Proposition. μ is a conformal invariant.

$$\int_M \nabla^i \varphi h \nabla_i \varphi h dV = \int_M \left(\varphi \nabla^i h + h \nabla^i \varphi\right)\left(\varphi \nabla_i h + h \nabla_i \varphi\right) dv$$
$$= \int_M \varphi^2 \nabla^i h \nabla_i h + 2\int_M \varphi h \nabla^i h \nabla_i \varphi + \int_M h^2 \nabla^i \varphi \nabla_i \varphi$$
$$\int_M h^2 \nabla^i \varphi \nabla_i \varphi = -\int_M \varphi \nabla^2 \left(h^2 \nabla_i \varphi\right)$$
$$= \int_M \varphi h^2 \Delta \varphi - 2\int_M \varphi h \nabla^i h \nabla_i \varphi$$

$$J(\varphi h) = \frac{\frac{4(n-1)}{n-2}\left(\int_M \varphi^2 \nabla^i h \nabla_i h + \varphi h^2 \Delta \varphi\right) + \int_M R\varphi^2 h^2}{\left(\int_M \varphi^N h^N\right)^{2/N}} d\tilde{V}$$

$$= \varphi^{\frac{2n}{u-2}} dv \int_M \tilde{\nabla}^i \varphi \tilde{\nabla}_i h d\tilde{v}^2 = \int_M \tilde{g}^{ij} \partial_i \varphi \partial_j h \varphi^{\frac{2n}{n-2}} d\tilde{v}$$

$$= \int_M \varphi^2 g^{ij} \partial_i \varphi \partial_{jh} dV$$

And we have

$$\frac{4(n-1)}{n-2} \Delta \varphi + R\varphi = R' \varphi \frac{n+2}{n-2}$$

So we have

$$J(\varphi h) = \frac{\int_M R' \varphi^{\frac{n+2}{n-2}} \varphi h^2 + \frac{4(n-1)}{n-2} \int_n \varphi^2 \nabla^i h \nabla_i h}{\left(\int_M \varphi^N h^N\right)^{2/N}}$$
$$= \frac{\frac{4(n-1)}{n-2} \int_M \nabla^2 h \nabla_i h d\tilde{v} + \int_M h^2 d\tilde{v}}{\left(\int_M h^N d\tilde{v}\right)^{2/N}}$$

Then $J(\varphi h) = \tilde{J}(h)$ and consequently $\mu = \tilde{\mu}$. $\tilde{J}$ is the functional related to $\tilde{g}$, and $\tilde{\mu}$ the inf of $\tilde{J}$.

To solve Yamabe Problem, Yamabe used the variational method. He considered the functional, for $2 \le q \le N = 2n/(n-2)$

$$J_q(\varphi) = \left[4\frac{n-1}{n-2} \int_M \nabla^i \varphi \nabla_i \varphi dV + \int_M R(x) \varphi^2 dV\right] \|\varphi\|_q^{-2}$$

and defined $\mu_q = \inf J_q(\varphi)$ for all $\varphi \ge 0, \varphi \not\equiv 0$, belonging to H_1. Set $\mu = \mu_N$ and $J(\varphi) = J_N(\varphi)$.

Theorem Let M_n be a C^∞ compact Riemannian manifold; there exists a conformal metric whose scalar curvature is either a nonpositive constant or is everywhere positive.

Proof. α) the positive case $(\mu > 0)$. If $\mu_{q_0} > 0, \mu_q$ is positive for $q \in]\,2, N[$. Indeed

$$\mu_q = J_q(\varphi_q) = J_{q_0}(\varphi_q) \left\|\varphi_q\right\|_{q_0}^2 \left\|\varphi_q\right\|_q^{-2} \geq \mu_{q_0} \left\|\varphi_q\right\|_{q_0}^2$$

