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Problem 1:

Consider the data set in the file speed.txt, available on Moodle as an attachment to the Final Project. It contains the times in seconds recorded by the winners in the finals of various men's running events (200, 400, 800 and 1500 metres) at each of the 21 Olympic Games from 1900 to 2000, along with the heights above sea level of the different venues.

Question 1:

Read the data and store it in a data frame called speed.data. Print the first 5 rows of speed.data.

Answer:

```r
# Read the data from file.txt
speed.data <- read.table("speed.txt", header = TRUE, sep = "\t")

# Display the data
print(head(speed.data , 5))
```

```
##   Year Distance.100 Time Altitude
## 1 1900            2 22.2       25
## 2 1904            2 21.6      455
## 3 1908            2 22.4        8
## 4 1912            2 21.7       46
## 5 1920            2 22.0        3
```

Question 2:

Calculate the average speed (in meters per second) of the winner of each race, and add this data as a new column of the data frame speed.data. Name this new column Speed. Print the first 5 rows of the modified data frame.

Answer:

```r
# Calculate speed (meters per second)
speed.data$Speed <- (speed.data$Distance * 100) / (speed.data$Time)

# Print the first 5 rows of the modified data frame
print(head(speed.data, 5))
```

```
##   Year Distance.100 Time Altitude    Speed
## 1 1900            2 22.2       25 9.009009
## 2 1904            2 21.6      455 9.259259
## 3 1908            2 22.4        8 8.928571
## 4 1912            2 21.7       46 9.216590
## 5 1920            2 22.0        3 9.090909
```

Question 3:

Sort the data by increasing value of year. Print the first 10 rows of the sorted data frame.

Answer:

```r
# Sort the data frame by increasing value of year
sorted_data <- speed.data[order(speed.data$Year), ]

# Print the first 10 rows of the sorted data frame
print(head(sorted_data, 10))
```

```
##    Year Distance.100  Time Altitude    Speed
## 1  1900            2  22.2       25 9.009009
## 24 1900            4  49.4       25 8.097166
## 47 1900            8 121.4       25 6.589786
## 70 1900           15 246.0       25 6.097561
## 2  1904            2  21.6      455 9.259259
## 25 1904            4  49.2      455 8.130081
## 48 1904            8 116.0      455 6.896552
## 71 1904           15 245.4      455 6.112469
## 3  1908            2  22.4        8 8.928571
## 26 1908            4  50.0        8 8.000000
```

Question 4:

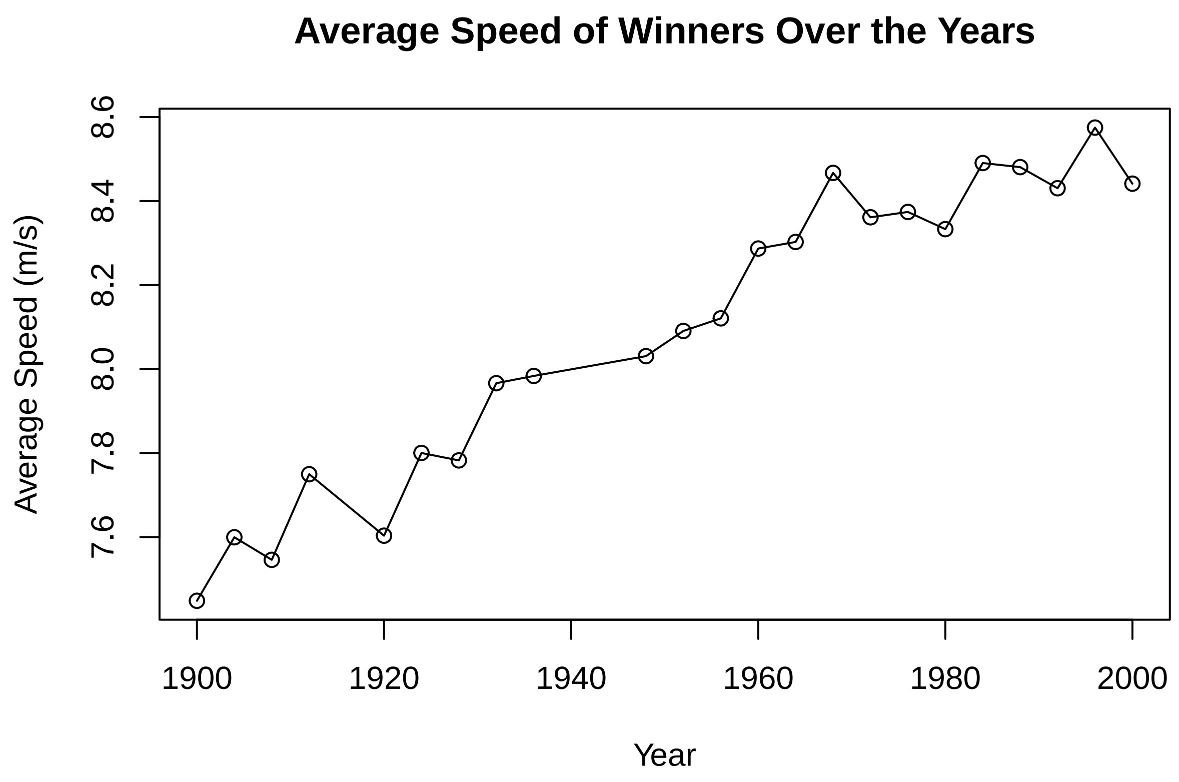
Create a new data frame called speed.year containing two columns: the year (called Year) and the average speed among all winners of that year (called Speed). Plot the speed as a function of the year for the new data frame speed. year. What do you obeserve?

Answer:

```r
# Calculate average speed for each year
average_speed <- aggregate(Speed ~ Year, data = speed.data, FUN = mean)

# Create new data frame
speed.year <- data.frame(Year = average_speed$Year, Speed = average_speed$Speed)

# Plot speed as a function of year
plot(speed.year$Year, speed.year$Speed, type = "o",
     xlab = "Year", ylab = "Average Speed (m/s)",
     main = "Average Speed of Winners Over the Years")
```



I see an upward raising trend.This means that the average speed for each year tends to increase from the last year. However, there are some datas where the average speed decreased from the last year.

Question 5:

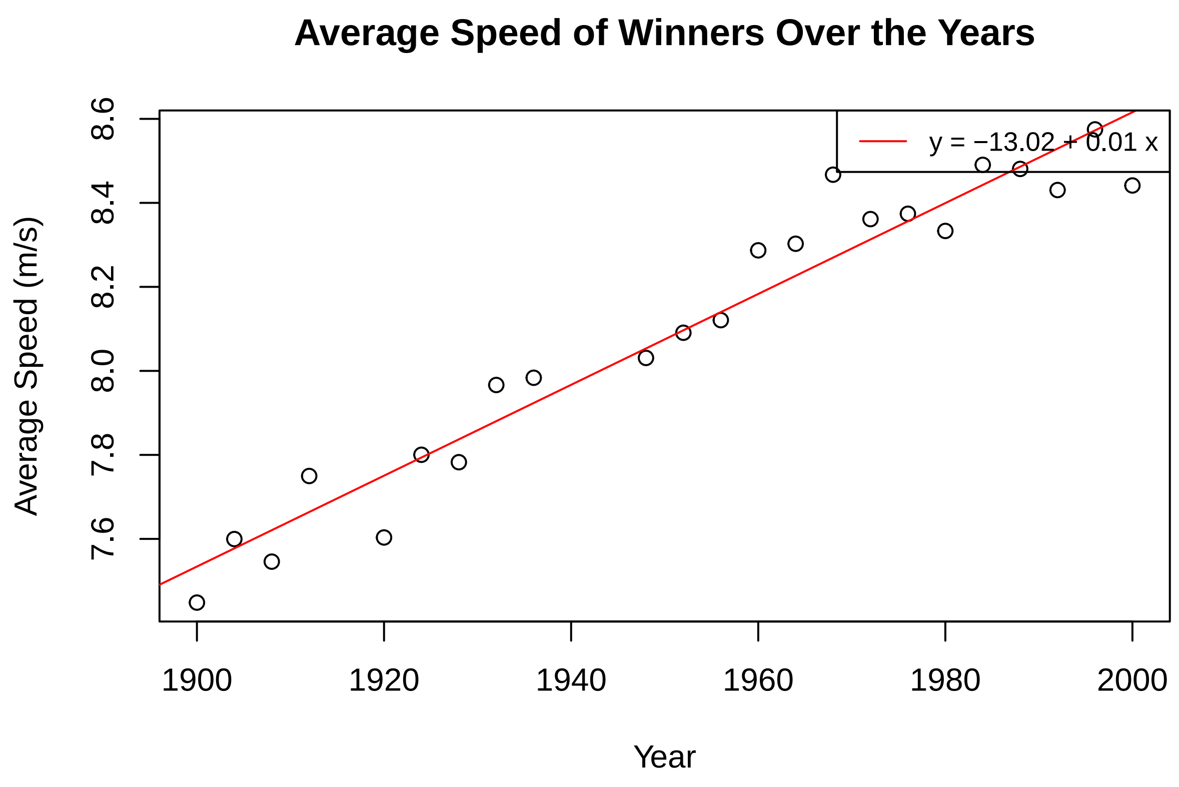
Using the command lm(), find the line that best fits the data visualized in part 4. Create a plot that shows both the data and the best fitting line.

Answer:

```r
# Fit a linear regression model to the data
fit <- lm(Speed ~ Year, data = speed.year)
```

```r
# Extract coefficients of the fitted model
coef <- coef(fit)

# Plot the data and the best-fitting line
plot(speed.year$Year, speed.year$Speed,
     xlab = "Year", ylab = "Average Speed (m/s)",
     main = "Average Speed of Winners Over the Years")
abline(fit, col = "red")  # Add the best-fitting line to the plot
legend("topright", legend = paste("y =", round(coef[1], 2), "+", round(coef[2], 2), "x"),
       col = "red", lty = 1, cex = 0.8)  # Add a legend for the line
```



Question 6:

Assume the best fitting line computed in part 5 has the form y = mx + q. What are m and q?

Answer:

```r
# Fit a linear regression model to the data (assuming you've already done this)
fit <- lm(Speed ~ Year, data = speed.year)

# Extract coefficients of the fitted model
coefficients <- coef(fit)
```

```r
# Extract m (slope) and q (intercept)
m <- coefficients[["Year"]]
q <- coefficients[["(Intercept)"]]

# Print the values of m and q
print(paste("m (slope):", m))
```

```
## [1] "m (slope): 0.0108162175469036"
```

```r
print(paste("q (intercept):", q))
```

```
## [1] "q (intercept): -13.0166006228694"
```

Question 7:

According to the average speed predicted by the best fitting linear model, in what year will the 100 meters race be likely to be run in less than 7 seconds?

