PDEs: Theory and Practice Coursework Assignment

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0.1 Pre-amble

For most of this assignment will will be considering the following PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - tu \tag{1}$$

which is defined over the interval $0 < x < 1$ and is subject to boundary conditions

$$u = 0 \text{ at } x = 0, \qquad \frac{\partial u}{\partial x} = 0 \text{ at } x = 1 \tag{2}$$

with initial conditions $u(x, 0) = f(x)$.

For notation purposes, partial derivatives may be re-written with subscripts (eg $\frac{\partial u}{\partial x} = u_x$, $\frac{\partial^2 y}{\partial t \partial x} = y_{tx}$, etc) for clarity in text. It is simply a notation change, so values are equivalent. This is just to avoid writing fractions in bodies of text, which can look small and confusing. If the function has only one variable a simple dash will show derivatives, where the number of dashes dictate how many times it has been differentiated, eg for $f(x)$

$$f'(x) = \frac{df}{dx} \qquad f''(x) = \frac{d^2 f}{dx^2} \tag{3}$$

If the subscript is not a variable in the PDE (eg n) then the subscript is used to specify a particular solution or value, not a derivative.

1 Series Solution Construction

Question: Use seperation of variables to construct a series solution to the initial value problem, for arbitrary $f(x)$. Briefly justify how you have ensured that you have found all the eigenfunctions.

1.1 Quick Observations

To begin, we must check that separation of variables is a legitimate strategy for finding a solution to the PDE. Since $u(x, t) = 0$ everywhere is a solution, we can say that the PDE is homogeneous, as is its boundary conditions. The PDE is also linear, 2nd order.

The linear and homogeneous traits of the PDE means that if, for example, we had two solutions to the PDE, say $u_1(x, t)$ and $u_2(x, t)$, then

$$u(x, t) = \alpha_1 u_1 + \alpha_2 u_2 \tag{4}$$

is also a solution to the PDE. In fact, if we had n number of solutions to the PDE and boundary conditions then we can construct a linear combination of solutions such that

$$u(x, t) = \sum_n \alpha_n u_n(x, t) \tag{5}$$

which is also a solution the the PDE and boundary conditions.

To use separation of variables, we say that each solution $u_n(x, t)$ can be written as a separable solution

$$u_n(x, t) = X_n(x) T_n(t) \tag{6}$$

where each function X_n is only a function of x, each T_n is only a function of t, and each function must individually satisfy the PDE and boundary conditions.

We therefore claim that the general solution to the PDE and boundary conditions is a linear combination of all possible separable solutions, which is countably infinite.

1.2 Seperation of Variables

1.2.1 The PDE

We start by rewriting the PDE as

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + tu = 0 \tag{7}$$

and assume that we have a separable solution such that

$$u(x,t) = X(x)T(t) \tag{8}$$

From here, we can make a substitution by differentiating the separable solution so we have

$$\frac{\partial u}{\partial t} = X(x)T'(t) \qquad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \tag{9}$$

and inserting these values into (7)

$$X(x)T'(t) - X''(x)T(t) + tX(x)T(t) = 0 \tag{10}$$

A little rearranging yields

$$\frac{X''(x)}{X(x)} = \frac{T'(t) + tT(t)}{T(t)} = -\lambda \tag{11}$$

Since one side of the equation says that λ is dependent on x and not t, and the other side says that λ is dependent on t and not x, we can say that λ is dependent on neither t nor x, and therefore it must be constant. So the separable solution can only be a solution to this PDE if the relationship established in (11) holds. Furthermore, this means we can write the two equations as a set of coupled ODEs

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T'(t) + tT(t) + \lambda T(t) = T'(t) + (t+\lambda)T(t) &= 0 \end{aligned} \tag{12}$$

which are coupled by λ. So if we find a particular value of λ for the $X(x)$ equation, we can use that to find the corresponding $T(t)$, and putting them together in a separable solution should satisfy the original PDE.

1.2.2 The Boundary Conditions

The boundary conditions state that we need $u(0,t) = 0$ and $u_x(1,t) = 0$. From (8) we can say that we expect $u_x(x,t) = X'(x)T(t)$, and (9) we can then say that

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \\ u_x(1,t) &= X'(1)T(t) = 0 \end{aligned} \tag{13}$$

for all t. So, either

$$X(0) = X'(1) = 0 \text{ or } T(t) = 0 \tag{14}$$

If we take that $T(t) = 0$ for all t, then from (8) we can say that $u(x,t) = 0$ for all t and x, which we already know is a solution since the original PDE is linear and homogeneous (trivial solution). We want to consider the non-trivial solution. We are therefore looking for values of $X(x)$ that satisfy

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) = X'(1) &= 0 \end{aligned} \tag{15}$$

which is an eigenvalue problem. Only particular values of λ, the eigenvalues, lead to non-zero $X(x)$.

1.2.3 Eigenvalue Problem

Claim: The value of λ must be real and positive.

Proof: Multiply (12) by $X(x)$ and integrate between the bounds the PDE is defined on $[0, 1]$ to get

$$\int_0^1 X(x)[X''(x) + \lambda X(x)]dx = 0 \implies \lambda = -\frac{\int_0^1 X(x)X''(x)dx}{\int_0^1 X(x)^2 dx} \tag{16}$$

For $X(x)$ to be an eigenfunction it must not be zero always, and therefore the denominator is non-zero. Integrating the numerator by parts gives

$$\int_0^1 X(x)X''(x)dx = [X(x)X'(x)]_0^1 - \int_0^1 X'(x)^2 dx \tag{17}$$

Since $X(0) = 0$ and $X'(1) = 0$, we can say that $[X(x)X'(x)]_0^1 = 0$. We can therefore re-write (16) as

$$\lambda = \frac{\int_0^1 X'(x)^2 dx}{\int_0^1 X(x)^2 dx} \tag{18}$$

If $X(x)$ is not zero everywhere, then

$$\int_0^1 X'(x)^2 dx \geq 0 \qquad \int_0^1 X(x)^2 dx > 0 \tag{19}$$

which means that λ cannot be negative. Furthermore, for $\lambda = 0$ to be a solution $X'(x)$ would have to equal 0 for all x. The boundary conditions only allow for $X(x) = 0$ for all x to be a solution, so $\lambda > 0$.

