(2.18) $\Delta'\left(\exp\{-C\varphi\}(m+\Delta\varphi)\right)$

$$\geq \exp\{-C\varphi\}\left(\Delta F - m^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}}\right) - C\exp\{-C\varphi\}\, m(m+\Delta\varphi)$$

$$+\left(C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\{-C\varphi\}(m+\Delta\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right)$$

(2.19) $$\sum_i \frac{1}{1+\varphi_{i\bar{i}}} \geq \left(\frac{\sum_i (1+\varphi_{i\bar{i}})}{\prod_i (1+\varphi_{i\bar{i}})}\right)^{1/m-1}$$

Finding solution when $g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}$ is positive definite Hermitian matrix

$\varphi_{i\bar{i}} \geq 0$

$m=4$ as an example:

$$\left(\frac{1}{1+\varphi_{1\bar{1}}}+\frac{1}{1+\varphi_{2\bar{2}}}+\frac{1}{1+\varphi_{3\bar{3}}}+\frac{1}{1+\varphi_{4\bar{4}}}\right)\left(\frac{1}{1+\varphi_{1\bar{1}}}+\frac{1}{1+\varphi_{2\bar{2}}}+\frac{1}{1+\varphi_{3\bar{3}}}+\frac{1}{1+\varphi_{4\bar{4}}}\right)\left(\frac{1}{1+\varphi_{1\bar{1}}}+\frac{1}{1+\varphi_{2\bar{2}}}+\frac{1}{1+\varphi_{3\bar{3}}}+\frac{1}{1+\varphi_{4\bar{4}}}\right)$$

$$\geq \sum_k \frac{1+\varphi_{k\bar{k}}}{\prod_{i=1}^{4}(1+\varphi_{i\bar{i}})}$$

(2.19) simply proved.

since $\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1} = \exp\{F\}$

(2.19) becomes $$\sum_i \frac{1}{1+\varphi_{i\bar{i}}} \geq \left(\frac{m+\Delta\varphi}{\exp\{F\}}\right)^{1/m-1} \quad (2.20)$$

chose C s.t. $C + \inf\limits_{i \neq l} R_{i\bar{i}l\bar{l}} > 1$

using (2.20), (2.18) becomes (2.22): $\Delta'\left(\exp\{-C\varphi\}(m+\Delta\varphi)\right)$

$$\geq \exp\{-\varphi\}\left(\Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right) - C\exp\{-C\varphi\}\, m\,(m+\Delta\varphi)$$

$$+ \left(C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\{-C\varphi\}\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{1+\frac{1}{m-1}}$$

using (2.22) to estimate $\exp\{-C\varphi\}(m+\Delta\varphi)$, φ attains maximum at p

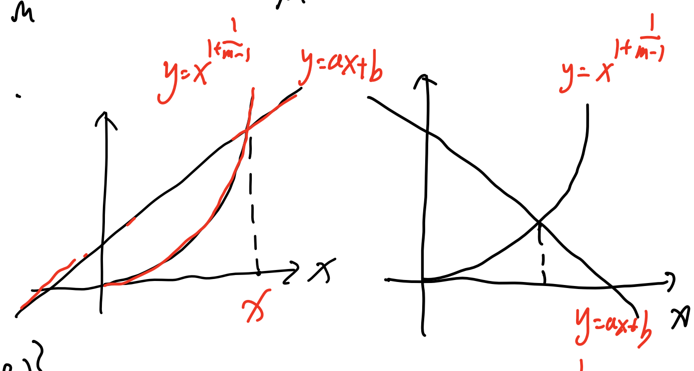
$$0 \geq \Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - Cm(m+\Delta\varphi) + \left(C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\left\{\frac{-F}{m-1}\right\}$$

$$(m+\Delta\varphi)^{1+\frac{1}{m-1}}.$$

$(m+\Delta\varphi)(p)$ depending $\sup\limits_M(-\Delta F)$, $\sup\limits_M\left|\inf\limits_{i\neq l} R_{i\bar{i}l\bar{l}}\right|$, C, m

and $\sup\limits_M |F|$ upper bound C_1.

$$x^{1+\frac{1}{m-1}} \leq ax+b:$$



(2.24) $0 < m+\Delta\varphi \leq C_1\exp\{C(\varphi - \inf\limits_M \varphi)\}$

estimate $\sup\limits_M \varphi$.