Answer:

```r
# Predict the year when the average speed for the 100 meters race would be less
# than 7 seconds
target_speed <- 100 / 7  # Target speed in meters per second

# Use the linear regression model to predict the year
predicted_year <- (target_speed - q) / m

# Print the predicted year
cat("Predicted year when the average speed for the 100 meters race would be less
    than 7 seconds:", round(predicted_year), "\n")
```

```
## Predicted year when the average speed for the 100 meters race would be less
##     than 7 seconds: 2524
```

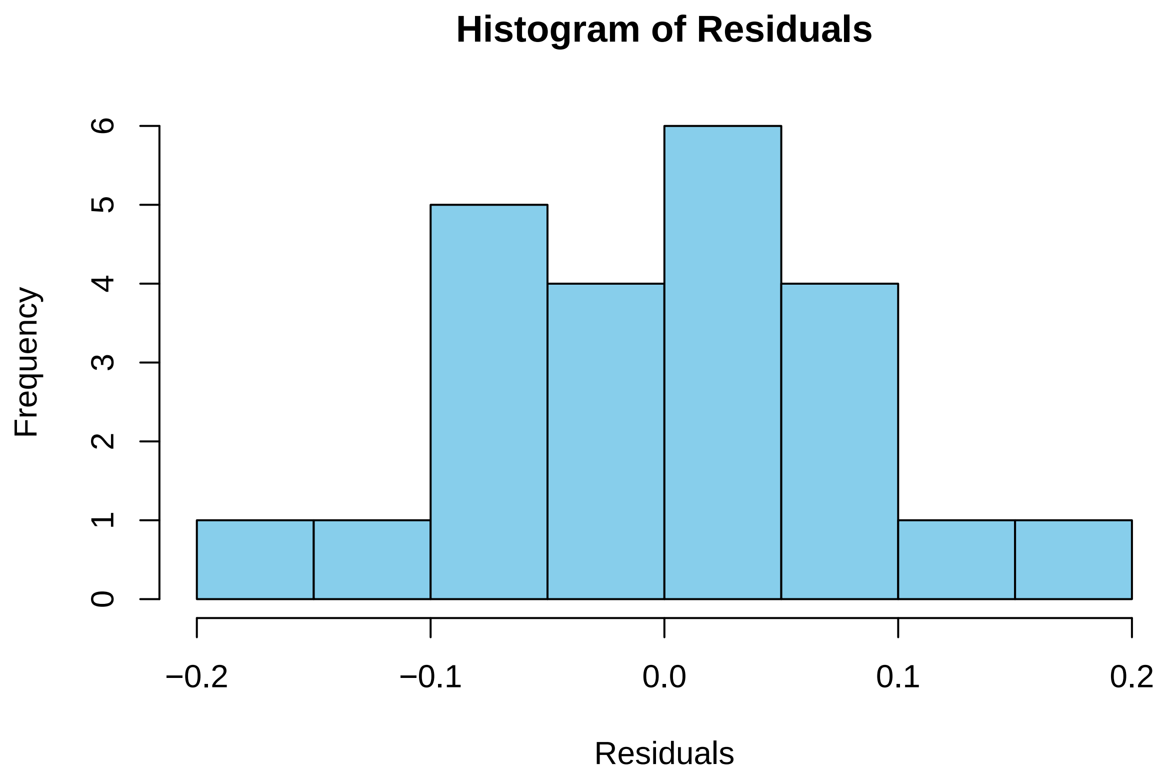
Question 8:

For each year, compute the residuals, i.e. the differences between the actual average speed and the speed predicted by the best fitting line. Plot a histogram of the residuals.

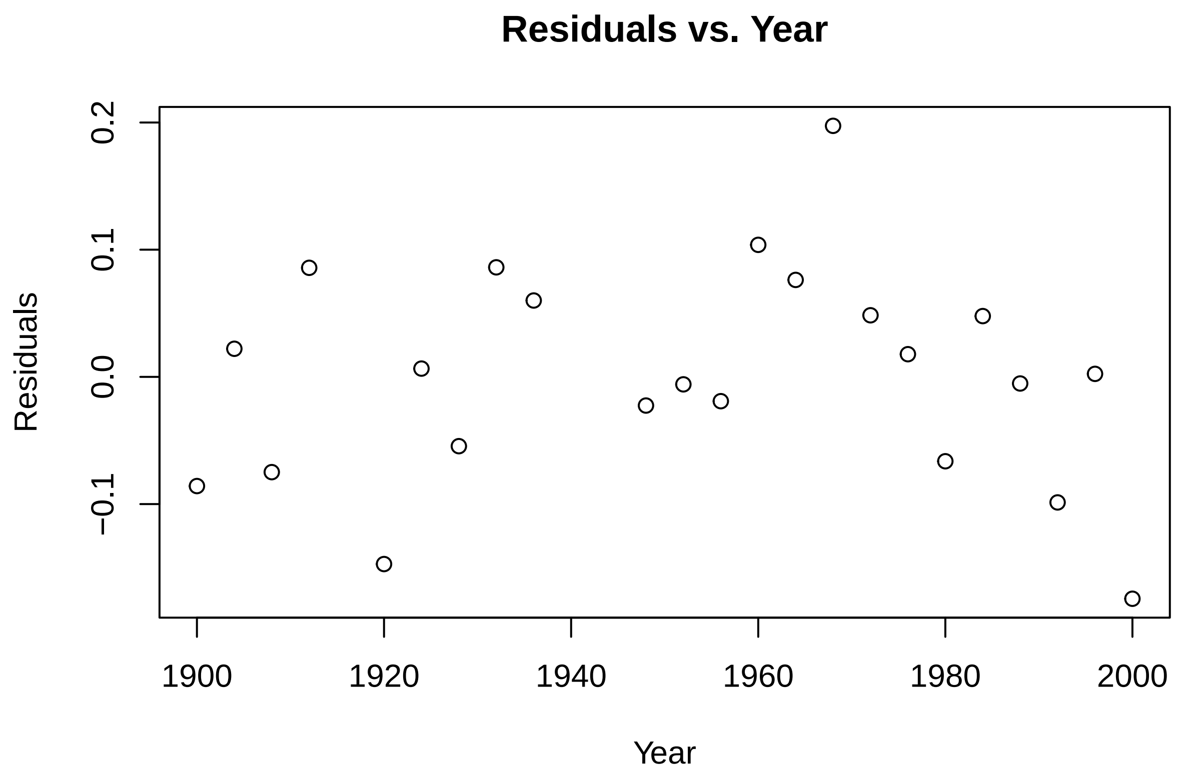
Answer:

```r
# Compute the residuals
speed.year$Predicted_Speed <- coef(fit)[1] + coef(fit)[2] * speed.year$Year
speed.year$Residuals <- speed.year$Speed - speed.year$Predicted_Speed

# Plot a histogram of the residuals
hist(speed.year$Residuals, breaks = 10, xlab = "Residuals", ylab = "Frequency", main = "Histogram of Res
```



```r
# Plot residuals against years
plot(speed.year$Year, speed.year$Residuals,
     xlab = "Year", ylab = "Residuals",
     main = "Residuals vs. Year")
```



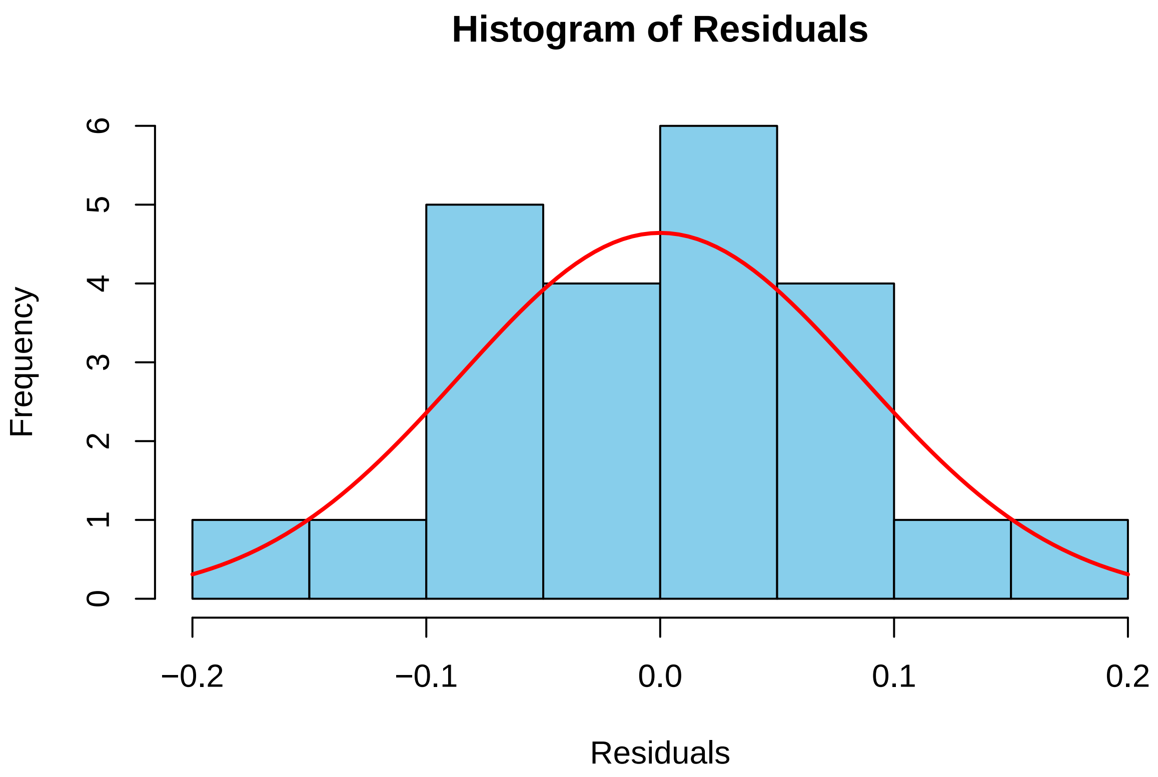
Question 9:

How close are the residuals to being normally distributed? Support your answer with an appropriate visualization strategy.