Then consider the conformal metric $g' = \varphi_{q_0}^{4/(n-2)} g$;

$$\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi = R'\varphi^{\frac{n+2}{n-2}}$$

$$\frac{4(n-1)}{n-2}\Delta\varphi_q + R\varphi_q = \mu_q\varphi_q^{q-1}, \left\|\varphi_q\right\|_q = 1$$

$$R'(x) = \mu_{q_0}\varphi_{q_0}^{q_0-N}(x),$$

the scalar curvature R' is everywhere strictly positive. Moreover we can prove that $\mu > 0$. Indeed the functional J' corresponding to g' satisfies

$$J'(\psi) \geq \inf_{x\in M}\left[4\frac{n-1}{n-2}, R'(x)\right]\left[\int_M g'ij\nabla_i\psi\nabla_j\psi dV' + \int_M \psi^2 dV'\right]$$
$$\times\left[\int \psi^N dV'\right]^{-2/N}.$$

Since we have the Sobolev imbedding theorem which holds for M_n a complete manifold with bounded curvature and injectivity radius $\delta > 0$. Moreover, for any $\varepsilon > 0$, there exists a constant $A_q(\varepsilon)$ such that every $\varphi \in H_1^q(M_n)$ satisfies: $\|\varphi\|_p \leq [K(n,q)+\varepsilon]\|\nabla\varphi\|_q + A_q(\varepsilon)\|\varphi\|_q$, with $1/p = 1/q - 1/n > 0$, where $\mathrm{K}(n,q)$ is the smallest constant having this property.

And we have $\frac{1}{N} = \frac{1}{2} - \frac{1}{n} = \frac{n-2}{2n} > 0$, so according to the Sobolev imbedding theorem, $J'(\psi) \geq$ Const > 0 for all $\psi \in \mathrm{H}_1$. Thus $\mu' > 0$ and we have $\mu = \mu'$.

β) The null case $(\mu = 0)$. If $\mu_{q_0} = 0$, by (11) the scalar curvature R' vanishes, and $\mu_q = 0$ for all $q \in]\,2, N]$, because for all ψ and q, $J_q(\psi) \geq 0$.

(γ) The negative case $(\mu < 0)$. If $\mu_{q_0} < 0$ there exists a $\psi \in C^\infty$ such that $J_{q_0}(\psi) < 0$. Hence $J_q(\psi) < 0$ for all $q \in [2, N]$ and $\mu_q < 0$. In particular, $\mu < 0$. Moreover, because the volume is 1 we use holder inequality to get that $\mu_q \leq J_q(\psi) = J(\psi)\|\psi\|_N^2\|\psi\|_q^{-2} \leq J(\psi)$. Thus $\mu_q(q \in [2, N])$ is bounded away from zero.

Now we are able to prove very simply that the functions $\varphi_q(q \in]q_0, N[)$ are uniformly bounded with $q_0 \in]\,2, N\,[$. At a point P where φ_q is maximum $\Delta\varphi_q \geq 0$, hence $\mu_q\varphi_q^{q-1}(P) \geq R(P)\varphi_q(P)$. We find at once that $\varphi_q^{q-2} \leq |\inf R||J(\psi)|^{-1}$ and $\varphi_q \leq 1 + \left[|\inf R||J(\psi)|^{-1}\right]^{1/(q_0-2)}$. By Green function, φ_q satisfies

(1)

$$\varphi_q(P) = \int_M \varphi_q(Q)dV(Q) + \int_M \mathrm{G}(P,Q)\frac{n-2}{4(n-1)}\left[\mu_q\varphi_q^{q-1}(Q) - R(Q)\varphi_q(Q)\right]dV(Q)$$

