Multi I/O Transformers

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I. Introduction

Let us first examine the use of the following two common equations with regards to an inductor [1]:

$$V = -\frac{d\phi}{dt} \tag{1}$$

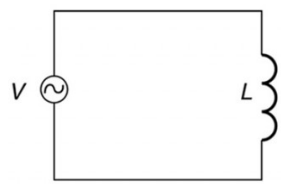
$$V = L\frac{dI}{dt} \tag{2}$$

Using the definition that inductance $L = \frac{\phi}{I}$ in (1), we can easily derive $V = -L\frac{dI}{dt}$, the opposite of (2). This is because the commonly known equation is for the **voltage drop** across the inductor. Similar to $V = IR$ and $I = C\frac{dV}{dt}$, we use $V = L\frac{dI}{dt}$ when discussing the inductor as a circuit component.

II. Inductor with AC Source

A. *Without Load*

We have a voltage source of $V = V_0 e^{i\omega t}$ connected to an inductor of inductance L.



It is clear that the current can be expressed as:

$$I = \frac{V}{Z} = \frac{V_0 e^{i\omega t}}{i\omega L} \tag{3}$$

This current will cause the inductor, in its nature a solenoid, to produce a time-varying magnetic field of $\phi = LI$. The solenoid will then react to its own magnetic field, using (1):

$$V = -\frac{d\phi}{dt} = -L\frac{dI}{dt} = -i\omega LI = -V_0 e^{i\omega t} \tag{4}$$

This will then be the potential difference across the inductor, which verifiably abides by Kirchhoff's voltage law.

But wait! Wasn't it the potential difference that caused the current that caused the magnetic flux that caused the potential difference that caused the I that caused the ϕ that caused the V that caused the I that caused the ϕ that caused the V that caused the I that caused the ϕ that caused the V that caused the I that caused the ϕ that caused the V…

The resolution to this issue is as follows: $Z = i\omega L$ is a conclusion from the above derivation, rather than an assumption. The only thing we know is the voltage source (as given) and L (because it is a material property). The only equations we know are: $V = IZ$, Faraday's Law (1), and Kirchhoff's Voltage Law.

From there, the logic is as follows:

1. The inductor is put under a potential difference $V_0 e^{i\omega t}$.
2. How much current should it draw?
3. The current drawn must cause the inductor to experience a voltage drop of exactly $V_0 e^{i\omega t}$.
    a. As the inductor is just a wire coil, it has no resistance of its own. Induced voltage from Faraday's Law acts as the voltage drop across this component.
4. This will lead us to write out exactly Equation (4);
5. And we will conclude that the amount of current drawn is as accordance with Eq. (3). This also allows us to conclude that $Z = i\omega L$.

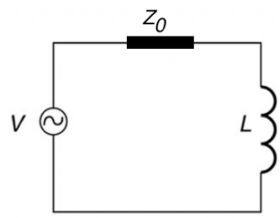
Here, let us calculate the flux in the core, which is the flux experienced by the coil divided by the number of turns:

$$\varphi = \frac{\phi}{N} = \frac{LI}{N} = \frac{V_0 e^{i\omega t}}{i\omega N} \tag{5}$$

This will prove useful later. ;)

B. *With Load*

Now, we add a load of Z_0 to the circuit.



Let us follow the same logic as above:

1. The inductor is put under a potential difference $V_0 e^{i\omega t} - IZ_0$.
2. How much current should the circuit draw?
3. The current drawn must cause the inductor to experience a voltage drop of exactly $V_0 e^{i\omega t} - IZ_0$
4. This will lead us to write out:

$$V = -\frac{d\phi}{dt} = -L\frac{dI}{dt} = -i\omega LI = -(V_0 e^{i\omega t} - IZ_0) \tag{6}$$

5. Which allows us to conclude:

$$I = \frac{V_0 e^{i\omega t}}{Z_0 + i\omega L} \tag{7}$$

We can see that $Z = i\omega L$ is generally true. In this case, we must be aware that we are not adding the self-regulatory role of the inductor onto the circuit: the voltage source is V, the induced voltage is V, and the voltage across the inductor is still V.

III. Magnetic Flux

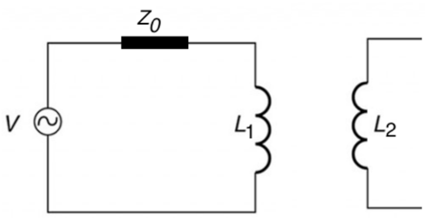
Before we discuss transformers, we muSt alter the definition of ϕ.

A. Number of Coils

In the traditional definition $\phi = LI$, ϕ stands for the magnetic flux the coil experiences, which is proportional to N^2. Here, I will define φ as the absolute (or single-coil) flux that is going around the transformer core, with the relation $\phi_1 = N_1\varphi$.

B. Open Secondary End

An AC source of $V = V_0 e^{i\omega t}$ is connected to a load of Z_0 and an inductor with number of coils N_1 and inductance L_1. A current of I_1 is present in the circuit. Not shown below is the iron core that the coils are wrapped around in a common transformer. On the right-hand side, we have an inductor of number of coils N_2 and inductance L_2.



The absolute flux going through the system is:

$$\varphi = \frac{\phi_1}{N_1} = \frac{L_1 I_1}{N_1} \tag{8}$$

Hence we can write the induced voltage on the second inductor as:

$$V_2 = -\frac{d\phi}{dt} = -N_2\frac{d\varphi}{dt} = -\frac{N_2 L_1}{N_1}\frac{dI_1}{dt} = -i\omega\frac{N_2 L_1 I_1}{N_1} \tag{9}$$

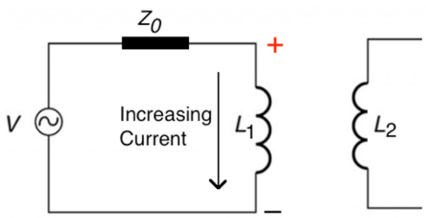
If we define $V_1 = V_0 e^{i\omega t} - I_1 Z_0$, we can see that:

$$V_2 = -i\omega\frac{N_2 L_1 I_1}{N_1} = -\frac{N_2}{N_1}\, i\omega L_1 I_1 = -\frac{N_2}{N_1}V_1 \tag{10}$$

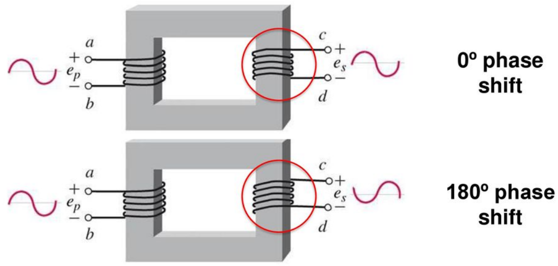
Which is a well-known result. But what does the negative sign here mean? Which side is + and which side is – ?