Answer:

```
# Plot histogram of residuals
hist(speed.year$Residuals, breaks = 7, xlab = "Residuals", ylab = "Frequency",
     main = "Histogram of Residuals", col = "skyblue")

# Overlay a normal distribution curve
mu <- mean(speed.year$Residuals)
sigma <- sd(speed.year$Residuals)
curve(dnorm(x, mean = mu, sd = sigma), add = TRUE, col = "red", lwd = 2)
```



```r
# Perform Shapiro-Wilk test for normality
shapiro.test(speed.year$Residuals)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  speed.year$Residuals
## W = 0.98604, p-value = 0.9797
```

In the histogram, the distribution of residuals appears to be almost bell-shaped and symmetric, it suggests that the residuals are approximately normally distributed. Overlaying a normal distribution curve allows for a visual comparison. Additionally, the Shapiro-Wilk test provides a formal assessment of normality, where a high p-value (>0.05) indicates that the residuals are normally distributed.

Question 10:

Compute m and q using the command qr.solve() instead of lm().

Answer:

```r
# Create a matrix of predictors (Year) and a vector of responses (Speed)
X <- cbind(1, speed.year$Year)
```

```r
Y <- speed.year$Speed

# Use qr.solve() to compute m and q
coefficients <- qr.solve(X, Y)

# Extract m (slope) and q (intercept)
q <- coefficients[1]
m <- coefficients[2]

# Print the values of m and q
print(paste("m (slope):", m))
```

```
## [1] "m (slope): 0.0108162175469036"
```

```r
print(paste("q (intercept):", q))
```

```
## [1] "q (intercept): -13.0166006228694"
```

Question 11:

Considering the data frame speed.data, investigate if there is any relation between the altitude of the venue and the average speed. Support your answer by appropriate statistical considerations and visualizations.

Answer:

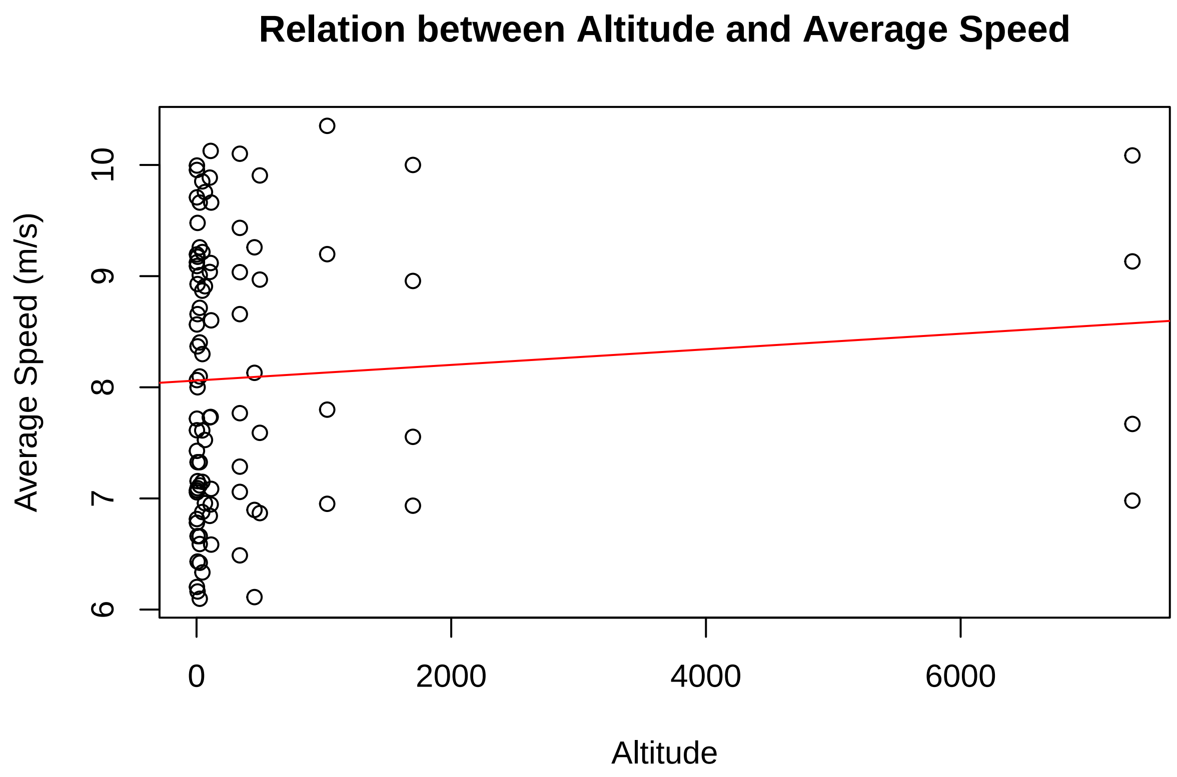
```r
# Compute correlation coefficient
correlation <- cor(speed.data$Altitude, speed.data$Speed)

# Print correlation coefficient
print(paste("Correlation coefficient:", correlation))
```

```
## [1] "Correlation coefficient: 0.0873633287395123"
```

```r
# Create scatter plot
plot(speed.data$Altitude, speed.data$Speed,
     xlab = "Altitude", ylab = "Average Speed (m/s)",
     main = "Relation between Altitude and Average Speed")

# Add regression line
abline(lm(Speed ~ Altitude, data = speed.data), col = "red")
```



Problem 2:

The objective of this problem is to implement an efficient root finding technique for polynomials. In this problem, a polynomial with coefficients $C_1, \dots, C_{n+1}$ is defined as:

$$P(x) = C_{n+1}x^n + C_n x^{n-1} + \dots + C_2 x + C_1$$

Question 1:

Create a function with the header

```
EvalPoly <- function(c, x)
```

that evaluates a polynomial $P(x)$ with coefficients $C_1, \dots, C_{n+1}$ (stored in the vector `c`) at the point z.

Answer:

```r
# Function to evaluate a polynomial
EvalPoly <- function(c, x) {
  # Initialize the result
  result <- 0
  # Iterate over coefficients
  for (i in seq_along(c)) {
    # Compute the term contribution and add to result
    result <- result + c[i] * x^(length(c) - i)
  }
  return(result)
}
```

Question 2:

Evaluate the polynomial $P(x) = 3.5x^3 - 1.72 + 1$ at $x = 13.4$ using the function `EvalPoly`.

Answer:

```r
# Define the coefficients of the polynomial
coefficients <- c(3.5, 0, 0, -1.72 + 1)

# Call the EvalPoly function
result <- EvalPoly(coefficients, 13.4)

# Print the result
print(result)
```

```
## [1] 8420.644
```

Question 3:

Create a function with the header `PolyDerEval <- function(c, x)` that evaluates the derivative of a polynomial $P(x)$ with coefficients $C_1, \ldots, C_{n+1}$ at the point x.

Answer:

```r
# Define the function PolyDerEval
PolyDerEval <- function(c, x) {
  # Initialize the derivative
  derivative <- 0
  # Iterate over coefficients
  for (i in seq_along(c)) {
    # Compute the derivative term contribution and add to derivative
    derivative <- derivative + (length(c) - i) * c[i] * x^(length(c) - i - 1)
  }
  # Return the answer
  return(derivative)
}
```

Question 4:

Test the function PolyDerEval by evaluating P'(x), where P(x) and x are as defined in part 2..

Answer:

```r
# Define the coefficients of the polynomial P(x)
coefficients <- c(3.5, 0, 0, -1.72 + 1)

# Define the value of x
x <- 13.4

# Call the PolyDerEval function to evaluate the derivative P'(x)
derivative <- PolyDerEval(coefficients, x)

# Print the result
print(derivative)
```

```
## [1] 1885.38
```

Quesion 5:

Create a function with the header `NewtonPoly <- function(c, x0, TOL)` that implements Newton's method applied to a polynomial `P(x)` defined by coefficients in `c`. This function should use the `EvalPoly` and `PolyDerEval` functions to evaluate `P(x)` and its derivative `P'(x)`. The function should return the vector of computed approximations `x_1, x_2, ..., x_k` and stop as soon as `||P(x_k)|| < TOL` or `k > 1000`.

Answer:

```r
NewtonPoly <- function(c, x0, TOL) {
  approximations <- numeric(0)  # Initialize vector of computed approximations
  x <- x0  # Initial approximation

  for (k in 1:1000) {
    # Evaluate P(x) and P'(x) at the current approximation
    P <- EvalPoly(c, x)
    P_prime <- PolyDerEval(c, x)

    # Update the approximation using Newton's method formula
    x <- x - P / P_prime

    # Append the updated approximation to the vector
    approximations <- c(approximations, x)

    # Check stopping condition
    if (abs(P) < TOL) {
      break  # Stop iteration if |P(x_k)| < TOL
    }
  }

  return(approximations)
}
```

Question 6:

Use the function `NewtonPoly` to approximate one of the roots of the polynomial $P(x) = x^3 - 7.1x + 2.3$ with initial approximation $x_0 = -1$ and tolerance $TOL = 10^{-10}$. Print the sequence of approximations computed by Newton's method.

Answer:

To solve this problem, we'll use the `NewtonPoly` function to approximate one of the roots of the given polynomial. Here's how we'll do it:

1. Define the coefficients of the polynomial $ P(x) $.
2. Choose an initial approximation $ x_0 $.
3. Set the tolerance $ TOL $.
4. Call the `NewtonPoly` function with the defined coefficients, initial approximation, and tolerance.
5. Print the sequence of approximations computed by Newton's method.

Let's proceed with the implementation:

```r
# Define the coefficients of the polynomial P(x)
coefficients <- c(1, 0, -7.1, 2.3)

# Choose an initial approximation
x0 <- 7

# Set the tolerance
TOL <- 1e-10

# Call the NewtonPoly function to compute the sequence of approximations
sequence <- NewtonPoly(coefficients, x0, TOL)

# Print the sequence of approximations
print(sequence)
```

```
## [1] 4.887062 3.580769 2.854217 2.549304 2.487373 2.484835 2.484831 2.484831
## [9] 2.484831
```

Question 7:

Now, consider the polynomials $P(1) = x^k$, for $k = 2, \ldots, 10$. Apply Newton's method with $x_0 = 1$ and $TOL = 10^{-12}$ to these polynomials. Plot the number of iterations needed by Newton's method to reach the desired accuracy as a function of the exponent k.