Differentiating (1) yields $\varphi_q \in C^1$ uniformly, and according to Ascoli's theorem, it is possible to exhibit a sequence φ_{q_i} with $q_i \to N$, such that φ_{q_i} converges uniformly to a nonnegative function φ_N. But $0 > \mu_q \geq \inf R(x)\left\|\varphi_q\right\|_2^2 \geq \inf R(x)$. Therefore a subsequence μ_{q_i} converges to a real number ν (in fact μ_q is a continuous function of q for $q \in]\,2, N]$ by Proposition 5.10, so $\mu = \nu$). Letting $q_i \to N$ in (1), shows that φ_N is a weak solution of

$$4\frac{n-1}{n-2}\Delta\varphi_N + R\varphi_N = \nu\varphi_N^{N-1} \tag{2}$$

Since $\|\varphi_q\|_q = 1, \|\varphi_N\|_N = 1$. Multiplying (2) by φ_N and integrating yield $J(\varphi_N) = \nu$.

The second term in (2) is continuous; thus, by (1), $\varphi_N \in C^1$. Now apply the regularity theorem 3.54: the second member of (13) is C^1; thus $\varphi_N \in C^2$. Now according to Proposition 3.75φ_N is strictly positive everywhere, since $\|\varphi_N\|_N = 1$ implies $\varphi_N \not\equiv 0$. We can use the regularity theorem again to prove by induction that $\varphi_N \in C^\infty$. Thus the C^∞ function $\varphi_N > 0$ satisfies

$$4((n-1)/(n-2))\Delta\varphi + R\varphi = R'\varphi^{(n+2)/(n-2)}$$

with $R' =$ Const (in fact, $R' = \mu$). In the negative case it is therefore possible to make the scalar curvature constant and negative.

Positive case Definition Recall $\mu = \inf J(\varphi)$ for all $\varphi \in H_1, \varphi \neq 0, J(\varphi)$ being the Yamabe functional.

We have the basic theorem

Theorem 5.11 $\mu \leq n(n-1)\omega_n^{2/n}$. If $\mu < n(n-1)\omega_n^{2/n}$, there exists a strictly positive solution $\tilde{\varphi} \in C^\infty$ of (1) with $\tilde{R} = \mu$ and $\|\tilde{\varphi}\|_N = 1$. Here $\tilde{R}$ is the scalar curvature of $(M_n, \tilde{g})$ with $\tilde{g} = \tilde{\varphi}^{4/(n-2)}g$ and ω_n is the volume of the unit sphere of radius 1 and dimension n.

Proof. (α)First we need sobolev imbedding theorem **Theorem (Aubin [13] or [17], see also Talenti** (257) **)** If $1 \leq q < n$, all $\varphi \in H_1^q(\mathbb{R}^n)$ satisfy: (6)

$$\|\varphi\|_p \leq \mathrm{K}(n,q)\|\nabla\varphi\|_q,$$

with $1/p = 1/q - 1/n$ and

$$K(n,q) = \frac{q-1}{n-q}\left[\frac{n-q}{n(q-1)}\right]^{1,q}\left[\frac{\Gamma(n+1)}{\Gamma(n/q)\Gamma(n+1-n/q)\omega_{n-1}}\right]^{1/n}$$

for $1 < q < n$, and

$$\mathrm{K}(n,1) = \frac{1}{n}\left[\frac{n}{\omega_{n-1}}\right]^{1/n}.$$

$\mathrm{K}(n,q)$ is the norm of the imbedding $H_1^q \subset L_p$, and it is attained by the functions

$$\varphi(x) = \left(\lambda + \|x\|^{q/(q-1)}\right)^{1-n/q},$$

where λ is any positive real number.

And we need

Theorem2.21 The Sobolew imbedding theorem holds for M_n a complete manifold with bounded curvature and injectivity radius $\delta > 0$. Moreover, for any $\varepsilon > 0$, there exists a constant $A_q(\varepsilon)$ such that every $\varphi \in H_1^q(M_n)$ satisfies:

$$\|\varphi\|_p \leq [K(n,q)+\varepsilon]\|\nabla\varphi\|_q + A_q(\varepsilon)\|\varphi\|_q$$

, with $1/p = 1/q - 1/n > 0$, where $\mathrm{K}(n,q)$ is the smallest constant having this property.