Since we know that λ is real and positive, let $\lambda = p^2$ where $p > 0$ and real. Therefore

$$X''(x) + p^2 X(x) = 0 \tag{20}$$

If $X(x) = e^{mx}$ then the above ODE becomes

$$m^2 e^{mx} + p^2 e^{mx} = (m^2 + p^2)e^{mx} = 0 \implies m^2 + p^2 = 0 \tag{21}$$

and so $m = \pm ip$, which is a complex conjugate pair of roots. The general solution of $X(x)$ is

$$X(x) = C\cos(px) + D\sin(px) \tag{22}$$

We can now apply the boundary conditions to find the value of coefficients. Firstly, if $X(0) = 0$ then

$$\begin{aligned} X(0) = C\cos(p0) + D\sin(p0) = 0 \\ C = 0 \end{aligned} \tag{23}$$

and we are left with $X(x) = D\sin(px)$. We can then differentiate this to find that $X'(x) = Dp\cos(px)$, and apply the second boundary condition of $X'(1) = 0$ to find

$$X'(1) = Dp\cos(p) = 0 \tag{24}$$

Here, $D = 0$ gives a trivial solution, so $D \neq 0$. Since we know that $p > 0$, we need to find values where $\cos(p) = 0$, which occurs when $p = \frac{\pi}{2} + n\pi$ for $n = 0, 1, 2, \ldots$. We can then say that

$$\lambda_n = p^2 = \left(\frac{\pi}{2} + n\pi\right)^2 \tag{25}$$

are the eigenvalues. Furthermore, we can then give a solution for each n for each $X(x)$ to be

$$X_n(x) = D\sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) \tag{26}$$

are the corresponding eigenfunctions. We know that these are all the eigenfunctions because of the proof that $\lambda > 0$ (shown above). We can now turn our attention to the $T(t)$ function. Each T that corresponds to the correct X can be written as

$$T_n'(t) + (t + \lambda_n)T_n(t) = 0 \tag{27}$$

which has the form

$$T_n(t) = Ge^{-\frac{1}{2}t(2\lambda_n + t)} \tag{28}$$

We can then substitute in the eigenvalue to get

$$T_n(t) = Ge^{-\frac{1}{2}t(t+2(n\pi + \frac{pi}{2})^2)} \tag{29}$$

1.2.4 Solution

Using the above work and the relationship established from (6), we can then say that, since $u_n(x,t) = X_n(x)T_n(t)$, that

$$u_n(x,t) = D_n \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right)G_n e^{-\frac{1}{2}t(t+2(n\pi + \frac{pi}{2})^2)} \tag{30}$$

which holds as long as $n = 0, 1, 2, ...$ We can do a sanity check by seeing if our boundary conditions hold for this solution. We can clearly see that

$$u_n(0,t) = D_n \sin(0) G_n e^{-\frac{1}{2}t(t+2(n\pi + \frac{pi}{2})^2)} = 0 \tag{31}$$

since $\sin(0) = 0$, so the whole equation vanishes. Differentiating the equation yields

$$\frac{\partial u_n}{\partial x}(x,t) = D_n \cos\left(x\big(n\pi + \frac{\pi}{2}\big)\right)\left(n\pi + \frac{\pi}{2}\right)G_n e^{-\frac{1}{2}t(t+2(n\pi + \frac{pi}{2})^2)} \tag{32}$$

which clearly goes to 0 when $x = 1$ as $\cos\left(n\pi + \frac{\pi}{2}\right) = 0$ for $n = 0, 1, 2, ...$ Finally, we can substitute in derivatives of the solution into the original form of the PDE and check that the relationship holds. From (30) we can show that

$$\frac{\partial u}{\partial t} = D_n G_n \sin\left(\big(\pi n + \frac{\pi}{2}\big)x\right)\left(-\big(\pi n + \frac{\pi}{2}\big)^2 - t\right)e^{-\frac{1}{2}t(t+2(n\pi + \frac{pi}{2})^2)} \tag{33}$$

and

$$\frac{\partial^2 u}{\partial x^2} = -D_n G_n \sin\left(\big(\pi n + \frac{\pi}{2}\big)x\right)\left(\pi n + \frac{\pi}{2}\right)^2 e^{-\frac{1}{2}t(2(\frac{\pi}{2} + n\pi)^2 + t)} \tag{34}$$

So, if we expect $= u_t - u_{xx} + tu = 0$ from (7), then the above substitutions must hold this relationship. Substituting in gives

$$\begin{aligned} D_n G_n \sin\left(\big(\pi n + \frac{\pi}{2}\big)x\right)\left(-\big(\pi n + \frac{\pi}{2}\big)^2 - t\right)e^{-\frac{1}{2}t(t+2(n\pi + \frac{pi}{2})^2)} + \\ D_n G_n \sin\left(\big(\pi n + \frac{\pi}{2}\big)x\right)\left(\pi n + \frac{\pi}{2}\right)^2 e^{-\frac{1}{2}t(2(\frac{\pi}{2} + n\pi)^2 + t)} + \\ tD_n \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right)G_n e^{-\frac{1}{2}t(2(\frac{\pi}{2} + n\pi)^2 + t)} = u_t - u_{xx} + tu \end{aligned} \tag{35}$$

We can take the constants, the sin and the exponential out as a common factor to get

$$D_n G_n \sin\left(\big(\pi n + \frac{\pi}{2}\big)x\right)e^{-\frac{1}{2}t(2(\frac{\pi}{2} + n\pi)^2 + t)}\left\{\big(\pi n + \frac{\pi}{2}\big)^2 - t - \big(\pi n + \frac{\pi}{2}\big)^2 + t\right\} = 0 \tag{36}$$

The component contained in the curly brackets is equal to 0, which vanishes the whole equation to 0, as expected.

From this, we can say that the general solution is the sum of all separable solutions (30). If we take $D_nG_n = A_n$, then the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) e^{-\frac{1}{2}t(2(\frac{\pi}{2}+n\pi)^2+t)} \tag{37}$$

We also have an initial condition to consider, which is that $u(x,0) = f(x)$, where $f(x)$ is (currently) some arbitrary function. Using the general solution, this means that

$$u(x,0) = \sum_{n=0}^{\infty} A_n \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) = f(x) \tag{38}$$

as $e^0 = 1$, and so

$$A_n = 2\int_0^1 f(x) \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) dx \tag{39}$$

2 Series Coefficients

Question: Calculate your series coefficients for

- $f(x) = 1$
- $f(x) = x$

For each case use a plotting package to plot $u(x,t)$ at $t = 0$, $t = 0.1$, and $t = 0.5$ for suitable truncations of the sum

2.1 $f(x) = 1$

If we take $f(x) = 1$ then (39) becomes

$$A_n = 2\int_0^1 \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) dx \tag{40}$$

and so

$$\begin{aligned} A_n &= -\frac{2}{\pi n + \frac{\pi}{2}}\left[\cos\left((\pi n + \frac{\pi}{2})x\right)\right]_0^1 \\ A_n &= -\frac{2}{\pi n + \frac{\pi}{2}}[-1] \\ A_n &= \frac{2}{\pi n + \frac{\pi}{2}} \end{aligned} \tag{41}$$

Therefore, for this initial condition, the solution looks like

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2}{\pi n + \frac{\pi}{2}} \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) e^{-\frac{1}{2}t(2(\frac{\pi}{2}+n\pi)^2+t)} \tag{42}$$

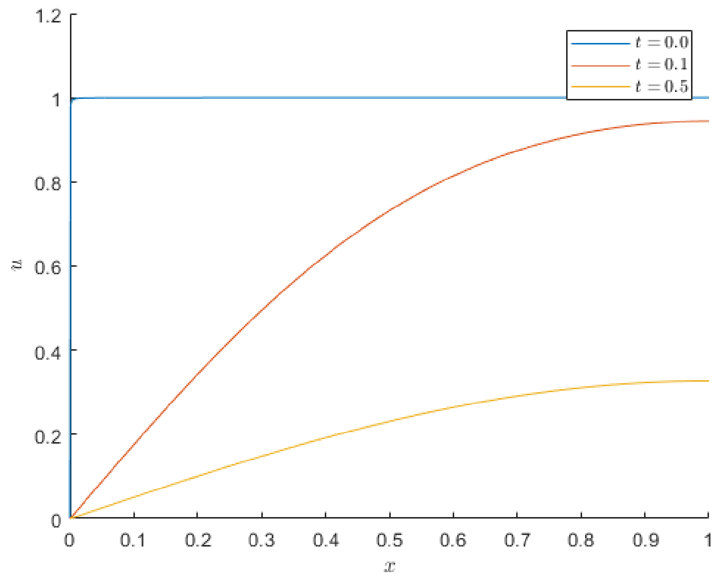

Figure 1: Shows how u behaves along x for $t = 0$, $t = 0.1$, $t = 0.5$ for initial condition $u(x, 0) = 1$

