Ω is Kahler form, $\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}=\exp\{F\}$

$$\Omega=\sum_{i,j} g_{i\bar{j}}\,dz^i\wedge d\bar{z}^j,\quad \Omega^m=m!\det(g_{i\bar{j}})\,dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^m\wedge d\bar{z}^m$$
$$=m!\,dVol_g$$

$$(\Omega+\partial\bar{\partial}\varphi)^m=m!\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^m\wedge d\bar{z}^m$$

PDE $\Longleftrightarrow$ $\underline{(\Omega+\partial\bar{\partial}\varphi)^m=\exp\{F\}\Omega^m}$ (4.1), integrate both side

since $\int_M \Omega\, d\bar{\partial}\varphi=-\int d\Omega\,\bar{\partial}\varphi=0$

$$\int_M \Omega^2\, d\bar{\partial}\varphi=-\int d\Omega^2\,\bar{\partial}\varphi=0$$

$$\int_M(\Omega\, d\bar{\partial}\varphi)\,d\bar{\partial}\varphi=-\int d(\Omega\, d\bar{\partial}\varphi)\,d\bar{\partial}\varphi=0$$

......

so $\int_M(\Omega+\partial\bar{\partial}\varphi)^m=\int_M\Omega^m=m!\,Vol(M)$

integrate both (4.1) we have $\int_M \exp\{F\}=Vol(M)$ (4.2)

Now prove if $F\in C^k(M)$ $(k\geq 3)$, F is (4.2). we can find solution to (4.2) $\varphi \in C^{k+1,\alpha}(M)$.

§ Continuity method

$$S = \left\{ t \in [0,1] \;\middle|\; \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} = \mathrm{Vol}(M) \left[\int_M \exp(tF)\right]^{-1} \exp(tF) \right\}$$

we want to prove $S = [0,1]$

since when $t=0$, $\varphi \equiv 0$ is a solution, only have to prove
so $0 \in S$
that S is both open and closed

■ open, inverse function theorem

$$\Theta = \left\{ \varphi \in C^{k+1,\alpha}(M) \;\middle|\; 1 + \varphi_{i\bar{i}} > 0 \text{ and } \int_M \varphi = 0 \right\}$$

$$B = \left\{ f \in C^{k-1,\alpha}(M) \;\middle|\; \int_M f = \mathrm{Vol}(M) \right\}$$

Θ open in $C^{k+1,\alpha}(M)$, B is hypersurface in $C^{k-1,\alpha}(M)$

so we have $G : \Theta \longrightarrow B$

$$\varphi \longmapsto \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} \quad (4.4)$$

(since $\int_M \exp(F) = \mathrm{Vol}(M)$)

dG

differentiate G : $\hat{\varphi}_t = \varphi_0 + t\eta$

$$\frac{d}{dt} G(\varphi_t)\Big|_{t=0} = \frac{d}{dt}\left[\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_t}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\right]$$

$$= \left[\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_t}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\right] tr\left[\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_t}{\partial z^i \partial \bar{z}^j}\right)^{-1}\left(\frac{\partial^2 \eta}{\partial z^i \partial \bar{z}^j}\right)\right]\Bigg|_{t=0}$$

$$= \left[\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\right] \Delta_{\varphi_0} \eta \qquad \left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right)^{-1}$$

(since $\frac{d|A|}{dt} = |A| \, tr\left(A^{-1} \cdot \frac{dA}{dt}\right)$, $\boxed{\Delta_{\varphi_0} = \sum_{i,j} \left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right)^{-1} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}}$

G's derivative at φ_0 $\boxed{\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1} \Delta_{\varphi_0}}$.

