Outline

① Setup (Coordinates + Integral full express)

② Find $\beta(t)$

③ Find $\hat{n}\cdot\hat{x} = n_x$

④ Use it to find $\theta \in [\omega_0, \omega_0]$

⑤ Integrate θ away

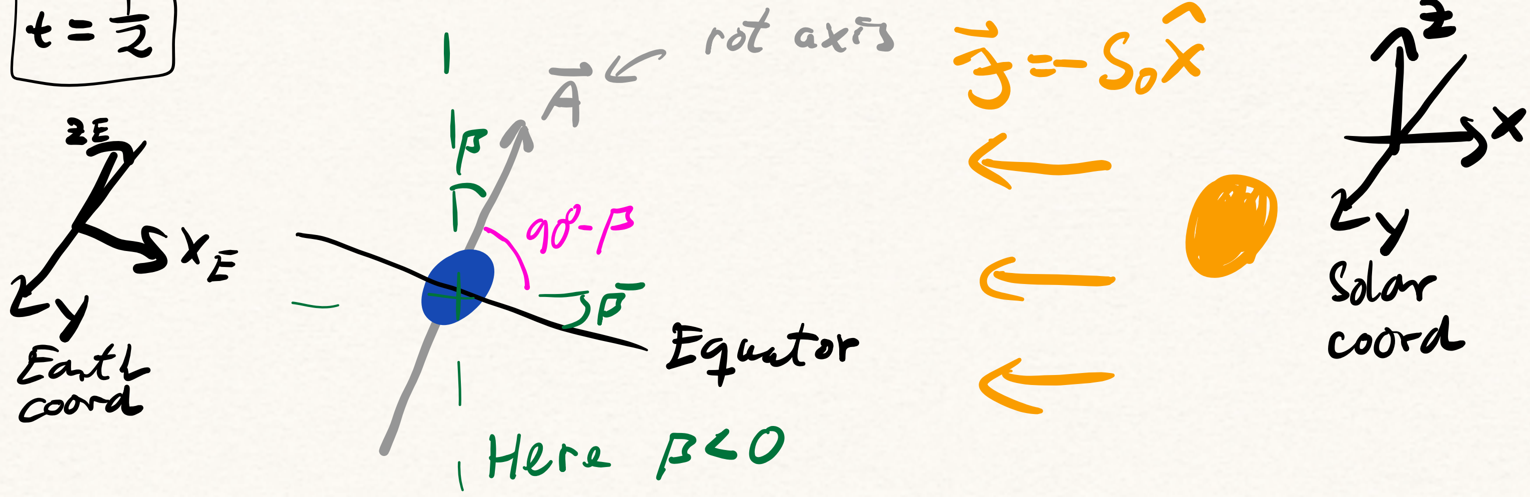
⑥ Year average

① Setup

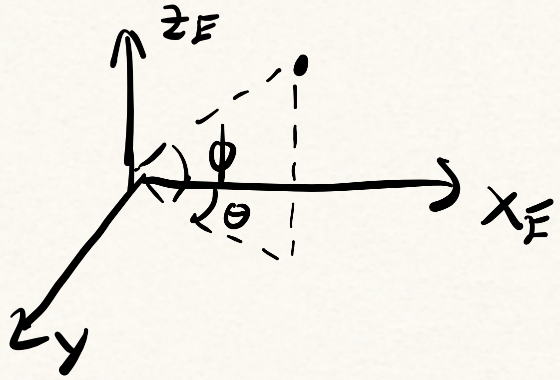
[Simplify: since only care about rot about x_E does not matter for solar exposition]

At time t, center as

$\boxed{t = \frac{1}{2}}$



Introduce ϕ, θ as follows:



Note: $\phi = \frac{\pi}{2} - \varphi$ (azimuth angle)

Hence

$\sin\varphi = \cos\phi$

$\cos\varphi = \sin\phi$

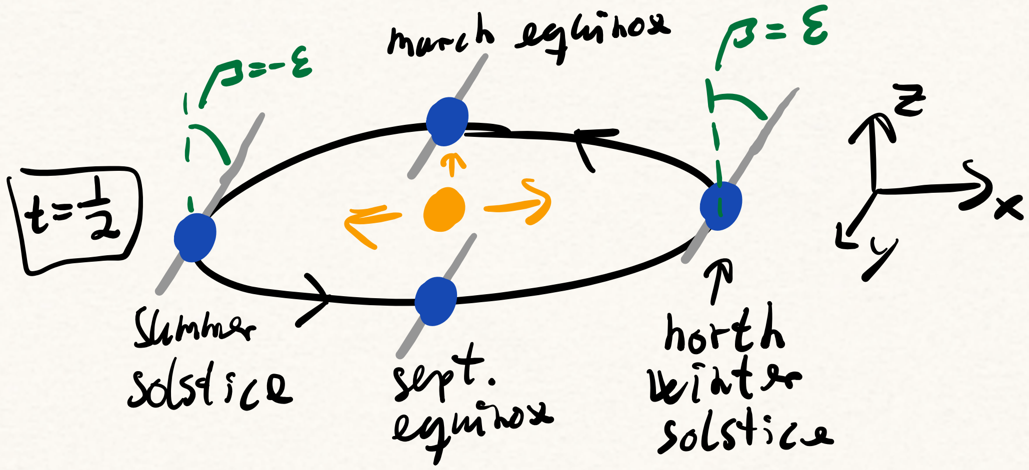
We want $S(x)$, the line-density of power with $x = \sin\phi$