We can study the behaviour of this PDE for the initial conditions and boundary conditions by plotting x against u for different fixed values of t. Choosing $N = 10000$ (a large, finite number instead of ∞) gives the behaviour in figure 1. To begin with $u = 1$ everywhere except at the specified boundary condition of $u(0, t) = 0$ (and we can see that this condition is held as we step through t). We also expect the curve to be flat at $x = 1$, which is specified as our secondary boundary condition $u_x(1, t) = 0$, which also holds. Code to generate the plot is in *A.1*

2.2 $f(x) = x$

If we take $f(x) = x$ then (39) becomes

$$A_n = 2\int_0^1 x \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) dx \tag{43}$$

and so

$$\begin{aligned} A_n &= \frac{2}{(\pi n + \frac{\pi}{2})^2}\big[\sin\big((\pi n + \frac{\pi}{2})x\big) - \big((\pi n + \frac{\pi}{2})x\big)x \cos\big((\pi n + \frac{\pi}{2})x\big)\big]_0^1 \\ A_n &= \frac{2}{(\pi n + \frac{\pi}{2})^2}\big[\sin\big((\pi n + \frac{\pi}{2})\big)\big] \\ A_n &= \frac{2\sin\big((\pi n + \frac{\pi}{2})\big)}{(\pi n + \frac{\pi}{2})^2} \\ A_n &= (-1)^n \frac{2}{(\pi n + \frac{\pi}{2})^2} \end{aligned} \tag{44}$$

Therefore, for this initial condition, the solution looks like

$$u(x, t) = \sum_{n=0}^{\infty} (-1)^n \frac{2}{(\pi n + \frac{\pi}{2})^2} \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) e^{-\frac{1}{2}t(2(\frac{\pi}{2}+n\pi)^2+t)} \tag{45}$$

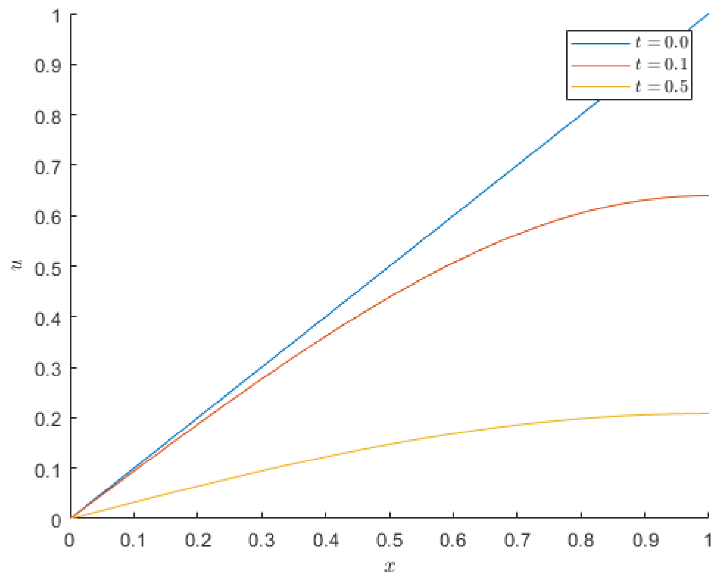

Figure 2: Shows how u behaves along x for $t = 0$, $t = 0.1$, $t = 0.5$ for initial condition $u(x, 0) = x$

We can study the behaviour of this PDE for the initial conditions and boundary conditions by plotting x against u for different fixed values of t. Choosing $N = 10000$ (a large, finite number instead of ∞) gives the behaviour in figure 1. This value of N is probably unnecessarily large, as the shape of the Fourier series suggests that it will converge quickly (more on this later). To begin with $u = x$, which allows for the boundary condition of $u(0, t) = 0$ (and we can see that this condition is held as we step through t). We also expect the curve to be flat at $x = 1$, which is specified as our secondary boundary condition $u_x(1, t) = 0$, which also holds as time progresses. Code to generate the plot is in *A.2*.

3 Periodic Extension

Question: Explain briefly how the boundary conditions have led to a periodic extension $f_{pe}(x)$ of $f(x)$ with period 4. Give a piece-wise definition of $f_{pe}(x)$ over the interval $-2 < x < 2$.

3.1 Definitions

We say that a function has a period of period T if

$$f(x) = f(x + T) \text{ for all } x \tag{46}$$

In other words, if we travel a distance T down the x axis, we would find that the function $f(x)$ repeats itself in every interval of length T. Clearly, sine and cosine are both examples of periodic functions, where $\sin(\omega x)$ and $\cos(\omega x)$ both have a period of $T = \frac{2\pi}{\omega}$.

We may also make some statements about general functions. A function is considered to be an even function if

$$f(-x) = f(x) \tag{47}$$

and it is considered odd if

$$f(-x) = -f(x) \tag{48}$$

A classic example of an even function is $f(x) = x^2$, but we can also see that $cos(x)$ would also be an even function. A classic odd function is $f(x) = x^3$, and $sin(x)$ is another example. We can use these facts about even and odd functions to say

$$\begin{aligned} \int_{-L}^{L} f(x)dx &= 2\int_{0}^{L} f(x)dx \qquad &\text{if } f(x) \text{ is even} \\ \int_{-L}^{L} f(x)dx &= 0 &\text{if } f(x) \text{ is odd} \end{aligned} \tag{49}$$

Finally, if a set of non-zero functions $\{f_i(x)\}$ are mutually orthogonal, then

$$\int_{a}^{b} f_i(x)f_j(x)dx = \begin{cases} 0 & i \neq j \\ c > 0 & i = j \end{cases} \tag{50}$$

So the set of functions $\{\sin(x(\frac{\pi}{2} + n\pi))\}_{n=0}^{\infty}$ are mutually orthogonal between -1 and 1.

3.2 Extension

At the moment, the functions is only defined in the interval $0 < x < 1$. From the question, we expect the function to have period 4. So if we define $f_{pe}(x)$ between $-2 < x < 2$ then we should be able to say that we have done enough to define the function over the entire interval.

Firstly, we already have a definition for $f(x)$ between $0 < x < 1$, so for this interval $f_{pe}(x) = f(x)$.

At $x = 0$ we have a bound which is described as $u(0, t) = 0$. Because it is described in terms of u, and we have shown the solution to u is a form of the Fourier sine series (and a sine series is a series of odd functions) then, from $-1 < x < 0$, $f_{pe}(x) = -f(-x)$, which can be seen as a rotation.