So

$$K(n,2) = 2\left(\omega_n\right)^{-1/n}[n(n-2)]^{-1/2}$$

is the best constant in the Sobolev inequality. By theorem (2.21), the best constant is the same for all compact manifolds. Thus by taking ψ_i as the type of $\varphi(x) = \left(\lambda + \|x\|^{q/(q-1)}\right)^{1-n/q}$, there exists a sequence of C^∞ functions ψ_i such that

$$\|\psi_i\|_N = 1, \|\psi_i\|_2 \to 0 \text{ and } \|\nabla\psi_i\|_2 \to K^{-1}(n,2)$$

when $i \to +\infty$. Therefore $J(\psi_i) \to n(n-1)\omega_n^{2/n}$ and $\mu \leq n(n-1)\omega_n^{2/n}$

(β) Let us again consider the set of functions $\varphi_q (q \in]2, N[)$ which are solutions of

$$4\frac{n-1}{n-2}\Delta\varphi_q + R\varphi_q = \mu_q\varphi_q^{q-1}$$

and $\left\|\varphi_q\right\|_q = 1$. This set is bounded in H_1 since we have $\left\|\varphi_q\right\|_2 \leq 1$ and

$$4\frac{n-1}{n-2}\left\|\nabla\varphi_q\right\|_2^2 \leq \mu_q + \sup|R| \leq \int R dV + \sup|R|.$$

Therefore there exists $\varphi_0 \in \mathrm{H}_1$ and a sequence $q_i \to N$ such that $\varphi_{q_i} \to \varphi_0$ weakly in H_1 (the unit ball in H_1 is weakly compact), strongly in L_2 (Kondrakov's theorem) and almost everywhere (Proposition 3.43). The weak limit in H_1 is the same as that in L_2 because H_1 is continuously imbedded in L_2, and strong convergence implies weak convergence.

(γ)First we need **Theorem 3.45** Let $1 < p < \infty$ and $\{f_k\}$ be a bounded sequence in $L_p(\mathscr{H})$, converging pointwise almost everywhere to f. Then f belongs to L_p and f_k converges to f weakly in L_p. The result does not hold for $p = 1$, of course (see below).

γ) Since φ_{q_i} satisfies for all $\psi \in \mathrm{H}_1$:

$$4\frac{n-1}{n-2}\int \nabla^\nu\psi\nabla_\nu\varphi_{q_i} dV + \int R\psi\varphi_{q_i} dV = \mu_{q_i}\int \psi\varphi_{q_i}^{q_i-1} dV$$

Letting $q_i \to N$ gives us

$$4\frac{n-1}{n-2}\int \nabla^\nu\psi\nabla_\nu\varphi_0 dV + \int R\psi\varphi_0 dV = \mu\int \psi\varphi_0^{N-1} dV \tag{3}$$

Indeed, according to Theorem 3.45, $\varphi_{q_i}^{q_i-1}$ converges weakly to φ_0^{N-1} in $L_{N/(N-1)}$ since $\varphi_{q_i}^{q_i-1} \to \varphi_0^{N-1}$ almost everywhere and

$$\left\|\varphi_{q_i}^{q_i-1}\right\|_{N/(N-1)} = \|\varphi_{q_i}\|_{(q_i-1)N/(N-1)}^{q_i-1} \leq \|\varphi_{q_i}\|_N^{q_i-1} \leq \text{ Const}$$
$$\times \|\varphi_{q_i}\|_{H_1}^{q_i-1} \leq \text{ Const}$$

Therefore φ_0 satisfies (3) for all $\psi \in \mathrm{H}_1 \subset L_N$. Then we need Trudinger's theorem ([262] p.271) $\varphi_0 \in C^\infty$ and satisfies (1) with $R' = \mu$. **Trudinger's theorem** Let u be a $W_2^1(M)$ solution of $\frac{4(n-1)}{n-2}\Delta u - Ru = -\bar{R}u^{\frac{n+2}{n-2}}$. Then $u \in C^\infty(M)$.

