Let (M, g) be a closed smooth Riemannian manifold, does there exists a positive and smooth function f on M such that the Riemannian metric fg has constant scalar curvature, this is the so-called Yamabe problem and we are going to transform the problem into solving the equation of type

$$\Delta\varphi + h(x)\varphi = \lambda f(x)\varphi^{N-1}$$

on a C^∞ compact Riemannian manifold M_n of dimension $n \geq 3$ where $h(x)$ and $f(x)$ are C^∞ functions on M_n, with $f(x)$ everywhere strictly positive and $N = 2n/(n-2)$.

First consider the conformal deformation. For convenience, write $\tilde{g} = ug$, where u is a positive smooth function on M and let $\{X_i\}$ be the normal frame at $p \in M$ with respect to g, i.e., $g(X_i, X_j) = \delta_{ij}$ in a neighborhood of p so that $\Gamma_{ij}^k = 0$ at p. So the Christoffel symbol of $\tilde{g}$ is given by

$$\begin{aligned}
\widetilde{\Gamma}_{ij}^k &= \frac{1}{2}\widetilde{g}^{kl}\left(\frac{\partial}{\partial x_i}\widetilde{g}_{jl} + \frac{\partial}{\partial x_j}\widetilde{g}_{il} - \frac{\partial}{\partial x_l}\widetilde{g}_{ij}\right) \\
&= \frac{1}{2}u^{-1}g^{kl}\left[\frac{\partial}{\partial x_i}(ug_{jl}) + \frac{\partial}{\partial x_j}(ug_{il}) - \frac{\partial}{\partial x_l}(ug_{ij})\right] \\
&= \frac{1}{2}u^{-1}g^{kl}u\left[\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{il} - \frac{\partial}{\partial x_l}g_{ij}\right] + \frac{1}{2}u^{-1}g^{kl}\left[g_{jl}\frac{\partial}{\partial x_i}u + g_{il}\frac{\partial}{\partial x_j}u - g_{ij}\frac{\partial}{\partial x_l}u\right] \\
&= \Gamma_{ij}^k + \frac{1}{2}\left(\delta_{jk}\frac{\partial}{\partial x_i}\log u + \delta_{ik}\frac{\partial}{\partial x_j}\log u - \delta_{ij}\frac{\partial}{\partial x_k}\log u\right).
\end{aligned}$$

From the assumption on normal frame. We have (*):
if i, j, k are all distinct

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$$

$$\widetilde{\Gamma}_{ii}^k = \Gamma_{ii}^k - \frac{1}{2}\frac{\partial}{\partial x_k}\log u$$

$$\widetilde{\Gamma}_{ij}^i = \Gamma_{ij}^i + \frac{1}{2}\frac{\partial}{\partial x_j}\log u = \widetilde{\Gamma}_{ji}^i$$

$$\widetilde{\Gamma}_{ii}^i = \Gamma_{ii}^i + \frac{1}{2}\frac{\partial}{\partial x_i}\log u$$

The curvature tensor of $\widetilde{g}$ is given by

$$\widetilde{R}_{ijk}^s = \widetilde{\Gamma}_{ik}^l\widetilde{\Gamma}_{jl}^s - \widetilde{\Gamma}_{jk}^l\widetilde{\Gamma}_{il}^s + \frac{\partial}{\partial x_j}\widetilde{\Gamma}_{ik}^s - \frac{\partial}{\partial x_i}\widetilde{\Gamma}_{jk}^s,$$

and the curvature tensor of g is given by

$$R_{ijk}^s = \Gamma_{ik}^l\Gamma_{jl}^s - \Gamma_{jk}^l\Gamma_{il}^s + \frac{\partial}{\partial x_j}\Gamma_{ik}^s - \frac{\partial}{\partial x_i}\Gamma_{jk}^s$$

In particular, at a point p,

$$R_{ijk}^s = \frac{\partial}{\partial x_j}\Gamma_{ik}^s - \frac{\partial}{\partial x_i}\Gamma_{jk}^s$$