However, the second bound at $x = 1$ is defined in terms of $u_x(1, 0)$ and so at this point we have to use the derivative. Because (38) is a series of sums, we can differentiate each part of the sum to get

$$\frac{\partial u}{\partial x}(x, 0) = \sum_{n=0}^{\infty} A_n(\frac{\pi}{2} + n\pi)\cos\left(x(\frac{\pi}{2} + n\pi)\right) \tag{51}$$

This is an even function (as it is a series of cosines), and so we need an even extension here. So, from $1 < x < 2$ we can say that $f_{pe}(x) = f(-x + 2)$, which can be seen as a reflection in the u axis, and then a shift along the u axis of 2. We apply the same process to get the extension for $-2 < x < -1$, which requires a reflection of the rotation with a shift, so $f_{pe} = -f(x + 2)$.

If we continued this process, we would find that the pattern repeats infinitely with period 4. This is also demonstrated in plots of the Fourier series for each of the initial conditions for question 2.

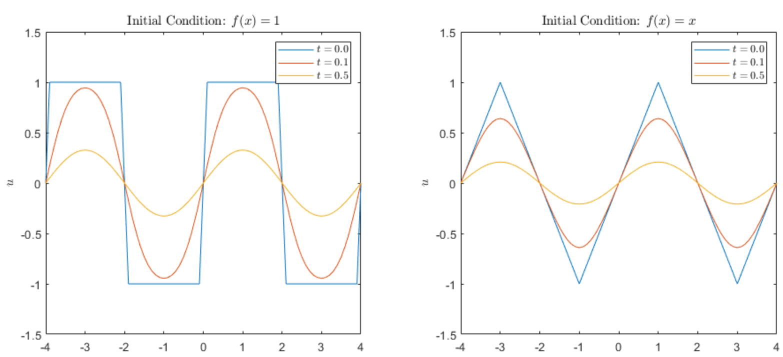

Figure 3: Shows the periodic extension of the Fourier series between -4 and 4. The function clearly repeats every $x + T$ where $T = 4$

From here we can say that the function has period 4, and that the piece-wise definition is

$$f_{pe}(x) = \begin{cases} -f(x+2) & -2 < x < -1 \\ -f(-x) & -1 < x < 0 \\ f(x) & 0 < x < 1 \\ f(-x+2) & 1 < x < 2 \end{cases} \tag{52}$$

and so $f_{pe}(x) = f_{pe}(x+4)$.

We can also see that some initial conditions give discontinuities in the function, whereas some don't. The smoothness and continuity of the function effects how quickly we achieve a good convergence, and so we would expect that $f(x) = x$ to converge quicker than $f(x) = 1$. More on this later.

4 Coefficient Decay

Question: The Dirichlet theorem states that if a periodic function $f(x)$ and its first derivative are piece-wise continuous with at most bounded discontinuities, its Fourier series must converge for every x. Typically this means the Fourier coefficients a_n and b_n should decay at least as fast as $O(1/n)$ when n is large.

Recall that we can differentiate the Fourier series for $f(x)$ term by term to obtain a Fourier series for $f'(x)$, so long as both series converge (the test is "uniform convergence").

What is the decay rate for the Fourier coefficients you have calculated in the two cases in Q2? How do these decay rates relate to the continuity of $f_{pe}(x)$ and its derivatives? (You should find that in (i), $f_{pe}(x)$ has jump discontinuities, while in (ii) both $f_{pe}(x)$ and $f'_{pe}(x)$ satisfy the Dirichlet conditions.)

What decay rate would you expect, and why, for the cases:

- $f(x) = 1 - (1-x)^2$
- $f(x) = \sinh\left(\sin(\frac{\pi x}{2})\right)$

4.1 Numerical Decay Rate

We can check the decay rate of both of the functions analysed in Question 2 reasonably easily using results we have already calculated. Since we have defined the Fourier series for each $f(x) = 1$ and $f(x) = x$ in equations (42) and (45), we can simply define

$$S_N = \sum_{n=0}^{N} A_n \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) \tag{53}$$

and calculate

$$\epsilon_N = \int_{-2}^{2} |f - S_N|^2 \tag{54}$$

to find the difference between the actual function f and the estimate of the function using the Fourier series up to N terms S_N. We should find that the difference between the functions decreases as N increase at some rate. Code for the left plots is just a slightly adapted version of A.1, and code for the error plot is in A.3

4.1.1 $f(x) = 1$

For this section, we use (53) and (54), and define

$$A_n = \frac{2}{\pi n + \frac{\pi}{2}} \tag{55}$$

We can then take $N = 1, 2, \ldots$ to get a sense of the decay.

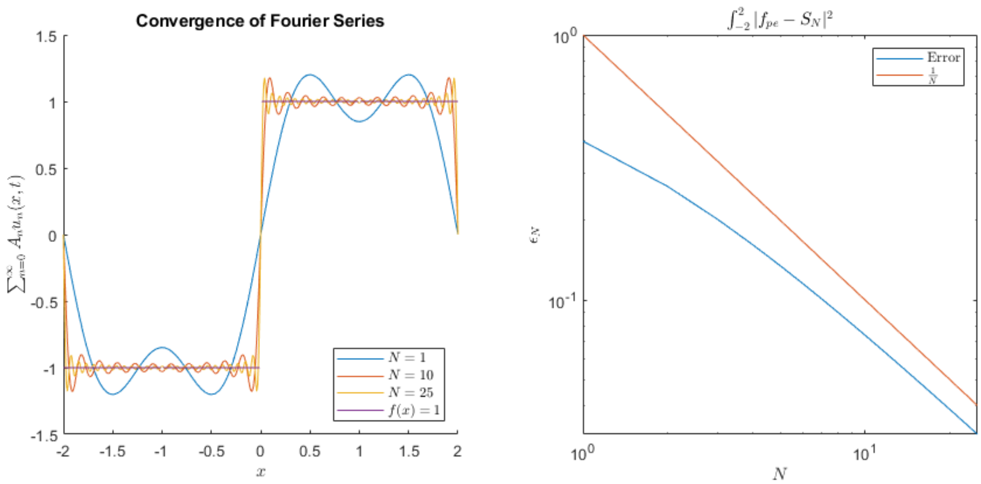

Figure 4: Graph to show how the difference between f and S_N changes as N increases (capped at N=25)

The error decays parallel to $\frac{1}{N}$, so we can say that the decay rate is proportional to this. Code to calculate this is in A.3. Plots were generated by modifying A.3 and A.1.

4.1.2 $f(x) = x$

For this section, we use (53) and (54), and define

$$A_n = (-1)^n \frac{2}{(\pi n + \frac{\pi}{2})^2} \tag{56}$$

We can then take $N = 1, 2, \ldots$ to get a sense of the decay.