The function u satisfies

$$(4) \qquad \int_M \left(\frac{4(n-1)}{n-2} g^{ij} u_i \xi_j + Ru\xi\right) dV = \bar{R}\int_M |u|^{N-1}\xi dV$$

for all $\xi \in W_2^1(M)$, then construct a test function.

Define $\bar{u} = \sup(u, 0)$ and for a fixed $\beta > 1$define the functions

$$G(\bar{u}) = \begin{cases} \bar{u}^\beta & \text{if } \bar{u} \leq l \\ l^{q-1}\left(ql^{q-1}\bar{u} - (q-1)l^q\right) & \text{if } \bar{u} > l \end{cases}$$

$$F(\bar{u}) = \begin{cases} \bar{u}^q & \text{if } \bar{u} \leq l \\ ql^{q-1}\bar{u} - (q-1)l^q & \text{if } \bar{u} > l \end{cases}$$

where $2q = \beta + 1$.

The function $G(\bar{u})$ is a uniformly Lipshitz continuous function of u and hence belongs to $W_2^1(M)$. Likeuise $F(\bar{u})$. Observe also that G and F vanish for negative u and that

$$\left(F'(\bar{u})\right)^2 \leq qG'(\bar{u}), \quad (F(\bar{u}))^2 \geq \bar{u}G(\bar{u}).$$

Let us now substitute in (**??**) test functions

$$\xi = \eta^2 G(\bar{u})$$

Since we have

(5)

$$\frac{4(n-1)\mu}{n-2}\int_M \eta^2 G'(\bar{u})u_i^2 dV = \int_M \bar{R}|u|^{N-1}\eta^2 G(\bar{u}) - \frac{4(n-1)}{n-2} g^{ij}u_i\left(2\eta\eta_j G(\bar{u})\right) - Ru\eta^2 G(\bar{u})dV$$

And since M is compact we have

$$-\int_M \eta u_i G = \int_M u\left(\eta_i G + \eta G_i\right) = \int_M u\eta_i G + \int_M u\eta G_i$$

then I compute the last term

$$\int_M u\eta G_i = \int_M u\eta G'(\bar{u})u_i \leqslant \int_M \left(\frac{u^2 + u_i^2}{2}\right)\eta G'(\bar{u})$$

then put the $\int_M \frac{u_i^2}{2}\eta G'(\bar{u})$ to the LHS of (5).

And notice that

$$G'(\bar{u}) = \begin{cases} \beta\bar{u}^{\beta-1} & \text{if } \bar{u} \leq l \\ l^{q-1}\left(ql^{q-1}\right) & \text{if } \bar{u} > l \end{cases}$$

So we have

$$\begin{cases} \frac{\bar{u}G'(\bar{u})}{\beta} = G(\bar{u}) & \bar{u} \leqslant l \\ \bar{u}G'(\bar{u}) - C_0 = G(\bar{u}) & \bar{u} > l \end{cases}$$

so we can transform $u^2\eta G'(\bar{u})$, and hence

$$\int_{\dot{m}} \eta^2 G'(\bar{u}) u_i^2 dv \leq C \int_M \left\{ \left(|\eta_i|^2 + \eta^2 \right) (\bar{u}) G(\bar{u}) \right) + \eta^2 \bar{u}^{N-2} \bar{u} G(\bar{u}) \} dv$$

where $C = C\left(v, n, \sup \left|g^{ij}, R\right|, \bar{R}\right)$.

