A SHORT STUDY IN AERODYNAMICS

Ong Hong Ming Teddy[1]

[1] NUS High School of Mathematics & Science, Clementi Avenue 1, Singapore 129957

I. Introduction

Why do planes fly? In this paper, we will demonstrate existing simplified aerodynamic models, concepts and calculations that are valid up to the level of the International Physics Olympiad (IPhO). Methods described in this paper are reasonably accurate for traditional model-sized remote-controlled aircraft. This document is prepared for the theory training in the Youth Flying Club and for IPhO students. The author, Ong Hong Ming Teddy, is a Gold Medallist at the 48th IPhO, and the president of YFC in 2017.

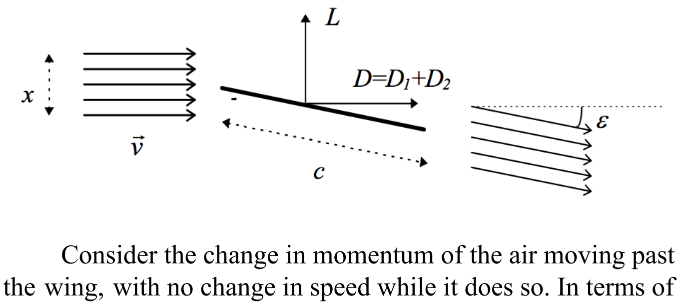
II. The Four Forces

Forces acting on an aeroplane can be put into four categories: thrust, lift, drag and weight (Fig. 1). In particular, lift can be generated via two mechanisms: deflection and pressure difference. Drag consists of induced drag, parasitic drag, form drag. Other types of drags not relevant to our regime include interference drag and wave drag. Reading through the following sections, even if you don't understand the equations, understanding the descriptions and the graphs is sufficiently rewarding.

III. A Basic Lift and Drag Model

A. *Lift by Angle of Attack*

We begin with IPhO 1997 Q3 [1]. Consider a rectangular wing of span *l*, chord *c*, wing area $S = cl$, and aspect ratio . We consider a slice of air of height *x* and length *l* being deflected downward at a small angle *ε* with only a very small change in speed. This simple model corresponds closely to reality if $x = \frac{\pi}{4}l$. The total mass of the aircraft is *M* and it flies horizontally with velocity *v* relative to the surrounding air.



Consider the change in momentum of the air moving past the wing, with no change in speed while it does so. In terms of wing dimensions, v, ε, and the air density ϱ, the lift is the vertical component of the rate of change of momentum,

$$\frac{dm}{dt} = \varrho x l v = \frac{\pi}{4}\varrho v l^2$$

$$L = \frac{dp_y}{dt} = \Delta v_y \frac{dm}{dt} = \frac{\pi}{4}\varrho l^2 v^2 \sin\varepsilon$$

B. *Induced Drag D_1*

Induced Drag is a direct consequence of the angle of attack as well, being the horizontal component of the rate of change of momentum. Hence, it can be expressed as

$$D_1 = \frac{dp_x}{dt} = \Delta v_x \frac{dm}{dt} = \frac{\pi}{4}\varrho l^2 v^2 (1 - \cos\varepsilon)$$

C. *Parasitic Drag D_2*

Parasitic Drag, also called skin friction drag, is caused by the interaction between the air and the wetted aircraft surface. Different sources may give different definitions and classifications of this drag. Anyway, the air slows slightly as it travels over the wing, with a change of speed, Δv

$$\frac{\Delta v}{v} = \frac{f}{A}$$

$$D_2 = \frac{dp}{dt} = \Delta v \frac{dm}{dt} = \frac{\pi f}{4A}\varrho l^2 v^2$$

Where A is the aspect ratio. Note that the direction of this drag is necessarily along the chord of the wing. We can either use the previous lift and drag equations separately, or we can also update them to include the parasitic drag effect. Either method is fine, just make sure you don't double count. Note that since Δv is very small, we assume that $\frac{dm}{dt}$ is constant. In essence, the only change is that the air exiting the wing at angle ε has velocity $v - \Delta v$.

