1. A interesting method to solve PDE $\Delta u = f(x,u)$

This is the chapter Nonlinear Elliptic Equations in Michael E. Taylor's book Partial Differential Equations III, this chapter looks at equations of the form

$$\Delta u = f(x,u) \quad \text{on } U \tag{1}$$

we consider the Dirichlet boundary condition

$$u|_{\partial U} = g \tag{2}$$

where U is a bounded domain in $\mathbb{R}^2$, we suppose $f \in C^\infty(\bar{U} \times \mathbb{R})$, we will treat (1)-(2) under the hypothesis that

$$\frac{\partial f}{\partial u} \geq 0 \tag{3}$$

Suppose $F(x,u) = \int_0^u f(x,s)ds$, so

$$f(x,u) = \partial_u F(x,u) \tag{4}$$

Then (3) is the hypothesis that $F(x,u)$ is a convex function of u. Let

$$I(u) = \frac{1}{2}\|du\|^2_{L^2(U)} + \int_U F(x,u(x))dx \tag{5}$$

We can see that a solution to (1)-(2) is a critical point of I on the space of functions u on U, equal to g on ∂U.

We make the following temporary restriction on F

$$\text{For } |u| \geq K, \partial_u f(x,u) = 0, \tag{6}$$

so $F(x,u)$ is linear in u for $u \geq K$ and for $u \leq -K$. Thus, for some constant L,

$$|\partial_u F(x,u)| \leq L \quad \text{ on } \bar{U} \times \mathbb{R} \tag{7}$$

Let

$$V = \left\{u \in H^1(U) : u = g \text{ on } \partial U\right\} \tag{8}$$

Lemma 1.1. *Under the hypotheses (4)-(7), we have the following results about the functional $I : V \to \mathbb{R}$:*

$$I \textit{ is strictly convex}, \tag{9}$$

$$I \textit{ is Lipschitz continuous}, \tag{10}$$

with the norm topology on V;

$$I \textit{ is bounded below}; \tag{11}$$

and

$$I(v) \to +\infty, \ \textit{as } \|v\|_{H^1} \to \infty \tag{12}$$

Proof. (9) is trivial. (10) follows from

$$|F(x,u) - F(x,v)| \leq L|u-v|$$

which follows from (7). The convexity of $F(x,u)$ in u implies

$$F(x,u) \geq -C_0|u| - C_1$$

Hence

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|du\|_{L^2}^2 - C_0\|u\|_{L^1} - C_1' \\ &\geq \frac{1}{4}\|du\|_{L^2}^2 + \frac{1}{2}B\|u\|_{L^2}^2 - C\|u\|_{L^2} - C' \end{aligned} \tag{13}$$

since

$$\frac{1}{2}\|du\|_{L^2}^2 \geq B\|u\|_{L^2}^2 - C'', \text{ for } u \in V$$

The last line in (13) clearly implies (11) and (12). ☐

Proposition 1.1. Under the hypotheses (4)-(7), $I(u)$ has a unique minimum on V.

Proof. Let $\alpha_0 = \inf_V I(u)$. By (11), α_0 is finite. Pick R such that $K = V \cap B_R(0) \neq \emptyset$, where $B_R(0)$ is the ball of radius R centered at 0 in $H^1(U)$, and such that $\|u\|_{H^1} \geq R \Rightarrow I(u) \geq \alpha_0 + 1$, which is possible by (12). Note that K is a closed, convex, bounded subset of $H^1(U)$. Let

$$K_\varepsilon = \{u \in K : \alpha_0 \leq I(u) \leq \alpha_0 + \varepsilon\}$$

For each $\varepsilon > 0$, K_ε is a closed, convex subset of K. By Mazur's Theorem K_ε is weakly closed in K, which is weakly compact. Hence

$$\bigcap_{\varepsilon > 0} K_\varepsilon = K_0 \neq \emptyset$$

Now inf $I(u)$ is assumed on K_0. By the strict convexity of $I(u)$, K_0 consists of a single point.

If u is the unique point in K_0 and $v \in C_0^\infty(U)$, then $u + sv \in V$, for all $s \in \mathbb{R}$, and $I(u+sv)$ is a smooth function of s which is minimal at $s = 0$, so $0 = \frac{d}{ds} I(u+sv)\big|_{s=0} = (-\Delta u, v) + \int_U f(x, u(x)) v(x) dV(x)$. Hence (1) holds. ☐

Proposition 1.2. Under the hypotheses (4)-(7), any solution $u \in V$ to (1)-(2) is actually a smooth solution $\in C^\infty(\bar{U})$

Proof. We start with $u \in H^1(U)$. Then the right side of (1) belongs to $H^1(U)$, if $f(x,u)$ satisfies (6). This gives $u \in H^2(U)$, provided $g \in H^{3/2}(\partial U)$. Additional regularity follows inductively. ☐

