Tutorial 2

- Please try the problems by yourself and do not look online for solutions. It defeats the purpose of these exercises if you directly search for the solutions.
- The sheet will be updated with a few problems from the next topic (Sets, functions and Relations)
- Not all questions are of the same difficulty level.

Topics: WOP, Mathematical Induction, Structural Recursion

1. (False Proof) Read Section 5.1.5 of the LLM book on tiling a square with a piece missing. Solve **Problem 5.2** based on it.

2. Suppose that we want to prove that:

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

   (a) Show that if we try to prove this using mathematical induction, the base case works, but the inductive step fails.

   (b) Show that you can prove the stronger inequality:

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

   for all integers $n > 1$, which together with a verification for the case $n = 1$, completes the proof of the original statement.

3. Prove that if a statement can be proved by (ordinary) mathematical Induction, it can be proved by the Well-Ordering Principle.

4. Prove using the Well-Ordering Principle that for all non-negative integers n,

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

5. Show that we can prove that $P(n, k)$ is true for all pairs of positive integers n, k if we show:

   (a) $P(1, 1)$ is true and $P(n, k) \implies [P(n+1, k) \wedge P(n, k+1)]$ is true for all positive integers n and k.

   (b) $P(1, k)$ is true for all positive integers k, and $P(n, k) \implies P(n+1, k)$ is true for all positive integers n and k.

   (c) $P(n, 1)$ is true for all positive integers n and $P(n, k) \implies P(n, k+1)$ is true for all positive integers n, k.

6. Use mathematical induction to show that a rectangular chessboard with an even number of cells and two squares of different colors missing : one W, one B, *can* be covered by dominoes each of size 2 squares. (Recall: We had shown in class that if the two missing squares are of the same color, we cannot cover the board using dominoes.)

7. (Invariants) Problem 5.42 from LLM MIT textbook.

Sets, Functions

1. Prove or disprove: for all sets A, B, C:

   (a) $A \times (B - C) = (A \times B) - (A \times C)$

   (b) $\overline{A} \times \overline{B \cup C} = \overline{A \times (B \cup C)}$

2. The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [**Hint**: First show that $a, a, b = c, c, d$ if and only if $a = c$ and $b = d$.]

3. Show that $(0, 1)$ and $\mathbb{R}$ have the same cardinality by

   (a) Showing that $f(x) = \frac{2x-1}{2x(1-x)}$ is a bijection from $(0, 1) \to R$.

   (b) Using the Schroeder-Bernstein theorem.

4. Show that the set of finite sequences of non-negative integers is countable.

5. A binary string is just a string consisting of 0's and 1's. e.g. the binary string 01110 is of length 5. You can also have an infinite binary string, e.g. $111111\ldots$. Denote the set of all finite length binary strings by $\{0, 1\}*$, and the set of all infinite length binary strings by $\{0, 1\}^\omega$. (The reason for this notation will become clear later in a subsequent class).

   (a) Show that the set $\{0, 1\}^*$ is countable.

   (b) Show that the set $\{0, 1\}^\omega$ is uncountable.

   (c) Let F be the set of all infinite binary strings, but with a finite number of 1's. Show that $F' := \{0, 1\}^\omega - F$ is uncountable. (Hint: Define a suitable function from $\{0, 1\}^\omega$ to F'. Alternatively, show that F is countable, and use it to conclude that F' is not.)