$$\frac{dm}{dt} = \varrho x l v = \frac{\pi}{4}\varrho v l^2$$

$$L' = \Delta v_y \frac{dm}{dt} = \frac{\pi}{4}\varrho l^2 v(v - \Delta v)\sin\varepsilon = \frac{\pi}{4}\varrho l^2 v^2 (1 - \tfrac{f}{A})\sin\varepsilon$$

$$D_1{}' = \frac{\pi}{4}\varrho l^2 v(v - \Delta v)(1 - \cos\varepsilon) = \frac{\pi}{4}\varrho l^2 v^2 (1 - \tfrac{f}{A})(1 - \cos\varepsilon)$$

D. *Form Drag D_3*

Form Drag is caused by the cross-section of the aircraft. This can be reduced by having a smaller and more streamlined profile. This drag can be generally given as such,

$$D_3 = \frac{1}{2} C_D \varrho A v^2$$

Where the drag coefficient, C_D is experimentally determined. This equation is applicable only at high reynold's number, otherwise stokes' drag may be more useful.

E. *Conditions for Level Flight*

For level flight to occur, lift must be equal to weight and thrust equal to drag. Hence, the first condition gives us

$$L' = L - D_2 \sin\varepsilon = \frac{\pi}{4}\varrho l^2 v^2 (1 - \tfrac{f}{A})\sin\varepsilon = Mg$$

$$\sin \varepsilon = \frac{4Mg}{\pi \varrho l^2 v^2 (1-\frac{f}{A})}$$

Hence, from this we can calculate the power needed,

$$P = (D_1 + D_2 \cos \varepsilon + D_3) v = \frac{\pi}{4} \varrho l^2 v^3 (1 - (1 - \frac{f}{A}) \cos \varepsilon) + \frac{1}{2} C_D \varrho A v^3$$

F. Minimum Power Required for Level Flight

We will first need to simplify these expressions by taking first- and second-order approximations. (We will ignore terms of $\varepsilon^2 f$ or higher.) Let us also ignore form drag. We have

$$\varepsilon v^2 = \frac{4Mg}{\pi \varrho l^2 (1-\frac{f}{A})}$$

$$P = \frac{\pi}{4} \varrho l^2 v^3 (1 - (1 - \frac{f}{A})(1 - \frac{\varepsilon^2}{2})) = \frac{\pi}{4} \varrho l^2 v^3 (\frac{\varepsilon^2}{2} + \frac{f}{A})$$

We can now substitute into the power equation the expression for either velocity or angle. Here, both methods are demonstrated. Firstly, the velocity method:

$$P = \frac{\pi}{4} \varrho l^2 v^3 \left(\frac{\left(\frac{4Mg}{\pi \varrho l^2 (1-\frac{f}{A}) v^2} \right)^2}{2} + \frac{f}{A} \right) = \frac{\pi}{4} \varrho l^2 v^3 \frac{f}{A} + \frac{2M^2 g^2}{\pi \varrho v l^2 (1-\frac{f}{A})^2}$$

$$\frac{dP}{dv} = \frac{3\pi}{4} \varrho l^2 v^2 \frac{f}{A} - \frac{2M^2 g^2}{\pi \varrho v^2 l^2 (1-\frac{f}{A})^2} = 0$$

$${v_0}^4 = \frac{8A}{3f} \left(\frac{Mg}{\varrho l^2 (1-\frac{f}{A})} \right)^2$$

Secondly, the angle method:

$$P = \frac{\pi}{4} \varrho l^2 \left(\frac{4Mg}{\pi \varrho l^2 (1-\frac{f}{A}) \varepsilon} \right)^{\frac{3}{2}} (\frac{\varepsilon^2}{2} + \frac{f}{A}) = \left(\frac{\varepsilon}{\pi \varrho l^2} \right)^{\frac{1}{2}} \left(\frac{Mg}{1-\frac{f}{A}} \right)^{\frac{3}{2}} + \frac{f}{A} \left(\frac{4}{\pi \varrho l^2} \right)^{\frac{1}{2}} \left(\frac{Mg}{\varepsilon (1-\frac{f}{A})} \right)^{\frac{3}{2}}$$

$$\frac{dP}{dv} = \frac{1}{2} \varepsilon^{-\frac{1}{2}} \left(\frac{1}{\pi \varrho l^2} \right)^{\frac{1}{2}} \left(\frac{Mg}{1-\frac{f}{A}} \right)^{\frac{3}{2}} - \frac{3f}{2A} \varepsilon^{-\frac{5}{2}} \left(\frac{4}{\pi \varrho l^2} \right)^{\frac{1}{2}} \left(\frac{Mg}{(1-\frac{f}{A})} \right)^{\frac{3}{2}} = 0$$

$${\varepsilon_0}^2 = \frac{6f}{A}$$

One other simplification we can make, as the IPhO answer sheet does, is to ignore the factor $(1 - \frac{f}{A})$. If we do so, we can arrive at the minimum power needed as,