B's tangent space: $C^{k-1,\alpha}(M)$ function satisfying $\int_M f = 0$

$\Delta_{\varphi_0} \varphi = g$ weakly sovable $\Leftrightarrow \int_M g \, dV_{\varphi_0} = 0$ | $dG|_{\varphi_0} \varphi = f$

so $\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1} \Delta_{\varphi_0} \varphi = f$ solvable $\Leftrightarrow$

$$0 = \int_M (\det(g_{\varphi_0}))^{-1} f \det(g_{i\bar{j}}) \, dV_{\varphi_0}$$

$$= \int_M f \, dV_g \qquad \left(\text{since } \frac{dV_g}{dV_{\varphi_0}} = \frac{\det(g_{i\bar{j}})}{\det(g_{\varphi_0})}\right)$$

weakly souable

$$\Delta_{\varphi_0}\varphi = f \Leftrightarrow \int f = 0 \Leftrightarrow dG|_{\varphi_0}\varphi = f$$

so $\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi_0}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\Delta_{\varphi_0}\varphi = f$ weakly soluable $\Leftrightarrow$

$$0 = \int_M (\det(g_{\varphi_0}))^{-1} f \det(g_{i\bar{j}})\, dV_{\varphi_0}$$

$$= \int_M f\, dV_g \quad \left(\text{since } \frac{dV_g}{dV_{\varphi_0}} = \frac{\det(g_{i\bar{j}})}{\det(g_{\varphi_0})}\right)$$

又有 Schauder 估计，$f \in C^{k-1,\alpha}(M)$，$\varphi \in C^{k+1,\alpha}(M)$，若 $\int_M \varphi = 0$

由极大值原理可得唯一解．（此即 $dG(\varphi) = f$）

$$\Theta = \left\{ \varphi \in C^{k+1,\alpha}(M) \,\middle|\, 1 + \varphi_{i\bar{i}} > 0 \text{ and } \int_M \varphi = 0 \right\}$$

$$B = \left\{ f \in C^{k-1,\alpha}(M) \,\middle|\, \int_M f = \mathrm{Vol}(M) \right\}$$

$$G : \Theta \longrightarrow B$$

$$\varphi \longmapsto \det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1} \quad (4.4)$$

dG 在 φ_0 处可逆，G 将 φ_0 一个开邻域映到 $G(\varphi_0)$ 的一个开邻域.

这说明 S 是开的. $G: \varphi_0 \to G(\varphi_0)$

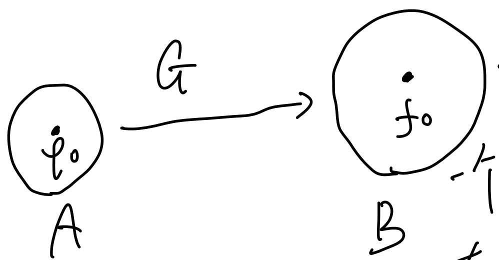
记 S 中描述的 $\int_M \varphi = 0$ 的解在 A 中.

记 $f_t = \mathrm{Vol}(M)\left[\int_M \exp(tF)\right]^{-1}\exp(tF)$ 在 B 中.

令 $t_0 \in S$，φ_0 是 $\det\left(g_{i\bar j} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar z^j}\right)\det(g_{i\bar j})^{-1} = f_{t_0}$ 的解.

满足 $|t-t_0|<\varepsilon$ 的 f_t 构成 B 中的一个以 f_{t_0} 为中心的邻域，于是是

A 中一开集在 G 下的像，这说明 $|t-t_0|<\varepsilon$ 在 S 中，S 开.



← 满足 $|t-t_0|<\varepsilon$ 的 f_t，是 φ_0 一个邻域在 G 下的像，即 $|t-t_0|<\varepsilon$ 也在 S 中.

■ 证 S 闭.

若 $\{t_q\}$ 是 S 中序列，即有一列 $\varphi_q \in C^{k+1,\alpha}(M)$

$$\det\left(g_{i\bar j} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar z^j}\right)\det(g_{i\bar j})^{-1} = \mathrm{Vol}(M)\left(\int_M \exp(t_q F)\right)^{-1}\exp(t_q F)$$

其中 $\int_M \varphi_q = 0$

对上式求导：$\frac{d|A|}{dt} = |A|\, tr\left(A^{-1}\cdot \frac{dA}{dt}\right)$

$$= \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j}\right) tr\left(\left[g_q^{\prime i\bar{j}}\right]\left[\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial \varphi_q}{\partial z^p}\right)\right]\right)$$

即得(4.5)：$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j}\right) \sum_{i,j}\left(g_q^{\prime i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial \varphi_q}{\partial z^p}\right)\right)$

$$= Vol(M)\left(\int_M \exp(t_q F)\right)^{-1} \frac{\partial}{\partial z^p}\left[\exp(t_q F)\det(g_{i\bar{j}})\right]$$

这里 $g_q^{\prime i\bar{j}}$ 是 $\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j}\right)$ 的逆矩阵.