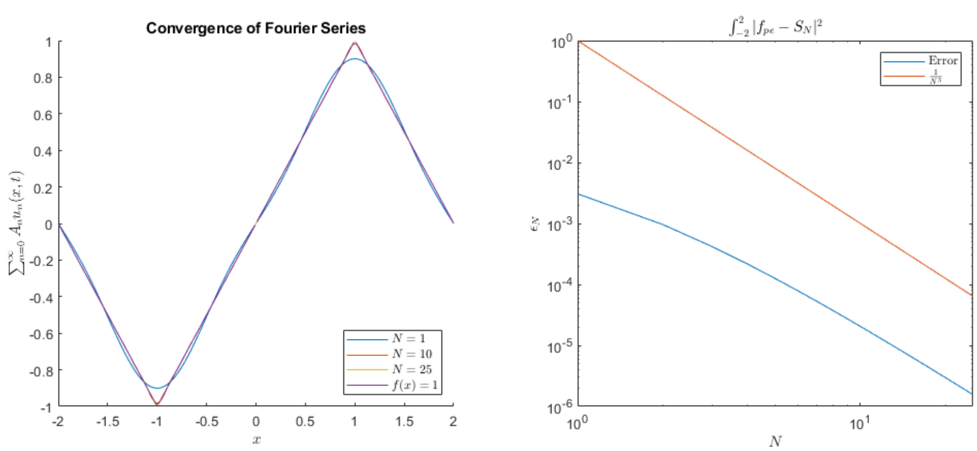

Figure 5: Graph to show how the difference between f and S_N changes as N increases (capped at N=25)

The error decays parallel to $\frac{1}{N^3}$, so we can say that the decay rate is proportional to this. Clearly the coefficients $f(x) = x$ decay substantially quicker then $f(x) = 1$. Code to calculate this is in A.4. Plots were generated by modifying A.4 and A.2.

4.1.3 Decay for $f(x) = 1 - (1-x)^2$ and $f(x) = \sinh\big(\sin(\frac{\pi x}{2})\big)$

The rate of decay of the coefficients is tied to discontinuities of the function over one period. More over, we can say that each coefficient A_n is bounded in such a way that, if f is continuous, and the derivatives up to $f^{(k-1)}$ are also continuous but the function has discontinuities for f^k then

$$|A_n| \leq \frac{M}{n^k} \tag{57}$$

where M is some constant. So the smoother a function is the quicker we should expect it to converge (hence why $f(x) = x$ converges quicker than $f(x) = 1$).

The worst case scenario, therefore, is that $f(x)$ has discontinuities, which leads to norm convergence

$$||f - S_N||^2 = B \sum_{n=N+1}^{\infty} A_n^2 \sim O\bigg(\frac{1}{N}\bigg) \tag{58}$$

which is what we have when $f(x) = 1$. Using these facts, we can simply plot $f(x) = 1 - (1-x)^2$ and $f(x) = \sinh\big(\sin(\frac{\pi x}{2})\big)$ over the interval $[-2, 2]$ using the piece-wise definition in (52). From there, we can continually differentiate them to find which differential creates discontinuities, and comment on the decay rate.

4.1.3.1 $f(x) = 1 - (1-x)^2$

Using the piece wise definition, and the derivative $f'(x) = 2(1-x)$, we get the following function behaviour

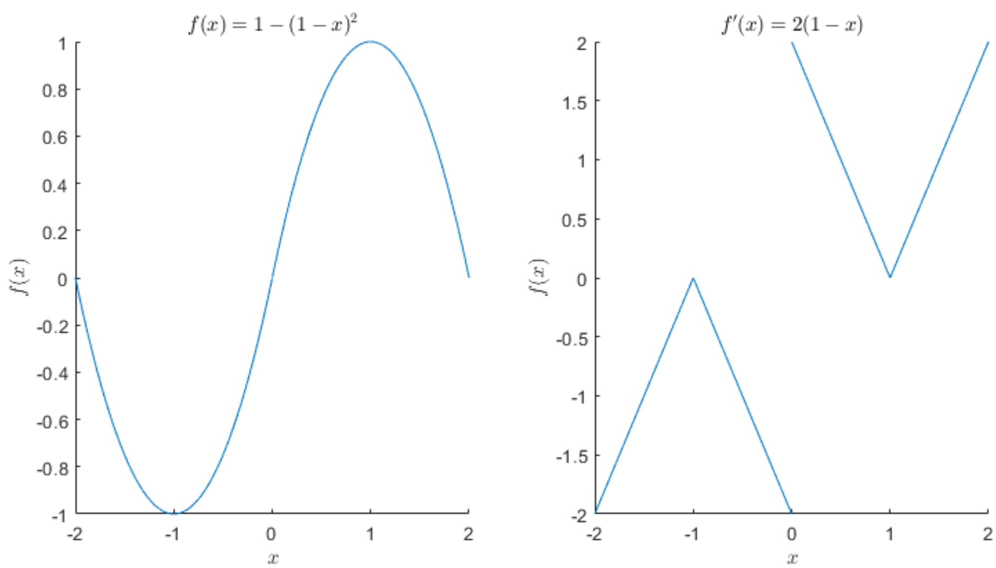

Figure 6: Function behaviour for the piece-wise definition using $f(x) = 1 - (1-x)^2$. It becomes discontinuous at f'

Because of the discontinuity in $f'(x)$, but continuity in $f(x)$, we would expect the error ϵ to decrease like

$$\epsilon = \sum_{n=N}^{\infty} \bigg(\frac{1}{n^2}\bigg)^2 \sim O\bigg(\frac{1}{N^3}\bigg) \tag{59}$$

4.1.3.2 $f(x) = \sinh\left(\sin(\frac{\pi x}{2})\right)$

Using the piece wise definition, and the derivative $f'(x) = \frac{1}{2}\pi\cos(\frac{\pi x}{2})\cosh(\sin(\frac{\pi x}{2}))$, we get the following function behaviour

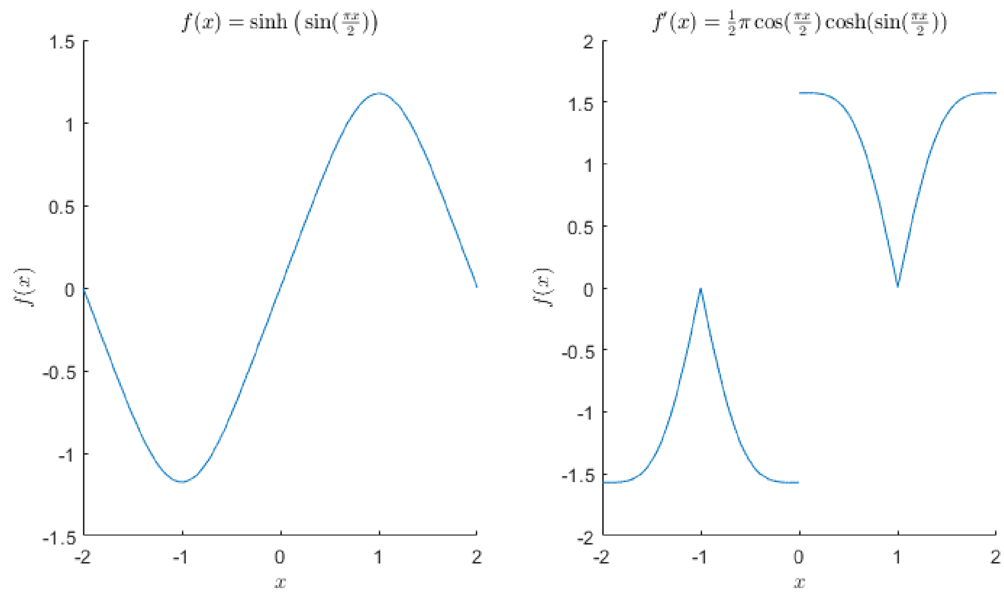

Figure 7: Function behaviour for the piece-wise definition using $f(x) = 1 - (1-x)^2$. It becomes discontinuous at f'

Again, due to the discontinuity in $f'(x)$, but continuity in $f(x)$, we would expect the error to decrease like

$$\epsilon = \sum_{n=N}^{\infty}\left(\frac{1}{n^2}\right)^2 \sim O\left(\frac{1}{N^3}\right) \tag{60}$$

Code for plotting the piece-wise functions is in A.5. Continuously differentiable, continuously continuous functions converge extremely quickly.

