when $m=2$, we estimate $|\inf_M \varphi|$, set N to be positive

from (2.18) $\Delta'\left(\exp\{-N\varphi\}(m+\Delta\varphi)\right) \geq$

$$\exp\{-N\varphi\}\left(\Delta F - m^2 \inf_{i\neq l} R_{i\bar i l\bar l}\right) - N\exp\{-N\varphi\}\, m(m+\Delta\varphi)$$

$$+\left(N + \inf_{i\neq l} R_{i\bar i l\bar l}\right)\exp\{-N\varphi\}(m+\Delta\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar i}}\right)$$

choose N s.t. $N + \inf_{i\neq l} R_{i\bar i l\bar l} \geq \frac{1}{2}N$

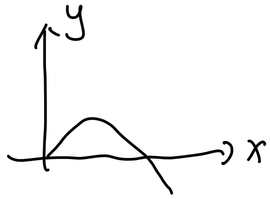
using (2.20) we get (2.49)

$$\left(N + \inf_{i\neq l} R_{i\bar i l\bar l}\right)(m+\Delta\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar i}}\right) \geq \frac{1}{2}N\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{m/(m-1)}$$

there is a C_9 only depends on $\sup|F|$ and m

$$\frac{1}{2}N\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{m/(m-1)} \geq 2Nm(m+\Delta\varphi) - NC_9$$

(consider $y = anx - bx^{\frac{n}{n-1}}$; $y' = an - \frac{nb}{n-1}x^{\frac{1}{n-1}}$)



$$2Nm(m+\Delta\varphi) - \frac{1}{2}N\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{m/(m-1)}$$

$$\leq NC_9$$

take (2.48) (2.49) (2.50) into 2.47

(2.47) $\Delta'\left(\exp\{-N\varphi\}(m+\Delta\varphi)\right) \geq$

$$\exp\{-N\varphi\}\left(\Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right) - N\exp\{-N\varphi\} m(m+\Delta\varphi)$$

$$+\left(N + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\{-N\varphi\}(m+\Delta\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right)$$

using $\overset{\frac{1}{2}N}{\sum_i \frac{1}{1+\varphi_{i\bar{i}}}} \geq (m+\Delta\varphi)^{\frac{1}{m-1}} \exp\left\{\frac{-F}{m-1}\right\}$

we get (2.51) $\Delta'\left(\exp\{-N\varphi\}(m+\Delta\varphi)\right)$ $\boxed{\times \exp\{F\} \text{ bothside}}$

$$\geq \exp\{-N\varphi\}\left(\Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_q\right) + N\exp\{-N\varphi\} m(m+\Delta\varphi)$$

we have $\exp\{F\}\Delta'\left(\exp\{-N\varphi\}(m+\Delta\varphi)\right)$

$$\geq \exp\{-N\varphi\}\exp\{F\}\left(\Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_q\right)$$

$$+ N\exp\{-N\varphi\}\exp\{\inf F\} m(m+\Delta\varphi)$$

$$\Delta \exp(-N\varphi) = \sum_i \left(\frac{\partial\, -N\exp\{-N\varphi\}\varphi_i}{\partial \bar{z}}\right)$$

$$= \sum_i \left(N^2\exp\{-N\varphi\}|\varphi_i|^2 - N\exp\{-N\varphi\}\varphi_{i\bar{i}}\right)$$

$$= \exp\{-N\varphi\} N^2 |\nabla\varphi|^2 - N \exp\{-N\varphi\} \Delta\varphi$$

$$= \exp\{-N\varphi\}\left[\exp\{F\}\left(\Delta f - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_9\right) + m^2 N \exp\left(\inf F\right)\right]$$

$$+ \; mN \exp\{\inf_M F\} \exp\{-N\varphi\} \Delta\varphi$$

$$= \exp\{-N\varphi\}\left[\exp\{F\}\left(\Delta f - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_9\right) + m^2 N \exp\left(\inf_M F\right)\right]$$

$$+ \; m \exp\left(\inf_M F\right)\left(- \Delta \exp(-N\varphi) + N^2 \exp\{-N\varphi\} |\nabla\varphi|^2\right)$$

$$\geq - C_{10} \exp\{-N\varphi\} + m \exp\{\inf F\}\left(- \Delta \exp(-N\varphi) + N^2 \exp\{-N\varphi\} |\nabla\varphi|^2\right)$$

