4.1

a) b) are in the zip file.

c)

Best - case, is dependent on the time complexity of insertion sort. For different increasing k values, it requires less time. When k becomes large insertion sort is applied, which has $\Theta(n)$.

Average case is dependendent on both merge & insertion sort, and on the k.

So, $\Theta(n) = \left[\frac{n}{k}\log\frac{n}{k} + \left(\frac{n}{k}\right)^2\right]$

Worst case is dependent on insertion sort because as array size gets too large $n^2 > n\log n$, hence $\Theta(n) = n^2$.

4.2

a) $T(n) = 36\,T(\frac{n}{6}) + 2n$

$a = 36,\ b = 6$

$n^{\log_b a} = n^2$

$f(n) = 2n.$

$\therefore f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

Since, $\varepsilon > 0$, $T(n) = \Theta(n^2)$

b) $T(n) = 5\,T(\frac{n}{3}) + 17n^{1.2}$

$a = 5,\ b = 3$

$n^{\log_3 5} = n^{1.46}$, $f(n) = n^{1.2}$

$\log_3 5 - \varepsilon = 1.2$

$\varepsilon = 0.26$

$f(n) = O(n^{\log_3 5 - \varepsilon})$

$\therefore T(n) = \Theta(n^{1.46})$.

c). $T(n) = 12\,T(\frac{n}{2}) + n^2 \log n.$

$a = 12,\ b = 2.$, $f(n) = n^2 \log n$

$n^{\log_2 12} = 3.58$

Next, case I:

$n^{3.58a} > n^2 \log n$ will be greater for some value.

$\therefore T(n) = \Theta(n^{3.58a})$.

d) $T(n) = 3T\left(\frac{n}{5}\right) + T\left(\frac{n}{2}\right) + \cancel{n}\, 2^n$

2^n — 2^n

$2^{\frac{n}{5}}$ $\quad 2^{\frac{n}{2}}$ — $\cancel{6^n\{2^{\frac{n}{5}} + 2^{\frac{n}{2}}\}}$ $2^{\frac{n}{2}+\frac{n}{5}}$

$2^{\frac{n}{25}}$ $\quad 2^{n/10}$ $\quad 2^{\frac{n}{10}}$ $\quad 2^{\frac{n}{4}}$. — $2^{\frac{n}{16}} + 2^{\frac{n}{8}+1} + 2^{\frac{n}{4}}$

$2^n + 2^{\frac{n}{2}} + 2^{\frac{n}{4}} + 2^{\frac{n}{16}} + 2^{\frac{n}{8}+1} + 2^{\frac{n}{4}}$

$\Rightarrow 2^n + (2^n)^{1/2} + 2(2^n)^{1/4} + (2^n)^{1/16}$

$+ (2^{n/8}) + \dots]$

$\theta(1)$.

Lower bound:

$2^n, 2^{\frac{n}{5}}, 2^{\frac{n}{25}} \cdot \cancel{(2^{n})} = (2^n)^{1/5^i}$

$3 = (2^n)^{1/5^i}$

$\Rightarrow i = \log_5\left(n \cdot \frac{\ln 2}{\ln 3}\right)$.

Next,

$cn\left(1 + \log_5 n \frac{\ln 2}{\ln 3}\right) \Rightarrow \Omega(n) = \Omega(\log_5 n)$

Upper bound

$(2^n)^{1/2^i} = 3$

$\Rightarrow i = \log_2\left(n\left(\frac{\ln 2}{\ln 3}\right)\right)$

Next,

$cn(1 + \log_2 n \frac{\ln 2}{\ln 3})$

$O(n) = n\log_2 n$.

Since bounds only differ by constant value

$\Theta(n) = n\log n$.

e) $T(n) = T\left(\frac{2n}{5}\right) + T\left(\frac{3n}{5}\right) + \Theta(n)$.

n — n

$\frac{2}{5}n$ $\quad \frac{3}{5}n$. — n.

$\frac{4}{25}n$ $\quad \frac{6}{25}n$ $\quad \frac{6}{25}n$ $\quad \frac{9}{25}n$ — n.

$n + n + n$

Lower bound: $\cancel{\frac{2}{5}n + 4\left(\frac{2}{5}\right)}$ (left path)

$n, \left(\frac{2}{5}\right)n, \left(\frac{2}{5}\right)^2 n, \dots, 1 = n\left(\frac{2}{5}\right)^i$

$1 = n\left(\frac{2}{5}\right)^i$

$n = \left(\frac{5}{2}\right)^i \Rightarrow i = \log_{\frac{5}{2}} n$.

upper bound: (right path)

$$n\left(\frac{3}{5}\right)^i = 1.$$

$$i = \log_{\frac{5}{3}} n.$$

~~Since, the~~ So, there are $1+\log_{\frac{5}{3}} n$ levels. (1 is for the top)

and each level will at most cn time.

$$\underbrace{cn + cn + \cdots + cn}_{1+\log_{\frac{5}{3}} n \text{ times}}$$

$$= cn\left(1+\log_{\frac{5}{3}} n\right)$$

$$= cn + cn\,\frac{\log n}{\log \frac{5}{3}}$$

~~which is~~

Ignoring lower order terms (cn) and constants $\left(\frac{c}{\log\frac{5}{3}}\right)$, we get ~~$\Theta(n$~~ upper bound = $n\log_{\frac{5}{3}} n$.

Same is applied for lower bound = $n\log_{\frac{5}{2}} n$.

Since, bounds only differ by constant value

$$\Theta(n) = n\log n$$