5 Change in Boundary Condition

Assume that the boundary conditions are now changed to

$$u(0,t) = 0 \qquad u_x(1,t) = -u(1,t) \tag{61}$$

but the initial condition remains as $u(x,0) = f(x)$ and the PDE is still the same.

Question: Derive the relevant eigenvalue problem for this case. How does it differ to the eigenvalue problem for Question 1?

Use a numerical root finder to evaluate the 10 smallest eigenvalues of this eigenproblem, and plot the eigenfunctions corresponding to the first few of these. How does the number of zeroes of each eigenfunction in the interval $0 < x < 1$ relate to Sturm-Liouville properties?

Derive an expression of an arbitrary function $f(x)$ as a linear combination of the eigenfunctions, and verify this graphically for $f(x) = \sin(4x)$ (you may calculate the relevant integrals numerically or analytically).

5.1 Separation

We begin, once again, by considering a separable solution, so

$$u = X(x)T(t) \tag{62}$$

and we actually end up with the same form of the PDE as we did before, so

$$\frac{X''(x)}{X(x)} = \frac{T'(t) + tT(t)}{T(t)} = -\lambda \tag{63}$$

which gives the same set of coupled ODEs

$$\begin{aligned} X''(x) + \lambda X(x) = 0 \\ T'(t) + tT(t) + \lambda T(t) = T'(t) + (t + \lambda)T(t) = 0 \end{aligned} \tag{64}$$

where we once again look for solutions not are non-trivial. What changes here is the boundary conditions. This time, our boundary conditions look like

$$X(0) = 0, \qquad X'(1) + X(1) = 0 \tag{65}$$

which gives a *Sturm-Liouville* eigenvalue problem, which is different to question 1. This takes the form of

$$L\big[X(x)\big] + \lambda r(x)X(x) = 0 \tag{66}$$

where $L\big[X(x)\big] = \frac{d}{dx}\big\{p(x)\frac{dX}{dx}\big\} + q(x)X$. Here $r(x) = p(x) = 1$, and $q(x) = 0$.

5.2 Sturm-Liouville Eigenvalue Problem

The coefficients of $X''(x)$ and λ do not vanish within the defined interval $[0, 1]$ (as they are constant), and so we can claim the following properties:

- All λ are real
- We have a countable sequence of corresponding eigenfunctions
- The eigenfunctions form a basis for piecewise continuous functions on $[0, 1]$
- The eigenfunctions are orthogonal (see (50) for what this property means for a set of non-zero functions) with weight function 1

This means that any function that is piece-wise on $[0, 1]$ is

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{67}$$

where ϕ_n is the set of orthonormal functions. This is a property formed by the fact that we have an orthogonal basis. This gives us the ability to find the all the coefficients of this function. Since

$$\int_0^1 X_m(x)X_n(x)dx = 0 \text{ when } m \neq n \tag{68}$$

we can say that

$$\int_0^l f(x)X_m(x)dx = \int_0^l X_m(x)\sum_{n=1}^{\infty} c_n X_n(x)dx \tag{69}$$

Since the functions are orthogonal, the only time that we have any contribution is when $m = n$, so

$$\int_0^l f(x)X_m(x)dx = c_m \int_0^1 X_m(x)^2 dx \tag{70}$$

This property will come in useful later.

5.3 λ

We now need to turn our attention to calculating the eigenfunction themselves. We first need to establish whether λ is positive, negative, or 0. We used a similar argument as before. Take the X ODE in (64), multiply it by X and integrate in the interval on which is its defined, so

$$\int_0^1 XX'' + \lambda X^2 dx = 0 \tag{71}$$

We can integrate this seperately to get

$$\begin{aligned} &\left[XX'\right]_0^l - \int_0^1 (X')^2 dx + \lambda \int_0^1 X^2 dx \\ &= X(1)X'(1) - X(0)x'(0) - \int_0^1 (X')^2 dx + \lambda \int_0^1 X^2 dx \end{aligned} \tag{72}$$

We know from the boundary conditions that $X(0) = 0$ and $X'(1) = -X(1)$, so we get

$$-X(1)^2 - \int_0^1 (X')^2 dx + \lambda \int_0^1 X^2 dx = 0 \tag{73}$$

which gives

$$\lambda = \frac{X(1)^2 + \int_0^1 (X')^2 dx}{\int_0^1 X^2 dx} \tag{74}$$

which means that $\lambda \geq 0$. If it were to equal 0, we would require that $x(1) = 0$ and $X'(x) = 0$ for all x, which is not true (since $X(x) = 0$ is not an eigenfunction), and so $\lambda > 0$.

5.4 Solution

Since we know $\lambda > 0$, let $\lambda = k^2$, which gives

$$X'' + k^2 X = 0 \tag{75}$$

which has a solution of $X(x) = A\cos(kx) + B\sin(kx)$. Applying the boundary condition at $x = 0$ means that

$$X(0) = A\cos(0) + B\sin(0) = 0 \implies A = 0 \tag{76}$$

so $X(x) = B\sin(kx)$. The other boundary condition establishes that $X'(1) + X(1) = 0$. Since $X'(x) = Bkcos(kx)$, this means that

$$X'(1) + X(1) = Bk\cos(k) + B\sin(k) = B(k\cos(k) + \sin(k)) = 0 \tag{77}$$

B=0 gives the trivial solution, and so we therefore require that $k\cos(k) + \sin(k) = 0$.

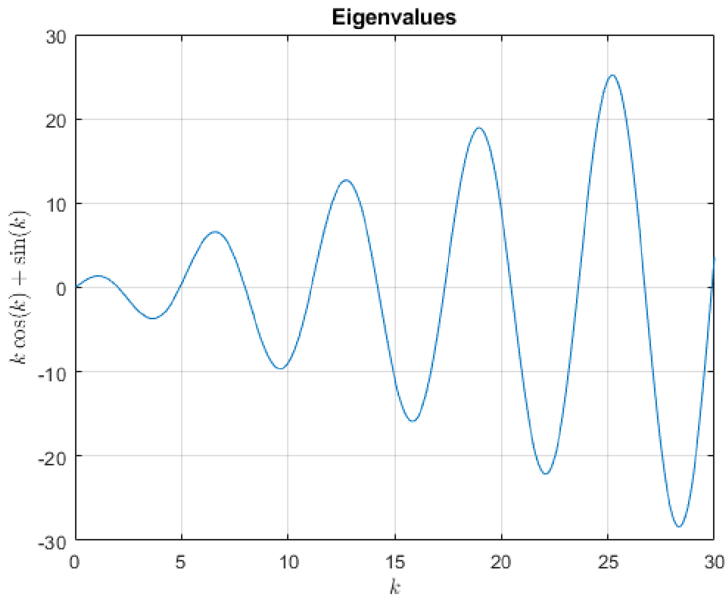

Figure 8: values that cross the k axis are where the function is 0

From here we can us a numerical root finder (such as fzero in matlab) to find the places where k returns a 0. Code for doing this for the first 10 roots is in A.6. Doing so returns that

$$\begin{aligned} k_n = \{&2.029, 4.913, 7.979, 11.086, 14.207, \\ &17.336, 20.469, 23.604, 26.741, 29.879\}_{n=1}^{10} \end{aligned} \tag{78}$$

Therefore, since $\lambda_n = k_n^2$ and $X_n(x) = \sin(k_n x)$. We can then plot the corresponding eigenfunctions.