$G(p,q_0)$ is Δ's Green function, K s.t. $G(p,q_0)+K\geq 0$

and require $\int_M \varphi = 0$ $\qquad \int_M \varphi = 0$

$$-\Delta G(p, q_\theta) = \delta(p - q_\theta)$$

$$-\int_M \Delta\big(G(p-q_\theta)+K\big)\varphi(q_\theta)\,dq_\theta = -\int_M \big(\Delta G(p, q_\theta)+K\big)\varphi(q_\theta)\,dq_\theta = \varphi(p)$$

$$= \int_M \nabla\big(G(p, q_\theta)+K\big)\nabla\varphi(q_\theta)\,dq_\theta = \varphi(p)$$

$$= -\int_M G(p, q_\theta)\,\Delta\varphi(q_\theta)\,dq_\theta = \varphi(p)$$

having (2.26) $\varphi(p) = -\int_M \big(G(p, q_\theta)+K\big)\Delta\varphi(q_\theta)\,dq_\theta$

$\Delta\varphi + m \geq 0$, $-\Delta\varphi \leq m$

having (2.27) $\sup\limits_M \varphi \leq m \sup\limits_{p\in M} \int_M \big(G(p, q_\theta)+K\big)\,dq_\theta$

$$\left|\sup_M \varphi\right| \leq m \sup_{p\in M} \int_M \left|G(p, q_\theta)+K\right|\,dq_\theta$$

renormalize φ s.t. $\sup\limits_M \varphi \leq -1$.

$$\int_M |\varphi| \leq \int_M \sup_M \varphi - \varphi + \int_M |\sup \varphi|$$

estimates of $\sup_M \varphi$, $\|\varphi\|_{L^1(M)}$ are given, we need $\inf_M \varphi$'s estimate.

First method works for $m=2$.

for $p \geq 1$: $\Delta'(-\varphi)^p = p(p-1)(-\varphi)^{p-2} \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} - p(-\varphi)^{p-1} \Delta' \varphi$

$$= p(p-1)(-\varphi)^{p-2} \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} - p(-\varphi)^{p-1}\left(m - \sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right)$$

$m=2$: $\Delta'(-\varphi)^p = p(p-1)(-\varphi)^{p-2} \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} - 2p(-\varphi)^{p-1}$

$$+ p(-\varphi)^{p-1}\left(\frac{2+\Delta\varphi}{(1+\varphi_{i\bar{i}})(1+\varphi_{i\bar{i}})}\right)$$

Since $p(-\varphi)^{p-1}\Delta\varphi = -\Delta(-\varphi)^p + p(p-1)(-\varphi)^{p-2}|\nabla\varphi|^2$

(differentiate $(-\varphi)^p$: $\frac{\partial(-\varphi)^p}{\partial z^i} = p(-\varphi)^{p-1}\frac{\partial(-\varphi)}{\partial z^i}$

$$\frac{\partial^2(-\varphi)^p}{\partial z^i \partial \bar{z}^i} = p(p-1)(-\varphi)^{p-2}\frac{\partial\varphi}{\partial z^i}\frac{\partial\varphi}{\partial \bar{z}^i} - p(-\varphi)^{p-1}\frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^i} \;)$$

$$\Delta'(-\varphi)^p = p(p-1)(-\varphi)^{p-2}\sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} - 2p(-\varphi)^{p-1} \quad (2\text{-}30)$$

$$+ 2p(-\varphi)^{p-1}\exp\{-F\} + \exp\{-F\}\left[-\Delta(-\varphi)^p + p(p-1)(-\varphi)^{p-2}|\nabla\varphi|^2\right]$$

for (2-30), multiply $\exp\{F\}$ on both side and integrate.

$\exp\{F\}$ * volume form of $\sum_{i,j} g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j$

= the volume form of $\sum_{i,j} g'_{i\bar{j}}\, dz^i \otimes d\bar{z}^j$

and $\int_M \Delta f = -\int_M \nabla f \cdot \nabla 1 = 0$

we get $2p \int_M (-\varphi)^{p-1}(1-\exp\{-F\}) = p(p-1)\int_M (-\varphi)^{p-2}\left(\sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}\right)\exp\{F\}$

$+ \ p(p-1)\int_M (-\varphi)^{p-2}|\nabla\varphi|^2 \qquad (2\text{-}31)$

and $\left|\nabla(-\varphi)^{\frac{p}{2}}\right|^2 = \frac{p^2}{4}(-\varphi)^{p-2}|\nabla\varphi|^2$