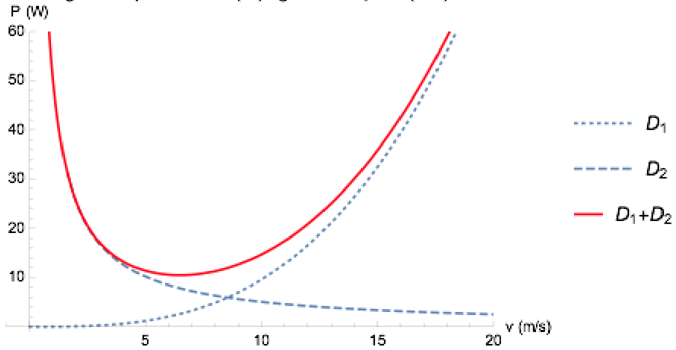
$$P_{min} = \pi \varrho l^2 {v_0}^3 \frac{f}{A}$$

These calculations imply that without parasitic drag, planes will fly at 0 angle at infinite speed, and need no power. Be thankful for drag because it makes flying a plane easier. Fig. 2 shows the relationship between power and speed.

G. Lift Profile along the Span

In this model, the lift is assumed to be constant across the span of the wing. We will see in other lift models that this is actually not the case. However, this is still reasonably and qualitatively accurate.

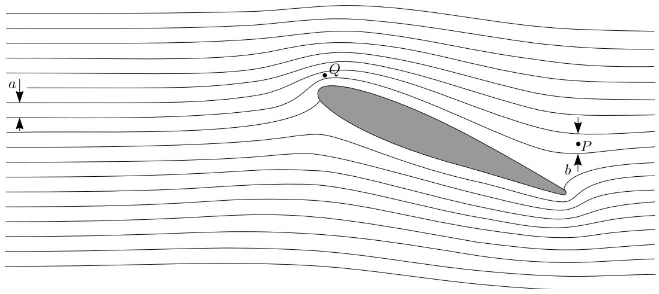
Fig. 2: Graph of Power (W) against Airspeed (m/s)



IV. Lift Profile along the Chord

Here, we reference IPhO 2012, Q1 [2]. The ability to read such a streamline diagram is a skill every YFC member should possess. The airspeed is v_1.

The way to read the pressure is to calculate the ratio of the spacing between the streamlines. Since fluid flows along a



streamline, fluid in between two streamlines are confined to this "pipe". The continuity equation then tells us that,

$$l\, h_1 v_1 = l\, h_2 v_2$$

For example, in the diagram above, the airspeed at point P would be $\frac{a}{b} v_1$. Hence, the point with the highest airspeed (hence the lowest pressure) is point Q. This lift distribution at a position y along the chord can be approximated as a modified Maxwell-Boltzmann distribution, where A is a constant to control the position of maximum lift y_{max}, and B is a normalization constant such that the total lift per length of span after integration is equal to $L_0 = \frac{Mg}{l}$.

$$L(y) = L_0\, B\, y\, e^{-Ay^2}$$

$$\frac{dL}{dy} = L_0\, B\, (1 - 2Ay^2)\, e^{-Ay^2} = 0$$

$$A = \frac{1}{2{y_{max}}^2}$$

$$L_0 = \int_0^c L\, dy \approx \int_0^\infty L\, dy = \frac{L_0 B}{2A}$$

$$B = 2A = \frac{1}{{y_{max}}^2}$$

$$L(y) = \frac{Mg}{l\, {y_{max}}^2}\, y\, e^{-\frac{y^2}{2{y_{max}}^2}}$$

We may naively assume that y_{max} coincides with where the airfoil is the thickest. For a flat-plate airfoil, we can assign this to be around ¼ to ⅕. In fact, this is generally true for all airfoils. Hence, we can calculate the center of pressure as a distance from the leading edge as