Prop 2.1 (C^0, C^1, C^2 估计) 说明 (4.5) 左边一致椭圆，Prop 3.1 (C^3 估计) 说明左边系数 Hölder 连续，则由 Schauder 估计，有一个 $\frac{\partial \varphi_q}{\partial z^p}$ $C^{2,\alpha}$ 估计，同理有 $\frac{\partial \varphi_q}{\partial \bar{z}^p}$ 的一个 $C^{2,\alpha}$ 估计.

又可得到左边系数 $C^{1,\alpha}$，这又有一个 $\frac{\partial \varphi_q}{\partial z^p}$ 与 $\frac{\partial \varphi_q}{\partial \bar{z}^p}$ 的 $C^{3,\alpha}$ 估计，重复此步骤可得 φ_q 的 $C^{k+1,\alpha}$ 估计，所以有一个子列在 $C^{k+1,\alpha}$ 中收敛到一个 φ (Arzela-Ascoli)，令 $q\to\infty$, $t_0 = \lim_{q\to\infty} t_q$

有 $\det\left(g_{i\bar j} + \frac{\partial^2\varphi}{\partial z^i \partial \bar z^j}\right)\det(g_{i\bar j})^{-1} = Vol(M)\left(\int_M \exp(t_0 F)\right)^{-1}\exp(t_0 F)$

这说明S闭. 则有:

Theorem 1. M 是紧Kähler流形, 有一度量 $g_{i\bar j}\, dz^i \otimes d\bar z^j$.

令 $F \in C^k(M)$ $(k \geq 3)$, $\int_M \exp\{F\} = Vol(M)$, 则有一个函数 $\varphi \in C^{k+1,\alpha}(M)$ $(0 \leq \alpha < 1)$ 使得 $\left(g_{i\bar j} + \frac{\partial^2\varphi}{\partial z^i \partial \bar z^j}\right) dz^i \otimes d\bar z^j$ 定义一个Kähler度量且 $\det\left(g_{i\bar j} + \frac{\partial^2\varphi}{\partial z^i \partial \bar z^j}\right) = \exp\{F\}\det(g_{i\bar j})$

证得此定理即有 Calabi-conjecture.

令 M 是一紧Kähler流形, 有Kähler度量 $g_{i\bar j}\, dz^i \otimes d\bar z^j$, 令 $\tilde R_{i\bar j}\, dz^i \otimes d\bar z^j$ 是一张量, $\frac{i}{2\pi}\tilde R_{i\bar j}\, dz^i \wedge d\bar z^j$ 代表 M 的第一陈类. 则可找到一个Kähler度量 $\tilde g_{i\bar j}\, dz^i \otimes d\bar z^j$ 与原度量上同调, 且它的Ricci张量是 $\tilde R_{i\bar j}\, dz^i \otimes d\bar z^j$.

(因为 $R_{i\bar j} = -\frac{\partial^2}{\partial z^i \partial \bar z^j}\log(\det(g_{i\bar j}))$, $\frac{i}{2\pi}\tilde R_{i\bar j}\, dz^i \wedge d\bar z^j$ 代表 M 的第一陈类, 则有 $\tilde R_{i\bar j} = R_{i\bar j} - \frac{\partial^2}{\partial z^i \partial \bar z^j} f$, 其中 f

是一个实函数，因为存一个光滑φ使$\left(g_{i\bar{j}}+\frac{\partial\varphi}{\partial z^i\partial\bar{z}^j}\right)dz^i\otimes d\bar{z}^j$

定义了一个Kahler度量. 则这时新的Ricci张量为.

$$\tilde{R}_{i\bar{j}}=-\frac{\partial^2}{\partial z^i\partial\bar{z}^j}\log[(\exp(f)\det(g_{i\bar{j}})]$$

$$=R_{i\bar{j}}-\frac{\partial^2 f}{\partial z^i\partial\bar{z}^j}$$

$Ric\, g=\lambda g$