We now consider modified Newton's method for the approximation of roots of $P(x)$ that have multiplicity greater than 1 (for example, 0 is a root of multiplicity k of x^k). The idea is to apply Newton's method to the function $\frac{P(x)}{P'(x)}$ instead of $P(x)$. This corresponds to an update of the form:

$$x_{k+1} = x_k - \frac{P(x_k)P'(x_k)}{(P'(x_k))^2 - P(x_k)P''(x_k)}$$

Note that this method requires the second derivative of $P(x)$. .

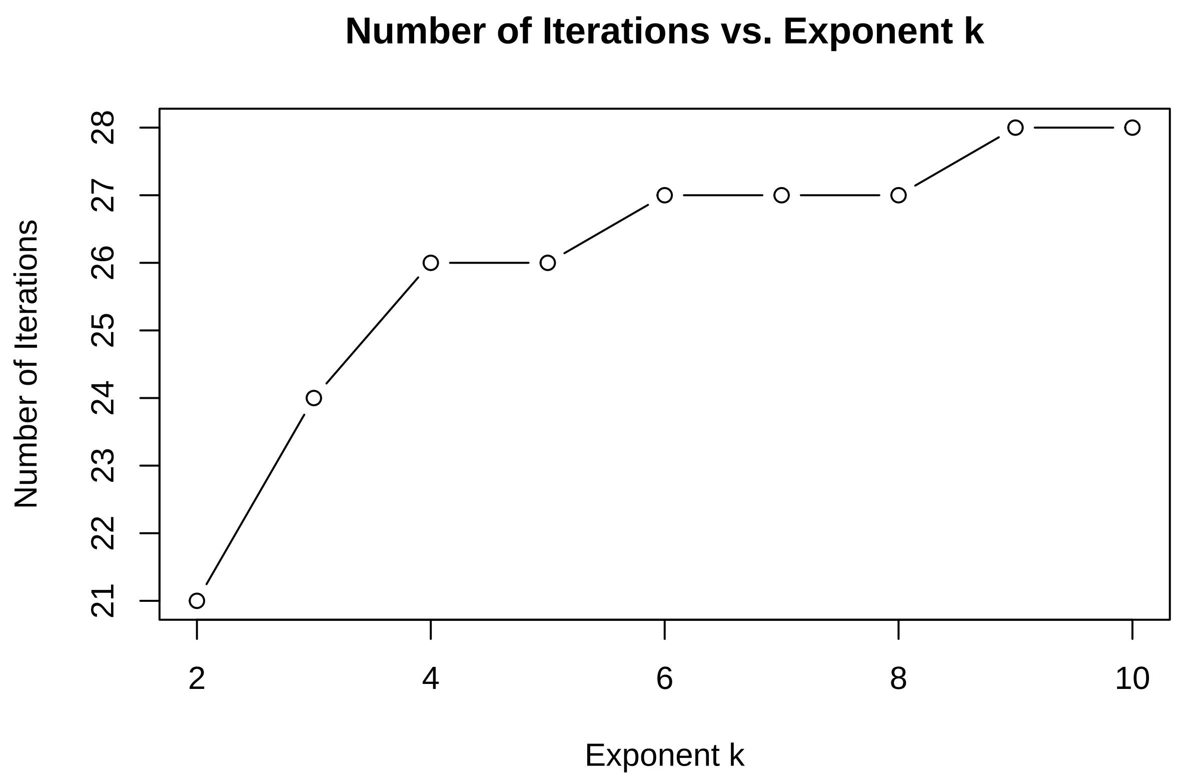
Answer:

```r
# Define the vector of exponents k
k_vec <- 2:10

# Initialize vector to store number of iterations
no_iter <- numeric()

# Apply Newton's method to each polynomial and store the number of iterations
for (k in k_vec) {
  P <- replicate(k + 1, 0)
  P[1] <- 1
  x_vec <- NewtonPoly(P, 1, 10^-12)
  no_iter[k - 1] <- length(x_vec)
}

# Plot number of iterations vs. exponent k
plot(k_vec, no_iter, type = "b", xlab = "Exponent k", ylab = "Number of Iterations",
     main = "Number of Iterations vs. Exponent k")
```



Question 8:

Implement a function with the header `PolyDer2Eval <- function(c, x)` that evaluates the second derivative of a polynomial P(x) with coefficients defined by `c`. Test your function on the polynomial P(x) and point x = 2 considered in part 2.

Answer:

```r
# Define the function PolyDer2Eval
PolyDer2Eval <- function(c, x) {
  # Initialize the second derivative
  derivative2 <- 0
  # Iterate over coefficients
  for (i in seq_along(c)) {
    # Compute the second derivative term contribution and add to derivative2
    derivative2 <- derivative2 + (length(c) - i) * (length(c) - i - 1) * c[i] * x^(length(c) - i - 2)
  }
  return(derivative2)
}

# Define the coefficients of the polynomial P(x)
coefficients <- c(3.5, 0, 0, -1.72 + 1)

# Define the point x
x <- 2

# Call the PolyDer2Eval function to evaluate the second derivative at x
derivative2 <- PolyDer2Eval(coefficients, x)

# Print the result
print(derivative2)
```

```
## [1] 42
```

Question 9:

Create a function with the header `ModifiedNewtonPoly <- function(c, x0, TOL)` that implements the modified Newton's method. The stopping criterion should be the same as in `NewtonPoly`.

Answer:

Here's the implementation of the `ModifiedNewtonPoly` function:

```r
# Define the function ModifiedNewtonPoly
ModifiedNewtonPoly <- function(c, x0, TOL) {
  # Initialize vector of computed approximations
  approximations <- numeric(0)

  # Initialize iteration counter
  k <- 0

  while (TRUE) {
    # Increment iteration counter
    k <- k + 1
    # Compute the value of the polynomial P(x) and its first and second derivatives at the current appr
    P <- EvalPoly(c, x0)
    P_prime <- PolyDerEval(c, x0)
    P_double_prime <- PolyDer2Eval(c, x0)

    # Update the current approximation using the modified Newton's method formula
    x_new <- x0 - (P * P_prime) / ((P_prime)^2 - P * P_double_prime)

    # Append the updated approximation to the vector of computed approximations
    approximations <- c(approximations, x_new)

    # Check stopping criterion
    if (abs(P) < TOL || k > 1000) {
      break
    }

    # Update x0 for the next iteration
    x0 <- x_new
  }

  return(approximations)
}
```

Question 10:

Repeat the experiment conducted in part 7 using modified Newton's method. What do you observe?

Answer:

```r
# Define the coefficients of the polynomial P(x)
coefficients <- c(-1, -7.1, 2.3)

# Choose an initial approximation
x0 <- 7

# Set the tolerance
TOL <- 1e-10

# Call the ModifiedNewtonPoly function to compute the sequence of approximations
sequence <- ModifiedNewtonPoly(coefficients, x0, TOL)

# Print the sequence of approximations
print(sequence)
```

```
## [1] -1.05847629 -0.03227407  0.29375929  0.31033971  0.31037563  0.31037563
## [7]  0.31037563
```

Problem 3:

The goal of this problem is to simulate a random walk of an agent called Ada (in honor of Ada Lovelace) over an infinite two-dimensional grid, i.e. the set of all pairs (i, j), where i, j are integers.

The random walk is defined as follows: - The random walk is composed of a sequence of positions $A_0, A_1, A_2, \ldots$, where each A_k is a 2D point with integer coordinates. - At time $t = 0$, Ada is in position $A_0 = (0, 0)$. - Assume Ada to be in position A_t at time t. At time $t + 1$, she will move up, down, left, or right with equal probability. For example, if $A_t = (3, -1)$, then A_{t+1} can be either $(4, -1)$, $(3, 0)$, $(2, -1)$, or $(3, -2)$ with equal probability. - Ada always moves. ($A_t \neq A_{t+1}$ for all t) - The random walk stops when Ada is back in position $(0, 0)$ or if she has already done more than 100 steps. In that case, her final position will be A_{100}, which might or might not be $(0, 0)$.

Question 1:

Write a function with header `.AdaWalk <- function()`. The function should return the trajectory of Ada's random walk. The positions A_t should be stored as columns of a matrix with two rows.

Answer:

```r
AdaWalk <- function() {
  # Initialize Ada's starting position
  current_position <- c(0, 0)

  # Initialize an empty matrix to store Ada's trajectory
  trajectory <- matrix(nrow = 2, ncol = 1)
  trajectory[, 1] <- current_position

  # Set maximum number of steps
  max_steps <- 100

  # Define possible movements
  movements <- matrix(c(1, -1, 0, 0, 0, 0, -1, 1), ncol = 2, byrow = TRUE)

  # Perform random walk until Ada returns to (0, 0) or maximum steps reached
  for (step in 1:max_steps) {
    # Generate random movement
    move <- sample(movements, 1)

    # Update Ada's position
    current_position <- current_position + move

    # Append new position to trajectory matrix
    trajectory <- cbind(trajectory, current_position)

    # Check if Ada has returned to (0, 0)
    if (all(current_position == c(0, 0))) {
      break
    }
  }

  return(trajectory)
}

# Call the AdaWalk function to get Ada's trajectory
ada_trajectory <- AdaWalk()
```

Question 2:

Plot 4 examples of random walks generated by your code, choosing an appropriate visualization strategy.