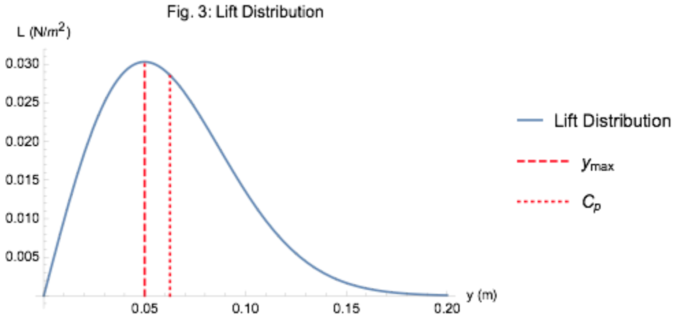
$$C_P \approx \frac{1}{L_0}\int_0^\infty y\,L(y)\,dy = \left[\frac{-yB}{2A}e^{-Ay^2}\right] - \int_0^\infty \frac{-B}{2A}e^{-Ay^2}\,dy$$

$$s = \int_0^\infty e^{-Ay^2}\,dy$$

$$s^2 = \int_0^\infty\int_0^\infty e^{-A(x^2+y^2)}\,dx\,dy = \int_0^\infty\int_0^{\pi/2} e^{-Ar^2}r\,dr\,d\theta = \frac{\pi}{4A}$$

$$C_P = \sqrt{\frac{\pi}{4A}} = y_{max}\sqrt{\frac{\pi}{2}} \Longrightarrow C_p \in [0.250c, 0.313c]$$

Fig. 3: Lift Distribution



Note that to calculate this value, we have used an aeromodelling rule of thumb. Results may vary depending on what kind of airfoil is used. Generally, the CG should be before the ⅓ line of the wing root. For Delta-wings, the CG should be placed before the ½ line. In Fig. 3, the chord length used is 0.20m.

A. *Contrails and Condensation*

Where do contrails form on a wing? Clearly at places of low pressure/low temperature, which corresponds to the top of the wing (such as point Q). We may use the following data:

Constant	Value
Relative Humidity, r	90%
Specific heat capacity of air at constant pressure c_p	1.00×10^3 J/kg K
Specific heat capacity of air at constant volume c_v	0.717×10^3 J/kg K
Pressure of saturated water vapour at T_a=293K, p_a	2.31kPa
Pressure of saturated water vapour at T_b=294K, p_b	2.46kPa

Humidity is defined as the ratio of the vapour pressure to the saturated vapour pressure at the given temperature. Saturated vapour pressure is defined as the vapour pressure by which vapour is in equilibrium with the liquid.

Firstly, let us calculate the temperature of the dew point. The vapour pressure is

$$p_w = p_a r = 2.08kPa$$

$$\frac{p_b - p_a}{T_b - T_a} = \frac{p_a - p_w}{T_a - T_w}$$

$$T_a - T_w = (T_b - T_a)\frac{(1-r)p_a}{p_b - p_a}$$

Next, let's assume that there is one mole of gas of molar mass μ.There are three components to its energy: its bulk movement kinetic energy, $\frac{1}{2}\mu v^2$; its heat energy, $C_v T$, and its pressure energy, PV . Incidentally, $PV = RT$, and $C_p = C_v + R$. Here, molar heat capacities $C_v = \mu c_v$ and $C_p = \mu c_p$. Hence, by conservation of energy,

$$\frac{1}{2}\mu v^2 + C_v T + RT = Constant$$

For condensation to occur, the speed at the point Q must be sufficient to cause the temperature to drop to the dew point. From the previous section, we know that $v_Q = \frac{a}{c}v_{crit}$, where a and c are the separation height between the streamlines at infinity and at Q, respectively. Hence,

$$\frac{1}{2}\Delta(v^2) = \frac{1}{2}{v_{crit}}^2\left(\frac{a^2}{c^2} - 1\right) = c_p\Delta T$$

$$v_{crit} = c\sqrt{\frac{2c_p\Delta T}{a^2 - c^2}} \approx 23\ m/s$$

Note that in reality, the required speed is probably somewhat higher, because for a fast condensation, a considerable over-saturation is needed. However, within an order of magnitude, this estimate remains valid. Also note that C_p in this section is different from the previous section.

V. The Dihedral Effect

The dihedral effect is the stabilizing effect brought primarily via having a dihedral (i.e. the wings intersect at an angle less than 180 degrees when viewed from the top). Suppose we have the following plane: it has wingspan l, dihedral angle θ on each wing, and a CG that is of height d lower than the joint of the wing.