C. Coiling Direction[2]

Suppose at the moment, the top of the inductor is positive, the bottom is negative, and the current is clockwise in the circuit and increasing.



Suppose that the left coil is wound anti-clockwise. This will produce an increasing flux upwards. This means the right-hand side coil will experience an increasing flux downwards, which prompts the induced voltage to generate an opposing flux upwards.



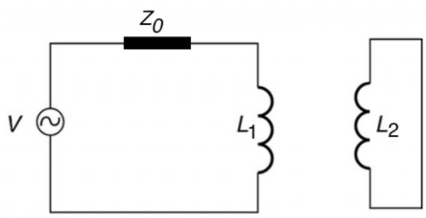
We have two cases here [3]:

1. The right coil is wound clockwise. This means the generated voltage would like to cause a current that runs anti-clockwise, from bottom to top. We can then conclude that as far as an external load is concerned, the top point is +, and the bottom point is –.
2. The right coil is wound anti-clockwise. This means the generated voltage would like to cause a current that runs anti-clockwise, from top to bottom. We can then conclude that as far as an external load is concerned, the top point is –, and the bottom point is +.

Here I must clarify that the aforementioned current occurs when the right-hand side circuit is completed, with the inductor acting as the voltage source for its circuit. The use of + and – is solely for demonstrative purposes, as the voltage is periodic. This is most useful in real transformers, where you need to know which is the correct end into which to plug your device.

D. Short-Circuited Secondary End

Here we run into mutual inductance: the reactionary current in the second circuit will cause the first inductor to react as well.



Here we must work out our logic before diving in:

1. The existing φ causes a potential difference V_2.
2. How much current should there be?
3. The current must cause the inductor to experience a voltage drop of exactly V_2.
   a. The voltage drop from Faraday's Law acting on the flux φ' created by the second inductor itself must be equal and opposite to V_2 (caused by the first inductor), in order for Kirchhoff's Law to hold.
4. This will lead us to write:

$$V = -\frac{d\varphi'}{dt} = -L_2\frac{dI_2}{dt} = -i\omega L_2 I_2 = -V_2 \tag{11}$$

5. Which allows us to conclude:

$$I_2 = \frac{V_2}{i\omega L_2} = -\frac{N_2 V_1}{N_1 i\omega L_2} = -\frac{N_1}{N_2}\frac{V_1}{i\omega L_1} = -\frac{N_1}{N_2}I_1 \tag{12}$$

But wait! This new flux will cause the first inductor to react, creating a new flux, which will cause the second inductor to react, creating a new flux, which will cause the first inductor to react…

The new flux is $\varphi' = \frac{L_2 I_2}{N_2} = -\frac{L_2 N_1 I_1}{{N_2}^2} = -\frac{L_1 I_1}{N_1}$. The original flux was $\varphi = \frac{L_1 I_1}{N_1}$. So it is really true from Lenz's law that an opposing flux is generated, and in the case for a perfect system, an exact and opposite flux!

Well, not really. This is only true if the first coil doesn't react to this flux. For example, the classic coil in a changing magnetic field problem.

Here, we must wipe the slate clean and perform all the calculations [4] from the start as I_1 and hence I_2 are not what we make them out to be previously, and we cannot use their previous relation.

$$\left(-\frac{d\phi}{dt}\right) + \left(-\frac{d\phi'}{dt}\right) = -(V_0 e^{i\omega t} - I_1 Z_0) \tag{13}$$

$$\frac{d\phi'}{dt} = N_1 \frac{d\varphi'}{dt} = \frac{N_1 L_2}{N_2}\frac{dI_2}{dt} = \frac{i\omega N_1 L_2 I_2}{N_2} \tag{14}$$

$$V_0 e^{i\omega t} = I_1 Z_0 + i\omega L_1 I_1 + \frac{i\omega N_1 L_2 I_2}{N_2} \tag{15}$$

And for the second coil we have:

$$\left(-\frac{d\phi}{dt}\right) + \left(-\frac{d\phi'}{dt}\right) = 0 \tag{16}$$

$$i\omega \frac{N_2 L_1 I_1}{N_1} + i\omega L_2 I_2 = 0 \tag{17}$$

In other words, there is no "going back and forth". The magnetic flux created by both of the coils are in such a way that the systems are already balanced (i.e. to say, I_1 already contains both the actual flux generation component and the reaction to I_2)

From (16), we have the current relation which still holds!

$$I_2 = -\frac{N_2 L_1 I_1}{N_1 L_2} = -\frac{N_1}{N_2} I_1 \tag{18}$$

(14) and (17) gives us:

$$V_0 e^{i\omega t} - I_1 Z_0 = i\omega L_1 I_1 - \frac{i\omega {N_1}^2 L_2}{{N_2}^2} I_1 = 0 \tag{19}$$

This implies $I_1 = \frac{V_0 e^{i\omega t}}{Z_0}$.

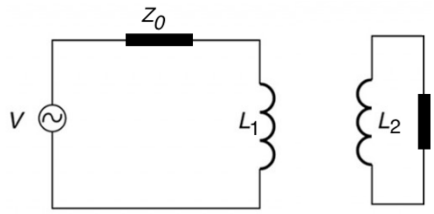
There is no voltage difference across the inductor!

The first inductor's production of magnetic flux is due to its reaction to the second inductor's flux, which was a reaction to the first's! There is no net flux going on!

If you think about it, it is not so ridiculous. The energy expenditure on the right: $\frac{1}{2} L_2 {I_2}^2 = \frac{1}{2} L_1 {I_1}^2$.

E. Secondary End with Load

Now we attach a load of Z_3 to the right-hand side. I have chosen to number it as such to avoid Z_1 and Z_2, in the case the reader would like to assign those for the inductors.



