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EE24BTECH11034 - K Teja Vardhan

1) Let $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be vectors in $\mathbb{R}^4$. Let U be the span of $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and let V be the span of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.
   Consider the following statements:
   (*I*) If the dimension of $U \cap V$ is 2 and the dimension of U is 3, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent.
   (*II*) If $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\} = \mathbb{R}^4$, then either $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is linearly independent or $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent.
   Which of the above statements is/are true?
   (*A*) Only (*I*)
   (*B*) Only (*II*)
   (*C*) Both (*I*) and (*II*)
   (*D*) Neither (*I*) nor (*II*)
2) Consider $\mathbb{R}^2$ with standard inner product. If $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is the vector in $\mathbb{R}^2$ such that the inner product of $\mathbf{u}$ with $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is 2 and with $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is -1, then which one of the following statements is true?
   (*A*) Inner product of $\mathbf{u}$ with $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is $\frac{1}{2}$
   (*B*) Inner product $\mathbf{u}$ with $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is $\frac{3}{5}$
   (*C*) Inner product of $\mathbf{u}$ with $\begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix}$ is $-\frac{6}{5}$
   (*D*) Inner product of $\mathbf{u}$ with $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is $\frac{4}{5}$
3) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ be a 2×3 real matrix, where $(a_1, a_2, a_3) \neq (0, 0, 0)$ and $(b_1, b_2, b_3) \neq (0, 0, 0)$. Assume that the rank of A is 1. Define the subspaces

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : Ax = 0 \right\},$$

$$W_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : a_1x_1 + a_2x_2 + a_3x_3 = 0 \right\},$$

   and

$$W_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : b_1 x_1 + b_2 x_2 + b_3 x_3 = 0 \right\}.$$

Consider the following statements:
$(I)\; W = W_1 \cap W_2$
$(II)\; W_1 = W_2$
Which of the above statements is/are true?

(A) Only (I)
(B) Only (II)
(C) Both (I) and (II)
(D) Neither (I) nor (II)

4) Let X be a random variable taking only two values, 1 and 2. Let $M_X(\cdot)$ be the moment generating function of X. If the expectation of X is $\frac{10}{7}$, then the fourth derivative of $M_X(\cdot)$ evaluated at 0 equals
   a) $\frac{52}{7}$
   b) $\frac{67}{7}$
   c) $\frac{48}{7}$
   d) $\frac{60}{7}$
5) Two fair dice, one having red and another having blue colour, are tossed independently once. Let A be the event that the die having red colour will show 5 or 6. Let B be the event that the sum of the outcomes will be 7 and let C be the event that the sum of the outcomes will be 8. Then which one of the following statements is true?
   a) A and B are independent as well as A and C are independent
   b) A and B are independent, but A and C are not independent
   c) A and C are independent, but A and B are not independent
   d) Neither A and B are independent, nor A and C are independent
6) Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

   where $\alpha > 0, \beta > 0$. If $E(X) = \frac{1}{3}$ and $E\left(X^2\right) = \frac{1}{6}$, then $\alpha + 3\beta$ equals
   a) 7
   b) 5
   c) 4
   d) 8
7) Let X and Y be two random variables with cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, respectively. Then which one of the following statements is NOT true?
   a) There exist X and Y such that $F_X(u) = F_Y(u)$ for all $u \in \mathbb{R}$, and $P(X \neq Y) > 0$.
   b) There exist X and Y such that $F_X(u) = F_Y(u)$ for all $u \in \mathbb{R}$, and $P(X = Y) = 0$.
   c) If X and Y are independent, then X^2 and Y^2 are also independent.

d) If X^2 and Y^2 are independent, then X and Y are also independent.

8) Let $\{F_n\}_{n\geq 1}$ be a sequence of cumulative distribution functions given by

$$F_n(x) = \begin{cases} 0 & \text{if } x < -n \\ \frac{x+n}{2n} & \text{if } -n \leq x < n \\ 1 & \text{if } x \geq n. \end{cases}$$

Which one of the following statements is true?

a) $F_n(x)$ converges for all $x \in \mathbb{R}$ and the limiting function is a cumulative distribution function.

b) $F_n(x)$ converges for all $x \in \mathbb{R}$, but the limiting function is not a cumulative distribution function.

c) $F_n(x)$ does not converge for any $x \in \mathbb{R}$.

d) There exist $x, y \in \mathbb{R}$ such that $F_n(x)$ converges but $F_n(y)$ does not converge.