Proposition 1.3. Let u and $v \in C^2(U) \cap C(\bar{U})$ satisfy (1), with $u = g$ and $v = h$ on ∂U. If (3) holds, then

$$\sup_U (u - v) \leq \sup_{\partial U} (g - h) \vee 0 \tag{14}$$

where $a \vee b = \max(a, b)$ and

$$\sup_U |u - v| \le \sup_{\partial U} |g - h| \tag{15}$$

Proof. Let $w = u - v$. Then, by (3),

$$\Delta w = \lambda(x) w, \quad w|_{\partial U} = g - h$$

where

$$\lambda(x) = \frac{f(x,u) - f(x,v)}{u - v} \ge 0$$

If $\mathcal{O} = \{x \in U : w(x) \ge 0\}$, then $\Delta w \ge 0$ on $\mathcal{O}$, so the maximum principle applies on $\mathcal{O}$, yielding (14). Replacing w by $-w$ gives (14) with the roles of u and v, and of g and h, reversed, and (15) follows. □

Corollary 1.1. Let $f(x, 0) = \varphi(x) \in C^\infty(\bar{U})$. Take $g \in C^\infty(\partial U)$, and let $\Phi \in C^\infty(\bar{U})$ be the solution to

$$\Delta \Phi = \varphi \text{ on } U, \quad \Phi = g \text{ on } \partial U \tag{16}$$

Then, under the hypothesis (3), a solution u to (1)-(2) satisfies

$$\sup_U u \le \sup_U \Phi + \left(\sup_U (-\Phi) \vee 0 \right) \tag{17}$$

and

$$\sup_U |u| \le \sup_U 2|\Phi| \tag{18}$$

Proof. We have

$$\Delta(u - \Phi) = f(x,u) - f(x,0) = \lambda(x) u$$

with $\lambda(x) = [f(x,u) - f(x,0)]/u \ge 0$. Thus $\Delta(u - \Phi) \ge 0$ on $\mathcal{O} = \{x \in U : u(x) > 0\}$, so

$$\sup_{\mathcal{O}} (u - \Phi) = \sup_{\partial \mathcal{O}} (u - \Phi) \le \sup_U (-\Phi) \vee 0$$

This gives (17). Also $\Delta(\Phi - u) \ge 0$ on $\mathcal{O}^- = \{x \in U : u(x) < 0\}$, so

$$\sup_{\mathcal{O}^-} (\Phi - u) = \sup_{\partial \mathcal{O}^-} (\Phi - u) \le \sup_U \Phi \vee 0$$

which together with (17) gives (18). □

Theorem 1.1. *Suppose $f(x,u)$ satisfies (3). Given $g \in C^\infty(\partial U)$, there is a unique solution $u \in C^\infty(\bar{U})$ to (1)-(2).*

Proof. Let $f_j(x,u)$ be smooth, satisfying

$$f_j(x,u) = f(x,u), \text{ for } |u| \le j$$

and be such that (4)-(7) hold for each f_j, with $K = K_j$. We have solutions $u_j \in C^\infty(\bar{U})$ to

$$\Delta u_j = f_j(x, u_j), \quad u_j|_{\partial U} = g$$

Now $f_j(x,0) = f(x,0) = \varphi(x)$, independent of j, and the estimate (18) holds for all u_j, so

$$\sup_U |u_j| \le \sup_U 2|\Phi|$$

where Φ is defined by (16). Thus the sequence (u_j) stabilizes for large j, and the proof is complete. $\square$

We next discuss a geometrical problem that can be solved using the results developed above. The problem we consider here is the following. Let $\bar{M}$ be a connected, compact, two-dimensional manifold, with nonempty boundary. We suppose that we are given a Riemannian metric g on $\bar{M}$, and we desire to construct a conformally related metric whose Gauss curvature $K(x)$ is a given function on $\bar{M}$. if $k(x)$ is the Gauss curvature of g and if $g' = e^{2u}g$, then the Gauss curvature of g' is given by

$$K(x) = (-\Delta u + k(x))e^{-2u},$$

where Δ is the Laplace operator for the metric g. Thus we want to solve the PDE

$$\Delta u = k(x) - K(x)e^{2u} = f(x,u),$$

for u. This is of the form (1). The hypothesis (3) is satisfied provided $K(x) \le 0$. Thus this yields that

If $\bar{M}$ is a connected, compact 2-manifold with nonempty boundary $\partial M, g$ a Riemannian metric on $\bar{M}$, and $K \in C^\infty(\bar{M})$ a given function satisfying

$$K(x) \le 0 \text{ on } M,$$

then there exists $u \in C^\infty(\bar{M})$ such that the metric $g' = e^{2u}g$ conformal to g has Gauss curvature K. Given any $v \in C^\infty(\partial M)$, there is a unique such u satisfying $u = v$ on ∂M.