(differentiate $(-\varphi)^{\frac{p}{2}}$: $\frac{\partial(-\varphi)^{\frac{p}{2}}}{\partial z^i} = \frac{p}{2}(-\varphi)^{\frac{p}{2}-1}\frac{\partial(-\varphi)}{\partial z^i}$)

(2-31) becomes $p(p-1)\int_M (-\varphi)^{p-2}\left(\sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}\right)\exp\{F\}$

$+ \ \frac{4(p-1)}{p}\int_M \left|\nabla(-\varphi)^{\frac{p}{2}}\right|^2$. Since $\varphi < 0$, from (2-31)

$$\int_M \left|\nabla(-\varphi)^{\frac{p}{2}}\right|^2 \le \frac{p^2}{2(p-1)}\left|\int_M (1-\exp\{F\})(-\varphi)^{p-1}\right|$$

$$\le pC_2\int_M |\varphi|^{p-1}$$

Sobolev inequality:

$$\left(\int_M |u|^{p^*} dv_g\right)^{p/p^*} \leq C\left(\int_M |\nabla u|^p dv_g + \int_M |u|^p dv_g\right)$$

$p=2$, $p^* = \frac{np}{n-p} = \frac{4\times 2}{2} = 4$, let $u = |\varphi|^{\frac{p}{2}}$

we have $\left(\int_M |\varphi|^{2p}\right)^{\frac{1}{2}} \leq C_3 \int_M |\varphi|^p + C_3 \int_M |\nabla(\varphi^{\frac{p}{2}})|^2$

$$\leq C_3 \int_M |\varphi|^p + p C_2 C_3 \int_M |\varphi|^{p-1} \quad (2\text{-}33)$$

since $\varphi \leq -1$, from (2-33)

$$\int_M |\varphi|^{2p} \leq C_4 p^2 \left(\int_M |\varphi|^p\right)^2 \quad (2\text{-}34)$$

since we already estimate $\int_M |\varphi|$, in order to use (2-34), we need $\int_M |\varphi|^2$

using (2-32) and Poincare inequality

$$\int_M |\varphi|^2 = \int_M |-\varphi|^2 \leq C_0 \int_M |\nabla(-\varphi)|^2$$

$$\leq 2 C_0 C_2 \int_M |\varphi|$$

we prove there is a C_5 (depending on M), $\int_M |\varphi|^p \leq C_5^p \left(\frac{p}{2}\right)^{\frac{p}{2}}$

p_0 is the first number that for $p \geq p_0$

$$C_4 \left(1+Vol(M)\right)\left(\frac{p+2}{4}\right)^{\frac{p+1}{2}} \leq \left(\frac{p+1}{2}\right)^{\frac{p-3}{2}} \quad (2\text{-}36)$$

$$\int_M |\varphi|^3 \leq C_4 \left(\frac{3}{2}\right)^2 \left(\int_M |\varphi|^{\frac{3}{2}}\right)^2$$

$$= C_4 \left(\frac{3}{2}\right)^2 \left(\int_M |\varphi|^{\frac{1}{2}} |\varphi|\right)^2$$

$$\leq C_4 \left(\frac{3}{2}\right)^2 \int_M |\varphi| \int_M |\varphi|^2$$

$$\int_M |\varphi|^4 \leq C_4\, 2^2 \left(\int_M |\varphi|^2\right)^2$$

$$\int_M |\varphi|^5 \leq C_4 \left(\frac{5}{2}\right)^2 \left(\int_M |\varphi|^{\frac{5}{2}}\right)^2$$

$$= C_4 \left(\frac{5}{2}\right)^2 \left(\int_M |\varphi|^{\frac{1}{2}} |\varphi|^2\right)^2$$

$$\leq C_4 \left(\frac{5}{2}\right)^2 \int_M |\varphi| \int_M |\varphi|^4$$

All come from Hölder inequality

if $\frac{p+1}{2} \in \mathbb{Z}$: $\displaystyle\int_M |\varphi|^{p+1} \leq C_4\left(\frac{p+1}{2}\right)^2 \left(\int_M |\varphi|^{\frac{p+1}{2}}\right)^2$

$$\leq C_4\left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+1}{4}\right)^{\frac{p+1}{2}}$$