Our new governing equations would be similar to (14) and (15), with the addition of the voltage drop for the load:

$$V_0 e^{i\omega t} = I_1 Z_0 + i\omega L_1 I_1 + \frac{i\omega N_1 L_2 I_2}{N_2} \tag{20}$$

$$\left(-\frac{d\phi}{dt}\right) + \left(-\frac{d\phi'}{dt}\right) - I_2 Z_3 = 0 \tag{21}$$

Equation (20) gives us:

$$i\omega \frac{N_2 L_1 I_1}{N_1} + i\omega L_2 I_2 + I_2 Z_3 = 0 \tag{22}$$

Hence, the current relation would be dependent on Z_3:

$$I_2 = -\frac{i\omega \frac{N_2 L_1}{N_1}}{i\omega L_2 + Z_3} I_1 \tag{23}$$

From (23), we can see that if we define $V_2 = (i\omega L_2 + Z_3) I_2$, and $V_1 = i\omega L_1 I_1$, we will get back the old voltage relation of:

$$V_2 = -\frac{N_2}{N_1} V_1 \tag{24}$$

Equation (19) can be simplified to:

$$V_0 e^{i\omega t} = I_1 Z_0 + i\omega L_1 I_1 - \left(\frac{i\omega N_1 L_2}{N_2}\right)\left(\frac{i\omega \frac{N_2 L_1}{N_1}}{i\omega L_2 + Z_3}\right) I_1 \tag{25}$$

$$V_0 e^{i\omega t} = I_1 Z_0 + i\omega L_1 I_1 + \left(\frac{\omega^2 L_2 L_1}{i\omega L_2 + Z_3}\right) I_1 \tag{26}$$

From here, let us find all the things we can find:

$$I_1 = \frac{V_0 e^{i\omega t}}{Z_0 + i\omega L_1 + \frac{\omega^2 L_2 L_1}{i\omega L_2 + Z_3}} \tag{27}$$

$$I_2 = -\frac{i\omega \frac{N_2 L_1}{N_1}}{i\omega L_2 + Z_3} I_1 = -\frac{i\omega \frac{N_2 L_1}{N_1} V_0 e^{i\omega t}}{(Z_0 + i\omega L_1)(i\omega L_2 + Z_3) + \omega^2 L_2 L_1} \tag{28}$$

$$I_2 = -\frac{i\omega N_2 L_1 V_0 e^{i\omega t}}{N_1 Z_0 Z_3 + i\omega N_1 L_2 Z_0 + i\omega N_1 L_1 Z_3} \tag{29}$$

$$V_1 = V_0 e^{i\omega t} - I_1 Z_0 = \frac{V_0 e^{i\omega t}\left(i\omega L_1 + \frac{\omega^2 L_2 L_1}{i\omega L_2 + Z_3}\right)}{Z_0 + i\omega L_1 + \frac{\omega^2 L_2 L_1}{i\omega L_2 + Z_3}} = \frac{V_0 e^{i\omega t}(i\omega L_1 Z_3)}{Z_0 Z_3 + i\omega L_2 Z_0 + i\omega L_1 Z_3} \tag{30}$$

$$V_2 = -I_2 Z_3 = \frac{i\omega N_2 L_1 Z_3 V_0 e^{i\omega t}}{N_1 Z_0 Z_3 + i\omega N_1 L_2 Z_0 + i\omega N_1 L_1 Z_3} \tag{31}$$

This is the actual "usable" voltage, which is measured on the load rather than the inductor. Hence you see that the relation $V_2 = \frac{N_2}{N_1} V_1$ still holds.

$$\varphi = \frac{\phi_1}{N_1} = \frac{L_1 I_1}{N_1} = \frac{L_1 V_0 e^{i\omega t}}{N_1 Z_0 + i\omega N_1 L_1 + \frac{\omega^2 N_1 L_2 L_1}{i\omega L_2 + Z_3}} \tag{32}$$

$$\varphi' = \frac{L_2 I_2}{N_2} = -\frac{i\omega L_2 L_1 V_0 e^{i\omega t}}{N_1 Z_0 Z_3 + i\omega N_1 L_2 Z_0 + i\omega N_1 L_1 Z_3} \tag{33}$$

$$\varphi + \varphi' = \cdots \tag{34}$$

Clearly, when there are loads on both sides, everything goes haywire. Let us consider a simpler case where $Z_0 = 0$.

$$I_1 = \frac{V_0 e^{i\omega t}}{i\omega L_1 + \frac{\omega^2 L_2 L_1}{i\omega L_2 + Z_3}} \tag{35}$$

$$I_2 = -\frac{N_2 V_0 e^{i\omega t}}{N_1 Z_3} \tag{36}$$

$$V_1 = V_0 e^{i\omega t} \tag{37}$$

$$V_2 = -I_2 Z_3 = \frac{N_2 V_0 e^{i\omega t}}{N_1} = \frac{N_2}{N_1} V_1 \tag{38}$$

$$\varphi = \frac{\phi_1}{N_1} = \frac{L_1 I_1}{N_1} = \frac{L_1 V_0 e^{i\omega t}}{i\omega N_1 L_1 + \frac{\omega^2 N_1 L_2 L_1}{i\omega L_2 + Z_3}} \tag{39}$$

$$\varphi' = \frac{L_2 I_2}{N_2} = -\frac{L_2 V_0 e^{i\omega t}}{N_1 Z_3} \tag{40}$$

$$\varphi + \varphi' = \frac{L_1 V_0 e^{i\omega t}(i\omega L_2 + Z_3)}{i\omega N_1 L_1 Z_3} - \frac{L_2 V_0 e^{i\omega t}}{N_1 Z_3} = \frac{V_0 e^{i\omega t}}{i\omega N_1} \tag{41}$$

Now, flip back to see Equation (5).

IV. THREE OUTPUTS

Now, what if I add a third inductor, with a third load of Z_4. Yes, I realize my number system has a flaw, but I've written way too much. The governing equations will hence be:

$$V_0 e^{i\omega t} = I_1 Z_0 + i\omega L_1 I_1 + \frac{i\omega N_1 L_2 I_2}{N_2} + \frac{i\omega N_1 L_3 I_3}{N_3} \tag{42}$$

$$i\omega \frac{N_2 L_1 I_1}{N_1} + i\omega L_2 I_2 + i\omega \frac{N_2 L_3 I_3}{N_3} + I_2 Z_3 = 0 \tag{43}$$

$$i\omega \frac{N_3 L_1 I_1}{N_1} + i\omega \frac{N_3 L_2 I_2}{N_2} + i\omega L_3 I_3 + I_3 Z_4 = 0 \tag{44}$$

Let's solve this, shall we? Let's do it in the most ridiculous form, which is of course, row-echelon. Of course, I will bore you with all the steps.