9) Let $\{W(t)\}_{t\geq 0}$ be a standard Brownian motion. Which one of the following statements is NOT true?

a) $E[W(7)] = 0$

b) $E[W(5)W(9)] = 7$

c) $2W(1)$ is normally distributed with mean 0 and variance 4

d) $E[W(5)|W(3) = 3] = 3$

10) Let X_1, X_2, X_3 be three independent and identically distributed binomial random variables with number of trials $n = 100$ and success probability $p\,(0 < p < 1)$, which is an unknown parameter. Let $T_1 = (X_1 + X_2, X_3)$ and $T_2 = X_1 + X_2 + X_3$. Consider the following statements:
(I) The distribution of T_2 given $T_1 = t_1$ is independent of p.
(II) The distribution of T_1 given $T_2 = t_2$ is independent of p.
Which of the above statements is/are true?

a) Only (I)

b) Only (II)

c) Both (I) and (II)

d) Neither (I) nor (II)

11) Let $X_1, X_2, ..., X_n$ be a random sample of size $n\,(\geq 2)$ from a population having probability density function

$$f(x;\theta) = \begin{cases} \theta(2x)^{\theta-1} & \text{if } 0 < x \leq \frac{1}{2} \\ \theta(2-2x)^{\theta-1} & \text{if } \frac{1}{2} < x \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Then which one of the following is a maximum likelihood estimator of θ?

a) $\left[\frac{1}{n}\sum_{\{i:X_i\leq\frac{1}{2}\}}\log_e 2X_i + \sum_{\{i:X_i>\frac{1}{2}\}}\log_e(2-2X_i)\right]^{-1}$

b) $-n\left[\sum_{\{i:X_i\leq\frac{1}{2}\}}\log_e 2X_i + \sum_{\{i:X_i>\frac{1}{2}\}}\log_e(2-2X_i)\right]^{-1}$

c) $n\left[\sum_{1\leq i\leq n}\log_e 2X_i + \sum_{1\leq i\leq n}\log_e(2-2X_i)\right]^{-1}$

d) $-n\left[\sum_{1\le i\le n}\log_e 2X_i + \sum_{1\le i\le n}\log_e(2-2X_i)\right]^{-1}$

12) In a testing of hypothesis problem, which one of the following statements is true?

a) The probability of the Type-I error cannot be higher than the probability of the Type-II error.

b) Type-II error occurs if the test accepts the null hypothesis when the null hypothesis is actually false.

c) Type-I error occurs if the test rejects the null hypothesis when the null hypothesis is actually false.

d) The sum of the probability of the Type-I error and the probability of the Type-II error should be 1.

13) A random sample of size 40 is drawn from a population having four distinct categories as $i = 1, 2, 3, 4$. The data are given as

Category	1	2	3	4
Observed Frequency	5	8	12	15

Let θ_i be the probability that an observation comes from the i-th category, $i = 1, 2, 3, 4$. If the chi-square goodness-of-fit test is used to test $H_0 : \theta_i = \frac{1}{4}, i = 1, 2, 3, 4$ against $H_1 : \theta_i \neq \frac{1}{4}$ for some $i = 1, 2, 3, 4$, then which one of the following statements is true?

a) Under H_0, the test statistic follows central chi-square distribution with 3 degrees of freedom and the observed value of the test statistic is 5.8.

b) Under H_0, the test statistic follows central chi-square distribution with 3 degrees of freedom and the observed value of the test statistic is 1.4.

c) Under H_0, the test statistic follows central chi-square distribution with 4 degrees of freedom and the observed value of the test statistic is 5.8.

d) Under H_0, the test statistic follows central chi-square distribution with 4 degrees of freedom and the observed value of the test statistic is 1.4.