(2.52), integrate on (2.52) (Both side)

$$\int_M |\nabla \exp\{-\tfrac{1}{2} N\varphi\}|^2 = \frac{1}{4} N^2 \int_M \exp\{-N\varphi\} |\nabla\varphi|^2$$

$$\leq \frac{1}{4} C_{10} m^{-1} \exp\{-\inf F\} \int_M \exp\{-N\varphi\} \qquad (2.53)$$

Then we say that (2.53) and (2.28) gives

$\int_M \exp\{-N\varphi\}$'s estimation. (depends on N, F, M)

Assume $\{\varphi_i\}$ satisfy (2-28), (2-53) such that:

$$\lim_{i\to\infty}\int_M \exp\{-N\varphi_i\} = \infty.$$

define $\exp\{-N\tilde{\varphi}_i\} = \exp\{-N\varphi_i\}\left[\int_M \exp(-N\varphi_i)\right]^{-1}$ (2-54)

from (2-53) $\int_M |\nabla \exp\{-\frac{1}{2}N\tilde{\varphi}_i\}|^2$ uniformly bounded.

means $\exp\{-\frac{1}{2}N\tilde{\varphi}_i\}$'s subsequence $\xrightarrow{\text{in } L^2(M)} f \in L^2(M)$

Assume subsequence is $\{\exp\{-\frac{1}{2}N\tilde{\varphi}_i\}\}$ its self.

on the other hand (2-55) $\int_M |\varphi| =$

$$\int_{\{|\varphi|(x)<\lambda\}} |\varphi| + \int_{\{|\varphi|(x)\geq\lambda\}} |\varphi| \geq \int_{\{x \mid \lambda\leq|\varphi|(x)\}} \lambda$$

$$= \lambda \operatorname{Vol}\{x \mid \lambda \leq |\varphi|(x)\}$$

we have (2.55) $\operatorname{Vol}\{x \mid \lambda\leq|\varphi|(x)\} \leq \frac{1}{\lambda}\int_M |\varphi|$

And $\exp\{-\frac{1}{2}N\tilde{\varphi}_i\} \geq \lambda$

$$\Leftrightarrow \quad \frac{\exp\{-\frac{1}{2}N\varphi_i\}}{\int_M \exp\{-\frac{1}{2}N\varphi_i\}} \geq \lambda$$

$$-\frac{1}{2}N\varphi_i - \log\int_M \exp\{-\frac{1}{2}N\varphi_i\} \geq \log\lambda$$

$$-\varphi_i \geq \frac{2}{N}\log\lambda + \frac{2}{N}\log\int_M \exp\{-\frac{1}{2}N\varphi_i\}$$

we have (2-56) $\mathrm{Vol}\{x \mid \lambda \leq \exp\{-\frac{1}{2}N\tilde{\varphi}_i\}\}$

$$= \mathrm{Vol}\{x \mid \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M \exp\{-N\varphi_i\} \leq -\varphi_i\}$$

since $\lim_{i\to\infty}\int_M \exp\{-\frac{1}{2}N\varphi_i\} = \infty$, for i sufficiently large we have

$$\mathrm{Vol}\{x \mid \lambda \leq \exp\{-\frac{1}{2}N\tilde{\varphi}_i\}\} \quad (2.57)$$

$$= \mathrm{Vol}\{x \mid 0 < \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M \exp\{-N\varphi_i\} \leq |\varphi_i|\}$$

$$\leq \left(\frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M \exp\{-N\varphi_i\}\right)^{-1}\int_M |\varphi_i|$$

which means $\lim_{i\to\infty}\mathrm{Vol}\{x \mid \lambda \leq \exp\{-\frac{1}{2}N\tilde{\varphi}_i\}\} = 0 \quad (2\text{-}58)$