$$\begin{pmatrix} Z_0 + i\omega L_1 & \frac{i\omega N_1 L_2}{N_2} & \frac{i\omega N_1 L_3}{N_3} \\ i\omega \frac{N_2 L_1}{N_1} & Z_3 + i\omega L_2 & i\omega \frac{N_2 L_3}{N_3} \\ i\omega \frac{N_3 L_1}{N_1} & i\omega \frac{N_3 L_2}{N_2} & Z_4 + i\omega L_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_0 e^{i\omega t} \\ 0 \\ 0 \end{pmatrix} \tag{45}$$

$$\begin{pmatrix} Z_0 + i\omega L_1 & \frac{i\omega N_1 L_2}{N_2} & \frac{i\omega N_1 L_3}{N_3} \\ i\omega \frac{N_2 L_1}{N_1} & Z_3 + i\omega L_2 & i\omega \frac{N_2 L_3}{N_3} \\ 0 & -\frac{N_3 Z_3}{N_2} & Z_4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_0 e^{i\omega t} \\ 0 \\ 0 \end{pmatrix} \tag{46}$$

$$\begin{pmatrix} Z_0 + i\omega L_1 & \frac{i\omega N_1 L_2}{N_2} & \frac{i\omega N_1 L_3}{N_3} \\ -\frac{N_2 Z_0}{N_1} & Z_3 & 0 \\ 0 & -\frac{N_3 Z_3}{N_2} & Z_4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_0 e^{i\omega t} \\ -\frac{N_2 V_0 e^{i\omega t}}{N_1} \\ 0 \end{pmatrix} \tag{47}$$

$$\begin{pmatrix} i\omega L_1 & \frac{N_1(i\omega L_2 + Z_3)}{N_2} & \frac{i\omega N_1 L_3}{N_3} \\ -\frac{N_2 Z_0}{N_1} & Z_3 & 0 \\ 0 & -\frac{N_3 Z_3}{N_2} & Z_4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{N_2 V_0 e^{i\omega t}}{N_1} \\ 0 \end{pmatrix} \tag{48}$$

$$\begin{pmatrix} i\omega N_2 N_3 L_1 & N_1 N_3 (i\omega L_2 + Z_3) & i\omega N_1 N_2 L_3 \\ -N_2 Z_0 & N_1 Z_3 & 0 \\ 0 & -N_3 Z_3 & N_2 Z_4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -N_2 V_0 e^{i\omega t} \\ 0 \end{pmatrix} \tag{49}$$

$$\begin{pmatrix} 0 & N_1 N_3 (i\omega L_2 + Z_3(1 + \frac{i\omega L_1}{Z_0})) & i\omega N_1 N_2 L_3 \\ -N_2 Z_0 & N_1 Z_3 & 0 \\ 0 & -N_3 Z_3 & N_2 Z_4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} -\frac{i\omega N_2 N_3 L_1}{Z_0} V_0 e^{i\omega t} \\ -N_2 V_0 e^{i\omega t} \\ 0 \end{pmatrix} \tag{50}$$

$$\begin{pmatrix} 0 & N_1 N_3 \left(i\omega L_2 + Z_3 \left(1 + \frac{i\omega L_1}{Z_0}\right)\right) + \frac{i\omega N_1 N_3 Z_3 L_3}{Z_4} & 0 \\ -N_2 Z_0 & N_1 Z_3 & 0 \\ 0 & -N_3 Z_3 & N_2 Z_4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} -\frac{i\omega N_2 N_3 L_1}{Z_0} V_0 e^{i\omega t} \\ -N_2 V_0 e^{i\omega t} \\ 0 \end{pmatrix} \tag{51}$$

We have our first conclusion:

$$\left(N_1 N_3 \left(i\omega L_2 + Z_3 \left(1 + \frac{i\omega L_1}{Z_0}\right)\right) + \frac{i\omega N_1 N_3 Z_3 L_3}{Z_4}\right) I_2 = -\frac{i\omega N_2 N_3 L_1}{Z_0} V_0 e^{i\omega t} \tag{52}$$

$$I_2 = -\frac{i\omega N_2 L_1 Z_4 V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{53}$$

And hence, allowing us to find I_1 and I_3 uneasily:

$$I_1 = \frac{N_2 V_0 e^{i\omega t} + N_1 Z_3 I_2}{N_2 Z_0} = \frac{V_0 e^{i\omega t}}{Z_0} - \frac{i\omega L_1 Z_3 Z_4 V_0 e^{i\omega t}}{Z_0(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{54}$$

$$I_1 = \frac{(i\omega L_2 Z_4 + Z_3 Z_4 + i\omega Z_3 L_3) V_0 e^{i\omega t}}{(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{55}$$

$$I_3 = \frac{N_3 Z_3 I_2}{N_2 Z_4} = -\frac{i\omega N_3 L_1 Z_3 V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{56}$$

And now, the voltages:

$$V_1 = V_0 e^{i\omega t} - I_1 Z_0 = \frac{i\omega L_1 Z_3 Z_4 V_0 e^{i\omega t}}{(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{57}$$

$$V_2 = -I_2 Z_3 = \frac{i\omega N_2 L_1 Z_3 Z_4 V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{58}$$

$$V_3 = -I_3 Z_4 = \frac{i\omega N_3 L_1 Z_3 Z_4 V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{59}$$

And now the flux:

$$\varphi = \frac{\phi_1}{N_1} = \frac{L_1 I_1}{N_1} = \frac{(i\omega L_1 L_2 Z_4 + L_1 Z_3 Z_4 + i\omega Z_3 L_1 L_3) V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{60}$$

$$\varphi' = \frac{L_2 I_2}{N_2} = -\frac{i\omega L_1 L_2 Z_4 V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{61}$$

$$\varphi'' = \frac{L_3 I_3}{N_3} = -\frac{i\omega L_1 L_3 Z_3 V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{62}$$

$$\varphi + \varphi' + \varphi'' = \frac{L_1 Z_3 Z_4 V_0 e^{i\omega t}}{N_1(i\omega L_2 Z_0 Z_4 + Z_0 Z_3 Z_4 + i\omega L_1 Z_3 Z_4 + i\omega Z_0 Z_3 L_3)} \tag{63}$$

And now let's examine the case where $Z_0 = 0$.

$$I_1 = \frac{(i\omega L_2 Z_4 + Z_3 Z_4 + i\omega Z_3 L_3) V_0 e^{i\omega t}}{(i\omega L_1 Z_3 Z_4)} \tag{64}$$

$$I_2 = -\frac{N_2 V_0 e^{i\omega t}}{N_1 Z_3} \tag{65}$$

$$I_3 = \frac{N_3 Z_3 I_2}{N_2 Z_4} = -\frac{N_3 V_0 e^{i\omega t}}{N_1 Z_4} \tag{66}$$

$$V_1 = V_0 e^{i\omega t} \tag{67}$$

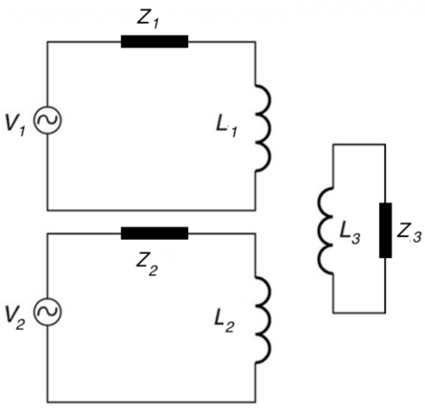
$$V_2 = -I_2 Z_3 = \frac{N_2}{N_1} V_0 e^{i\omega t} \tag{68}$$

$$V_3 = -I_3 Z_4 = \frac{N_3}{N_1} V_0 e^{i\omega t} \tag{69}$$

$$\varphi + \varphi' + \varphi'' = \frac{V_0 e^{i\omega t}}{i\omega N_1} \tag{70}$$

V. MULTIPLE INPUTS

Allow me to rename everything.



