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EE24BTECH11034 - K Teja Vardhan

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1. Given two $n \times n$ matrices A and B with entries in $\mathbb{C}$, consider the following statements:

   P: If A and B have the same minimal polynomial, then A is similar to B.

   Q: If A has n distinct eigenvalues, then there exists $u \in \mathbb{C}^n$ such that $u, Au, \ldots, A^{n-1}u$ are linearly independent.

   Which of the above statements hold TRUE?

   (a) Both P and Q
   (b) Only P
   (c) Only Q
   (d) Neither P nor Q

2. Let $A = (a_{ij})$ be a 10×10 matrix such that $a_{ij} = -1$ for $i \neq j$ and $a_{ii} = \alpha + 1$, where $\alpha > 0$. Let λ and μ be the largest and the smallest eigenvalues of A, respectively. If $\lambda + \mu = 24$, then α equals

3. Let C be the simple, positively oriented circle of radius 2 centered at the origin in the complex plane. Then $\frac{2}{\pi i} \int_C \left[\frac{e^{i\theta}}{2} + \tan\left(\frac{z}{2}\right) + \frac{1}{(z-1)(z-3)^2} \right] dz =$

4. Let $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$, respectively, denote the real part and the imaginary part of a complex number z. Let $T : \mathbb{C} \cup [\infty] \to \mathbb{C} \cup [\infty]$ be the bilinear transformation such that $T\,(6) = 0$, $T\,(3 - 3i) = i$, and $T\,(0) = \infty$. Then, the image of $D = [z \in \mathbb{C} : |z - 3| < 3]$ under the mapping $w = T\,(z)$ is

   (a) $[w \in \mathbb{C} : \mathrm{Im}\,(w) < 0]$
   (b) $[w \in \mathbb{C} : \mathrm{Re}\,(w) < 0]$
   (c) $[w \in \mathbb{C} : \mathrm{Im}\,(w) > 0]$
   (d) $[w \in \mathbb{C} : \mathrm{Re}\,(w) > 0]$

5. Let (x_n) and (y_n) be two sequences in a complete metric space (X, d) such that $d\,(x_n, x_{n+1}) \leq \frac{1}{n^2}$ and $d\,(y_n, y_{n+1}) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

   (a) both (x_n) and (y_n) converge

(b) (x_n) converges but (y_n) need NOT converge

(c) (y_n) converges but (x_n) need NOT converge

(d) neither (x_n) nor (y_n) converges

6. Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = 0$ if x is rational, and if x is irrational then $f(x) = 9^n$, where n is the number of zeros immediately after the decimal point in the decimal representation of x. Then the Lebesgue integral $\int_0^1 f(x)\, dx$ equals

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{\sqrt{x^2+y^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ Then, at $(0,0)$,

   (a) f is continuous and the directional derivative of f does NOT exist in some direction

   (b) f is NOT continuous and the directional derivatives of f exist in all directions

   (c) f is NOT differentiable and the directional derivatives of f exist in all directions

   (d) f is differentiable

8. Let D be the region in $\mathbb{R}^2$ bounded by the parabola $y^2 = 2x$ and the line $y = x$. Then $\iint_D 3xy\, dx\, dy =$

9. Let $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ for $x \in \mathbb{R}$. Consider the following statements: P: $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$ on $\mathbb{R}$. Q: The Wronskian $y_1(x)\frac{dy_2}{dx}(x) - y_2(x)\frac{dy_1}{dx}(x) = 0$ for all $x \in \mathbb{R}$. Which of the above statements hold TRUE?

   (a) Both P and Q

   (b) Only P

   (c) Only Q

   (d) Neither P nor Q

10. Let α and β with $\alpha > \beta$ be the roots of the indicial equation of $\left(x^2-1\right)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} - y = 0$ at $x = -1$. Then $\alpha - 4\beta$ equals

11. Let S_9 be the group of all permutations of the set $[1,2,3,4,5,6,7,8,9]$. Then the total number of elements of S_9 that commute with $\tau = (123)(4567)$ in S_9 equals

12. Let $\mathbb{Q}[x]$ be the ring of polynomials over $\mathbb{Q}$. Let $x^4 - 1 = p_1 p_2 \cdots p_n$ be the prime factorization of $x^4 - 1$ into irreducible polynomials in $\mathbb{Q}[x]$ over $\mathbb{Q}$. Then the total number of maximal ideals in the quotient ring $\mathbb{Q}[x]/\left(x^4-1\right)$ equals

13. Let $[e_n]_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space H. Let $T : H \to H$ be given by $Tx = \sum_{n=1}^{\infty} \frac{1}{n}\langle x, e_n\rangle e_n$. For each $n \in \mathbb{N}$, define $T_n : H \to H$ by $T_n x = \sum_{j=1}^{n} \frac{1}{j}\langle x, e_j\rangle e_j$.

    Then

    (a) $|T_n - T| \to 0$ as $n \to \infty$

    (b) $|T_n - T| \to 0$ as $n \to \infty$ but for each $x \in H$, $|T_n x - Tx| \to 0$ as $n \to \infty$

    (c) for each $x \in H$, $|T_n x - Tx| \to 0$ as $n \to \infty$ but the sequence $(|T_n|)$ is unbounded

    (d) there exist $x, y \in H$ such that $\langle T_n x, y\rangle \nrightarrow \langle Tx, y\rangle$ as $n \to \infty$