Graph Theory Definitions

1 Introduction to Graph Theory

- **Graph**

- **Vertex**

- **Edge set**

- **Order**

- **Size**

- **Neighborhood**

- **Degree**

- **Complete Graph**

- **Empty Graph**

- **Complement**

- **Path Graph**

- **Cycle Graph**

- **K-Regular Graph**

- **Handshaking Lemma**

- **Subgraph**

- **Induced Subgraph**

- **Isomorphism**

- **Clique**

- **Independent Set**

- **Connected Components**

- **Maximum Degree ($\Delta(G)$)**

2 Trees and Distance

- **Tree**

- **Forest**

- **Leaf**

- **Spanning Subgraph**

- **Spanning Tree**

3 Hamiltonian Cycles and Eulerian Walks

- **Hamiltonian Graph**

- **Hamiltonian Cycle**

- **Bondy-Chvátal Theorem**

- **Eulerian Circuit**

- **Eulerian Trail**

4 Matchings

- **Bipartite Graph**

- **Proper Bicoloring**

- **Vertex Cover**

- **Independent Set**

- **Matching**

- **Edge Cover**

- **Hall's Theorem**

- **Hall's Marriage Theorem**

- **Defect Hall's Marriage Theorem**

- **König's Theorem**

5 Max Flow Min Cut

- **Directed Graph**

- **Flow Network**

- **Capacity**

- **Flow**

- **Cut**

- **Maximum Flow = Minimum Cut Theorem**

6 Connectivity

- **Vertex Connectivity ($\kappa(G)$)**

- **Vertex-Separator**

- **k-vertex connected**

- **State what happen to the vertex connectivity by removing an arbitrary edge, which is analysis $K(G-e)$**

- **Edge Connectivity ($\lambda(G)$)**

- **Bridge**

- **k-Edge connected**

- **Line Graph** A Line graph $L(G)$ is a graph where treat the edges in G as vertices.

  $ab \in V(L(G)), a = uv \in E(G), b = vw \in E(G)$

- **Vertex Menger's Theorem**

  The smallest s, t separable is the largest number of vertex disjiont set between s, t

- **Vertex Menger's Corollary**

  Graph is k vertex connected $\Leftrightarrow$ there exists k vertex disjoint path between s, t

- **Edge Menger's Theorem**

- **Edge Menger's Corollary**

- **State the raltionship between $k(G)$ and $\lambda(G)$**

$$k(G) \leq \lambda(G)$$

The vertex connected is less or equal to edge connectivity.

7 Planar Graphs and Coloring

- **Planar Graph**

- **Plane Graph**

- **Face**

- **Boundary of a Face** (∂F)

- **Chromatic Number** ($\chi(G)$)

- **k-Partite Graph**

  V can be parition into K different components such that $G[V_i]$ has no edges.

- **Clique Number ($\omega(G)$)**

- **Euler's Formula**

  For connected and planar graph, $n - m + f = 2$

- **k-Degenerate Graph**

  G is $k-$degenerate graph if exists a vertex sequences $v_1, \cdots, v_n$ such that $deg(v_i) \leq k$ in graph $G \setminus \{v_1, \cdots, v_{i-1}\}$

- $r-$ **Proper Edge Coloring**

Let $C : E \to \{r, b, w, \cdots\}$, $C(e_1) \neq C(e_2)$, if e_1, e_2 sharing the same vertex.

- **Kuratowski's theorem**

- $r - b$ **kempe chains**

  A connected component on induced subgraph with vertices with color *red* and *blue*

- **Vizing-theorem**

$$\Delta - 1 \leq \chi'(G) \leq \Delta$$

  The number of smallest edge coloring is either $\Delta - 1$ or Δ

- **Brief Explain how fix color theorem being proved**

- **Prove for planar graph** $n \geq 3, m \leq 3n - 6$

- **Show that for every planar graph,** $\exists v \in V(G), deg_G(v) \leq 5$

- **Six Color Theorem Proof**

- **Prove $K_{3,3}$ is not planar**

- **State 5 Color Theorem Proof Technique**

8 Chromatic Polynomial

- **Chromatic Polynomial ($P_G(r)$)**

- **Edge Contraction**

- **Chromatic Polynomial Recursion Formula**
  $P_G(r) = P_{G\backslash e}(r) - P_{G/e}$

- **Proper r-Coloring**

- **Clique Polynomial**

- **Acyclic Orientation**

  For graph G, A_G is a choice of direction for each edges no directed cycle occurs

- A_G

  The number of possilbe acyclic orientation for graph G.

- **How to compute** A_G

  $A_G = (-1^{|G|} P_G(-1)$

- **GHRV Theorem**

$$\min_{\text{Orientation of the graph}\mathcal{O}} \max_{\text{directed path } P_i} \text{number of vertices for longest directed path P}$$

9 Ramsey Theory

- **Ramsey Number**

- **Ramsey Theory Application**

- **k-Uniform Hypergraph**

  A graph G with a finite set of vertices named V. And a family of size k subsets of V is called edges.

- **Complete k-Uniform Hypergraph**

  A graph G contains a complete $k-$uniform hypergraph if $A \subset V$ and $|A| = k$. The graph G contains all subset of A with length k.

- **k-Colored Ramsey Number**

- **Proof technique about** $R(a, b) = n$

  $R(a, b) \geq n$, we need to show that for $n - 1$ vertices there is no way for finding a red K_a or a blue K_b.

  $R(a, b) \leq n$, we need to show that for any coloring of K_n, we could find a red K_a and a blue K_b

10 Linear Algebra and Graph Theory

- **Adjacency Matrix** A_G

- $(A_G^k)_{ij}$

- **What will be return from** $A_G[1, \cdots, 1]$

- **What will be return from** $A_G[v_1, \cdots, v_n]$

- **Explain what can we get from spectral theorem**

- **Relationship between eigenvalue and maximum degree**

  $|\lambda| < \Delta$

- **For a connected graph, what will happen if G is regular, what will happen if G is regular and bipartite**

  If Δ is the eigen-value, then G is regular.

  If $-\Delta$ is the eigen-value, then G is regular and bipartite

- **Relationship between λ_{max} and σ the smallest degree**

  $\lambda_{max} \geq \sigma$

- **Relationship between $\chi(G)$ and λ_{max}**

  $\chi(G) \leq \lambda_{max} + 1$

- **Diameter $diam(G)$**

  The size of longest distance between two vertices in G

- **Moore graph**

  A $k-$regular graph with $diam(G) = 2$ and with vertices $k^2 + 1$

- **Moore graph exists on which k**

Possible k values are $2, 3, 7$ and maybe 57