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A SEMILINEAR PARTIAL DIFFERENTIAL EQUATION INDUCED BY HERMITIAN YANG-MILLS METRICS *

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Abstract This paper will discuss a class of semilinear partial differential equations induced by studying the limiting behaviour of Hermitian Yang-Mills metrics. We will study the radial symmetry of the C^2 global solution of this equation in $\mathbb{R}^2$ and the existence of $C^{2,\alpha}$ solution of the Dirichlet boundary value problem in any bounded domain.

Key words Hermitian Yang-Mills metric; C^k-estimates; Boundary value problems

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1 Introduction

Let X be a Kähler manifold with a family of Kähler metrics ω_ε, and let V be a slope stable holomorphic vector bundle over X. According to the Donaldson-Uhlenbeck-Yau theorem [5], V admits unique fully irreducible Hermitian-Yang-Mills metrics H_ε associated to each ω_ε. Similar to study the limiting behaviour Ricci flat metrics, Professor Jixiang Fu [2] studied the limiting behaviour of Hermitian Yang-Mills metrics H_ε when ω_ε goes to a large Kähler metric limit. A critical step in [2] is to explicitly construct a family of Hermitian Yang-Mills metrics by solving the following semilinear partial differential equation in unit ball $B_1(0)$ of $\mathbb{R}^2$

$$\begin{cases}\Delta u = \varepsilon^{-2}\left(e^u - \left(x^2+y^2\right)e^{-u}\right) & \text{in } B_1(0),\\ u = 0 & \text{on } \partial B_1(0).\end{cases} \tag{1.1}$$

Here ε is a constant and (x, y) is the coordinate of $\mathbb{R}^2$. By the symmetry of the domain $B_1(0)$ and using reference [3], Jixiang Fu proved the equation (1.1) has a unique radially symmetric solution. However, this method can not be applied to non-symmetric domain Ω in $\mathbb{R}^2$.

In this paper, we first study the following equation defined in a bounded connected domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary value

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$$\begin{cases} \Delta u = \varepsilon^{-2}\left(e^u - \left(x^2+y^2\right)e^{-u}\right) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

For existence and uniqueness of the solution of (1.2), we have the following theorem

Theorem 1.1 If $\partial\Omega$ is $C^{2,\alpha}$ and $g \in C^{2,\alpha}(\partial\Omega)$, there is a unique solution $u \in C^{2,\alpha}(\Omega)$ to equation (1.2) . Especially if $\partial\Omega$ and g is smooth, the solution u is smooth.

This theorem will give a Hermitian Yang-Mills metric on a certain Kähler manifold given by [2]. On the other hand, the equation (1.1) can be defined in whole space $\mathbb{R}^2$. It is natural to explore whether the global solution of (1.1) is radially symmetric. The symmetry of global solutions of some semilinear equations has been investigated in [1] and [3] under the assumption $u(x,y)$ decays to zero at a certain rate as $r^2 = x^2+y^2 \to +\infty$. But they do not fit the equation(1.1) since one can see the global solution u is not bounded. Similar to [1, 3], by using moving plane method and maximum principle, we get the following theorem

Theorem 1.2 For any given constant c, if the global C^2 solution u of

$$\Delta u = \varepsilon^{-2}\left(e^u - \left(x^2+y^2\right)e^{-u}\right) \qquad \text{in } \mathbb{R}^2 \tag{1.3}$$

satisfies

$$u(s) - u(t) \to 0 \quad \text{as} \quad |s|, |t| \to \infty \quad \text{and} \quad |s| - |t| = c,$$

then u is radially symmetric and $\frac{\partial u}{\partial r} \geqslant 0$. Here $s, t \in \mathbb{R}^2$.

One may observe $\frac{1}{2}\log(x^2+y^2)$ is a singular solution to the equation (1.3) and also satisfies $\log(|s|) - \log(|t|) \to 0$ as $|t| - |s| = c$ and $|s|, |t| \to \infty$, so the assumption of Theorem 1.2 is natural and reasonable.

The next part of this paper will give the detailed proof of Theorem 1.1 and Theorem 1.2.

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2 Existence of solution of the Dirichlet boundary value problem

In this section we will prove Theorem 1.1. One can use Chapter 14 in [6] to show the existence of the equation 1.2 by using variational method. Here we take Leray-Schauder existence theorem to prove it.

Let Ω be a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^2$ and $g \in C^{2,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$. We first have a $C^0(\Omega)$ estimate.

