ASSIGNMENT 12

EE24BTECH11034 - K Teja Vardhan

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1) If $F(s) = \tan^{-1}(s) + k$ is the Laplace transform of some function $f(t)$, $t \geq 0$, then k is:

   a) $-\pi$ b) $-\frac{\pi}{2}$ c) 0 d) $\frac{\pi}{2}$

2) Suppose $y_p(x) = x\cos(2x)$ is a particular solution of $y'' + \alpha y = -4\sin(2x)$. Then the constant α equals:

   a) -4 b) -2 c) 2 d) 4

3) Let $S = [(0,1,1), (1,0,1), (-1,2,1)] \subseteq \mathbb{R}^3$. Suppose $\mathbb{R}^3$ is endowed with the standard inner product $\langle \cdot, \cdot \rangle$. Define $M = \left[x \in \mathbb{R}^3 : \langle x, y \rangle = 0 \text{ for all } y \in S\right]$. Then the dimension of M equals:

   a) 0 b) 1 c) 2 d) 3

4) Let X be an uncountable set and let $\mathscr{T} = [U \subseteq X : U = \emptyset \text{ or } U^c \text{ is finite}]$. Then the topological space $(X, \mathscr{T})$:
   a) is separable
   b) is Hausdorff
   c) has a countable basis
   d) has a countable basis at each point
5) Suppose $(X, \mathscr{T})$ is a topological space. Let $[S_n]_{n \geq 1}$ be a sequence of subsets of X. Then:
   a) $(S_1 \cup S_2)^c = S_1^c \cup S_2^c$
   b) $\left(\bigcup_n S_n\right)^c = \bigcup_n S_n^c$
   c) $\overline{\bigcup_n S_n} = \bigcup_n \overline{S_n}$
   d) $\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$
6) Suppose (X, d) is a metric space. Consider the metric ρ on X defined by:

$$\rho(x, y) = \min\left[\frac{1}{2}, d(x, y)\right], \quad x, y \in X.$$

   Suppose $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on X defined by d and ρ, respectively. Then:
   a) $\mathscr{T}_1$ is a proper subset of $\mathscr{T}_2$
   b) $\mathscr{T}_2$ is a proper subset of $\mathscr{T}_1$
   c) Neither $\mathscr{T}_1 \subseteq \mathscr{T}_2$ nor $\mathscr{T}_2 \subseteq \mathscr{T}_1$
   d) $\mathscr{T}_1 = \mathscr{T}_2$

7) A basis of $V = \left[(x,y,z,w) \in \mathbb{R}^4 : x+y-z=0,\ y+z+w=0,\ 2x+y-3z-w=0\right]$ is:

a) $[(1,1,-1,0),(0,1,1,1)]$

b) $[(1,-1,0,1)]$

c) $[(1,0,2,1,-1,3)]$

d) $[(1,-1,0,1),(1,0,1,-1)]$

8) Consider $\mathbb{R}^3$ with the standard inner product. Let

$$S = [(1,1,1),(2,-1,2),(1,-2,1)].$$

For a subset W of $\mathbb{R}^3$, let $L(W)$ denote the linear span of W in $\mathbb{R}^3$. Then an orthonormal set T with $L(S) = L(T)$ is

a) $\left[\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{6}}(1,-2,1)\right]$

b) $[(1,0,0),(0,1,0),(0,0,1)[$

c) $\left[\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,-1,0)\right]$

d) $\left[\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(0,1,-1)\right]$

9) Let A be a 3×3 matrix. Suppose that the eigenvalues of A are $-1,0,1$ with respective eigenvectors $(1,-1,0)^T$, $(1,1,-2)^T$, and $(1,1,1)^T$. Then $6A$ equals:

a) $\begin{bmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} -2 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

d) None of the above

10) A matrix is called row stochastic if the sum of each of its row entries is 1. Suppose P is a row stochastic matrix of size 4×4. Then for any $\mathbf{x} \in \mathbb{R}^4$ and for any $n \geq 0$,

$$\mathbf{1}^T P^n \mathbf{x} =$$

a) 1

b) 0

c) $\mathbf{1}^T\mathbf{x}$

d) $\mathbf{x}$

11) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let $B = A^n$ for $n \geq 1$. Then B equals:

a) $2^{n-1}A$

b) nA

c) $2^{n-2}A$

d) A^2

12) Let

$$u(x,y) = f(xe^x) + g\left(y^2\cos(y)\right),$$

where f and g are infinitely differentiable functions. Then the partial differential equation of minimum order satisfied by u is

a) $u_{xx} + xu_{xy} = u_x$
b) $u_{yy} + xu_{xx} = xu_x$
c) $u_{yy} - xu_{xy} = u_x$
d) $u_{yy} - xu_{xx} = xu_x$

13) Let C be the boundary of the triangle formed by the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Then the value of the line integral

$$\int_C -2y\,dx + \left(3x - 4y^2\right)\,dy + \left(z^2 + 3y\right)\,dz$$

is

a) 0
b) 1
c) 2
d) 4

14) Let X be a complete metric space and let $E \subseteq X$. Consider the following statements:

a) E is compact,
b) E is closed and bounded,
c) E is closed and totally bounded,
d) Every sequence in E has a subsequence converging in E.

Which one of the above statements does NOT imply all the other statements?

a) S_1 b) S_2 c) S_3 d) S_4

15) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin\left(nx\right).$$

Then the series

a) converges uniformly on $\mathbb{R}$
b) converges pointwise but NOT uniformly on $\mathbb{R}$
c) converges in L^1 norm to an integrable function on $[0, 2\pi]$ but does NOT converge uniformly on $\mathbb{R}$
d) does NOT converge pointwise

16) Let $f(z)$ be an analytic function. Then the value of

$$\int_0^{2\pi} f\left(e^{it}\right) \cos\left(t\right) dt$$

equals

a) 0
b) $2\pi f(0)$
c) $2\pi f'(0)$
d) $\pi f'(0)$

17) Let G_1 and G_2 be the images of the disc $[z \in \mathbb{C} : |z+1| < 1]$ under the transformations

$$w = \frac{(1-i)\,z+2}{(1+i)\,z+2}$$

and

$$w = \frac{(1+i)\,z+2}{(1-i)\,z+2}$$

respectively. Then

a) $G_1 = [w \in \mathbb{C} : \mathrm{Im}\,(w) < 0]$ and $G_2 = [w \in \mathbb{C} : \mathrm{Im}\,(w) > 0]$
b) $G_1 = [w \in \mathbb{C} : \mathrm{Im}\,(w) > 0]$ and $G_2 = [w \in \mathbb{C} : \mathrm{Im}\,(w) < 0]$
c) $G_1 = [w \in \mathbb{C} : |w| > 2]$ and $G_2 = [w \in \mathbb{C} : |w| < 2]$
d) $G_1 = [w \in \mathbb{C} : |w| < 2]$ and $G_2 = [w \in \mathbb{C} : |w| > 2]$