Now the sectional curvature of $\widetilde{g}$ in $\{X_i, X_j\}$ at p is given by

$$\begin{aligned}\widetilde{K}\left(X_i, X_j\right) &= \frac{\widetilde{R}^s_{iji}\widetilde{g}_{sj}}{\widetilde{g}_{ii}\widetilde{g}_{jj} - \widetilde{g}^2_{ij}} \\ &= \frac{\widetilde{R}^s_{iji}ug_{sj}}{\left(ug_{ii}\right)\left(ug_{jj}\right) - \left(ug_{ij}\right)^2} \\ &= u^{-1}\widetilde{R}^j_{iji}.\end{aligned}$$

This implies

$$u\widetilde{K}\left(X_iX_j\right) = \widetilde{R}^j_{iji}$$

$$= \widetilde{\Gamma}^l_{ii}\widetilde{\Gamma}^j_{jl} - \widetilde{\Gamma}^l_{ji}\widetilde{\Gamma}^j_{il} + \frac{\partial}{\partial x_j}\widetilde{\Gamma}^j_{ii} - \frac{\partial}{\partial x_i}\widetilde{\Gamma}^j_{ji}$$

Note that

$$\sum_{l=1, l\neq i\neq j}^{n} \widetilde{\Gamma}^l_{ji}\widetilde{\Gamma}^j_{il} = \sum_{l=1, l\neq i\neq j}^{n} \Gamma^l_{ij}\Gamma^j_{il} = 0$$

since $l \neq i \neq j$ and $\Gamma^l_{ij} = 0$ at p.

Now from (*) we have:

$$\begin{aligned}u\widetilde{K}\left(X_i, X_j\right) = &\sum_{l=1, l\neq i}^{n} \widetilde{\Gamma}^l_{ii}\widetilde{\Gamma}^j_{jl} + \widetilde{\Gamma}^i_{ii}\widetilde{\Gamma}^j_{ji} \\ &- \left(\widetilde{\Gamma}^j_{ji}\widetilde{\Gamma}^j_{ij} + \widetilde{\Gamma}^i_{ji}\widetilde{\Gamma}^j_{ii}\right) + \frac{\partial}{\partial x_j}\widetilde{\Gamma}^j_{ii} - \frac{\partial}{\partial x_j}\widetilde{\Gamma}^j_{ji} \\ = &\sum_{l=1, l\neq i}^{n} \left(-\frac{1}{2}(\log u)_l\right)\left(\frac{1}{2}(\log u)_l\right) + \left(\frac{1}{2}(\log u)_i\right)\left(\frac{1}{2}(\log u)_i\right) \\ &- \left(\frac{1}{2}(\log u)_i\right)^2 - \left(\frac{1}{2}(\log u)_j\right)\left(-\frac{1}{2}(\log u)_j\right) \\ &+ \frac{\partial}{\partial x_j}\left(\Gamma^j_{ii} - \frac{1}{2}(\log u)_j\right) - \frac{\partial}{\partial x_i}\left(\Gamma^j_{ji} + \frac{1}{2}(\log u)_i\right) \\ = &-\frac{1}{4}\sum_{l=1}^{n}\left[(\log u)^2_l + \frac{1}{4}\left[(\log u)_j\right]^2 + \frac{1}{4}\left[(\log u)_i\right]^2\right. \\ &- \frac{1}{2}(\log u)_{jj} - \frac{1}{2}(\log u)_{ii} + K\left(X_i, X_j\right)\end{aligned}$$