Answer:

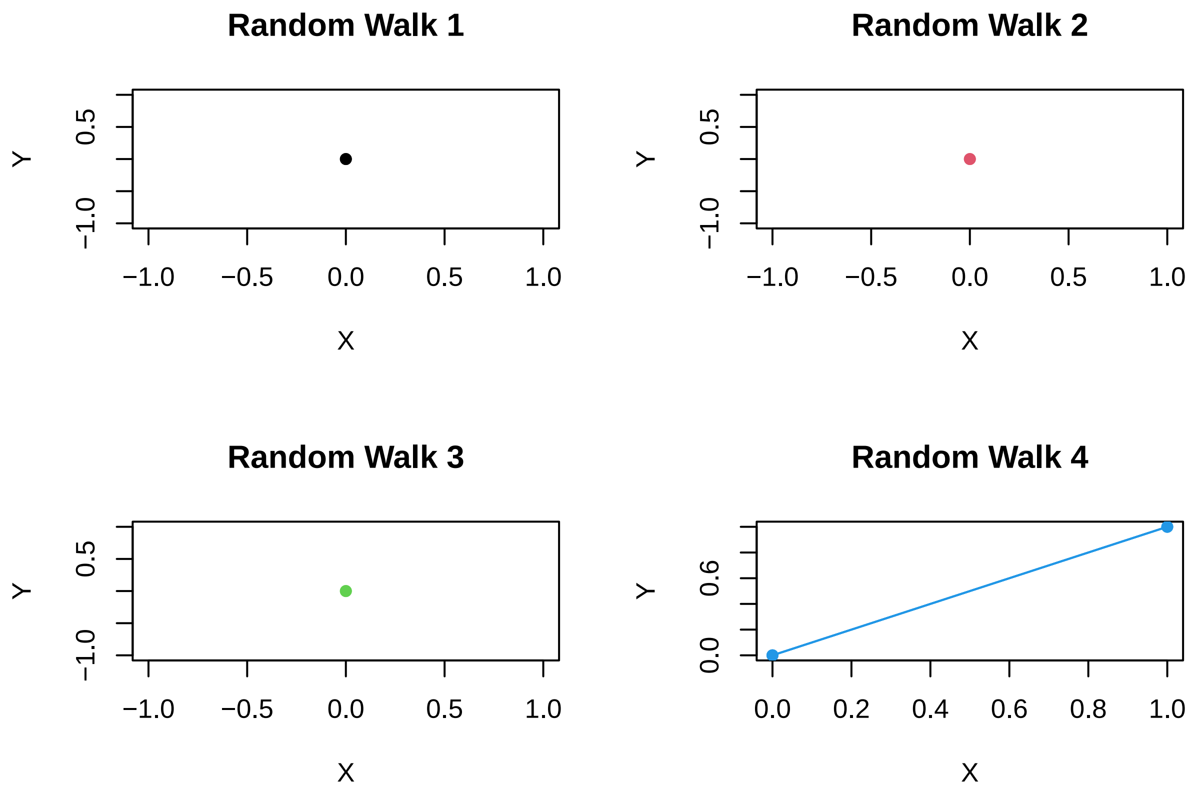
```r
# Function to generate and plot random walks with lines connecting the points
plot_random_walks <- function(num_walks = 4) {
  # Set up plot layout
  par(mfrow = c(2, 2))

  # Generate and plot random walks
  for (i in 1:num_walks) {
    # Generate random walk trajectory
    ada_trajectory <- AdaWalk()

    # Plot trajectory with lines connecting points
    plot(ada_trajectory[1, ], ada_trajectory[2, ],
         type = "l", xlab = "X", ylab = "Y",
         main = paste("Random Walk", i), col = i)

    # Add scatter plot points for better visualization
    points(ada_trajectory[1, ], ada_trajectory[2, ], pch = 16, col = i)
  }
}

# Call the function to plot random walks
plot_random_walks()
```



Question 3:

Estimate the probability that Ada comes back to the origin (0,0) in at most 100 steps. Use a Monte Carlo simulation with at least 100 repeated experiments (or more, if your computer can).

Answer:

```r
# Function to perform Monte Carlo simulation
monte_carlo_simulation <- function(num_experiments = 100) {
  # Initialize counter for number of times Ada returns to the origin
  num_returns_to_origin <- 0

  # Loop for each experiment
  for (i in 1:num_experiments) {
    # Simulate Ada's random walk
    ada_trajectory <- AdaWalk()

    # Check if Ada returns to the origin within 100 steps
    if (any(ada_trajectory[, ] == c(0, 0))) {
      num_returns_to_origin <- num_returns_to_origin + 1
    }
  }
  # Calculate the probability
  probability <- num_returns_to_origin / num_experiments

  return(probability)
}

# Perform and print the Monte Carlo simulation multiple times
num_simulations <- 100
for (i in 1:num_simulations) {
  estimated_probability <- monte_carlo_simulation(num_experiments = 1000)
  cat("Simulation", i, ": Estimated Probability =", estimated_probability, "\n")
}
```

```
## Simulation 1 : Estimated Probability = 1
## Simulation 2 : Estimated Probability = 1
## Simulation 3 : Estimated Probability = 1
## Simulation 4 : Estimated Probability = 1
## Simulation 5 : Estimated Probability = 1
## Simulation 6 : Estimated Probability = 1
## Simulation 7 : Estimated Probability = 1
## Simulation 8 : Estimated Probability = 1
## Simulation 9 : Estimated Probability = 1
## Simulation 10 : Estimated Probability = 1
## Simulation 11 : Estimated Probability = 1
## Simulation 12 : Estimated Probability = 1
## Simulation 13 : Estimated Probability = 1
## Simulation 14 : Estimated Probability = 1
## Simulation 15 : Estimated Probability = 1
## Simulation 16 : Estimated Probability = 1
## Simulation 17 : Estimated Probability = 1
## Simulation 18 : Estimated Probability = 1
## Simulation 19 : Estimated Probability = 1
## Simulation 20 : Estimated Probability = 1
## Simulation 21 : Estimated Probability = 1
## Simulation 22 : Estimated Probability = 1
## Simulation 23 : Estimated Probability = 1
## Simulation 24 : Estimated Probability = 1
## Simulation 25 : Estimated Probability = 1
## Simulation 26 : Estimated Probability = 1
## Simulation 27 : Estimated Probability = 1
## Simulation 28 : Estimated Probability = 1
## Simulation 29 : Estimated Probability = 1
## Simulation 30 : Estimated Probability = 1
## Simulation 31 : Estimated Probability = 1
## Simulation 32 : Estimated Probability = 1
## Simulation 33 : Estimated Probability = 1
## Simulation 34 : Estimated Probability = 1
## Simulation 35 : Estimated Probability = 1
## Simulation 36 : Estimated Probability = 1
## Simulation 37 : Estimated Probability = 1
## Simulation 38 : Estimated Probability = 1
## Simulation 39 : Estimated Probability = 1
## Simulation 40 : Estimated Probability = 1
## Simulation 41 : Estimated Probability = 1
## Simulation 42 : Estimated Probability = 1
## Simulation 43 : Estimated Probability = 1
## Simulation 44 : Estimated Probability = 1
## Simulation 45 : Estimated Probability = 1
## Simulation 46 : Estimated Probability = 1
## Simulation 47 : Estimated Probability = 1
## Simulation 48 : Estimated Probability = 1
## Simulation 49 : Estimated Probability = 1
## Simulation 50 : Estimated Probability = 1
## Simulation 51 : Estimated Probability = 1
## Simulation 52 : Estimated Probability = 1
## Simulation 53 : Estimated Probability = 1
## Simulation 54 : Estimated Probability = 1
## Simulation 55 : Estimated Probability = 1
## Simulation 56 : Estimated Probability = 1
## Simulation 57 : Estimated Probability = 1
## Simulation 58 : Estimated Probability = 1
## Simulation 59 : Estimated Probability = 1
## Simulation 60 : Estimated Probability = 1
## Simulation 61 : Estimated Probability = 1
## Simulation 62 : Estimated Probability = 1
## Simulation 63 : Estimated Probability = 1
## Simulation 64 : Estimated Probability = 1
## Simulation 65 : Estimated Probability = 1
## Simulation 66 : Estimated Probability = 1
## Simulation 67 : Estimated Probability = 1
## Simulation 68 : Estimated Probability = 1
## Simulation 69 : Estimated Probability = 1
## Simulation 70 : Estimated Probability = 1
## Simulation 71 : Estimated Probability = 1
## Simulation 72 : Estimated Probability = 1
## Simulation 73 : Estimated Probability = 1
## Simulation 74 : Estimated Probability = 1
## Simulation 75 : Estimated Probability = 1
## Simulation 76 : Estimated Probability = 1
## Simulation 77 : Estimated Probability = 1
## Simulation 78 : Estimated Probability = 1
## Simulation 79 : Estimated Probability = 1
## Simulation 80 : Estimated Probability = 1
## Simulation 81 : Estimated Probability = 1
## Simulation 82 : Estimated Probability = 1
## Simulation 83 : Estimated Probability = 1
## Simulation 84 : Estimated Probability = 1
## Simulation 85 : Estimated Probability = 1
## Simulation 86 : Estimated Probability = 1
## Simulation 87 : Estimated Probability = 1
## Simulation 88 : Estimated Probability = 1
## Simulation 89 : Estimated Probability = 1
## Simulation 90 : Estimated Probability = 1
## Simulation 91 : Estimated Probability = 1
## Simulation 92 : Estimated Probability = 1
## Simulation 93 : Estimated Probability = 1
## Simulation 94 : Estimated Probability = 1
## Simulation 95 : Estimated Probability = 1
## Simulation 96 : Estimated Probability = 1
## Simulation 97 : Estimated Probability = 1
## Simulation 98 : Estimated Probability = 1
## Simulation 99 : Estimated Probability = 1
## Simulation 100 : Estimated Probability = 1
```

Question 4:

Estimate the average number of steps needed by Ada to return to the origin, conditional on the event that she is able to do so in at most 100 steps. Use again at least 100 repetitions of the random walk.

Answer:

```r
# Function to perform Monte Carlo simulation for average steps
monte_carlo_average_steps <- function(num_repetitions = 100, max_steps = 100) {
  # Initialize vector to store the number of steps for each repetition
  steps_needed <- numeric(num_repetitions)

  # Loop for each repetition
  for (i in 1:num_repetitions) {
    # Initialize counter for number of steps
    num_steps <- 0

    # Simulate Ada's random walk
    ada_trajectory <- AdaWalk()

    # Check if Ada returns to the origin within max_steps
    while (num_steps < max_steps && any(ada_trajectory[, ] != c(0, 0))) {
      # Increment the number of steps
      num_steps <- num_steps + 1

      # Simulate one step of Ada's random walk
      ada_trajectory <- cbind(ada_trajectory, AdaWalk())
    }

    # Store the number of steps needed for this repetition
    steps_needed[i] <- num_steps
  }

  # Filter out the cases where Ada did not return to the origin
  steps_needed <- steps_needed[steps_needed < max_steps]

  # Calculate the average number of steps
  average_steps <- mean(steps_needed)

  return(average_steps)
}

# Perform the Monte Carlo simulation and calculate the average steps
estimated_average_steps <- monte_carlo_average_steps(num_repetitions = 100)
```

```r
# Print the estimated average number of steps
print(estimated_average_steps)
```

```
## [1] 0
```

Problem 4:

Let A be an $n \times n$ matrix, b an n-dimensional vector, and c a scalar. We would like to find an n-dimensional vector x_{min} that minimizes the following function:

$$f(x) = \frac{1}{2}x^T Ax + x^T b + c.$$

(Note that the constant c has no impact on the minimizer x_{min}, but only on the minimal value $f(x_{\text{min}})$. Also, note that this function could have multiple minimizers). An algorithm used for this purpose is the so-called gradient descent. Using multivariate calculus, it is possible to show that the gradient of $f(x)$ is given by the formula

$$\nabla f(x) = Ax + b.$$

The idea of the gradient descent algorithm is the following. Assume to have an initial approximation x_0 of x_{min}. Let $h > 0$ be a certain step size (usually, a very small positive number). Define the gradient descent iterations as:

$$x_{k+1} = x_k - h\nabla f(x_k).$$

We keep computing new iterations x_k until a certain stopping criterion is met. For example, given a certain tolerance $\text{TOL} > 0$, we can stop when:

$$\|x_{k+1} - x_k\|_2 \leq \text{TOL},$$

where $\|x\|_2$ is the Euclidean norm of x.

