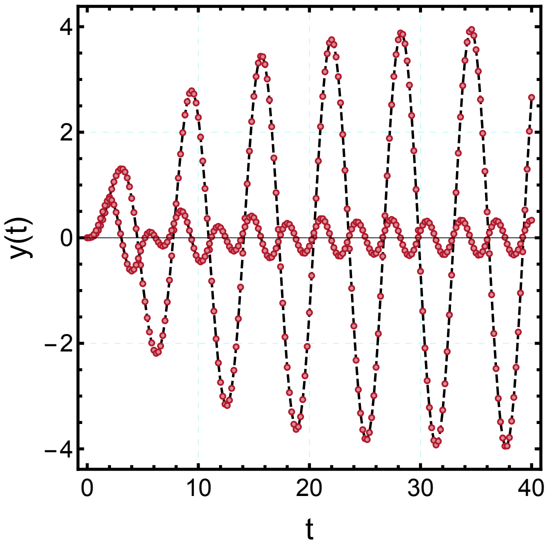

Figure 3: $\gamma = 0.25$

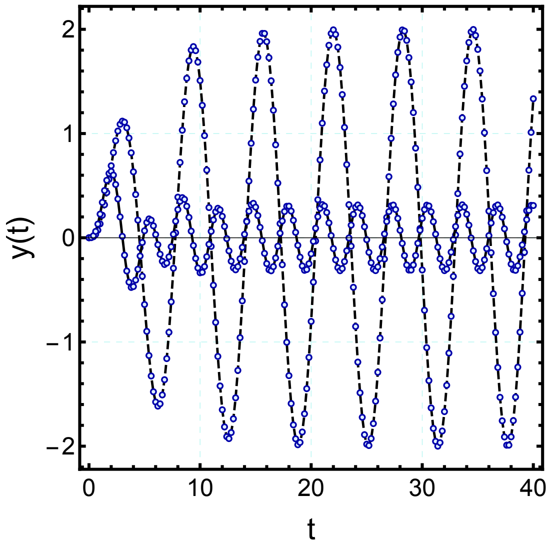

Figure 4: $\gamma = 0.5$

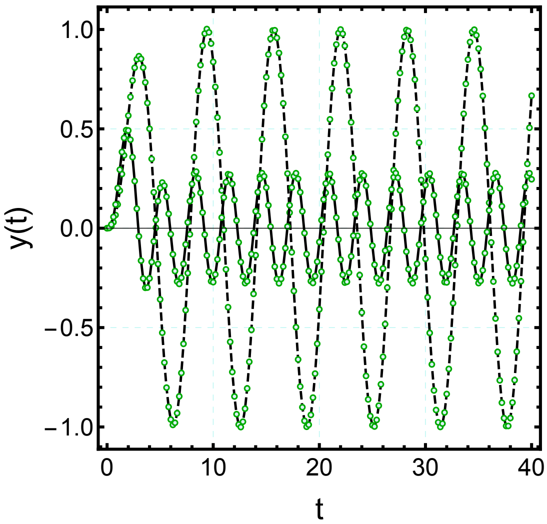

Figure 5: $\gamma = 1.0$

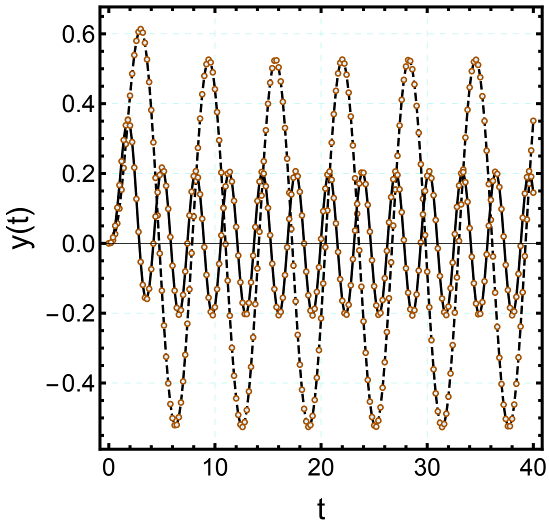

Figure 6: $\gamma = 1.9$

The amplitude of oscillations is roughly inversely proportional to γ, as can be seen on the vertical scale (when γ is doubled, the amplitude is halved).

When $\omega = 1$, the first oscillation has lower amplitude than the steady-steady state solution when $\gamma = 0.25, 0.5, 1.0$ but it has higher amplitude when $\gamma = 1.9$. The number of cycles needed to reach steady-state decreases when γ increases, which is because the transient solution decays as $e^{-\gamma t/2}$

The amplitude of oscillations decreases significantly when ω increases from 1 to 2. In addition, the ratio of amplitudes $\frac{A(\omega=2)}{A(\omega=1)}$ increases as γ increases.

$$\frac{A(\omega=2)}{A(\omega=1)} = \frac{\sqrt{(\Omega^2-1^2)^2+\gamma^2\cdot 1^2}}{\sqrt{(\Omega^2-2^2)^2+\gamma^2\cdot 2^2}} = \frac{\gamma}{\sqrt{9+4\gamma^2}} \tag{13}$$

Which is an increasing function that asymptotically reaches $\frac{1}{2}$.