So we have (**)

$$
\begin{aligned}
uR_{\widetilde{g}} =& \sum_{i\neq j} u\widetilde{K}\left(X_i, X_j\right) \\
=& \sum_{i\neq j} -\frac{1}{4}\sum_{l=1}^{n}\left[(\log u)_l\right]^2 + \frac{1}{4}\left[(\log u)_j\right]^2 + \frac{1}{4}\left[(\log u)_i\right]^2 \\
& -\frac{1}{2}(\log u)_{jj} - \frac{1}{2}(\log u)_{ii} + K\left(X_i, X_j\right) \\
=& -\frac{1}{4}\left(n^2 - n\right)\sum_{l=1}^{n}\left[(\log u)_l\right]^2 + \frac{1}{2}(n-1)\sum_{i=1}^{n}\left[(\log u)_i\right]^2 - (n-1)\sum_{i=1}^{n}(\log u)_{ii} + R_g
\end{aligned}
$$

In normal coordinate, we see that

$$\nabla_g(\log u) = \nabla(\log u), \quad \triangle_g u = \Delta u$$

Thus, we get

$$uR_{\widetilde{g}} = -\frac{1}{4}(n-2)(n-1)\left|\nabla_g(\log u)\right|^2 - (n-1)\triangle_g(\log u) + R_g$$

If $n = 2$, then

$$uR_{\tilde{g}} = R_g - \triangle_g(\log u)$$

when $\tilde{g} = ug$. Write $u = e^v$. Then

$$e^v R_{\widetilde{g}} = R_g - \Delta_g v$$

so that

$$e^v \frac{1}{2} K_{\tilde{g}} = \frac{1}{2} K_g - \Delta_g v$$

where K_g is a Gaussian curvature. If $n \geq 3$, then we let $u = v^{\frac{4}{n-2}}$, i.e., $\tilde{g} = v^{\frac{4}{n-2}} g$, then

$$
\begin{aligned}
\log u &= \frac{4}{n-2}\log v \\
\nabla_g(\log u) &= \frac{4}{n-2}\nabla_g(\log v) = \frac{4}{n-2}\frac{\nabla_g v}{v} \\
\Delta_g(\log u) &= -\frac{4}{n-2}\frac{\left|\nabla_g v\right|^2}{v^2} + \frac{4}{n-2}\frac{\Delta_g v}{v}
\end{aligned}
$$

Put these this into (**), we obtain

$$v^{\frac{4}{n-2}} R_{\widetilde{g}} = R_g - \frac{4(n-1)}{n-2}\frac{\triangle_g v}{v},$$

which is what we want.

Let v=φ, we have

$$4((n-1)/(n-2))\Delta\varphi + R\varphi = R'\varphi^{(n+2)/(n-2)} \tag{1}$$

where $\Delta\varphi = -\nabla^\nu\nabla_\nu\varphi$

So the Yamabe problem is equivalent to solving equation (1) with $R' =$ Const, and the solution φ must be smooth and strictly positive.

On a C^∞ compact Riemannian manifold M_n of dimension $n \geq 3$, consider the general case which is the differential equation

$$\Delta\varphi + h(x)\varphi = \lambda f(x)\varphi^{N-1} \tag{2}$$

where $h(x)$ and $f(x)$ are C^∞ functions on M_n, with $f(x)$ everywhere strictly positive and $N = 2n/(n-2)$.

The problem is to prove the existence of a real number λ and of a C^∞ function φ, everywhere strictly positive, satisfying (1).

Yamabe considered, for $2 < q \leq N$, the functional

$$I_q(\varphi) = \left[\int_M \nabla^i\varphi\nabla_i\varphi dV + \int_M h(x)\varphi^2 dV\right]\left[\int_M f(x)\varphi^q dV\right]^{-2/q} \tag{3}$$

where $\varphi \not\equiv 0$ is a nonnegative function belonging to H_1, the first Sobolev space. The denominator of $I_q(\varphi)$ makes sense since $\mathrm{H}_1 \subset L_N \subset L_q$. Define

$$\mu_q = \inf I_q(\varphi) \text{ for all } \varphi \in \mathrm{H}_1, \varphi \geq 0, \varphi \equiv 0.$$

It is impossible to prove directly that μ_N is attained and thus to solve Equation (3). (We shall soon see why.) This is the reason why Yamabe considered the approximate equations for $q < N$:

$$\Delta\varphi + h(x)\varphi = \lambda f(x)\varphi^{q-1} \tag{4}$$

5.5 Theorem. For $2 < q < N$, there exists a C^∞ strictly positive function φ_q satisfying Equation (4) with $\lambda = \mu_q$ and $I_q(\varphi_q) = \mu_q$.