We want S(x); the year-averaged irradiance. It should be the density function such that $\int_{x_1}^{x_2} S(x)\mathrm{d}x$ gives the yearly-averaged total power radiated into the lattitudes $x \in [x_1, x_2]$. In our coordinate system we have that the flux of power P radiated into the domain D is:

$$P = -\int_D \boldsymbol{J} \cdot \boldsymbol{dS} = \int_D S_0 \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}} R^2 \cos\phi \mathrm{d}\theta \mathrm{d}\phi = S_0 R^2 \int_D \hat{n}_x \mathrm{d}\theta \mathrm{d}x$$

② Finding $\beta(t)$

Variation of β due to orbit



Rays from sun:

$$\hat{r} = [\cos 2\pi t, \sin 2\pi t, 0]$$

Earth axis

$$\hat{a} = [\sin \varepsilon, 0, \cos \varepsilon]$$

Hence

$$\sin(\beta) = \cos(90° - \beta) = \cos(\angle(\hat{a}, \hat{r}))$$

$$= \frac{\hat{a} \cdot \hat{r}}{|\hat{a}||\hat{r}|} = \sin \varepsilon \cos(2\pi t)$$

So, since $\beta \in [-\varepsilon, \varepsilon] \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}] = \text{range}(\arcsin)$

$$\boxed{\beta(t) = \arcsin(\sin \varepsilon \cos(2\pi t))}$$

③ Finding n_x

In earth coordinates (x_E, y, z_E) we have

$$\hat{n}(\theta, \varphi) = [\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi]$$

To find $\hat{n}_x(\theta, \varphi)$ on solar coordinates:

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\hat{n}_x = (R_y(\beta)\,\hat{n}_E)_x = \boxed{\cos\theta\cos\phi\cos\beta + \sin\phi\sin\beta}$$

④ Finding angle of day $\theta \in [-\omega_0, \omega_0]$

To get sunlight, need $\hat{n}_x > 0$:

$$\cos\theta\,\underbrace{\cos\phi^{\in[-\frac{\pi}{2}, \frac{\pi}{2}]}}_{\geq 0}\,\underbrace{\cos\beta^{\in[-\varepsilon, \varepsilon]}}_{\geq 0} + \sin\phi\sin\beta > 0$$

$$\boxed{\cos\theta > -\tan\phi\tan\beta}$$

Hence

$$\boxed{\omega_0 = \begin{cases} 0 \\ \arccos(-\tan\phi\tan\beta) \\ \pi \end{cases}}$$

$\overbrace{\text{if } \tan\phi\tan\beta \leq -1}^{\text{* Probably in terms of } \phi,\ \varepsilon \ldots}$

else

$\overbrace{\text{if } \tan\phi\tan\beta \geq 1}^{*}$

⑤ Get rid of θ

$$\int_{-w_0}^{w_0} n_x \, d\theta = 2\int_0^{w_0} \cos\theta\cos\phi\cos\beta + \sin\phi\sin\beta$$

$$= \boxed{2\left(\sin w_0 \cos\phi\cos\beta + w_0 \sin\phi\sin\beta\right)}$$

⑥ Average on $t \in [-\frac{1}{2}, \frac{1}{2}]$ (1 year)

Recall

$$\beta(t) = \arcsin\left(\sin\varepsilon\cos(2\pi t)\right)$$

$$w_0(t) = \begin{cases} 0 & \text{if } \tan\phi\tan\beta \le -1 \\ \arccos(-\tan\phi\tan\beta) & \text{else} \\ \pi & \text{if } \tan\phi\tan\beta \ge 1 \end{cases}$$

Note that $\beta(-t) = \beta(t)$. Hence $w_0(t) = w_0(-t)$. Also this means $\sin(\beta(t)) = \sin(\beta(-t))$ and so on.
I.e. all is symmetric.

$$\int_{-1/2}^{1/2} 2\left(\sin w_0 \cos\phi\cos\beta + w_0 \sin\phi\sin\beta\right) dt$$

$$= 4\int_0^{1/2} \sin w_0 \cos\phi\cos\beta + w_0 \sin\phi\sin\beta \, dt$$