When $\omega = \Omega = 1$, the system is close to resonance (damping causes the resonance frequency to become $< \Omega$), and amplitude for the resulting motion is $\frac{a}{\sqrt{(\Omega^2-\omega^2)^2+\gamma^2\omega^2}} = \frac{a}{\gamma\omega} = \frac{1}{\gamma}$, hence is inversely proportional to γ.

The system is resonant when $(\Omega^2 - \omega^2)^2 + \gamma^2\omega^2 = \omega^4 - (2\Omega^2 - \gamma^2)\omega^2 + \Omega^4$ is minimised, which happens when $\omega^2 = \Omega^2 - \frac{\gamma^2}{2}$.

When $\omega = 2$, the system is not in resonance and therefore has much lower amplitude. The initial part of the solution closely matches that of $\omega = 1$. The steady-state solution oscillates at twice the frequency, following the driving force.

8 Q8

Plotted below is the numerical solution for $\omega = 1$, $\gamma = 0, \delta = 0.25, 0.5, 1.0, 20$ (in gray solid line and square markers), overlaid on plots for $\omega = 1, \gamma = 0.25, 0.5, 1.0, 1.9, \delta = 0$ (coloured, dashed lines) as in Q7.

To generate the first three graphs, $h = 0.3$ was used as it worked and computed quickly. However, I ran into significant difficulty for the fourth. For $h = 0.1, 0.05$, solutions diverged at $x = 2, x = 4$. However, I verified that $h = 0.03, 0.04, 0.045, 0.048, 0.049$ all succeeded with essentially the same accuracy. This seems to imply that $h < \frac{1}{\delta}$ works.

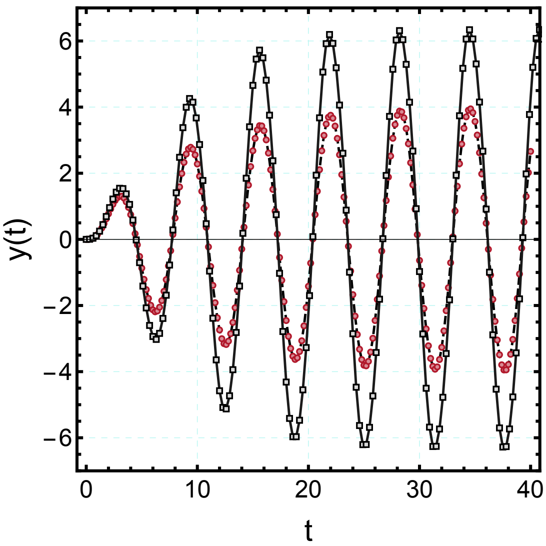

Figure 7: $\gamma = 0.25$ (dashed); $\delta = 0.25$ (solid)

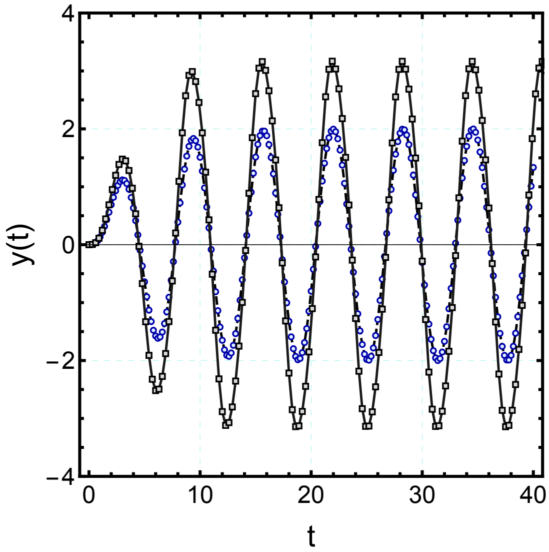

Figure 8: $\gamma = 0.5$ (dashed); $\delta = 0.5$ (solid)

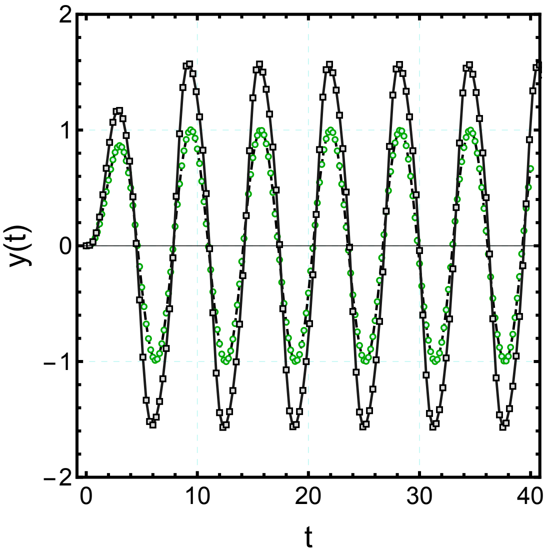

Figure 9: $\gamma = 1.0$ (dashed); $\delta = 1.0$ (solid)