Proof. (a) For $2 < q \leq N, \mu_q$ is finite. Indeed

First, $\mu_q \leq I_q(1) = \left[\int_M h(x)dV\right]\left[\int_M f(x)dV\right]^{-2/q}$.

And,

$$I_q(\varphi) \geq \left[\inf_{x\in M}(0, h(x))\right]\left[\sup_{x\in M} f(x)\right]^{-2/q} \|\varphi\|_2^2\|\varphi\|_q^{-2}$$

$$\int \varphi^2 = \int (\varphi^q)^{\frac{2}{q}} = \int (\varphi^q)^{\frac{2}{q}} \cdot 1$$

$$\leq \left(\int \left[(\varphi^q)^{\frac{2}{q}}\right]^{\frac{q}{2}}\right)^{\frac{2}{q}} V^{\frac{q-2}{q}}$$

Then we get

$$\|\varphi\|_2^2\|\varphi\|_q^{-2} \leq V^{1-2/q} \leq \sup\left(1, V^{2/n}\right)$$

(b) Let $\{\varphi_i\}$ be a minimizing sequence such that $\int_M f(x)\varphi_i^q dV = 1$:

$$\varphi_i \in \mathrm{H}_1, \varphi_i \geq 0, \lim_{i\to\infty} I_q(\varphi_i) = \mu_q.$$

First we prove that the set of the φ_i is bounded in H_1,

$$\|\varphi_i\|_{H_1}^2 = \|\nabla\varphi_i\|_2^2 + \|\varphi_i\|_2^2 = I_q(\varphi_i) - \int_M h(x)\varphi_i^2 dV + \|\varphi_i\|_2^2.$$

Since we can suppose that $I_q(\varphi_i) < \mu_q + 1$, then

$$\|\varphi_i\|_{H_1}^2 \leq \mu_q + 1 + \left[1 + \sup_{x\in M} |h(x)|\right] \|\varphi_i\|_2^2$$

and

$$\|\varphi_i\|_2^2 \leq [V]^{1-2/q} \|\varphi_i\|_q^2 \leq [V]^{1-2/q} \left[\inf_{x\in M} f(x)\right]^{-2/q}$$

(c)First we need **Kondrakov Theorem**. For the compact Riemannian manifolds W_n with C^1-boundary. Namely, the following imbeddings are compact

(a) $H_k^q(M_n) \subset L_p(M_n)$ and $H_k^q(W_n) \subset L_p(W_n)$, with $1 \geq 1/p > 1/q - k/n > 0$.

(b) $H_k^q(M_n) \subset C^2(M_n)$ and $H_k^q(W_n) \subset C^\alpha\left(\bar{W}_n\right)$, if $k - \alpha > n/q$, with $0 \leq \alpha < 1$.

If $2 < q < N$, there exists a nonnegative function $\varphi_q \in \mathrm{H}_1$, satisfying

$$I_q(\varphi_q) = \mu_q \quad \text{and} \quad \int_M f(x)\varphi_q^q dV = 1$$

Indeed, for $2 < q < N$, the imbedding $\mathrm{H}_1 \subset L_q$ is compact by Kondrakov's theorem and, since the bounded closed sets in H_1 are weakly compact, there exists $\{\varphi_j\}$ a subsequence of $\{\varphi_i\}$, and a function $\varphi_q \in H_1$ such that

$(\alpha)\varphi_j \to \varphi_q$ in L_q

$(\beta)\varphi_j \to \varphi_q$ weakly in H_1

$(\gamma)\varphi_j \to \varphi_q$ almost everywhere.