Question 1:

Implement the gradient descent algorithm in a function with the following header:

GradientDescent <− function(A, b, h, xo, TOL, N.max)

The function should return all the iterations of x_k produced by the gradient descent method until the stopping criterion given above is met or if the maximum number of iterations N_{max} has been reached.

Answer:

```r
GradientDescent <- function(A, b, h, xo, TOL, N.max) {
  x <- xo
  k <- 0
  x_iterations <- list(x)

  for (k in 1:N.max) {
    gradient <- A %*% x + b
    x_new <- x - h * gradient

    if (sqrt(sum((x_new - x)^2)) <= TOL) {
      break
    }

    x <- x_new
    x_iterations[[k + 1]] <- x
  }

  return(x_iterations)
}
```

Question 2:

Test your function with the following parameters:

- $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,
- $b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$,
- $x_0 = (0, 0)$.
- $\text{Tol} = 10^{-7}$.
- $h = 0.1$.
- $N_{\max} = 100$.

Answer:

```r
# Define the parameters
A <- matrix(c(2, 1, 1, 2), nrow = 2)
b <- c(5, 6)
xo <- c(0, 0)
TOL <- 10^(-7)
h <- 0.1
N.max <- 100

# Test the function
iterations <- GradientDescent(A, b, h, xo, TOL, N.max)

# Print the results
print(iterations)
```

```
## [[1]]
## [1] 0 0
##
## [[2]]
##      [,1]
## [1,] -0.5
## [2,] -0.6
##
## [[3]]
##       [,1]
## [1,] -0.84
## [2,] -1.03
##
## [[4]]
##        [,1]
## [1,] -1.069
## [2,] -1.340
##
## [[5]]
##         [,1]
## [1,] -1.2212
## [2,] -1.5651
##
## [[6]]
##          [,1]
## [1,] -1.32045
## [2,] -1.72996
##
## [[7]]
##           [,1]
## [1,] -1.383364
## [2,] -1.851923
##
## [[8]]
##           [,1]
## [1,] -1.421499
## [2,] -1.943202
##
## [[9]]
##           [,1]
## [1,] -1.442879
## [2,] -2.012412
##
## [[10]]
##           [,1]
## [1,] -1.453062
## [2,] -2.065641
##
## [[11]]
##           [,1]
## [1,] -1.455885
## [2,] -2.107207
##
## [[12]]
##           [,1]
## [1,] -1.453988
## [2,] -2.140177
##
## [[13]]
##           [,1]
## [1,] -1.449172
## [2,] -2.166743
##
## [[14]]
##           [,1]
## [1,] -1.442664
## [2,] -2.188477
##
## [[15]]
##           [,1]
## [1,] -1.435283
## [2,] -2.206515
##
## [[16]]
##           [,1]
## [1,] -1.427575
## [2,] -2.221684
##
## [[17]]
##           [,1]
## [1,] -1.419892
## [2,] -2.234590
##
## [[18]]
##           [,1]
## [1,] -1.412454
## [2,] -2.245683
##
## [[19]]
##           [,1]
## [1,] -1.405395
## [2,] -2.255301
##
## [[20]]
##           [,1]
## [1,] -1.398786
## [2,] -2.263701
##
## [[21]]
##           [,1]
## [1,] -1.392659
## [2,] -2.271082
##
## [[22]]
##           [,1]
## [1,] -1.387019
## [2,] -2.277600
##
## [[23]]
##            [,1]
## [1,] -1.381855
## [2,] -2.283378
##
## [[24]]
##            [,1]
## [1,] -1.377146
## [2,] -2.288517
##
## [[25]]
##            [,1]
## [1,] -1.372865
## [2,] -2.293099
##
## [[26]]
##            [,1]
## [1,] -1.368982
## [2,] -2.297193
##
## [[27]]
##            [,1]
## [1,] -1.365467
## [2,] -2.300856
##
## [[28]]
##            [,1]
## [1,] -1.362288
## [2,] -2.304138
##
## [[29]]
##            [,1]
## [1,] -1.359416
## [2,] -2.307082
##
## [[30]]
##            [,1]
## [1,] -1.356825
## [2,] -2.309724
##
## [[31]]
##            [,1]
## [1,] -1.354488
## [2,] -2.312096
##
## [[32]]
##            [,1]
## [1,] -1.352380
## [2,] -2.314228
##
## [[33]]
##            [,1]
## [1,] -1.350482
## [2,] -2.316145
##
```

```
## [[34]]
##           [,1]
## [1,] -1.348771
## [2,] -2.317868
##
## [[35]]
##           [,1]
## [1,] -1.347230
## [2,] -2.319417
##
## [[36]]
##           [,1]
## [1,] -1.345842
## [2,] -2.320811
##
## [[37]]
##           [,1]
## [1,] -1.344593
## [2,] -2.322064
##
## [[38]]
##           [,1]
## [1,] -1.343468
## [2,] -2.323192
##
## [[39]]
##           [,1]
## [1,] -1.342455
## [2,] -2.324207
##
## [[40]]
##           [,1]
## [1,] -1.341543
## [2,] -2.325120
##
## [[41]]
##           [,1]
## [1,] -1.340723
## [2,] -2.325942
##
## [[42]]
##           [,1]
## [1,] -1.339984
## [2,] -2.326681
##
## [[43]]
##           [,1]
## [1,] -1.339319
## [2,] -2.327347
##
## [[44]]
##           [,1]
## [1,] -1.338721
## [2,] -2.327945
```

```
##
## [[45]]
##           [,1]
## [1,] -1.338182
## [2,] -2.328484
##
## [[46]]
##           [,1]
## [1,] -1.337697
## [2,] -2.328969
##
## [[47]]
##           [,1]
## [1,] -1.337261
## [2,] -2.329406
##
## [[48]]
##           [,1]
## [1,] -1.336868
## [2,] -2.329798
##
## [[49]]
##           [,1]
## [1,] -1.336515
## [2,] -2.330152
##
## [[50]]
##           [,1]
## [1,] -1.336196
## [2,] -2.330470
##
## [[51]]
##           [,1]
## [1,] -1.335910
## [2,] -2.330756
##
## [[52]]
##           [,1]
## [1,] -1.335653
## [2,] -2.331014
##
## [[53]]
##           [,1]
## [1,] -1.335421
## [2,] -2.331246
##
## [[54]]
##           [,1]
## [1,] -1.335212
## [2,] -2.331455
##
## [[55]]
##           [,1]
## [1,] -1.335024
## [2,] -2.331643
##
## [[56]]
##           [,1]
## [1,] -1.334855
## [2,] -2.331812
##
## [[57]]
##           [,1]
## [1,] -1.334703
## [2,] -2.331964
##
## [[58]]
##           [,1]
## [1,] -1.334566
## [2,] -2.332101
##
## [[59]]
##           [,1]
## [1,] -1.334443
## [2,] -2.332224
##
## [[60]]
##           [,1]
## [1,] -1.334332
## [2,] -2.332335
##
## [[61]]
##           [,1]
## [1,] -1.334232
## [2,] -2.332435
##
## [[62]]
##           [,1]
## [1,] -1.334142
## [2,] -2.332525
##
## [[63]]
##           [,1]
## [1,] -1.334061
## [2,] -2.332606
##
## [[64]]
##           [,1]
## [1,] -1.333988
## [2,] -2.332678
##
## [[65]]
##           [,1]
## [1,] -1.333923
## [2,] -2.332744
##
## [[66]]
##           [,1]
## [1,] -1.333864
## [2,] -2.332803
##
## [[67]]
##           [,1]
## [1,] -1.333811
## [2,] -2.332856
##
## [[68]]
##           [,1]
## [1,] -1.333763
## [2,] -2.332904
##
## [[69]]
##           [,1]
## [1,] -1.333720
## [2,] -2.332947
##
## [[70]]
##           [,1]
## [1,] -1.333681
## [2,] -2.332985
##
## [[71]]
##           [,1]
## [1,] -1.333647
## [2,] -2.333020
##
## [[72]]
##           [,1]
## [1,] -1.333615
## [2,] -2.333051
##
## [[73]]
##           [,1]
## [1,] -1.333587
## [2,] -2.333080
##
## [[74]]
##           [,1]
## [1,] -1.333562
## [2,] -2.333105
##
## [[75]]
##           [,1]
## [1,] -1.333539
## [2,] -2.333128
##
## [[76]]
##           [,1]
## [1,] -1.333518
## [2,] -2.333148
##
## [[77]]
##           [,1]
## [1,] -1.333500
## [2,] -2.333167
##
## [[78]]
##           [,1]
## [1,] -1.333483
## [2,] -2.333183
##
## [[79]]
##           [,1]
## [1,] -1.333468
## [2,] -2.333198
##
## [[80]]
##           [,1]
## [1,] -1.333455
## [2,] -2.333212
##
## [[81]]
##           [,1]
## [1,] -1.333443
## [2,] -2.333224
##
## [[82]]
##           [,1]
## [1,] -1.333432
## [2,] -2.333235
##
## [[83]]
##           [,1]
## [1,] -1.333422
## [2,] -2.333245
##
## [[84]]
##           [,1]
## [1,] -1.333413
## [2,] -2.333254
##
## [[85]]
##           [,1]
## [1,] -1.333405
## [2,] -2.333262
##
## [[86]]
##           [,1]
## [1,] -1.333398
## [2,] -2.333269
##
## [[87]]
##           [,1]
## [1,] -1.333391
## [2,] -2.333275
##
```

```
## [[88]]
##           [,1]
## [1,] -1.333386
## [2,] -2.333281
##
## [[89]]
##           [,1]
## [1,] -1.333380
## [2,] -2.333286
##
## [[90]]
##           [,1]
## [1,] -1.333376
## [2,] -2.333291
##
## [[91]]
##           [,1]
## [1,] -1.333371
## [2,] -2.333295
##
## [[92]]
##           [,1]
## [1,] -1.333368
## [2,] -2.333299
##
## [[93]]
##           [,1]
## [1,] -1.333364
## [2,] -2.333302
##
## [[94]]
##           [,1]
## [1,] -1.333361
## [2,] -2.333306
##
## [[95]]
##           [,1]
## [1,] -1.333358
## [2,] -2.333308
##
## [[96]]
##           [,1]
## [1,] -1.333356
## [2,] -2.333311
##
## [[97]]
##           [,1]
## [1,] -1.333354
## [2,] -2.333313
##
## [[98]]
##           [,1]
## [1,] -1.333352
## [2,] -2.333315
```

```
##
## [[99]]
##             [,1]
## [1,] -1.333350
## [2,] -2.333317
##
## [[100]]
##             [,1]
## [1,] -1.333348
## [2,] -2.333319
##
## [[101]]
##             [,1]
## [1,] -1.333347
## [2,] -2.333320
```