Lemma 2.1 Let Φ be the $C^{2,\alpha}$solution of Dirichlet boundary value problem

$$\begin{cases} \Delta\Phi = \varepsilon^{-2}(1 - \left(x^2+y^2\right)) & \text{in } \Omega, \\ \Phi = g & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then a solution u to (1.2) satisfies

$$\sup_{\bar{\Omega}} |u| \leqslant \sup_{\bar{\Omega}} 2|\Phi|. \tag{2.2}$$

Proof The existence of Φ is from Green formula (one can see [4]). Consider $u - \Phi$ for $u > 0$. Note that on $\mathcal{O} = \{x \in \Omega : u(x) > 0\}$

$$\Delta(u - \Phi) = \varepsilon^{-2}(e^u - 1 + \left(x^2+y^2\right)\left(1 - e^{-u}\right)) > 0.$$

By maximum principle, we have

$$\sup_{\mathcal{O}}(u-\Phi)=\sup_{\partial\mathcal{O}}(u-\Phi)\leqslant\sup_{\Omega}\{-\Phi,0\}\leqslant\sup_{\Omega}|\Phi|.$$

It follows

$$\sup_{\Omega}u\leqslant\sup_{\Omega}2|\Phi|. \tag{2.3}$$

Similarly, if $u<0$,

$$\Delta(u-\Phi)=\varepsilon^{-2}(e^{u}-1+\left(x^2+y^2\right)\left(1-e^{-u}\right))<0.$$

Hence we obtain on $\mathcal{O}^-=\{x\in\Omega:u(x)<0\}$

$$\sup_{\mathcal{O}^-}(\Phi-u)=\sup_{\partial\mathcal{O}^-}(\Phi-u)\leqslant\sup_{\Omega}\{\Phi,0\}\leqslant\sup_{\Omega}|\Phi|$$

which implies

$$\sup_{\Omega}-u\leqslant\sup_{\Omega}2|\Phi|. \tag{2.4}$$

Therefore from (2.3) and (2.4) one can get the estimate (2.2).

Second, we give the gradient estimate of u.

Lemma 2.2 Suppose $u\in C^2(\Omega)$ satisfies the equation (1.2) in Ω, then there is positive constant C depend only on Ω and g such that

$$\sup_{\bar{\Omega}}|\nabla u|\leqslant C. \tag{2.5}$$

Proof From the equation (1.2) and by standard regularity, one can see u is C^4 since u and g are C^2. Then we have

$$\begin{aligned}\Delta|\nabla u|^2&=<\nabla\Delta u,\nabla u>+|\nabla^2u|^2\\&=\varepsilon^{-2}(e^{u}+r^2e^{-u})|\nabla u|^2-\varepsilon^{-2}e^{-u}<\nabla r^2,\nabla u>+|\nabla^2 u|^2.\end{aligned} \tag{2.6}$$

If $|\nabla u|^2$ attains its maximum on the boundary $\partial\Omega$, we have $\sup|\nabla u|=\sup|\nabla g|$ which leads to (2.5). Now assume $|\nabla u|^2$ attains its maximum at $z_0\in\Omega$. Then from (2.6), at the point z_0 we have

$$\varepsilon^{-2}(e^{u}+r^2e^{-u})|\nabla u|^2-\varepsilon^{-2}e^{-u}<\nabla r^2,\nabla u>\leqslant 0$$

or

$$(e^{u}+r^2e^{-u})|\nabla u|^2\leqslant e^{-u}|\nabla r^2||\nabla u|.$$

Since $|u|$ is bounded from Lemma 2.1, there is a constant C dependent on Ω and g such that

$$|\nabla u|(z_0)\leqslant C.$$

Then we finish the proof.

Now we give the proof of Theorem 1.1.

Proof Let $\sigma\in[0,1]$, we claim if $u_\sigma\in C^{2,\alpha}(\Omega)$ is the solution of boundary value problem

$$\begin{cases}\Delta u=\sigma\varepsilon^{-2}\left(e^{u}-\left(x^2+y^2\right)e^{-u}\right) & \text{in }\Omega,\\ u=\sigma g & \text{on }\partial\Omega,\end{cases} \tag{2.7}$$

then there is a constant M independent of u_σ and σ such that

$$||u_\sigma||_{C^{1,\alpha}(\bar{\Omega})} \leqslant M. \tag{2.8}$$

Then one can use the Leray-Schauder existence theorem (see Theorem 6.23 in [4]) to show the Dirichlet problem (1.2) is solvable in $C^{2,\alpha}(\bar{\Omega})$.

