2b) :-

Assume we have an ordinal response variable Y with ~~rak~~ k categories.

Proportional odds model, which is a common approach in ordinal logistic Regression.

$$\log\left[\frac{P(Y\le K)}{1-P(Y\le K)}\right] = \alpha_K + \beta_1 X_1 + \beta_2 X_2 + \cdots \beta_p x_p$$

$$\text{Logit}(P\le j) = \theta_j - \beta^T x_i$$

$P(Y\le j)$: Cummalative probability

θ_j : threshold parameter

$\beta^T x_i$: linear combinations

$$\therefore\ \text{Logit}(P(Y\le K)) = \log\left[\frac{P(Y\le K)}{1-P(Y\le K)}\right]$$

$$P(Y_i\le K) = \frac{1}{1+e^{-(\alpha_K+\beta_1 X_1+\cdots \beta_p X_p)}}$$

The Contribution from a single Multinomial observation Likelihood function

$$\pi_1^{n_1}, \pi_2^{n_2} \ldots\ldots \pi_K^{n_K}$$

$R_1 = n_1$ $\quad\quad$ $Z_1 = R_1/n$

$R_2 = n_1 + n_2$ $\quad\quad$ $Z_2 = R_2/n$

$\vdots$ $\quad\quad$ $\vdots$

$R_K = \Sigma n_j = n_j$ $\quad\quad$ $Z_K = R_K/n = 1$

The Likelihood function can be expressed as the product of K-1 distinct quantities.

$$\left\{\left(\frac{\gamma_1}{\gamma_2}\right)^{R_1}\left(\frac{\gamma_2-\gamma_1}{\gamma_2}\right)^{R_2-R_1}\right\}\left\{\left(\frac{\gamma_2}{\gamma_3}\right)^{R_2}\left(\frac{\gamma_3-\gamma_2}{\gamma_3}\right)^{R_3-R_2}\right\}$$

$$\cdots\left(\frac{\gamma_{K-1}}{\gamma_K}\quad\frac{\gamma_K-\gamma_{K-1}}{\gamma_K}\right)^{R_K-R_{K-1}}$$

$$\phi_i = \log\left(\frac{\gamma_j}{\gamma_{j+1}-\gamma_j}\right) = \text{Logit}\left(\frac{\gamma_j}{\gamma_{j+1}}\right)$$

$$= \text{Log}\left[\left(\frac{\gamma_{i+1}-\gamma_j+\gamma_j}{\gamma_{j+1}-\gamma_j}\right)\right] = \text{Log}\left[\frac{\gamma_{j+1}}{\gamma_{j+1}-\gamma_j}\right]$$

$$L = \eta\left[Z_1\phi_1 - Z_2 g(\phi_1)\right] + \cdots$$

$$+\left\{Z_{K-1}\phi_{K-1} - g(\phi_{K-1})\right\}$$