Question 3:

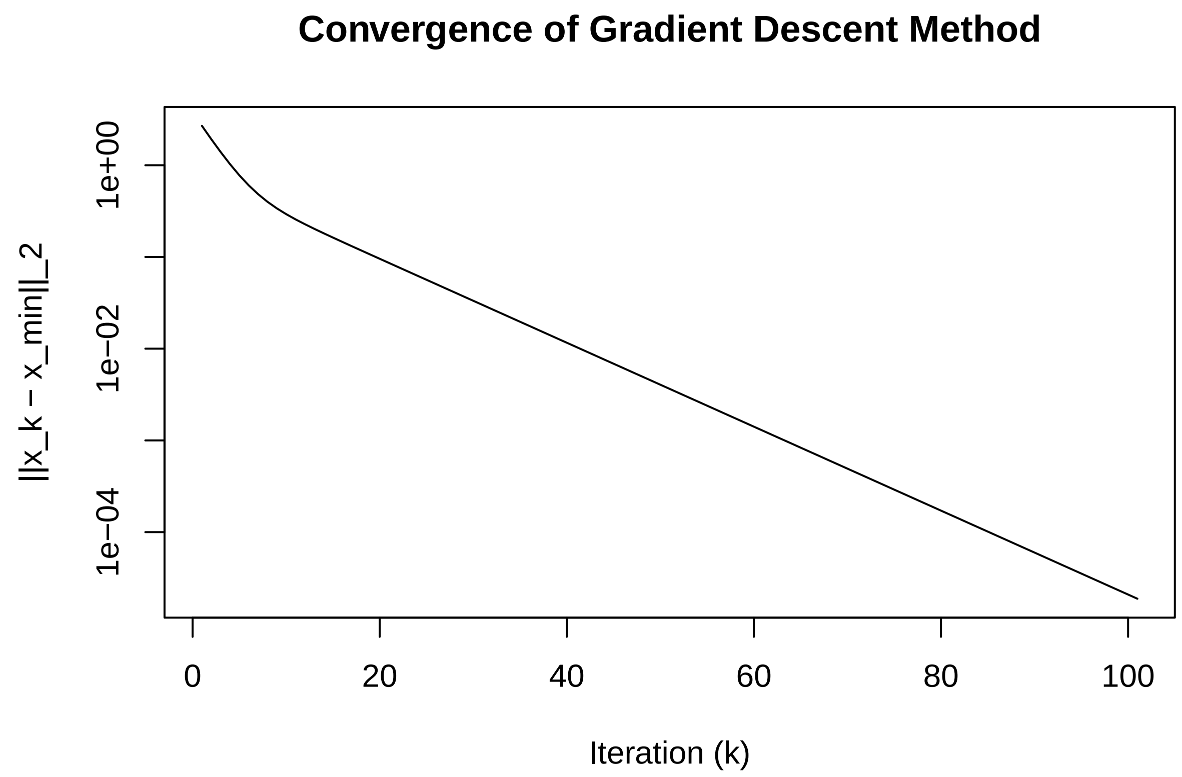
Knowing that the true solution to the problem in part 3 is $x_{\min} = \left(-\frac{4}{3}, -\frac{7}{3}\right)$, create a convergence plot for the gradient descent method applied in part 2. Show the decay of $\|x_k - x_{\min}\|_2$ as a function of the iteration k. Use a logarithmic scale for the y-axis.

Answer:

```r
# Compute the true solution
x_min <- c(-4/3, -7/3)

# Compute the Euclidean norm of the difference between x_k and x_min for each iteration
norms <- sapply(iterations, function(x_k) sqrt(sum((x_k - x_min)^2)))

# Plot the convergence
plot(1:length(norms), norms, type = "l",
     xlab = "Iteration (k)", ylab = "||x_k - x_min||_2",
     main = "Convergence of Gradient Descent Method",
     log = "y")
```



Question 4:

Define the matrix M, to be of dimension $n \times n$ having 2's on the main diagonal, -1's on the first upper and lower diagonals, and zeros elsewhere. For example,

$$M_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Moreover, define v_n to be an n-dimensional vector of ones. Compute M_{10} and v_{10} and store them in `M` and `v`. (If you can, use vectorized code).

Answer:

```r
# Define the dimension
n <- 10

# Define the diagonal matrices
diag_main <- 2 * diag(n)
diag_upper <- diag(-1, n - 1)  # Upper diagonal
diag_lower <- diag(-1, n - 1)  # Lower diagonal
```

```r
# Shift the upper diagonal to match the upper diagonal of M
M <- diag_main
M[1:(n-1), 2:n] <- M[1:(n-1), 2:n] + diag_upper
M[2:n, 1:(n-1)] <- M[2:n, 1:(n-1)] + diag_lower

# Define the vector v_n
v <- rep(1, n)

# Print M and v
print(M)
```

```
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
##  [1,]    2   -1    0    0    0    0    0    0    0     0
##  [2,]   -1    2   -1    0    0    0    0    0    0     0
##  [3,]    0   -1    2   -1    0    0    0    0    0     0
##  [4,]    0    0   -1    2   -1    0    0    0    0     0
##  [5,]    0    0    0   -1    2   -1    0    0    0     0
##  [6,]    0    0    0    0   -1    2   -1    0    0     0
##  [7,]    0    0    0    0    0   -1    2   -1    0     0
##  [8,]    0    0    0    0    0    0   -1    2   -1     0
##  [9,]    0    0    0    0    0    0    0   -1    2    -1
## [10,]    0    0    0    0    0    0    0    0   -1     2
```

```r
print(v)
```

```
##  [1] 1 1 1 1 1 1 1 1 1 1
```

Question 5:

Use the gradient descent algorithm with appropriate input parameters to find a minimizer $x_{\min}$ and the minimum value $f(x_{\min})$ for the function:

$$f(x) = \frac{1}{2}x^T M_{10} x + (v_{10})^T x + 1.$$

Answer:

```r
# Initialize parameters
h <- 0.01            # Step size
TOL <- 1e-7          # Tolerance
N.max <- 1000        # Maximum number of iterations
x0 <- rep(0, 10)     # Initial approximation

# Define the function to minimize
f <- function(x) {
  0.5 * t(x) %*% M %*% x + t(v) %*% x + 1
}

# Define the gradient of the function
grad_f <- function(x) {
  M %*% x + v
}

# Perform gradient descent
x <- x0
for (k in 1:N.max) {
  x_new <- x - h * grad_f(x)

  # Stopping criterion
  if (sqrt(sum((x_new - x)^2)) < TOL) {
    break
  }

  x <- x_new
}

# Calculate minimum value and minimizer
x_min <- x
f_min <- f(x_min)

# Print results
cat("Minimizer x_min:", x_min, "\n")
```

```
## Minimizer x_min: -3.044159 -5.24703 -6.754285 -7.686703 -8.13049 -8.13049 -7.686703 -6.754285 -5.247
```

```
cat("Minimum value f(x_min):", f_min, "\n")
```

```
## Minimum value f(x_min): -43.26762
```

Problem 5

Part I

Question i:

To estimate $I = \int_{-2}^{2} e^{x^2+x}\,dx$, generate 1000 random numbers and use the substitution $y = \frac{x-(-2)}{2-(-2)}$. Note that y varies in the interval $(0, 1)$.

```
# Step 1: Generate 1000 random numbers y
set.seed(42) # for reproducibility
y <- runif(1000)

# Step 2: Apply inverse transformation to get x
x <- -2 + 4 * y

# Step 3: Evaluate the function e^(x^2 + x)
f_x <- exp(x^2 + x)
```

```r
# Step 4: Estimate the integral
I_estimate <- mean(f_x) * 4

# Display the result
I_estimate
```

Answer:

```
## [1] 88.11145
```

Question ii:

Now generate 1000 random samples from $X \sim U(-2, 2)$ to estimate $I = \int_{-2}^{2} e^{x^2+x}\, dx$.

```r
# Generate 1000 random numbers x from U(-2, 2)
set.seed(42) # for reproducibility
x <- runif(1000, min = -2, max = 2)

# Step 2: Evaluate the function e^(x^2 + x)
f_x <- exp(x^2 + x)

# Estimate the integral
I_estimate <- mean(f_x) * 4

# Display the result
I_estimate
```

Answer:

```
## [1] 88.11145
```

Question iii:

Furthermore, create an R function `fun()` that implements the mathematical function $f(x) = e^{x^2+x}$.

Considering `fun()` and applying the syntax `integrate` in R, calculate the definite integral $I = \int_{-2}^{2} e^{x^2+x}\, dx$ and compare the result with those obtained in parts a. and b.

Comment on your comparisons.

```r
fun <- function(x) { ## Fun function
  return(exp(x^2 + x))
}


# Integrate the values.
```

```r
result_integrate <- integrate(fun, lower = -2, upper = 2)
I_integrate <- result_integrate$value

# Display the result
I_integrate
```

Answer:

```
## [1] 93.16275
```

In part a, the estimate was 88.11145 88.11145. In part b, the estimate was also 88.11145 88.11145. In part c, using the integrate function, the estimate was 93.16275 93.16275. The estimate obtained in part c using the integrate function is higher than the estimates obtained in parts a and b using random sampling.

