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1. Given two $n \times n$ matrices A and B with entries in \mathbb{C} , consider the following statements:

P : If A and B have the same minimal polynomial, then A is similar to B .

Q : If A has n distinct eigenvalues, then there exists $u \in \mathbb{C}^n$ such that $u, Au, \dots, A^{n-1}u$ are linearly independent.

Which of the above statements hold TRUE?

- (a) Both P and Q
 - (b) Only P
 - (c) Only Q
 - (d) Neither P nor Q
2. Let $A = (a_{ij})$ be a 10×10 matrix such that $a_{ij} = -1$ for $i \neq j$ and $a_{ii} = \alpha + 1$, where $\alpha > 0$. Let λ and μ be the largest and the smallest eigenvalues of A , respectively. If $\lambda + \mu = 24$, then α equals
 3. Let C be the simple, positively oriented circle of radius 2 centered at the origin in the complex plane. Then $\frac{2}{\pi i} \int_C \left[\frac{e^{i\theta}}{2} + \tan\left(\frac{z}{2}\right) + \frac{1}{(z-1)(z-3)^2} \right] dz =$
 4. Let $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$, respectively, denote the real part and the imaginary part of a complex number z . Let $T : \mathbb{C} \cup [\infty] \rightarrow \mathbb{C} \cup [\infty]$ be the bilinear transformation such that $T(6) = 0$, $T(3 - 3i) = i$, and $T(0) = \infty$. Then, the image of $D = \{z \in \mathbb{C} : |z - 3| < 3\}$ under the mapping $w = T(z)$ is
 - (a) $\{w \in \mathbb{C} : \operatorname{Im}(w) < 0\}$
 - (b) $\{w \in \mathbb{C} : \operatorname{Re}(w) < 0\}$
 - (c) $\{w \in \mathbb{C} : \operatorname{Im}(w) > 0\}$
 - (d) $\{w \in \mathbb{C} : \operatorname{Re}(w) > 0\}$
 5. Let (x_n) and (y_n) be two sequences in a complete metric space (X, d) such that $d(x_n, x_{n+1}) \leq \frac{1}{n^2}$ and $d(y_n, y_{n+1}) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Then
 - (a) both (x_n) and (y_n) converge

- (b) (x_n) converges but (y_n) need NOT converge
- (c) (y_n) converges but (x_n) need NOT converge
- (d) neither (x_n) nor (y_n) converges
6. Let $f : [0, 1] \rightarrow \mathbb{R}$ be given by $f(x) = 0$ if x is rational, and if x is irrational then $f(x) = 9^n$, where n is the number of zeros immediately after the decimal point in the decimal representation of x . Then the Lebesgue integral $\int_0^1 f(x) dx$ equals
7. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{\sin(x^2 + y^2)}{\sqrt{x^2 + y^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ Then, at $(0, 0)$,
- (a) f is continuous and the directional derivative of f does NOT exist in some direction
- (b) f is NOT continuous and the directional derivatives of f exist in all directions
- (c) f is NOT differentiable and the directional derivatives of f exist in all directions
- (d) f is differentiable
8. Let D be the region in \mathbb{R}^2 bounded by the parabola $y^2 = 2x$ and the line $y = x$. Then $\iint_D 3xy dx dy =$
9. Let $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ for $x \in \mathbb{R}$. Consider the following statements: P : $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0$ on \mathbb{R} . Q : The Wronskian $y_1(x) \frac{dy_2}{dx}(x) - y_2(x) \frac{dy_1}{dx}(x) = 0$ for all $x \in \mathbb{R}$. Which of the above statements hold TRUE?
- (a) Both P and Q
- (b) Only P
- (c) Only Q
- (d) Neither P nor Q
10. Let α and β with $\alpha > \beta$ be the roots of the indicial equation of $(x^2 - 1)^2 \frac{d^2 y}{dx^2} + (x + 1) \frac{dy}{dx} - y = 0$ at $x = -1$. Then $\alpha - 4\beta$ equals
11. Let S_9 be the group of all permutations of the set $[1, 2, 3, 4, 5, 6, 7, 8, 9]$. Then the total number of elements of S_9 that commute with $\tau = (123)(4567)$ in S_9 equals

12. Let $\mathbb{Q}[x]$ be the ring of polynomials over \mathbb{Q} . Let $x^4 - 1 = p_1 p_2 \cdots p_n$ be the prime factorization of $x^4 - 1$ into irreducible polynomials in $\mathbb{Q}[x]$ over \mathbb{Q} . Then the total number of maximal ideals in the quotient ring $\mathbb{Q}[\frac{[x]}{(x^4-1)}]$ equals
13. Let $[e_n]_{n \in \mathbb{N}}$ be an orthonormal basis of a Hilbert space H . Let $T : H \rightarrow H$ be given by $Tx = \sum_{n=1}^{\infty} \frac{1}{n} \langle x, e_n \rangle e_n$. For each $n \in \mathbb{N}$, define $T_n : H \rightarrow H$ by $T_n x = \sum_{j=1}^n \frac{1}{j} \langle x, e_j \rangle e_j$.

Then

- (a) $|T_n - T| \rightarrow 0$ as $n \rightarrow \infty$
- (b) $|T_n - T| \rightarrow 0$ as $n \rightarrow \infty$ but for each $x \in H$, $|T_n x - Tx| \rightarrow 0$ as $n \rightarrow \infty$
- (c) for each $x \in H$, $|T_n x - Tx| \rightarrow 0$ as $n \rightarrow \infty$ but the sequence $(|T_n|)$ is unbounded
- (d) there exist $x, y \in H$ such that $\langle T_n x, y \rangle \nrightarrow \langle Tx, y \rangle$ as $n \rightarrow \infty$