对所有 $\lambda>0$，有 (2.59)

$$\mathrm{Vol}\{x\mid\lambda\le f\}\le \mathrm{Vol}\{x\mid \tfrac12\lambda\le|f-\exp\{-\tfrac12 N\tilde\varphi_i\}|\}$$

$$+\mathrm{Vol}\{x\mid \tfrac12\lambda\le\exp\{-\tfrac12 N\tilde\varphi_i\}\}$$

$$\le\frac{4}{\lambda^2}\int_M|f-\exp\{-\tfrac12 N\tilde\varphi_i\}|^2+\mathrm{Vol}\{x\mid \tfrac12\lambda\le\exp\{-\tfrac12 N\tilde\varphi_i\}\}$$

则由 (2.58)，(2.59)，对于所有 $\lambda>0$ 有 $\mathrm{Vol}\{x\mid\lambda\le f\}=0$

而 f 是 $\exp\{-\tfrac12 N\tilde\varphi_i\}$ 的 L^2 极限，(2.59) 说明 $f=0$ a.e.

而 $\int_M f^2=1$，矛盾。 $\int|\varphi|$

则有了一个 $\int_M\exp\{-N\varphi\}$ 的估计。 $\int\varphi=0$

$\varphi\le-1$

接下来用与 $m=2$ 类似的方法即可得到 $|\inf_M\varphi|$ 的估计。

找一个以 $-\tfrac12\inf\varphi\, C_7^{-1}(\exp\{-C\inf\varphi\}+1)^{-1}$ 为半径的 geodesic ball，找一个足够大的 N，使 $N+\inf_{i\in I}R_{i\bar i i\bar i}\ge\tfrac12 N$

$$C_{12}\exp\{-\tfrac12 N\inf\varphi\}\left(-\tfrac12\inf\varphi\right)^{2m}C_7^{-1}\left(\exp\{-C\inf\varphi\}+1\right)^{-2m}$$

$\underline{\sup |}$ $\underline{\int |\varphi|}$

$\le \int_M \exp\{-N\varphi\} \le C_{N,F,M}$

同理可得到 $\boxed{\sup_M |\varphi|}$，$\boxed{\sup_M |\nabla\varphi|}$，$\boxed{\sup_M (m + \Delta\varphi)}$ 的估计。

则可找到 $1+\varphi_{i\bar{i}}$ 的上界（对每个 i 来说）（M_0）

$\prod_{i=1}^{m} (1+\varphi_{i\bar{i}}) = \exp\{F\}$ 又可给出一个 $1+\varphi_{i\bar{i}}$ 的正下界

因为 $\prod_{i=1}^{m} (1+\varphi_{i\bar{i}}) \ge M_1$（取决于 F），若有一个 $1+\varphi_{i\bar{i}} < M_2$，

使 $M_2 M_0^{m-1} < M_1$，则矛盾，则有一个正下界 M_2.

则有如下结论：$\boxed{\text{Prop 2.1}}$. 令 M 是一个紧 Kahler 流形，

给定Kahler度量 $\sum_{i,j} g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j$，令 φ 是一个实值 $C^4(M)$ 函数，

$\int_M \varphi = 0$，$\sum_{i,j}\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) dz^i \otimes d\bar{z}^j$ 定义了 M 上另一个度量.

若 $\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} = \exp\{F\}$. 则有正数

C_1, C_2, C_3, C_4, depending on $\inf_M F$, $\sup_M F$, $\inf \Delta F$ and M,

使得 $\sup_M |\varphi| \le C_1$，$\sup_M |\nabla\varphi| \le C_2$，$0 < C_3 \le 1+\varphi_{i\bar{i}} \le C_4$

for all i.