The last assertion is true by the fact that let $\{f_k\}$ be a sequence in L_p (or in L_∞) which converges in L_p to $f \in L_p$. Then there exists a subsequence converging pointwise almost everywhere to f.

$(\alpha) \Longrightarrow \int_M f(x)\varphi_q^q dV = 1$; $(\gamma) \Rightarrow \varphi_q \geq 0$, and $(\beta$) implies

$$\|\varphi_q\|_{H_1} \leq \liminf_{i\to\infty} \|\varphi_j\|_{\mathrm{H}_1}$$

From the fact that a weakly convergent sequence $\{x_i\}$ in a normed space $\tilde{F}$ has a unique limit x, is bounded, and

$$\|x\| \leq \lim_{i\to\infty} \inf \|x_i\|$$

Hence $I_q(\varphi_q) \leq \lim_{j\to\infty} I_q(\varphi_j) = \mu_q$ because $\varphi_j \to \varphi_q$ in L_2, according to (α) since $q \geq 2$. Therefore, by definition of $\mu_q, I_q(\varphi_q) = \mu_q$.

(d) φ_q satisfies Equation (4) weakly in H_1. Just simply compute $\frac{d}{dt} I_q(\varphi_q + t\psi)$ when $t = 0$ we can get that φ_q satisfies for all $\psi \in \mathrm{H}_1$

$$\int_M \nabla^\nu \varphi_q \nabla_\nu \psi dV + \int_M h(x)\varphi_q \psi dV = \mu_q \int_M f(x)\varphi_q^{q-1}\psi dV$$

.

To check that the preceding computation is correct, we note that since $\mathscr{D}(M)$ is dense in H_1 and $\varphi \not\equiv 0$, then

$$\inf_{\varphi\in \mathrm{H}_1} I_q(\varphi) = \inf_{\varphi\in C^\infty} I_q(\varphi) = \inf_{\varphi\in C^\infty} I_q(|\varphi|) \geq \inf_{\varphi\in \mathrm{H}_1}^{\varphi\geq 0} I_q(\varphi) \geq \inf_{\varphi\in \mathrm{H}_1} I_q(\varphi).$$

$I(\varphi) = I(|\varphi|)$ when $\varphi \in C^\infty$ because of the fact that let M_n be a C^∞ Riemannian manifold and $\varphi \in H_1^p(M)$; then almost everywhere $|\nabla|\varphi|| = |\nabla\varphi|$.

We need **Theorem 4.13.** Let M be a compact C^∞ Riemannian manifold. There exists $G(P,Q)$ a Green's function of the Laplacian which has the following properties:

1. For all functions $\varphi \in C^2$:

$$\varphi(P) = V^{-1}\int_M \varphi(Q)dV(Q) + \int_M G(P,Q)\Delta\varphi(Q)dV(Q)$$

2. $G(P,Q)$ is C^∞ on $M\times M$ minus the diagonal (for $P \neq Q$).
3. There exists a constant k such that:

$$|G(P,Q)| < k(1+|\log r|) \text{ for } n=2 \text{ and}$$
$$|G(P,Q)| < kr^{2-n} \text{ for } n>2, |\nabla_Q G(P,Q)| < kr^{1-n}$$
$$\left|\nabla_Q^2 G(P,Q)\right| < kr^{-n} \text{ with } r = d(P,Q)$$

And a corollary of sobolev inequality: Let λ be a real number, $0<\lambda<n$, and $q'>1$. If r, defined by $1/r = \lambda/n + 1/q' - 1$, satisfies $r>1$, then

$$h(y) = \int_{\mathbb{R}^n} \frac{f(x)}{\|x-y\|^\lambda}dx \text{ belongs to } L_r, \text{ when } f\in L_q\left(\mathbb{R}^n\right).$$

Moreover, there exists a constant $C\left(\lambda, q', n\right)$ such that for all $f \in L_{q'}\left(\mathbb{R}^n\right)$

$$\|h\|_r \leq C\left(\lambda, q', n\right)\|f\|_{q'}$$

(e) $\varphi_q \in C^\infty$ for $2\leq q<N$ and the functions φ_q are uniformly bounded for $2\leq q\leq q_0<N$.