Firstly, we note that we will decompose the airflow over the wing into two parts: headwind and crosswind. Headwind generates the main lift that counters weight, and crosswind causes the dihedral effect. From conservation of energy, when the aircraft is at a height y above its equilibrium position, its headwind speed will be

$$v_{head} = \sqrt{{v_0}^2 - 2gy}$$

Where v_0 is the headwind speed of the aircraft at equilibrium. Assuming that at v_0, the wings generate lift equals to the weight of the aircraft, the total lift vector generated by headwind will be at an angle α and have a magnitude of

$$L_0 = Mg\left(1 - \frac{2gy}{{v_0}^2}\right)$$

Where α is the angle at which the aircraft has deviated from being upright. Next, let us consider the two halves of the wings. Since they are in crosswind, they now have a length of c and aspect ratio of $\frac{2c}{l}$. Hence, we can denote the lift and induced drag generated as

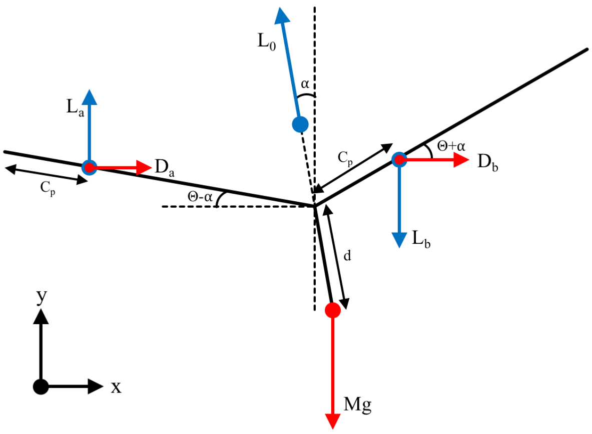
$$L(v, \varepsilon) = \frac{\pi}{4}\varrho l^2 v^2 (1 - \frac{fl}{2c}) \sin\varepsilon$$

$$D(v,\ \varepsilon) = \frac{\pi}{4}\varrho l^2 v^2 (1 - \frac{fl}{2c})(1 - \cos\varepsilon)$$

$$L_a = L(\dot{x}, \theta - \alpha);\ L_b = L(\dot{x}, \theta + \alpha)$$

$$D_a = D(\dot{x}, \theta - \alpha);\ D_b = D(\dot{x}, \theta + \alpha)$$

Here we have made three naïve assumptions: the crosswind is entirely characterised by the velocity of the aircraft in the x-direction, i.e. the velocity in the y-direction is negligible; and that hence the crosswind is assumed to be along the x-axis and makes angles $\theta - \alpha$ and $\theta + \alpha$ with the two halves of the wing; and that the airflow over one wing does not affect the airflow over the other, i.e. both halves of the wing are assumed to be in clean, horizontal crosswind. Clearly, these assumptions do not hold when: the plane is has significant in the y-direction; the dihedral angle is large, or the deviation angle is large. In essence, our model fails when the plane is meant to be unstable and will hence crash. Since we are only concerned with the stable cases, this is acceptable. The force diagram is shown below.



Here, C_p denotes the center of pressure for the crosswind lift vectors as previously integrated. Here, we shall simply assume it to be ¼ of the half-wing chord, which is $\frac{l}{8}$. The equations of motion can be written for torque and force as follows:

$$M\ddot{x} = -L_0 \sin\alpha - sign(\dot{x})\,(D_a + D_b) - D_x$$

$$M\ddot{y} = -Mg + L_0 \cos\alpha - sign(\dot{x})\,(L_a + L_b) - D_y$$

$$I\ddot{\alpha} = sign(\dot{x})\,(L_a l_{La} + L_b l_{Lb} + D_a l_{Da} + D_b l_{Db}) - D_\alpha$$

$$l_{La} = d\ \sin\alpha + \left(\frac{l}{4} + sign(\dot{x})(C_p - \frac{l}{4})\right)\cos(\theta - \alpha)$$

$$l_{Lb} = -d\ \sin\alpha + \left(\frac{l}{4} - sign(\dot{x})(C_p - \frac{l}{4})\right)\cos(\theta + \alpha)$$

$$l_{Da} = d\ \cos\alpha + \left(\frac{l}{4} + sign(\dot{x})(C_p - \frac{l}{4})\right)\sin(\theta - \alpha)$$

$$l_{Db} = -d\ \cos\alpha + \left(\frac{l}{4} - sign(\dot{x})(C_p - \frac{l}{4})\right)\sin(\theta + \alpha)$$