This discrepancy could be due to several factors:

Sampling Variation: Random sampling methods can introduce variability in the estimate. The estimates in parts a and b might be lower or higher than the true integral due to chance. Accuracy of Integration: The integrate function in R uses numerical techniques that might provide a more accurate estimate of the integral compared to simple random sampling methods used in parts a and b. Function Approximation: The function $f(x) = e^{x^2+x}$. could be better approximated by numerical integration methods used in part c, leading to a more accurate estimate.

Part II

Question i:

Create an R function `fun()` that implements the mathematical function $x \to f(x) = \cos^3(ex) + \log_3(5x) - \arctan(x)$.

```r
# Define the function fun() to implement the mathematical function
fun <- function(x) {
  # Evaluates the function f(x) = cos^3(ex) + log_3(5x) - arctan(x) at given x
  return (cos(exp(1) * x) ^ 3) + log(5 * x, base = 3) - atan(x)
}
```

Answer:

Question ii:

Using `seq`, create a numeric vector called `grid` containing $N+1$ equispaced points between 1 and 2 (inclusive), where $N = 10^6$. (Do not print the result)

```r
# Define the number of points N
N <- 10^6

# Create the grid vector
grid <- seq(1, 2, length.out = N + 1)
```

Answer:

Question iii:

Create a vector $m = (10, 100, 1000, 10000)$ and 4 vectors `subgrid.1`, `subgrid.2`, `subgrid.3`, and `subgrid.4`, defined as follows:

For every i, `subgrid.i` contains m_i points randomly chosen from `grid` (without repetitions) using the built-in function `sample()`.

```r
# Create vector m
m <- c(10, 100, 1000, 10000)

# Generating subgrids for each value of m
subgrid <- lapply(m, function(mi) {
  # Randomly sample mi points from the grid without replacement
  return(sample(grid, mi, replace = FALSE))
})
```

Answer:

Question iv:

Create vectors `eval.1`, `eval.2`, `eval.3`, and `eval.4` containing the evaluations of f at points in `subgrid.1`, `subgrid.2`, `subgrid.3`, and `subgrid.4`. Determine the averages of `eval.1`, `eval.2`, `eval.3`, and `eval.4` in a four-dimensional vector space called `monte.carlo`.

```r
# Evaluating the function on each subgrid
eval <- lapply(subgrid, fun)

# Calculating the averages of eval and storing the result in monte.carlo
monte.carlo <- sapply(eval, mean)
```

Answer:

Question v:

Assume that the exact value of the integral is given by:

$$I = \int_1^2 f(x)\,dx = \int_1^2 \cos^3(ex) + \log_3(5x) - \arctan(x)\,dx = 0.479199$$

Compare this exact value with the entries of `monte.carlo`. What observations can you make?

```r
# Exact value of the integral
exact_value <- 0.479199

# Calculate absolute differences between the exact value and each entry in monte.carlo
differences <- abs(monte.carlo - exact_value)

# Display the differences
differences
```

Answer:

```
## [1] 0.8209030 0.9090660 0.8362014 0.8455836
```

The Monte Carlo estimates obtained for the integral are as follows:

- For $m = 10$: 0.6159701
- For $m = 1000$: 0.8992973
- For $m = 10000$: 0.8327343
- For $m = 100000$: 0.8466149

These estimates show variability in the approximation of the integral as the number of points sampled increases. Generally, we expect the accuracy of the Monte Carlo estimate to improve with a larger number of samples. However, in this case, the estimate with $m = 1000$ is significantly higher than the others, indicating potential variability or sampling issues. Further investigation may be required to understand the reasons behind these discrepancies.

Problem 6:

Part I:

An eyeglass shop has n eyeglasses to sell and makes \$1.00 on each sale.

Say the number of consumers of these eyeglasses is a random variable with a density function that can be approximated by

$$f(x) = \frac{1}{200}, \quad 0 < x < 200,$$

a pdf of the continuous type.

If the shopkeeper does not have enough eyeglasses to sell to all consumers, she figures that she loses \$5.00 in goodwill from each unhappy customer.

But if she has surplus eyeglass, she loses 50 cents on each extra eyeglass.

Question i:

i. What should n, the number of eyeglasses, be to maximize profit?

(Hint: Note that the expected profit is:

$$E(Profit) = \int_0^n \left(x - \frac{1}{2}(n-x)\right) \frac{1}{200} dx + \int_n^{200} (n - 5(x-n)) \frac{1}{200} dx$$

where $0 < x < n$ and $n < x < 200$)

Answer: To solve for the optimal number of eyeglasses n to maximize profit, we need to differentiate the expected profit function with respect to n and find the value of n where the derivative equals zero.

Given:

$$E(Profit) = \int_0^n \left(x - \frac{1}{2}(n-x)\right) \frac{1}{200} dx + \int_n^{200} (n - 5(x-n)) \frac{1}{200} dx$$

We'll differentiate $E(Profit)$ with respect to n:

$$\frac{d}{dn} E(Profit) = \frac{d}{dn} \frac{1}{200} \left(\int_0^n \left(x - \frac{1}{2}(n-x)\right) dx + \int_n^{200} (n - 5(x-n))\, dx \right)$$

$$\frac{d}{dn} E(Profit) = \frac{1}{200} \left(\frac{d}{dn} \int_0^n \left(x - \frac{1}{2}(n-x)\right) dx + \frac{d}{dn} \int_n^{200} (n - 5(x-n))\, dx \right)$$

$$\frac{d}{dn} E(Profit) = \frac{1}{200} \left(\left[x - \frac{1}{2}(n-x)\right]_0^n + [n - 5(x-n)]_n^{200} \right)$$

$$\frac{d}{dn} E(Profit) = \frac{1}{200} \left(\left(n - \frac{1}{2}n\right) - \left(0 - \frac{1}{2}n\right) + (200 - 5(200-n)) - (n - 5(n-n)) \right)$$

$$\frac{d}{dn} E(Profit) = \frac{1}{200} \left(\frac{1}{2}n + 200 - 5(200-n) - n \right)$$

$$\frac{d}{dn} E(Profit) = \frac{1}{200} \left(\frac{1}{2}n + 200 - 1000 + 5n - n \right)$$

$$\frac{d}{dn} E(Profit) = \frac{1}{200} (4.5n - 800)$$

Now, we set the derivative equal to zero and solve for n:

$$\frac{1}{200} (4.5n - 800) = 0$$

$$4.5n - 800 = 0$$

$$4.5n = 800$$

$$n = \frac{800}{4.5}$$

$$n \approx 177.78$$

Therefore, to maximize profit, the shop should have approximately 178 eyeglasses.

Question ii:

Simulate 100 sales of the shop.

```r
# Define parameters
n <- 178  # Number of eyeglasses
price_per_sale <- 1.00
loss_per_unhappy_customer <- 5.00
loss_per_extra_eyeglass <- 0.50
density_function <- function(x) ifelse(x > 0 & x < 200, 1/200, 0)  # Density function

# Simulate 100 sales
sales <- replicate(100, {
  # Generate random number of customers
  customers <- rpois(1, lambda = n)

  # Calculate expected number of sales
  expected_customers <- integrate(density_function, lower = 0, upper = Inf)$value
  expected_sales <- min(n, expected_customers)

  # Calculate profit and loss
  actual_sales <- min(n, customers)
  profit <- actual_sales * price_per_sale
  loss_unhappy_customers <- max(0, customers - actual_sales) * loss_per_unhappy_customer
  loss_surplus_eyeglasses <- max(0, n - customers) * loss_per_extra_eyeglass

  # Total profit after losses
  total_profit <- profit - loss_unhappy_customers - loss_surplus_eyeglasses

  return(total_profit)
})

# Calculate total profit over 100 sales
total_profit <- sum(sales)

# Print total profit
print(paste("Total profit over 100 sales:", total_profit))
```

Answer:

```
## [1] "Total profit over 100 sales: 14464.5"
```

Part ii:

For uniform (0,1) random variables $U_1, U_2, U_3, \ldots$, define

$$N = \text{minimum}\{n : \sum_{i=1}^{n} U_i > 1\}$$

That is, N is equal to the number of random numbers that must be summed to exceed 1.

Question i:

Estimate $E(N)$ by generating 100 values of N.

```r
# Define the number of simulations
num_simulations <- 100

# Function to simulate N
simulate_N <- function() {
  sum_U <- 0
  N <- 0
  while (sum_U <= 1) {
    sum_U <- sum_U + runif(1)
    N <- N + 1
  }
  return(N)
}

# Generate 100 values of N
N_values <- replicate(num_simulations, simulate_N())

# Estimate E(N)
estimated_mean_N <- mean(N_values)

# Print the estimated mean
print(paste("Estimated E(N) based on 100 simulations:", estimated_mean_N))
```

Answer:

```
## [1] "Estimated E(N) based on 100 simulations: 2.61"
```

Question ii:

Calculate the exact value of $E(N)$.

Answer: To calculate the exact value of $E(N)$, we need to find the expected value of the random variable N. Given the definition of N as the minimum number of random variables that must be summed to exceed 1, we can derive its probability distribution and then calculate the expected value.

Let's denote X_i as the i-th uniform (0,1) random variable, and let p_k be the probability that $N = k$. Then p_k is the probability that the sum of the first k random variables exceeds 1 while the sum of the first $k-1$ random variables is less than or equal to 1.

Therefore, we have:

$$p_k = P(N = k) = P(X_1 + X_2 + \ldots + X_k > 1, X_1 + X_2 + \ldots + X_{k-1} \leq 1)$$

$$= P(X_1 + X_2 + \ldots + X_k > 1) - P(X_1 + X_2 + \ldots + X_{k-1} > 1)$$

$$= (1 - P(X_1 + X_2 + \ldots + X_k \leq 1)) - (1 - P(X_1 + X_2 + \ldots + X_{k-1} \leq 1))$$

$$= P(X_1 + X_2 + \ldots + X_{k-1} < 1) - P(X_1 + X_2 + \ldots + X_{k-1} \leq 1)$$

$$= P(X_1 + X_2 + \ldots + X_{k-1} < 1) - P(X_1 + X_2 + \ldots + X_{k-1} < 1) + P(X_k < 1)$$

$$= P(X_k < 1)$$

Since X_i follows a uniform distribution on the interval (0,1), $P(X_i < 1) = 1$ for all i.

Therefore, $p_k = 1$ for all k.

Now, we have a geometric distribution with parameter $p = 1$, where p is the probability of success (i.e., $N = k$). In a geometric distribution, the expected value $E(N)$ is given by $E(N) = \frac{1}{p}$.

Since $p = 1$, we have $E(N) = \frac{1}{1} = 1$.

So, the exact value of $E(N)$ is 1.