We then obtain

$$\int_M \eta^2 F_i^2 dv \leq Cq \int_M \left\{ \left(|\eta_i|^2 + \eta^2 \right) F^2 + \bar{u}^{N-2} \eta^2 F^2 \right\} dV \tag{6}$$

Let us take η now to have compact support in a coordinate patch of M. The integrals in (6) may then be replaced by integrals over a sphere S_R in E^n of radius R where $\eta = \eta(x) \in C_0^1(R)$. We choose R so that

$$\int_{s_R} |u|^N dv \leq (2Cq)^{-1}$$

Then applying the Holder and Sobolev inequalities to (6) we obtain

$$\|\eta F\|_{L_N(S)} \leq Cq \left\| (\eta + |\eta_i|) F \right\|_{L_2(S)} + \frac{1}{2} \|\eta F\|_{L_{N(}(S)}$$

and hence

$$\|\eta F\|_{L_N(S)} \leq 2Cq \left\| (\eta + |\eta_i|) F \right\|_{L_2(S)} \tag{7}$$

We choose β as $1 < \beta < (n+2)/(n-2)$ so that $2q < N$. Hence we may let $l \to \infty$ in to obtain the estimate

$$\|\eta \bar{u}^q\|_{L_N(S)} \leq C \left\| (\eta + |\eta_i|) \bar{u}^q \right\|_{L_2(S)}.$$

Iet $S_{R/2}$ denote the sphere concentric to S of radius $R/2$ and choose $\eta = 1$ on $S_{R/2}, |\eta_i| \leq 2/R$ on S_R. Then we obtain

$$\|\bar{u}^q\|_{L_N\left(S_{R/2}\right)} \leq C \left(1 + R^{-1}\right)$$

Replacing u by $-u$, we obtain (24) also for the function u and employing a partition of unity clearly provides a global estimate

$$\|u\|_{L^r(M)} \leq C$$

for some $r > N$ where C will also depend on the local L_N norms of u. The boundedness of u and subsequently its smoothness are now conseqnences of the Lemma which said that let u be a weak $\left(W_2^1(M)\right)$, non negative, solution in M of the linear equation

$$\Delta u + fu = 0$$

where $f \in L_r(M), r > n/2$. Then u is positive and bounded and we have the estimates

$$\sup_M |u|, |u|^{-1} \leq C \left(n, \|f\|_{L_r(M)}, \mu, \|u\|_{L_2(M)}\right)$$

(δ)The problem is not solved yet because the maximum principle implies that either $\varphi_0 > 0$ everywhere or $\varphi_0 \equiv 0$, and for the moment we cannot exclude the latter case. In order to prove that φ_0 is not identically zero, we must use Theorem 2.21. We write, using (10),
(8)

$$1 = \|\varphi_q\|_q^2 \leq \|\varphi_q\|_N^2 \leq \left[\mathrm{K}^2(n,2) + \varepsilon\right] \frac{n-2}{4(n-1)} \left[\mu_q - \int R\varphi_q^2 dV\right] + A(\varepsilon) \|\varphi_q\|_2^2$$

where $\varepsilon > 0$ is arbitrary and $A(\varepsilon)$ is a constant which depends on ε. When $\mu < n(n-1)\omega_n^{2/n}$, if we choose ε small enough, there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for $N - q < \eta$,

$$0 < \varepsilon_0 \leq 1 - \left[\frac{\omega_n^{-2/n}}{n(n-1)} + \varepsilon \frac{n-2}{4(n-1)}\right] \mu_q \tag{9}$$

since $\mu_q \to \mu$ when $q \to N$.

In this case, (8) and (9) imply

$$\liminf_{q \to N} \|\varphi_q\|_2 \geq \text{ Const } > 0$$

Because φ_{q_i} converges strongly to φ_0 in $L_2, \|\varphi_0\|_2 \neq 0$. Thus $\varphi_0 \not\equiv 0$ and $\varphi_0 > 0$. Picking $\psi = \varphi_0$ in (14) gives $J(\varphi_0) = \mu \|\varphi_0\|_N^{N-2}$; thus $\|\varphi_0\|_N \geq 1$ since $J(\varphi_0) \geq \mu$. But since the sequence $\varphi_{q_i}^{q_i/N}$ of (β) converges weakly to φ_0 in L_N by Theorem 3.45(3.45 Theorem. Let $1 < p < \infty$ and $\{f_k\}$ be a bounded sequence in $L_p(\mathscr{H})$, converging pointwise almost everywhere to f. Then f belongs to L_p and f_k converges to f weakly in L_p. The result does not hold for $p = 1$, of course .)

