Magnetic Field Spectrum

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A statistically isotropic and homogeneous, Gaussian-distributed magnetic field can be described by the equal time correlation function

$$\langle B_i(\mathbf{x}+\mathbf{r},t)B_j(\mathbf{x},t)\rangle \tag{1}$$

Using the Fourier transform

$$b_i(\mathbf{k},t) = \int d^3x B_i(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2}$$

the k-space correlation function can be defined as

$$\langle b_i^*(\mathbf{k},t)b_j^*(\mathbf{k}',t)\rangle = (2\pi)^6\delta^{(3)}(\mathbf{k}-\mathbf{k}')F_{ij}(\mathbf{k},t), \tag{3}$$

where the spectrum $F_{ij}(\mathbf{k},t)$ is given by

$$\frac{F_{ij}(\mathbf{k},t)}{(2\pi)^3} = (\delta_{ij}-\hat{k}_i\hat{k}_j)\frac{E_M(k,t)}{4\pi k^2} + i\epsilon_{ijk}k_l\frac{H_M(k,t)}{8\pi k^2}\,. \tag{4}$$

Here, $E_M(k)$ is the power spectrum and $H_M(k)$ is the helicity spectrum. Thus the mean magnetic energy density is given by

$$\rho_B(t) = \frac{1}{2}\langle \mathbf{B}^2(\mathbf{x},t)\rangle = \int_0^\infty E_M(k,t)dk \tag{5}$$

For numerical simulations, the ensemble average $\langle\cdot\rangle$ can be expressed as a volume average $\frac{1}{V}\int d^3x(\cdot)$ $\langle \mathbf{B}^2(\mathbf{x},t)\rangle$ can be expressed as

$$\langle \mathbf{B}^2(\mathbf{x},t)\rangle = \frac{1}{V}\int_V d^3x B_i(\mathbf{x},t)B_i(\mathbf{x},t) \tag{6}$$

$$= \frac{1}{V(2\pi)^6}\int_V d^3x \int\int d^3k d^3k' b_i^*(\mathbf{k},t)b_i(\mathbf{k}',t)\exp(i\mathbf{x}.(\mathbf{k}-\mathbf{k}'))$$

$$= \frac{1}{V(2\pi)^3}\int\int d^3k d^3k' b_i^*(\mathbf{k},t)b_i(\mathbf{k}',t)\delta(\mathbf{k}-\mathbf{k}')$$

$$= \frac{1}{V}\int \frac{d^3k}{(2\pi)^3}b_i^*(\mathbf{k},t)b_i(\mathbf{k},t)$$

On a 3D lattice with length $L = N\delta x$, the coordinates are given by $X_i\delta x$ for $i = 1, 2, 3$, corresponding to x,y and z, respectively. The corresponding discrete wavenumbers are given by

$$k_i = 2\pi K_i'/L, \tag{7}$$

where,

$$K_i' = \begin{cases} K_i & K_i \leq N/2 \\ K_i - N & K_i > N/2 \end{cases} \tag{8}$$

The discrete fourier transform can thus be written in terms of the discrete coordinates $\mathbf{X}$ and $\mathbf{K}$ as

$$b_i(\mathbf{K}) = (\delta x)^3\sum_{\mathbf{X}} B_i(\mathbf{X})\exp\left(2\pi\frac{\mathbf{K}\cdot\mathbf{X}}{N}\right) \tag{9}$$

$$B_i(\mathbf{X}) = \frac{1}{L^3}\sum_{\mathbf{K}} b_i(\mathbf{K})\exp\left(-2\pi\frac{\mathbf{K}\cdot\mathbf{X}}{N}\right) \tag{10}$$

Using $k_i = 2\pi \mathbf{K}'_i/L$, the ensemble average in (6) can now be discretized as

$$\langle \mathbf{B}^2(\mathbf{x},t)\rangle = \frac{1}{L^6}\sum_{\mathbf{K}'} b_i^*(\mathbf{K}',t) b_i(\mathbf{K}',t) \tag{11}$$

The discretized mean magnetic energy density (5) can be expressed as

$$\rho_B(t) = \frac{1}{2}\langle \mathbf{B}^2(\mathbf{x},t)\rangle = \sum_{K'} E_M(K',t)\Delta K', \tag{12}$$

where $K' = |\mathbf{K}'|$. Thus the one dimensional power spectrum $E_M(K')$ can be written as

$$E_M(K',t) = \frac{1}{2\Delta K'}\frac{1}{L^6}\sum_{K'=|\mathbf{K}'|} b_i^*(\mathbf{K}',t) b_i(\mathbf{K}',t) \tag{13}$$

One thing to note that the power spectrum component in (3) is $E_M(k,t)/(4\pi k^2)$. The analogical discrete version would be given by $E_M(K',t)L^2/16\pi^3 K'^2$.

To make quantitative estimates, one would need to compute the average of the estimated power spectrum over multiple realizations of the magnetic field. The mean power across all realizations, $\bar{E_M}(k',t)$ is given by

$$\bar{E_M}(K',t) = \frac{1}{N_r}\sum_{i=1}^{N_r} E_M^{(i)}(K',t), \tag{14}$$

where N_r is the number of realizations and the $E_M^{(i)}(K',t)$ is the estimated power spectrum of the i-th realization. The variance in $E_M(K',t)$ is given by

$$\sigma^2(\bar{E_M}(K',t)) = \frac{1}{N_r - 1}\sum_{i=1}^{N} [E_M(K',t) - \bar{E_M}(K',t)]^2 \tag{15}$$

The estimator of error is the expected uncertainty given by

$$(\Delta \bar{E_M}(K',t))^2 \equiv \frac{1}{N_r}\sigma^2(\bar{E_M}(K',t)) \tag{16}$$