In fact, one can see $\sigma\Phi$ solves

$$\begin{cases} \Delta\Phi = \sigma\varepsilon^{-2}(1-(x^2+y^2)) & \text{in } \Omega, \\ \Phi = \sigma g & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Then from Lemma 2.1 , we have

$$||u_\sigma||_{C^0(\bar{\Omega})} \leqslant \sup_\Omega 2|\sigma\Phi| \leqslant \sup_\Omega 2|\Phi|. \tag{2.10}$$

Therefore, from (2.7), there is a constant C independent on σ and u_σ, such that

$$|\Delta u| \leqslant C.$$

This means $|\nabla^2 u|$ is also bounded. From Lemma 2.2 and using interpolation inequality in Hölder space, there is a constant M independent on u and σ such that (2.8) is satisfied.

In the end, by standard bootstrap argument of the regularity we have u is smooth if Ω and g are smooth. This finishes the proof.

3 Radial symmetry of the global C^2 solution of the equation in $\mathbb{R}^2$

In this section we will prove Theorem 1.2. In [3] and [1], the radially symmetry of the C^2 positive solutions of the following second order elliptic equation is studied

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^n$$

under the assumption on f and u. For example, they assumed $u(x) \to 0$ as $x \to \infty$. Obviously, our equation (1.2) is different from this type since $e^u - r^2e^{-u}$ has the term r^2. Also we cannot assume $|u| \to 0$ as $r \to +\infty$. In fact, it will lead $\Delta u \to -\infty$ and then u is unbounded. It contradicts the hypothesis. In this paper, we assume for any finite constant c

$$u(s) - u(t) \to 0 \quad \text{and} \quad |s| - |t| = c, \quad \text{as} \quad |s|, |t| \to \infty \tag{3.1}$$

where $s, t \in \mathbb{R}^2$.

Proof **Proof of Theorem1.2** Since the partial differential equation (1.3) is rotationally symmetric, we only have to prove the symmetry about a line across origin. Here we choose the line y axis. Define

$$\Sigma(\lambda) = \left\{(x, y) \in \mathbb{R}^2 \mid x < \lambda\right\}$$

and let

$$v = u(2\lambda - x, y),\ x^\lambda = 2\lambda - x.$$

In $\Sigma(\lambda)$ we define

$$w = v(x) - u(x).$$

When $\lambda = 0$ and $x \in \Sigma(\lambda)$, we have $x + x^\lambda = 0$ and $x < x^\lambda$. Then

$$x^2 = (x^\lambda)^2$$

and

$$\Delta v - \varepsilon^{-2}\left(e^{v} - \left((x^{\lambda})^{2} + y^{2}\right)e^{-v}\right) = \Delta v - \varepsilon^{-2}\left(e^{v} - \left(x^{2} + y^{2}\right)e^{-v}\right) = 0. \tag{3.2}$$

By the mean value theorem, we have

$$\Delta w + \bar{c}w = 0$$

where

$$\bar{c} = -\int_{0}^{1} \varepsilon^{-2}(e^{u+tw} + r^{2}e^{-u-wt})dt < 0.$$

Then from the assumption (3.1) and $w(0,0) = 0$ on the y axis, we have by maximum principle and minimum principle

$$w = 0$$

in $\Sigma(0)$. That's to say the global solution of (1.3) in $\mathbb{R}^2$ is symmetric about y axis.

In the end, assuming $\lambda > 0$ and $x \in \Sigma(\lambda)$, then we have $x + x^{\lambda} > 0$ and $x < x^{\lambda}$. It follows

$$x^{2} < (x^{\lambda})^{2}$$

which implies

$$\Delta v - \varepsilon^{-2}\left(e^{v} - \left(x^{2} + y^{2}\right)e^{-v}\right) < 0. \tag{3.3}$$

Then by mean value theorem, in $\Sigma(\lambda)$

$$\Delta w + \bar{c}w < 0$$

with $c < 0$. Using the infinite boundary condition (3.1) and $w(\lambda, \lambda) = 0$, we have by maximum principle, in $\Sigma(\lambda)$

$$w \geqslant 0.$$

Then if $x > 0$ and let $x_{\lambda} \to x$, we have $\frac{\partial u}{\partial x} \geqslant 0$. Since u is radially symmetric and from

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r}$$

it follows $\frac{\partial u}{\partial r} \geqslant 0$ and we finish the proof.

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