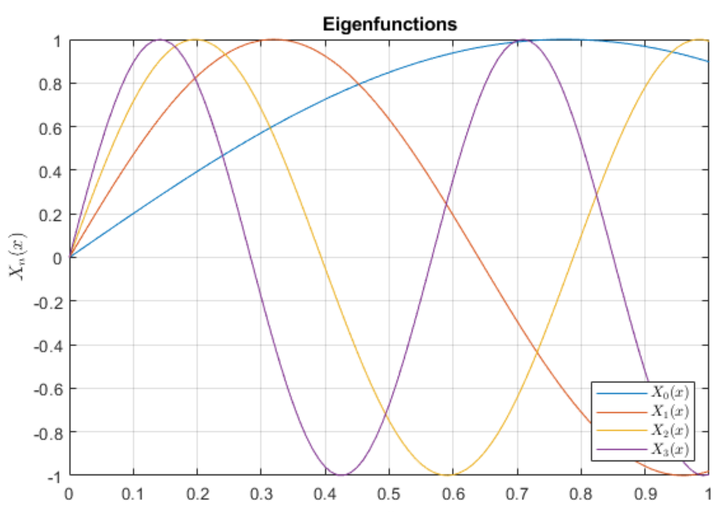

Figure 9: plots of the eigenfunctions over the interval $x \in (0, 1)$

Each $\lambda_0 < \lambda_1 < ... < \lambda_n < ...$ has a corresponding eigenfunction with n roots in $x \in (0, 1)$. Now that we have defined the eigenfunctions and eigenvalues (and even calculated a few) we can define the normalised eigunfunctions of this system to be

$$\phi_n(x) = B_n \sin(k_n x) \tag{79}$$

and, since $r(x) = 1$, we choose B_n such that

$$(\phi_n, \phi_n) = \int_0^1 B_n^2 \sin^2(k_n x) dx = 1 \tag{80}$$

and therefore

$$\frac{B_n^2}{2} \big\{ 1 - \frac{1}{2k_n} \sin(2k_n) \big\} = 1 \tag{81}$$

which means

$$B_n = \left[\frac{2}{1 - \frac{1}{2k_n} \sin(2k_n)} \right]^{\frac{1}{2}} \tag{82}$$

This gives us a definition for the normalised eigenfunction, which can be used to define

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{83}$$

where $c_n = \int_0^1 r(x) f(x) \phi_n(x)$, and $f(x)$ is an arbitrary function.

5.5 $f(x) = \sin(4x)$

As an example, take $f(x) = \sin(4x)$. This means that

$$\begin{aligned} c_n &= \int_0^1 \sin(4x) B_n \sin(k_n x) dx \\ &= \frac{B_n}{2} \big(- \frac{1}{4 + k_n} \sin(k_n + 4) + \frac{1}{4 - k_n} \sin(-k_n + 4) \big) \end{aligned} \tag{84}$$

which gives

$$\sin(4x) = \sum_{n=1}^{\infty} \frac{B_n}{2} \big(- \frac{1}{4 + k_n} \sin(k_n + 4) + \frac{1}{4 - k_n} \sin(-k_n + 4) \big) B_n \sin(k_n x) \tag{85}$$

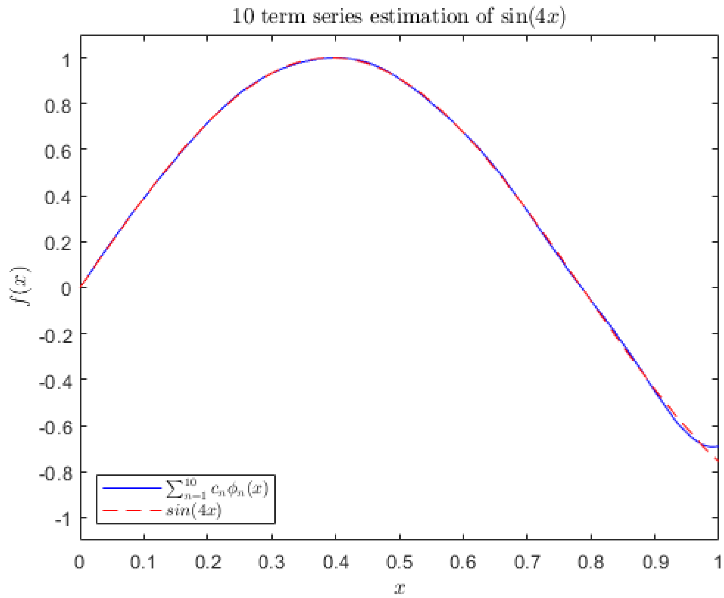

Figure 10: Calculating just 10 terms of this series solution gives a good estimate of $f(x) = sin(4x)$

As shown in figure 10, this estimates $\sin(4x)$ reasonably well over the interval $x \in [0, 1]$ for few terms in the series. Adding even more terms showed that it estimated $\sin(4x)$ even more accurately.

A Code for Series Coefficient Plots

A.1 $f(x) = 1$

```
L = 1; %boundary for PDE

% x is a vector, starting at 0, going to L
x = (0:0.001:1)*L;

%% include t
hold on;
% plot the behaviour of the pde at different values of t
for t = [0, 0.1, 0.5]
    u = 0*x;
    % set how many values of the sum to use
    N = 100000;
    for n = 0:N
        % calculate coeffecient
        a_n = 2/(pi*n+(pi/2));
        % Add on an appropriate multiple of the separable solution
        u = u + a_n*sin((pi/2 + n*pi)*x/L)*exp((-1/2)* ...
            t * (2*(pi/2 + n*pi)^2+t));
    end
    % plot the result
    plot(x,u)
end
% beautify the plot
xlabel('$x$', 'Interpreter' ,'Latex');
ylabel('$u$', 'Interpreter' ,'Latex')
legend('$t=0.0$', '$t=0.1$', '$t=0.5$', 'Interpreter' ,'Latex')
```

A.2 $f(x) = x$

```
L = 1; %boundary for PDE

% x is a vector, starting at 0, going to L
x = (0:0.001:1)*L;
hold on;
% plot the behaviour of the pde at different values of t
for t = [0, 0.1, 0.5]
    u = 0*x;
    % set how many values of the sum to use
    N = 10000;
    for n = 0:N
        % calculate coeffecient
        a_n = (-1)^n*(2)/(pi * n + (pi/2))^2;
        % Add on an appropriate multiple of the separable solution
        u = u + a_n.*sin((pi/2 + n*pi)*x/L)*exp((-1/2)* ...
            t * (2*(pi/2 + n*pi)^2+t));
    end
    % plot the result
    plot(x,u)
end
% beautify the plot
xlabel('$x$', 'Interpreter' ,'Latex');
ylabel('$u$', 'Interpreter' ,'Latex')
legend('$t=0.0$', '$t=0.1$', '$t=0.5$', 'Interpreter' ,'Latex')
```