And so we have: (Well, at least the right hand side doesn't have sources. This is going to kill me.)

$$\begin{pmatrix} Z_1 + i\omega L_1 & \frac{i\omega N_1 L_2}{N_2} & \frac{i\omega N_1 L_3}{N_3} \\ i\omega \frac{N_2 L_1}{N_1} & Z_2 + i\omega L_2 & i\omega \frac{N_2 L_3}{N_3} \\ i\omega \frac{N_3 L_1}{N_1} & i\omega \frac{N_3 L_2}{N_2} & Z_3 + i\omega L_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_1 e^{i\omega t} \\ V_2 e^{i\omega t} \\ 0 \end{pmatrix} \tag{71}$$

$$\begin{pmatrix} Z_1 & 0 & -\frac{N_1}{N_3} Z_3 \\ i\omega \frac{N_2 L_1}{N_1} & Z_2 + i\omega L_2 & i\omega \frac{N_2 L_3}{N_3} \\ i\omega \frac{N_3 L_1}{N_1} & i\omega \frac{N_3 L_2}{N_2} & Z_3 + i\omega L_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_1 e^{i\omega t} \\ V_2 e^{i\omega t} \\ 0 \end{pmatrix} \tag{72}$$

$$\begin{pmatrix} Z_1 & 0 & -\frac{N_1}{N_3} Z_3 \\ i\omega \frac{N_2 L_1}{N_1} & Z_2 + i\omega L_2 & i\omega \frac{N_2 L_3}{N_3} \\ 0 & -\frac{N_3}{N_2} Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_1 e^{i\omega t} \\ V_2 e^{i\omega t} \\ -\frac{N_3}{N_2} V_2 e^{i\omega t} \end{pmatrix} \tag{73}$$

$$\begin{pmatrix} N_3 Z_1 & 0 & -N_1 Z_3 \\ N_2 N_3 i\omega L_1 & N_1 N_3 Z_2 + N_1 N_3 i\omega L_2 & N_1 N_2 i\omega L_3 \\ 0 & -N_3 Z_2 & N_2 Z_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} N_3 V_1 e^{i\omega t} \\ N_1 N_3 V_2 e^{i\omega t} \\ -N_3 V_2 e^{i\omega t} \end{pmatrix} \tag{74}$$

$$\begin{pmatrix} N_3 Z_1 & 0 & -N_1 Z_3 \\ 0 & N_1 N_3 Z_2 + N_1 N_3 i\omega L_2 & N_1 N_2 i\omega L_3 + \frac{N_1 N_2 i\omega L_1 Z_3}{Z_1} \\ 0 & -N_3 Z_2 & N_2 Z_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} N_3 V_1 e^{i\omega t} \\ N_1 N_3 V_2 e^{i\omega t} - \frac{N_2 N_3 i\omega L_1 V_1 e^{i\omega t}}{Z_1} \\ -N_3 V_2 e^{i\omega t} \end{pmatrix} \tag{75}$$

$$\begin{pmatrix} N_3 Z_1 & 0 & -N_1 Z_3 \\ 0 & 0 & N_1 N_2 \left(Z_3 + i\omega L_3 + \frac{i\omega L_1 Z_3}{Z_1} + \frac{i\omega L_2 Z_3}{Z_2}\right) \\ 0 & -N_3 Z_2 & N_2 Z_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} N_3 V_1 e^{i\omega t} \\ -\frac{N_1 N_3 i\omega L_2}{Z_2} V_2 e^{i\omega t} - \frac{N_2 N_3 i\omega L_1}{Z_1} V_1 e^{i\omega t} \\ -N_3 V_2 e^{i\omega t} \end{pmatrix} \tag{76}$$

As you can see, row-echelon has no fixed method. We have:

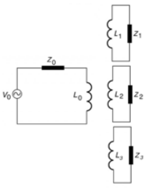
$$I_3 = -\frac{N_1 N_3 i\omega L_2 Z_1 V_2 e^{i\omega t} + N_2 N_3 i\omega L_1 Z_2 V_1 e^{i\omega t}}{N_1 N_2 (Z_1 Z_2 Z_3 + i\omega L_3 Z_1 Z_2 + i\omega L_1 Z_2 Z_3 + i\omega L_2 Z_1 Z_3)} \tag{77}$$

$$I_2 = \frac{N_1 (Z_1 Z_3 + i\omega L_3 Z_1 + i\omega L_1 Z_3) V_2 e^{i\omega t} - N_2 i\omega L_1 Z_3 V_1 e^{i\omega t}}{N_1 (Z_1 Z_2 Z_3 + i\omega L_3 Z_1 Z_2 + i\omega L_1 Z_2 Z_3 + i\omega L_2 Z_1 Z_3)} \tag{78}$$

$$I_1 = -\frac{N_1 i\omega L_2 Z_3 V_2 e^{i\omega t} + N_2 (Z_2 Z_3 + i\omega L_3 Z_2 + i\omega L_2 Z_3) V_1 e^{i\omega t}}{N_2 (Z_1 Z_2 Z_3 + i\omega L_3 Z_1 Z_2 + i\omega L_1 Z_2 Z_3 + i\omega L_2 Z_1 Z_3)} \tag{79}$$

VI. GENERALISED OUTPUT

Let's say I have one source and N outputs.



Let us describe the matrix from A_{00} to A_{nn} . $A_{Row\ Column}$. Well, we have $A_{ij} = \frac{i\omega N_i L_j}{N_j}, i \neq j, A_{ii} = Z_i + i\omega L_i$.

$$A\begin{pmatrix} I_0 \\ \vdots \\ \vdots \\ I_n \end{pmatrix} = \begin{pmatrix} V_0 e^{i\omega t} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{80}$$