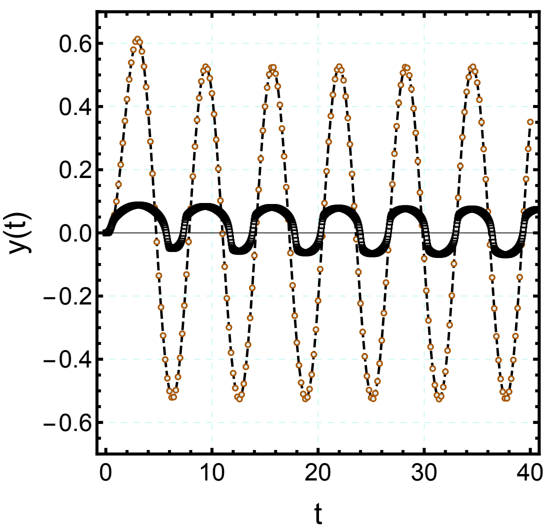

Figure 10: $\gamma = 1.9$ (dashed); $\delta = 20$ (solid)

We observe that for the first 3 graphs, when γ and δ takes the same values $0.25, 0.5, 1.0$, the amplitude of oscillations is always higher for δ-non-linear damping compared γ linear damping. This shows the non-linear damping is lighter in this regime.

When $\delta = 20$, the resulting motion is non-sinusoidal and has asymmetrical behaviour at positive and negative $y(t)$. The amplitude is much lower than $\gamma = 1.9$, which is expected as δ^3 becomes a large damping coefficient. In the initial motion, the amplitude and duration of positive displacement is greater than that for the subsequent negative displacement. Eventually the oscillation becomes 2π periodic, although the shape is still not sinusoidal.

We observe that when δ is small, the solution reaches steady-state oscillation with period 2π, and we conjecture it to be of the form

$$y = \sum_{n=-1}^{\infty} \delta^n y_n(t) = \delta^{-1} A \cos t + (B \sin t + C \sin 3t) + \delta y_1(t) + ... \tag{14}$$

Substituting this into the original equation and keeping only terms of order δ^0 and δ^1, we obtain:

$$-9C \sin 3t + \delta \ddot{y}_1 + \frac{d}{dt}\left(\frac{1}{3}\left(A \cos t + \delta(B \sin t + C \sin 3t)\right)^3\right) + C \sin 3t + \delta y_1 = \sin t \tag{15}$$

$$\frac{1}{3}\frac{d}{dt}\left(A^3 \cos^3 t + 3A^2 \cos^2 t \delta(B \sin t + C \sin 3t)\right) = \sin t + 8C \sin 3t - \delta(\ddot{y}_1 + y_1) \tag{16}$$

Comparing terms of order δ^0, we obtain:

$$\frac{1}{3}\frac{d}{dt}\left(A^3 \cos^3 t\right) = \sin t + 8C \sin 3t \tag{17}$$

$$\frac{A^3}{3}\frac{d}{dt}\left(\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right) = \sin t + 8C \sin 3t \tag{18}$$

Hence

$$A = -4^{1/3}, \quad C = \frac{1}{8} \tag{19}$$

Comparing terms of order δ^1, we obtain the differential equation:

$$\ddot{y}_1 + y_1 + \frac{d}{dt}\left(A^2 \cos^2 t(B \sin t + C \sin 3t)\right) = 0 \tag{20}$$

$$\ddot{y}_1 + y_1 + A^2(B(-2\cos t \sin^2 t + \cos^3 t) + C(-2\cos t \sin t \sin 3t + 3\cos^2 t \cos 3t) = 0 \tag{21}$$

Using trigonometric identities, can be simplified to

$$\ddot{y}_1 + y_1 + \frac{A^2}{2}\cos t\left(-B + (3B + C)\cos 2t + 5C \cos 4t\right) = 0 \tag{22}$$

$$\ddot{y}_1 + y_1 + \frac{A^2}{2}\left(-B \cos t + \frac{3B + C}{2}(\cos t + \cos 3t) + \frac{5C}{2}(\cos 3t + \cos 5t)\right) = 0 \tag{23}$$

Observe that the driving term contains $\frac{A^2(B+C)}{2}\cos t$, which causes resonance with the system as its natural frequency is $\Omega = 1$. This means that the solution will contain $t \sin t$ and $t \cos t$ term (in fact it would be $-\frac{B+C}{8} t \sin t$) which is (a) not 2π periodic and (b) unbounded.

Therefore to obtain 2π periodic solution for $y_1(t)$, we must have $B = -C = -\frac{1}{8}$ in order for the $\cos t$ driving term to disappear.

When δ is large, then we need to compare terms of δ^n $\forall$ $n > 0$ in the equation as those become significant. It means that the coefficients in the Fourier Series expansion need to decay faster than $\sim \delta^{-n}$ in order for the sum to converge.