Here, I is the rotational inertia of the airplane, D_x, D_y, and D_α are appropriate drag terms to stabilize the plane. Without them, the plane will crash in all cases. Drag terms are

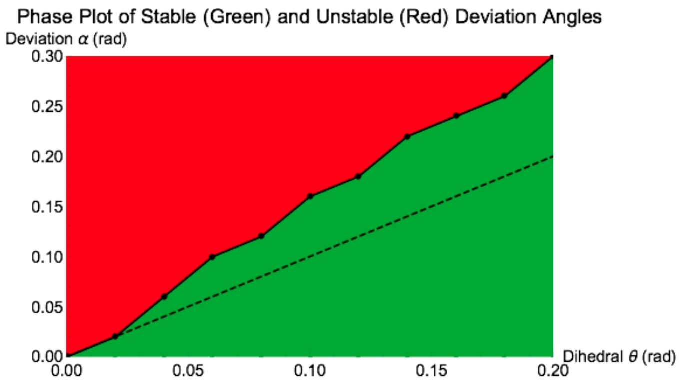
$$D_x = -sign(\dot{x})\ k(\dot{x})^2 - k\dot{x}$$

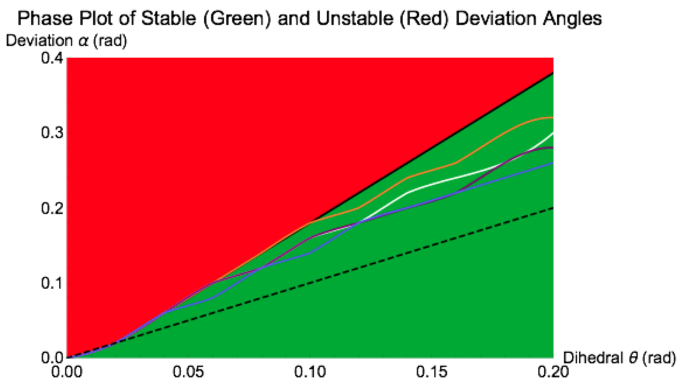
$$D_y = -k\dot{y}$$

$$D_\alpha = -k\dot{\alpha}$$

Where k is an appropriate constant. This is reasonable accurate for the stokes' regime, given that the drift speed in the y direction is considered small. For the x direction, a quadratic drag is also considered. We can put these equations into MATHEMATICA, NDSolve for the motion and post-process to find the following phase diagrams.

The first phase plot is for a high-wing configuration, yet when we look at the second phase plot, we see an interesting phenomenon.





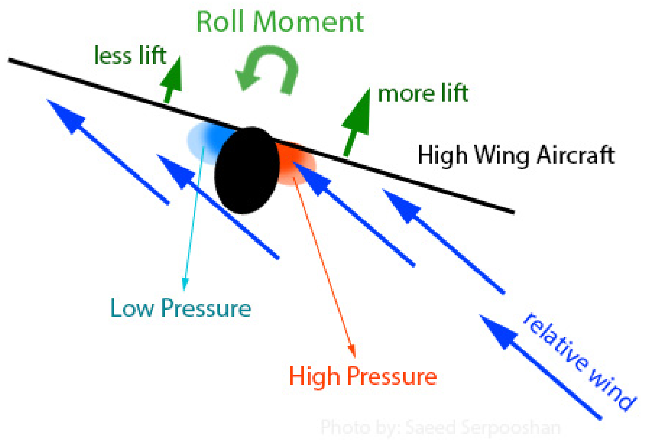
The colour code is as follows,

Line	**Height of Wing above CG**
Black, Solid	0.0 m
Orange	0.2 m
White	0.4 m
Purple	0.6 m
Blue	0.8 m
Black, Dotted	"Neutral" Stability, $\alpha = \theta$