Allow me to venture a guess:

$$I_0 = \frac{\left(\prod_{i=1}^{n} Z_i\right)\left(1+\sum_{i=1}^{n}\frac{i\omega L_i}{Z_i}\right)V_0 e^{i\omega t}}{\left(\prod_{i=0}^{n} Z_i\right)\left(1+\sum_{i=0}^{n}\frac{i\omega L_i}{Z_i}\right)} = \frac{\left(1+\sum_{i=1}^{n}\frac{i\omega L_i}{Z_i}\right)}{\left(1+\sum_{i=0}^{n}\frac{i\omega L_i}{Z_i}\right)}\frac{V_0 e^{i\omega t}}{Z_0} \tag{81}$$

$$I_n = -\frac{N_n}{N_1}\frac{\left(\prod_{i=0}^{n} Z_i\right)\frac{i\omega L_0}{Z_0 Z_n}V_0 e^{i\omega t}}{\left(\prod_{i=0}^{n} Z_i\right)\left(1+\sum_{i=0}^{n}\frac{i\omega L_i}{Z_i}\right)} = -\frac{N_n}{N_1}\frac{\frac{i\omega L_0}{Z_0}}{\left(1+\sum_{i=0}^{n}\frac{i\omega L_i}{Z_i}\right)}\frac{V_0 e^{i\omega t}}{Z_n} \tag{82}$$

Let us denote an operation: [i] + [j] as adding to the row i each element of row j. Let us generate an algorithm to solve this matrix. Here goes:

A. Simplification

1. $[n] - [n-1] \times \frac{N_n}{N_{n-1}} = \begin{pmatrix} 0 & \dots & 0 & -\frac{N_n}{N_{n-1}} Z_{n-1} & Z_n \end{pmatrix}$
2. $[n-1] - [n-2] \times \frac{N_{n-1}}{N_{n-2}} = \begin{pmatrix} 0 & \dots & 0 & -\frac{N_{n-1}}{N_{n-2}} Z_{n-2} & Z_1 & 0 \end{pmatrix}$
3. …

n. $[1] - [0] \times \frac{N_1}{N_0} = \begin{pmatrix} -\frac{N_1}{N_0} Z_0 & Z_1 & 0 & \dots & 0 \end{pmatrix}$

At this point, it is necessary to mention that the right-hand side matrix becomes $\begin{pmatrix} V_0 e^{i\omega t} \\ -\frac{N_1}{N_0} V_0 e^{i\omega t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

B. Manipulation

1. $[0] - [n] \times \frac{i\omega \frac{N_0}{N_n} L_n}{Z_n} = \begin{pmatrix} \dots & i\omega \frac{N_0}{N_{n-2}} L_{n-2} & i\omega \frac{N_0}{N_{n-2}}\left(L_{n-1} + \frac{Z_{n-1}}{Z_n} L_n\right) & 0 \end{pmatrix}$
2. $[0] - [n-1] \times \frac{i\omega \frac{N_0}{N_{n-1}}\left(L_{n-1} + \frac{Z_{n-1}}{Z_n} L_n\right)}{Z_{n-1}} = \begin{pmatrix} \dots & i\omega \frac{N_0}{N_{n-2}}\left(L_{n-2} + \frac{Z_{n-2}}{Z_{n-1}} L_{n-1} + \frac{Z_{n-2}}{Z_n} L_n\right) & 0 & 0 \end{pmatrix}$
3. ... (Here we can tell that the thing in the bracket can be written as $\sum_{j=i}^{n} \frac{Z_i}{Z_j} L_j$.)
4. $[0] - [2] \times \frac{i\omega \frac{N_0}{N_2}\left(\sum_{j=2}^{n} \frac{Z_2}{Z_j} L_j\right)}{Z_2} = \begin{pmatrix} Z_0 + i\omega L_0 & i\omega \frac{N_0}{N_1}\left(\sum_{j=1}^{n} \frac{Z_1}{Z_j} L_j\right) & 0 & \dots \end{pmatrix}$
5. $[0] - [1] \times \frac{i\omega \frac{N_0}{N_1}\left(\sum_{j=1}^{n} \frac{Z_1}{Z_j} L_j\right)}{Z_1} = \begin{pmatrix} Z_0 + i\omega \sum_{j=0}^{n} \frac{Z_0}{Z_j} L_j & 0 & 0 & \dots \end{pmatrix}$

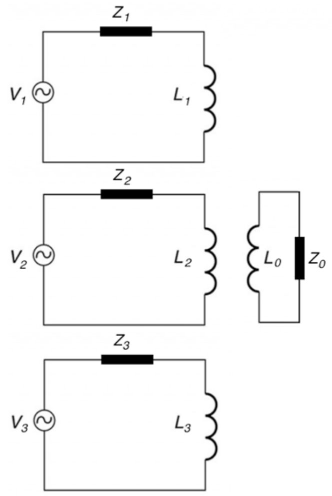
Here, we note that the right-hand side for row 0 becomes $V_0 e^{i\omega t}\left(1 + i\omega \sum_{j=1}^{n} \frac{1}{Z_j} L_j\right)$. Hence, we can conclude:

$$I_0 = -\frac{V_0 e^{i\omega t}\left(1 + i\omega \sum_{j=1}^{n} \frac{1}{Z_j} L_j\right)}{Z_0 + i\omega \sum_{j=0}^{n} \frac{Z_0}{Z_j} L_j} \tag{83}$$

Which is, of course, exactly our prediction in (81)! Simply substitute in this value of I_0, we should be able to get (82) and all other things.

VII. GENERALISED INPUT

Let's say I have N sources and 1 output.



Let us describe the matrix from A_{00} to A_{NN}. $A_{Row\ Column}$. Well, we have $A_{ij} = \frac{i\omega N_i L_j}{N_j}, i \neq j, A_{ii} = Z_i + i\omega L_i$.

$$A\begin{pmatrix} I_0 \\ \vdots \\ \vdots \\ I_n \end{pmatrix} = \begin{pmatrix} 0 \\ V_1 e^{i\omega t + \phi_1} \\ \vdots \\ V_n e^{i\omega t + \phi_n} \end{pmatrix} \tag{84}$$

A. Simplification

1. $[1] - [0] \times \frac{N_1}{N_0} = \begin{pmatrix} -\frac{N_1}{N_0} Z_0 & Z_1 & 0 & \dots & 0 \end{pmatrix}$
2. $[2] - [0] \times \frac{N_2}{N_0} = \begin{pmatrix} -\frac{N_2}{N_0} Z_0 & 0 & Z_2 & \dots & 0 \end{pmatrix}$
3. … (we remark that this is surprisingly similar)
4. $[n] - [0] \times \frac{N_n}{N_0} = \begin{pmatrix} -\frac{N_n}{N_0} Z_0 & 0 & \dots & 0 & Z_n \end{pmatrix}$

Allow me to mention that the right-hand side hasn't changed.