Let $\mathrm{G}(P,Q)$ be the Green's function. φ_q satisfies the integral equation (5)

$$\varphi_q(P) = V^{-1}\int_M \varphi_q(Q)dV(Q) + \int_M \mathrm{G}(P,Q)\left[\mu_q f(Q)\varphi_q^{q-1} - h(Q)\varphi_q\right]dV(Q)$$

We know that $\varphi_q \in L_{r_0}$ with $r_0 = N$.

Since, by **Theorem 4.13** there exists a constant B such that $|\mathrm{G}(P,Q)| \leq B[d(P,Q)]^{2-n}$ Notice that the estimate on the Green's function means that

$$\int |G(P,Q)|^\alpha dQ$$

is bounded whenever $\alpha < \frac{n}{n-2}$ (the function $1/|x|^\alpha$ is locally integrable in $\mathbb{R}^d$ when $d>\alpha$ and the manifold M is compact, so local integrability gives integrability, and the bound can be taken to be uniform; that is independent of P). By Holder's inequality, we have

$$\int G(P,Q)F(Q)dQ \leq \|G(P,-)\|_{L^\alpha}\|F\|_{L^\beta}$$

when $\beta^{-1} = 1 - \alpha^{-1}$. To ensure that the first term is bounded, we need $\alpha < \frac{n}{n-2}$ as described above; this means that $\beta^{-1} < 1 - \frac{n-2}{n} = \frac{2}{n}$.

So applying to equation (2) (noting that h and f are smooth, bounded, and can be discarded), we see that if φ_q is such that both $\varphi_q \in L^\beta$ and $\varphi_q^{q-1} \in L^{\beta'}$ with $\beta, \beta' > \frac{n}{2}$, then we can apply the chain of reasoning above to conclude that φ_q is uniformly bounded.

By assumption $q-1 > 1$; if $\varphi_q^{q-1} \in L^{\beta'}$ then $\varphi_q \in L^{(q-1)\beta'}$. This means that we only need: if $\varphi_q^{q-1} \in L^{\beta'}$ with $\beta' > \frac{n}{2}$, then the equation will guarantee that $\varphi_q \in L^\infty$. This final condition can be re-written as $\varphi_q \in L^\beta$ for some $\beta > \frac{n}{2}(q-1)$. So you just need to argue somehow that φ_q can be taken to have this degree of integrability.

Iteration process

If we know that φ_q is a priori in L^{β_0}, with $\beta_0 < \frac{n}{2}(q-1)$, then the Sobolev inequality will tell you that

$$\varphi_q \in L^{\beta_k}, \quad \beta_k = \frac{n\beta_{k-1}}{n(q-1) - 2\beta_{k-1}}$$

Our hope is that starting with $\varphi_q \in L^{\beta_0}$, we can improve it to some L^{β_1} for $\beta_1 > \beta_0$. And repeating this improvement should eventually get us above the threshold $\frac{n}{2}(q-1)$.

Examining this β_1, we see that (under still the assumption that $\beta_0 < \frac{n}{2}(q-1)$)

$$\beta_1 > \beta_0 \Longleftrightarrow \frac{n}{n(q-1) - 2\beta_0} > 1 \Longleftrightarrow 2 + \frac{2\beta_0}{n} > q$$

starting from $\beta_0 = \frac{2n}{n-2}$ we have $q < 2 + \frac{2\beta_0}{n} = 2 + \frac{4}{n-2} = \frac{2n}{n-2}$