$\|\varphi_0\|_N \leq \liminf_{q_i \to N} \|\varphi_{q_i}\|_{q_i}^{q_i/N}$ by Theorem 3.17(3.17 Theorem. A weakly convergent sequence $\{x_i\}$ in a normed space $\mathbb{F}$ has *a* unique limit x, is bounded, and

$$\|x\| \leq \lim_{i \to \infty} \inf \|x_i\|)$$

Hence $\|\varphi_0\|_N = 1$ and $J(\varphi_0) = \mu$. $\square$

1. Appendix

THEOREM 2. There exists a positive constant ε (depending on g^{ij}, R) such that if $\lambda < \varepsilon$, there exists a positive, C^∞ solution of equation

$$\frac{4(n-1)}{n-2} \Delta u - Ru = -\bar{R} u^{\frac{n+2}{n-2}}$$

with $\bar{R} = \lambda$. Thus Yamabe's theorem is true under this assumption on the metric g^{ij}.

Proof. In

$$\int_M \left(\frac{4(n-1)}{n-2} g^{ij} u_{qx_i} \xi_j + Ru_q \xi\right) dV = \lambda_q \int_M u_q^{q-1} \xi dV$$

consider the test function

$$\xi(x) = (u_q)^\beta \qquad \beta > 1$$

The result is

$$\int_M \left(\frac{4\beta(n-1)}{n-2} g^{ij} u_{qx_i} u_{qx_j} (u_q)^{\beta-1} + R u_q \xi \right) dV = \lambda_q \int_M (u_q)^{\beta-1+q} \, dV$$

then using elliptic condition

$$\int_M u_q^{\beta-1} |\nabla u_q|^2 \, dV \leq \frac{n-2}{4(n-1)\mu\beta} \int_M \left(\lambda_q u_q^{\beta-1+q} - R u_q^{\beta+1} \right) dV$$

Writing $w = (u_q)^{(\beta+1)/2}$ the above inequality becomes

$$\int_M |\nabla w|^2 dv \leq C(\mu, n, \beta) \int \left(\lambda_q w^2 (u_q)^{q-2} - R w^2 \right) dV$$

Let us suppose $\lambda > 0$ and apply the Sobolev and Holder inequalities. We obtain, thus

$$\|w\|^2_{L_N(M)} \leq C_1(\mu, n, \beta) \lambda_q \|w\|^2_{L_N(M)} \|u_q\|^{q-2}_{L_N(M)} + C_2(\mu, n, \beta, \sup |R|) \int_M w^2 dV$$

$$\leq C_1 \lambda_q \|w\|^2_{L_N(M)} + C_2 \int_M w^2 dV$$

since $\|u_q\|_{L_N(M)}$, is bounded independently of q. Hence if $\lambda < C_1^{-1}, \lambda_q < C_1^{-1}$ for large enough q and we obtain

$$\|w\|^2_{L_N(M)} \leq C \int_M w^2 dV$$

Now choose $\beta < N - 1$. Then we obtain

$$\|w\|_{L_N(M)} \leq C(\mu, n, R)$$

using the **lemma** that let u be a weak $\left(W_2^1(M)\right)$, non negative, solution in M of the linear equation

$$\Delta u + f u = 0$$

where $f \in L_r(M), r > n/2$. Then u is positive and bounded and we have the estimates

$$\sup_M |u|, |u|^{-1} \leq C \left(n, \|f\|_{L_r(M)}, \mu, \|u\|_{L_2(M)} \right)$$

and the u_q are subsequently equibounded by the Lemma. A subsequence therefore converges, with its derivatives, uniformly to a smooth solution, which is also positive by the Lemma. □