B. Manipulation

1. $[0] - [1] \times \frac{i\omega \frac{N_0}{N_1} L_1}{Z_1} = \begin{pmatrix} Z_0 + i\omega\left(L_0 + \frac{Z_0}{Z_1} L_1\right) & 0 & \dots \end{pmatrix}$
2. $[0] - [2] \times \frac{i\omega \frac{N_0}{N_2} L_2}{Z_2} = \begin{pmatrix} Z_0 + i\omega\left(L_0 + \frac{Z_0}{Z_1} L_1 + \frac{Z_0}{Z_2} L_2\right) & 0 & \dots \end{pmatrix}$
3. … (This is a lot easier! Surprising, huh?)
4. $[0] - [n] \times \frac{i\omega \frac{N_0}{N_n} L_n}{Z_n} = \begin{pmatrix} Z_0 + i\omega\left(\sum_{i=0}^{n} \frac{Z_0}{Z_i} L_i\right) & 0 & \dots \end{pmatrix}$

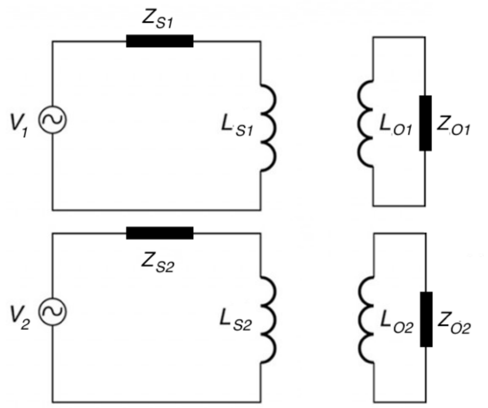
And here the right-hand side is: $-\sum_{i=1}^{n}\left(\frac{i\omega \frac{N_0}{N_i} L_i}{Z_i} V_i e^{i\omega t + \phi_i}\right)$. If all the sources are in phase, then we have $-\sum_{i=1}^{n}\left(i\omega \frac{N_0 V_i}{N_i Z_i} L_i\right) e^{i\omega t}$. Anyways, we will have:

$$I_0 = -\frac{\sum_{i=1}^{n}\left(\frac{i\omega \frac{N_0}{N_i} L_i}{Z_i} V_i e^{i\omega t + \phi_i}\right)}{Z_0 + i\omega\left(\sum_{i=0}^{n} \frac{Z_0}{Z_i} L_i\right)} \tag{85}$$

Which agrees with I_3 in (77).

VIII. Generalised Input and Output

We have **n** sources and **m** outputs.



$$\begin{pmatrix} Z_{S1}+i\omega L_{S1} & \cdots & \frac{i\omega N_{S1}L_{Sn}}{N_{Sn}} & \frac{i\omega N_{S1}L_{O1}}{N_{O1}} & \cdots & \frac{i\omega N_{S1}L_{Om}}{N_{Om}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{i\omega N_{Sn}L_{S1}}{N_{S1}} & \cdots & Z_{Sn}+i\omega L_{Sn} & \frac{i\omega N_{Sn}L_{O1}}{N_{O1}} & \cdots & \frac{i\omega N_{Sn}L_{Om}}{N_{Om}} \\ \frac{i\omega N_{O1}L_{S1}}{N_{S1}} & \cdots & \frac{i\omega N_{O1}L_{Sn}}{N_{Sn}} & Z_{O1}+i\omega L_{O1} & \cdots & \frac{i\omega N_{O1}L_{Om}}{N_{Om}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{i\omega N_{Om}L_{S1}}{N_{S1}} & \cdots & \frac{i\omega N_{Om}L_{Sn}}{N_{Sn}} & \frac{i\omega N_{Om}L_{O1}}{N_{O1}} & \cdots & Z_{Om}+i\omega L_{Om} \end{pmatrix} \begin{pmatrix} I_{S1} \\ \vdots \\ I_{Sn} \\ I_{O1} \\ \vdots \\ I_{Om} \end{pmatrix} = \begin{pmatrix} V_1 e^{i\omega t+\phi_1} \\ \vdots \\ V_2 e^{i\omega t+\phi_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (86)$$

A. Simplification

We can't surrender here! We've come so far.

1. $[Om]-[Om-1]\times\frac{N_{Om}}{N_{Om-1}} = \begin{pmatrix} 0 & \dots & 0 & -\frac{N_{Om}}{N_{Om-1}}Z_{Om-1} & Z_{Om} \end{pmatrix}$
2. $[Om-1]-[Om-2]\times\frac{N_{Om-1}}{N_{Om-2}} = \begin{pmatrix} 0 & \dots & -\frac{N_{Om-1}}{N_{Om-2}}Z_{Om-2} & Z_{Om-1} & 0 \end{pmatrix}$
3. …
4. $[O2]-[O1]\times\frac{N_{O2}}{N_{O1}} = \begin{pmatrix} \dots & -\frac{N_{O2}}{N_{O1}}Z_{O1} & Z_{O2} & \dots & 0 \end{pmatrix}$

At this point, we have cleaned up the bottom half of the matrix, less [O1].

5. $[Sn]-[O1]\times\frac{N_{Sn}}{N_{O1}} = \begin{pmatrix} \dots & Z_{Sn} & -\frac{N_{Sn}}{N_{O1}}Z_{O1} & \dots & 0 \end{pmatrix}$
6. $[Sn-1]-[O1]\times\frac{N_{Sn-1}}{N_{O1}} = \begin{pmatrix} \dots & Z_{Sn-1} & -\frac{N_{Sn-1}}{N_{O1}}Z_{O1} & \dots & 0 \end{pmatrix}$
7. …
8. $[S1]-[O1]\times\frac{N_{S1}}{N_{O1}} = \begin{pmatrix} Z_{S1} & \dots & -\frac{N_{S1}}{N_{O1}}Z_{O1} & \dots & 0 \end{pmatrix}$

At this point, the matrix looks like this:

$$\begin{pmatrix} Z_{S1} & \cdots & 0 & -\frac{N_{S1}}{N_{O1}}Z_{O1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & Z_{Sn} & -\frac{N_{Sn}}{N_{O1}}Z_{O1} & \cdots & 0 \\ \frac{i\omega N_{O1}L_{S1}}{N_{S1}} & \cdots & \frac{i\omega N_{O1}L_{Sn}}{N_{Sn}} & Z_{O1}+i\omega L_{O1} & \cdots & \frac{i\omega N_{O1}L_{Om}}{N_{Om}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & -\frac{N_{Om}}{N_{Om-1}}Z_{Om-1} & Z_{Om} \end{pmatrix} \begin{pmatrix} I_{S1} \\ \vdots \\ I_{Sn} \\ I_{O1} \\ \vdots \\ I_{Om} \end{pmatrix} = \begin{pmatrix} V_1 e^{i\omega t+\phi_1} \\ \vdots \\ V_2 e^{i\omega t+\phi_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (87)$$

[O1] is the only complete row. The right-hand side hasn't changed.