$$\leq C_4\left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+1}{4}\right)^{\frac{1}{2}} \times \frac{\left(\frac{p}{2}\right)^{\frac{p-4}{2}}}{C_4(1+Vol(M))}$$

$$\leq C_5^{p+1} \left(\frac{p+1}{2}\right)^{\frac{p+1}{2}}$$

if $\frac{p+1}{2} \notin \mathbb{Z}$: $\displaystyle\int_M |\varphi|^{p+1} \leq C_4\left(\frac{p+1}{2}\right)^2 \left(\int_M |\varphi|^{\frac{p+1}{2}}\right)^2$

$$\leq C_4\left(\frac{p+1}{2}\right) \left[\int_M |\varphi|^{\frac{p+2}{2}}\right]^{\frac{2(p+1)}{p+2}}$$

(since $\displaystyle\int_M ab \leq \left(\int_M a^p\right)^{1/p} \left(\int_M b^q\right)^{1/q}$

$$\left(\int_M |\varphi|^{\frac{p+1}{2}} \cdot 1\right)^2 \leq \left\{\left[\int_M \left(|\varphi|^{\frac{p+1}{2}}\right)^{\frac{p+2}{p+1}}\right]^{\frac{p+1}{p+2}} \left(\int_M 1\right)^{\frac{1}{p+2}}\right\}^2$$

$$\leq C_4\left(\frac{p+1}{2}\right)^2 \left(\int_M |\varphi|^{\frac{p+2}{2}}\right)^{\frac{2(p+1)}{p+2}} Vol(M)^{\frac{2}{p+2}}$$

$$\leq C_4 \left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+2}{4}\right)^{\frac{p+2}{4} \cdot \frac{2(p+1)}{p+2}} \mathrm{Vol}(M)^{\frac{2}{p+2}}$$

$$= C_4 \left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+2}{4}\right)^{\frac{p+1}{2}} \mathrm{Vol}(M)^{\frac{2}{p+2}} \qquad (2\text{-}38)$$

Taking (2-36) in

$$\leq C_4 \left(\frac{p+1}{2}\right)^2 C_5^{p+1} \frac{\left(\frac{p+1}{2}\right)^{\frac{p-3}{2}}}{C_4(1+\mathrm{Vol}\,M)} \mathrm{Vol}(M)^{\frac{2}{p+2}}$$

$$\leq C_5^{p+1} \left(\frac{p+1}{2}\right)^{\frac{p+1}{2}}$$

So, $\int_M |\varphi|^p \leq C_5^p \left(\frac{p}{2}\right)^{\frac{p}{2}}$ exists for all p.

(2-39) expands $\exp\{k\varphi^2\}$: $\int_M \exp\{k\varphi^2\} \leq \sum_{p=0}^{\infty} \frac{k^p}{p!} \int_M |\varphi|^{2p}$

$$\leq \sum_{p=0}^{\infty} \frac{k^p}{p!} (C_5^2)^p p^p$$

Stirling forcular (2-40): $p! \geq \sqrt{2\pi}\, p^{p+\frac{1}{2}} e^{-p}$

(2-41): $\int_M \exp\{k\varphi^2\} \leq \sum_{p=0}^{\infty} (2\pi)^{-\frac{1}{2}} (k C_5^2 e)^p p^{-\frac{1}{2}}$

when $k < C_5^{-2} e^{-1}$, RHS of (2-41) is bonded.

using (2-24) : $0 < m + \Delta\varphi \le C_1 \exp\{C(\varphi - \inf_M \varphi)\}$

rewrite (2-24) as $\Delta\varphi = f$

$$(2.44) \qquad -m \le f \le C_1 \exp\{C \sup_M \varphi\} \exp\{-\inf_M \varphi\}$$

From Schauder estimate

$$\|\varphi\|_{C^{2,\alpha}(M)} \le C'\left(\|f\|_{C^{0,\alpha}(M)} + \|\varphi\|_{0,M}\right)$$

$$\sum_{|k|\le 2} \sup_M |D^k\varphi| + \sup_{\substack{x\neq y \\ M}} \frac{|D^2\varphi(x) - D^2\varphi(y)|}{|x-y|^\alpha}$$

$$\le C'\left(\exp\{-c \inf_M \varphi\} + \sup_M |\varphi|\right)$$

$$(2.45) \quad \sup_M |\Delta\varphi| \le C_6\left(\exp\{-C \inf \varphi\} + \int_M |\varphi|\right)$$

$$(2.46) \quad \sup_M |\Delta\varphi| \le C_7\left(\exp\{-C \inf \varphi\} + 1\right)$$

q_k is the point s.t. $\varphi(q_k) = \inf_M \varphi$. in the ball centered at p.