A.3 $f(x) = 1$ Decay

```
%%
L = 1;
syms x
```

```
pw(x) = piecewise(x<-1, -1, -1<x<0, -1, 0<x<1, 1, x>1, 1);

%%%% x is a vector, starting at 0, going to L
x = (-2:0.01:2)*L;
syms x
error = @(x) abs(pw(x) - u);

%%% Calculate a Fourier series
P = 25;
diff = zeros(P, 1);
for N = 1:P
    u = 0*x;
    for n = 0:N
        a_n = 2/(pi*n+(pi/2));
        fourier = a_n*sin((pi/2 + n*pi)*x/L);
        u = u + fourier;
    end
    diff(N) = int((u-pw(x))^2,x, -2, 2);
    disp(N)
end

%%
figure(2)
subplot(1, 2, 2)
loglog(1:P, diff)
hold on
loglog(1:P, (ones(1,P)./(1:P)))
xlabel('$N$', 'Interpreter' ,'Latex');
ylabel('$\epsilon_N$', 'Interpreter' ,'Latex')
legend('Error', '$\frac{1}{N}$', 'Interpreter' ,'Latex')
title('$\int_{-2}^2|f_{pe}-S_N|^2$', 'Interpreter' ,'Latex')
```

A.4 $f(x) = x$ Decay

```
%%
L = 1;
syms x
f(x) = x;
pw(x) = piecewise(x<-1, -f(x+2), -1<x<0, -f(-x), 0<x<1, f(x), x>1, f(-x+2));

%%%% x is a vector, starting at 0, going to L
x = (-2:0.01:2)*L;
syms x
error = @(x) abs(pw(x) - u);

%%% Calculate a Fourier series
P = 25;
diff = zeros(P, 1);
for N = 1:P
    u = 0*x;
    for n = 0:N
        a_n = (-1)^n*(2)/(pi * n + (pi/2))^2;
        fourier = a_n*sin((pi/2 + n*pi)*x/L);
        u = u + fourier;
    end
    diff(N) = int((u-pw(x))^2,x, -2, 2);
    disp(N)
end

%%
figure(1)
subplot(1, 2, 2)
loglog(1:P, diff)
hold on
loglog(1:P, (ones(1,P)./(1:P).^3))
```

```
xlabel('$N$', 'Interpreter' ,'Latex');
ylabel('$\epsilon_N$', 'Interpreter' ,'Latex')
legend('Error', '$\frac{1}{N^3}$', 'Interpreter' ,'Latex')
title('$\int_{-2}^2|f_{pe}-S_N|^2$', 'Interpreter' ,'Latex')
```

A.5 Piece-wise Plotting

```
%%
L = 1;
syms x;
%Define the function
f(x) = sinh(sin(pi*x/2));
%Define the piecewise function
pw(x) = piecewise(x<-1, -f(x+2), -1<x<0, -f(-x), ...
    0<x<1, f(x), x>1, f(-x+2));

%plot over the interval
figure(2);
subplot(1, 2, 1);
hold on
plot([-2:0.001:2], pw([-2:0.001:2]));
xlabel('$x$', 'Interpreter' ,'Latex');
ylabel('$f(x)$', 'Interpreter' ,'Latex');
title('$f(x)=\sinh\big(\sin(\frac{\pi x}{2})\big)$', 'Interpreter' ,'Latex');

%repeat for differential
fdash(x) = (pi*cos(pi*x/2)*cosh(sin(pi*x/2)))/2;
pwdash(x) = piecewise(x<-1, -fdash(x+2), -1<x<0, -fdash(-x), ...
    0<x<1, fdash(x), x>1, fdash(-x+2));
subplot(1, 2, 2);
hold on
plot([-2:0.001:2], pwdash([-2:0.001:2]));
xlabel('$x$', 'Interpreter' ,'Latex');
ylabel('$f(x)$', 'Interpreter' ,'Latex');
title("$f'(x)=\frac{1}{2} \pi \cos(\frac{\pi x}{2})\cosh(\sin(\frac{\pi x}{2}))$", '
    Interpreter' ,'Latex');
```

A.6 Sturm-Liouville Routine

```
%plot the function for values of k
f = @(k) k.*cos(k) + sin(k);

x = [0:0.01:30];
figure(3)
plot(x, f(x))
grid on
xlabel('$k$', 'Interpreter' ,'Latex');
ylabel('$k\cos(k)+\sin(k)$', 'Interpreter' ,'Latex')
title('Eigenvalues')

%% find first 10 roots, excluding negatives and k=0
lambda = zeros(1,12);

lambda(1) = fzero(f, [2 2.2]);
lambda(2) = fzero(f, [4.5 5]);
lambda(3) = fzero(f, [7.9 8]);
lambda(4) = fzero(f, [11 11.2]);
lambda(5) = fzero(f, [14 14.4]);
lambda(6) = fzero(f, [17.2 17.4]);
lambda(7) = fzero(f, [20.4 20.5]);
lambda(8) = fzero(f, [23.4 23.8]);
lambda(9) = fzero(f, [26.6 26.8]);
lambda(10) = fzero(f, [29.8 30]);
lambda(11) = fzero(f, [33 33.1]);
```

```
lambda(12) = fzero(f, [36.1 36.2]);

%% plot first 4 eigenfunctions for 0 < x < 1
x=[0:0.01:1]
figure(1)
for n=1:4
    X = @(x) sin(lambda(n)*x);
    plot(x, X(x))
    hold on
end
grid on
xlabel('$x$', 'Interpreter' ,'Latex');
ylabel('$X_n(x)$', 'Interpreter' ,'Latex')
legend('$X_0(x)$', '$X_1(x)$', '$X_2(x)$', '$X_3(x)$', ...
    'Interpreter' ,'Latex', 'Location', 'Southeast')
title('Eigenfunctions')

%% plot  sine(4x) and the estimate with 10 eigenfunctions
x=[0:0.001:1];

B = @(k) ((2)/(1-(1/(2*k))*sin(2*k)))^0.5;
c = @(k) (B(k)/2)*((-1/(4+k))*sin(k+4)+(1/(4-k))*sin(-k+4));
phi = @(k, x) B(k)*sin(k*x);

u=0;

for n=1:10
    c(lambda(n));
    phi(lambda(n), x);
    u = u + c(lambda(n))*phi(lambda(n), x);
end

figure(2)
plot(x, u, '-b')
hold on
plot(x, sin(4*x), '--r')
title('10 term series estimation of $\sin(4x)$', 'Interpreter', 'Latex')
xlabel('$x$', 'Interpreter', 'Latex')
ylabel('$f(x)$', 'Interpreter', 'Latex')
legend('$\sum_{n=1}^{10}c_n\phi_n(x)$', '$sin(4x)$',...
    'Location', 'Southwest', 'Interpreter', 'Latex')
ylim([-1.1, 1.1])
```