B. Manipulation

1. $[O1]-[S1]\times\frac{\frac{i\omega N_{O1}L_{S1}}{N_{S1}}}{Z_{S1}} = \left(0 \quad \dots \quad Z_{O1}+i\omega\left(L_{O1}+\frac{Z_{O1}}{Z_{S1}}L_{S1}\right) \quad \dots\right)$
2. $[O1]-[S2]\times\frac{\frac{i\omega N_{O1}L_{S2}}{N_{S2}}}{Z_{S2}} = \left(0 \quad \dots \quad Z_{O1}+i\omega\left(L_{O1}+\frac{Z_{O1}}{Z_{S1}}L_{S1}+\frac{Z_{O1}}{Z_{S2}}L_{S2}\right) \quad \dots\right)$
3. …
4. $[O1]-[Sn]\times\frac{\frac{i\omega N_{O1}L_{Sn}}{N_{Sn}}}{Z_{Sn}} = \left(0 \quad \dots \quad Z_{O1}+i\omega\left(L_{O1}+\sum_{i=1}^{n}\frac{Z_{O1}}{Z_{Si}}L_{Si}\right) \quad \dots\right)$

Here, the right-hand side has become:

$-\sum_{i=1}^{n}\left(\frac{i\omega\frac{N_{O1}}{N_{Si}}L_{Si}}{Z_{Si}}V_i e^{i\omega t+\phi_i}\right)$Now, let's tackle the bottom right.

5. $[O1]-[Om]\times\frac{i\omega\frac{N_{O1}}{N_{Om}}L_{Om}}{Z_{Om}} =$
$\left(\dots \quad Z_{O1}+i\omega\left(L_{O1}+\sum_{i=1}^{n}\frac{Z_{O1}}{Z_{Si}}L_{Si}\right) \quad \dots \quad \frac{i\omega N_{O1}L_{Om-1}}{N_{Om-1}}+\frac{N_{O1}i\omega L_{Om}}{Z_{Om}N_{Om-1}}Z_{Om-1} \quad 0\right)$
6. $[O1]-[Om-1]\times\frac{i\omega\frac{N_{O1}}{N_{Om-1}}\left(L_{Om-1}+\frac{Z_{Om-1}}{Z_{Om}}L_{Om}\right)}{Z_{Om-1}} =$
$\left(\dots \quad \dots \quad \frac{i\omega N_{O1}}{N_{Om-2}}\left(L_{Om-2}+\frac{Z_{Om-2}}{Z_{Om-1}}L_{Om-1}+\frac{Z_{Om-2}}{Z_{Om}}L_{Om}\right) \quad 0 \quad 0\right)$
7. …
8. $[O1]-[O3]\times\frac{i\omega\frac{N_{O1}}{N_{O3}}\left(\sum_{j=3}^{m}\frac{Z_{O3}}{Z_{Oj}}L_{Oj}\right)}{Z_{O3}} =$
$\left(\dots \quad Z_{O1}+i\omega\left(L_{O1}+\sum_{i=1}^{n}\frac{Z_{O1}}{Z_{Si}}L_{Si}\right) \quad \frac{i\omega N_{O1}}{N_{O2}}\left(\sum_{j=2}^{m}\frac{Z_{O2}}{Z_{Oj}}L_{Oj}\right) \quad 0 \quad \dots\right)$

And now, the piece-de-resistance.

9. $[O1]-[O2]\times\frac{i\omega\frac{N_{O1}}{N_{O2}}\left(\sum_{j=2}^{m}\frac{Z_{O2}}{Z_{Oj}}L_{Oj}\right)}{Z_{O2}} =$
$\left(\dots \quad Z_{O1}+i\omega\left(\sum_{i=1}^{n}\frac{Z_{O1}}{Z_{Si}}L_{Si}+\sum_{j=1}^{m}\frac{Z_{O1}}{Z_{Oj}}L_{Oj}\right) \quad 0 \quad 0 \quad \dots\right)$

Yes! We have successfully cleared out [O1]. This means

$$I_{O1} = -\frac{\sum_{i=1}^{n}\left(\frac{i\omega\frac{N_{O1}}{N_{Si}}L_{Si}}{Z_{Si}}V_i e^{i\omega t+\phi_i}\right)}{Z_{O1}+i\omega\left(\sum_{i=1}^{n}\frac{Z_{O1}}{Z_{Si}}L_{Si}+\sum_{j=1}^{m}\frac{Z_{O1}}{Z_{Oj}}L_{Oj}\right)} \quad (88)$$

$$I_{Ok} = -\frac{\frac{N_{Ok}Z_{O1}}{N_{O1}Z_{Ok}}\sum_{i=1}^{n}\left(\frac{i\omega\frac{N_{O1}}{N_{Si}}L_{Si}}{Z_{Si}}V_i e^{i\omega t+\phi_i}\right)}{Z_{O1}+i\omega\left(\sum_{i=1}^{n}\frac{Z_{O1}}{Z_{Si}}L_{Si}+\sum_{j=1}^{m}\frac{Z_{O1}}{Z_{Oj}}L_{Oj}\right)} \quad (89)$$

$$I_{Ok} = -\frac{\sum_{i=1}^{n}\left(\frac{i\omega\frac{N_{Ok}}{N_{Si}}L_{Si}}{Z_{Si}}V_i e^{i\omega t+\phi_i}\right)}{Z_{Ok}+i\omega\left(\sum_{i=1}^{n}\frac{Z_{Ok}}{Z_{Si}}L_{Si}+\sum_{j=1}^{m}\frac{Z_{Ok}}{Z_{Oj}}L_{Oj}\right)} \quad (90)$$

$$I_{Sp} = \frac{V_p e^{i\omega t+\phi_p}}{Z_{Sp}} - \frac{\sum_{i=1}^{n}\left(\frac{i\omega\frac{N_{Sp}}{N_{Si}}L_{Si}}{Z_{Si}}V_i e^{i\omega t+\phi_i}\right)}{Z_{Sp}+i\omega\left(\sum_{i=1}^{n}\frac{Z_{Sp}}{Z_{Si}}L_{Si}+\sum_{j=1}^{m}\frac{Z_{Sp}}{Z_{Oj}}L_{Oj}\right)} \quad (91)$$

Hooray.

IX. CONCLUSION

We learnt that the convenient shorthand below only works when there is no load on the source.

$$V_2 = -\frac{N_2}{N_1} V_1 \quad (92)$$

VERY HELPFUL LINKS

The following websites proved helpful in understand how transformers work.

[1] https://en.wikipedia.org/wiki/Transformer

[2] https://www.electricaltechnology.org/2013/12/transformer-phasing-the-dot-notation-and-dot-convention.html

[3] https://slideplayer.com/slide/11863962/66/images/6/Transformer+Overview+A+time-varying+current+in+the+primary+windings+induces+a+magnetic+flux+

[4] http://www.insula.com.au/physics/1221/L17.html