$-\frac{1}{2}(\inf \varphi) C_7^{-1}\left(\exp\{-c\inf\varphi\} + 1\right)^{-1}$ as radius.

$\varphi \leq \frac{1}{2}\inf_M \varphi$ (mean value theorem)

assme $-\inf_M \varphi$ large. So the radius of the ball is smaller than injective radius. $\exp\{k\varphi^2\}$'s integration in Ball

$$\int_{Ball} \exp\{k\varphi^2\} \geq C_8 \exp\{\tfrac{1}{4}k(\inf \varphi)^2\}\left(-\tfrac{1}{2}\inf \varphi\right)^{2m} C_7^{-2m}$$

$$\left(\exp\{-C\inf\varphi\}+1\right)^{-2m}.$$

(In this ball : $\exp(k\varphi^2) \geq \exp\left(\frac{k}{4}(\inf_M \varphi)^2\right)$)

considering : $\exp\{\frac{1}{4}k(\inf\varphi)^2\}\left(\frac{1}{2}\inf\varphi\right)^4\left(\exp\{-C\inf\varphi\}+1\right)^{-4}$

consider : $\dfrac{\exp\{\frac{1}{4}kx^2\}x^4}{(\exp\{cx\}+1)^4} = h(x)$, $h(0)=0$.

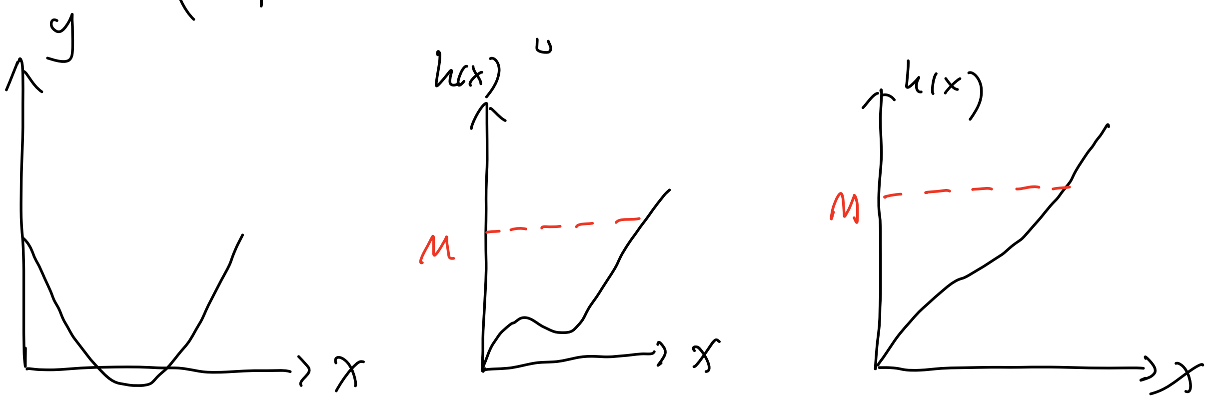
differentiate: $$\frac{\left(\frac{1}{2}kx^5\exp\{\frac{1}{4}kx^2\}+4x^3\exp\{\frac{1}{4}kx^2\}\right)(\exp\{cx\}+1)^4}{(\exp\{cx\}+1)^8}$$

$$-\frac{4c(\exp\{cx\}+1)^3\exp\{cx\}\exp\{\frac{1}{4}kx^2\}x^4}{(\exp\{cx\}+1)^8}$$

$$= \frac{\left(\frac{1}{2}kx^5+4x^3\right)\left(\exp\{cx\}+1\right) - 4c\exp\{cx\}x^4}{\left(\exp\{cx\}+1\right)^8}$$

$\nearrow y$

$$\geq \frac{\frac{1}{2}kx^5 - 4cx^4 + 4x^3}{\left(\exp\{cx\}+1\right)^7} = \frac{x^3\left(\frac{1}{2}kx^2 - 4cx + 4\right)}{\left(\exp\{cx\}+1\right)^7}$$



All get the boundness of x.