

1D Heat Eq. Solver

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

FDM:

Taylor series expansions:

$$u(x + \Delta x) = u(x) + \Delta x \cdot u'(x) + \frac{\Delta x^2}{2} u''(x) + \frac{\Delta x^3}{6} u'''(x) \dots$$

$$u(x - \Delta x) = u(x) - \Delta x \cdot u'(x) + \frac{\Delta x^2}{2} u''(x) - \frac{\Delta x^3}{6} u'''(x) \dots$$

Δx^3 and up are negligible error

$$u(x + \Delta x) = u(x) + \Delta x u' + \frac{\Delta x^2}{2} u'' + \dots + \frac{\Delta x^4}{4!} u'''$$

$$u(x - \Delta x) = u(x) - \Delta x u' + \frac{\Delta x^2}{2} u'' - \dots - \frac{\Delta x^4}{4!} u'''$$

$$= 2u(x) + \underbrace{\Delta x^2 u'' + \frac{\Delta x^4}{4!} u''' + \dots}_{\text{negligible error}}$$

$$u(x + \Delta x) + u(x - \Delta x) = 2u(x) + \Delta x^2 u''(x)$$

$$u''(x) = \frac{u(x + \Delta x) + u(x - \Delta x) - 2u(x)}{\Delta x^2}$$

$$\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

from Taylor series exp.

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+\Delta x) + u(x-\Delta x) - 2u(x)}{\Delta x^2}$$

$$\text{and } \frac{\partial u}{\partial t} \approx \frac{u(t+\Delta t) - u(t)}{\Delta t}$$

$$\Rightarrow u(t+\Delta t) \approx u(t) + \frac{\partial u}{\partial t} \Delta t + u(p)$$

$$\Rightarrow u(t+\Delta t) \approx u(t) + \left(\frac{u(x+\Delta x) + u(x-\Delta x) - 2u(x)}{\Delta x^2} \right) \Delta t + u(p)$$

$$\Rightarrow u_i^{n+1} \approx u_i^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{i+1}^n + u_{i-1}^n - 2u_i^n)$$

\hookrightarrow in my program =

$$a = \frac{\alpha \Delta t}{\Delta x^2}, \text{ surrounding} = u_{i-1}^n + u_i^n + u_{i+1}^n$$

$$\text{so } u_{\text{next}}[i] = u[i] - (\text{surrounding} * a)$$

$$= u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{i+1}^n + u_{i-1}^n - 2u_i^n)$$

$$\text{surrounding} = \text{diff_left} + \text{diff_right}$$

$$= u[i] - u[\text{left}[i]] + u[i] - u[\text{right}[i]]$$

$$= 2u[i] - u[\text{left}[i]] - u[\text{right}[i]]$$

$$\Rightarrow -(-2u_i^n + u_{i+1}^n + u_{i-1}^n)$$

1D Heat Flow chart

zoom
20/10 = 2

Periodic Boundary:

$\frac{L}{k}$ \Rightarrow $x[0]$ left neighbor is $x[k-1]$, loops around

use $u_p, roll(k)$
 $k=-1$: shift array elements left
one entry, wraps

Domain: $x \in [0, L]$

$u[0]$ left is $u[L]$
 $u[L]$ right is $u[0]$

$x = \text{array} [k]$ ~~entries~~
k entries, $\Delta x = \frac{L}{k}$

$$\frac{\partial u}{\partial t}(x, t_n) = \frac{u_{i+1} - u_i}{\Delta t}$$

$u(x)$ = heat at a point during i timestep

goal: $t: i \rightarrow i+1$

$\hookrightarrow u[k] \rightarrow u_{i+1}[k]$

where u at each $x[k]$ is updated based on its neighbors

$$\text{Surrounding} = u(x) - u(x+\Delta x) + u(x) - u(x-\Delta x)$$

debugging:
 $\rightarrow 2u(x) - u(x+\Delta x) - u(x-\Delta x)$

after 60 iterations, u starts increasing

$$u = [.5, 1, .5, 0, -.5, -1, -5, 0] \quad L=20 \quad k=20$$

$$u_{\text{right}} = [1, .5, 0, -.5, -1, -5, 0, .5] \quad dx=1$$

$$u_{\text{left}} = [0, -.5, 1, -.5, 0, -.5, -1, -.5]$$

$$\text{diff-right} = .5 - 1 \quad \text{diff-left} = .5 - 0$$

$$\text{surrounding} = 0$$

$$u[1] = .5 \quad u[2] = 1 \quad \text{diff-left} = 1 - .5 = .5 \quad \text{right} = .5$$

$$\text{surrounding} = 1$$

$$u[2] = 1 - (1 - .1) = -.9$$