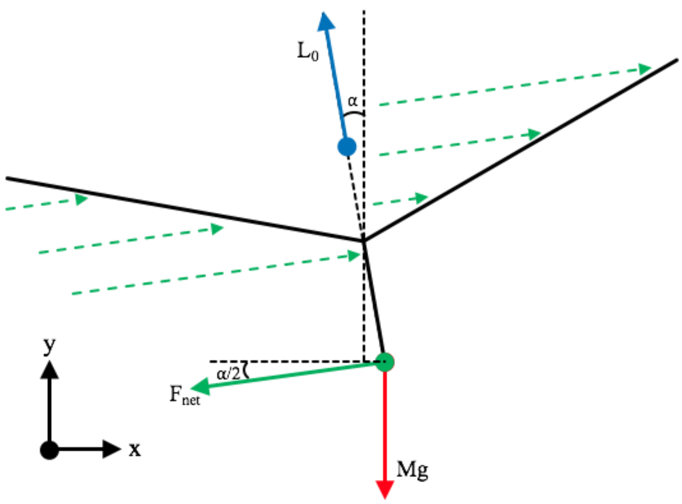
We can see that as the distance between the wing and the CG increases, the stability of the plane actually decreases, as far as rotation is concerned.

A. High-wing Effect

It is apparent that the high-wing actually reduces the rotational stability of the aircraft. Therefore, we distinguish rotational from translational stability. When the aircraft is displaced without rotation, a high pressure zone will be created under the wing, providing the restoring force required.



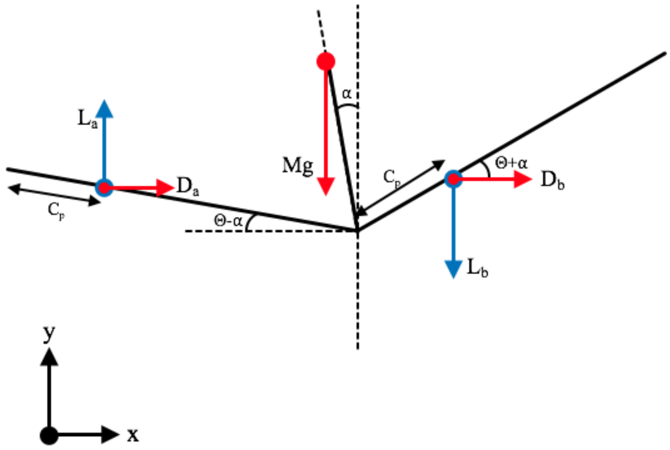
We see that when the aircraft is rotated, the sum of the forces L_0 and Mg (in the figure below) creates a resultant force towards the left, angled downwards by angle $\frac{\alpha}{2}$. Of course, this means that the "relative wind", which is in the opposite direction of the displacement, will be in the opposite direction of the net force. However, the dihedral means that the sideways angle of attack on the right (starboard) side is higher



than that of the left (port) side. This would be usually stabilizing. However, we see that in the force diagram on the previous page, that all 4 forces L_a, L_b, D_a, and D_b are stabilizing (they cause a torque to rotate back to equilibrium), if the location of the CG is too low, then it would be on the right-hand-side of L_b, then L_b would be destabilizing (causing a torque anti-clockwise).

B. Low-wing Effect

We can also see that a low wing would cause D_a and D_b to be destabilizing, creating an anti-clockwise torque.



Hence, we conclude that a high-wing that is too high doesn't work, but low-wing generally doesn't as well. Therefore, it is prudent to have a reasonably low or centered CG, without intentionally making it too low, and to let the dihedral effect do the stabilizing work. From my personal experience, planes with high wings do not tend to be exceptionally more stable than those without such high CGs, as long as they have a healthy amount of dihedral.

Here are the values used in the simulation,

Variable / Constant	**Value**
M	0.4 kg
g	9.81 m/s^2
l	1 m
c	0.2 m
f	0.01
ρ	1.225 kg/m^3
I_s	0.5 kg m^2
v_0	10 m/s
k	2

VI. Conclusion

Planes are very hard to fly and even harder to understand. I hope these calculations have helped you grasp a bit about the mechanics of aircrafts. The dihedral effect is great, but try not to make your CG too low, or worse too high. Have fun!

VII. References

[1] http://ipho.org/problems-and-solutions/1997/IPhO_1997_Theoretical%20Questions.pdf

[2] http://www.ipho2012.ee/wp-content/uploads/2012/07/IPhO2012_Theoretical_problem.pdf

[3] https://aviation.stackexchange.com/questions/26759/how-does-the-dihedral-angle-work