If $\varphi \in L^\beta$ for some $\beta > \frac{n}{2}(q-1)$ then we can get that $\varphi \in L^\infty$. So we start with $L^{\beta 0}$, with $\beta_0 < \frac{n}{2}(q-1)$. And the iteration step is

$$\beta_k = \frac{n\beta_{k-1}}{n(q-1) - 2\beta_k - 1}$$

then

$$\frac{1}{\beta_k} = (q-1)^k \left[\frac{1}{\beta_0} - \frac{2}{n(q-2)}\right] + \frac{2}{n(q-2)}$$

and from the fact that $q < \frac{2n}{n-2}$ we have $\frac{1}{\beta_0} - \frac{2}{n(q-2)} < 0$ we can find a k such that $\beta_k > \frac{n}{2}(q-1)$

If we want the uniform bound, we should prove $\varphi_q \in L^\beta$ for some $\beta > \frac{n}{2}(q_0-1)$, then according to the iteration process and corollary of the sobolev inquality, $\varphi_q \in L_{r_1}$, for $2 < q \le q_0$ with

$$\frac{1}{r_1} = \frac{n-2}{n} + \frac{q_0-1}{r_0} - 1 = \frac{q_0-1}{r_0} - \frac{2}{n}$$

starting from $f \in L^{\frac{r_0}{q_0-1}}$, $\frac{1}{r_1} = \frac{n-2}{n} + \frac{q_0-1}{r_0} - 1$
we have

$$\|\varphi_q\|_{r_1} \le C \left\|\varphi_q^{q-1}\right\|_{\frac{r_0}{q_0-1}}$$

consider $(\int(\varphi^{\frac{q-1}{q_0-1}})^{r_0})^{\frac{q_0-1}{r_0}}$ and by holder inequality we have

$$\int \varphi_q^{\frac{q-1}{q_0-1}\cdot r_0} \cdot 1 \le C_1(\int \varphi_q^{r_0})^{\frac{q-1}{q_0-1}}$$

then there exists a constant A_1 such that $\|\varphi_q\|_{r_1} \le A_1 \|\varphi_q\|_{r_0}^{q-1}$. Then by induction we have $\|\varphi_q\|_{r_k} \le A_k \|\varphi_q\|_{r_0}^{(q-1)^k}$ and the sobolev inequality which said that the Sobolev imbedding theorem holds for M_n a complete manifold with bounded curvature and injectivity radius $\delta > 0$. Moreover, for any $\varepsilon > 0$, there exists a constant $A_q(\varepsilon)$ such that every $\varphi \in H_1^q(M_n)$ satisfies:

$$\|\varphi\|_p \le [K(n,q)+\varepsilon]\|\nabla\varphi\|_q + A_q(\varepsilon)\|\varphi\|_q$$

with $1/p = 1/q - 1/n > 0$, where K(n,q) is the smallest constant having this property.

Thus the functions φ_q are uniformly bounded. Since $\varphi_q \in L_\infty$, differentiating (5) yields $\varphi_q \in C^1, \varphi_q$ satisfies (4); thus $\Delta\varphi_q$ belongs to C^1 and $\varphi_q \in C^2$ according to the fact that let Ω be an open set of $\mathbb{R}^n$ and $A = a_\ell \nabla^\ell$ a linear elliptic operator of order $2m$ with C^∞ coefficients ($a_\ell \in C^\infty(\Omega)$ for $0 \le \ell \le 2m$). Suppose u is a distribution solution of the equation $A(\mathrm{u}) = f$ and $f \in C^{k,\alpha}(\Omega)$ (resp., $C^\infty(\Omega)$). Then u $\in C^{k+2m,\alpha}(\Omega)$, (resp., $C^\infty(\Omega)$) with $0 < \alpha < 1$. If f belongs to $H_k(\Omega), 1 < p < \infty$, then u belongs locally to H^p_{k+2m}.

(f)From the proposition that let M_n be a compact Riemannian manifold. If a function $\psi \ge 0$, belonging to $C^2(M)$, satisfies an inequality of the type $\Delta\psi \ge \psi f(P,\psi)$, where $f(P,t)$ is a continuous numerical function on $M \times \mathbb{R}$, then either ψ is strictly positive, or ψ is identically zero. We can simply get the proof. □