CALCULUS

1. Limits and Continuity
2. Differentiation
3. Definite Integrals

    Improper Integrals

4. Partial Differentiation
5. Multiple Integrals
6. Vector Differentiation
7. Vector Integration
8. Fourier Series

Calculus is defined as science of acceleration, retardation and variation.

<u>Function</u> :→ A relation between two sets 'A' & 'B' if $\forall \in A$ $\exists$ a unique $y \in B$ s.t. $f(x) = y$

(i) <u>Explicit Function</u>:→ $z = f(x_1, x_2, ---, x_n)$

$$\underset{\text{Dependent variable}}{z} = f(\underset{\text{Independent variable}}{x_1, x_2, ---, x_n})$$

(ii) <u>Implicit function</u>:→ $\phi(z, x_1, x_2, ---, x_n) = C$

(iii) <u>Composite function</u>:→ If $z = f(x,y)$ where $x = \phi(t)$ & $y = \psi(t)$

i.e. z is function of some function.

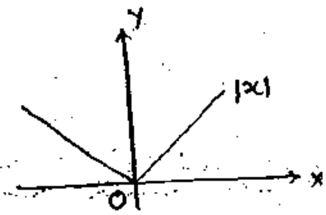
Some Special functions →

(i) Even Function :→ $f(-x) = f(x)$ Eg:- $\cos x$, $|x|$, ...

(ii) Odd function :→ $f(-x) = -f(x)$ Eg:- $\sin x$, x, ...

(iii) Modulus function :→

$$f(x) = |x| = \begin{cases} x & ; & x > 0 \\ -x & ; & x < 0 \\ 0 & ; & x = 0 \end{cases}$$



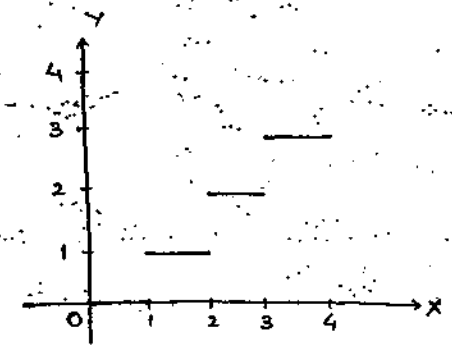
$$\frac{d}{dx}|x| = \frac{|x|}{x} \quad \text{for } x \neq 0$$

(iv) Step - Function / Greatest Integer function /

$f(x) = [x] = n \in \mathbb{Z}$

where, $n \leq x < n+1$

Eg:- $[7.2] = 7$; $[7.999] = 7$; $[-1.2] = -2$



Symmetric Properties of the curve :→

Let $f(x,y) = c$ be the eq^n of the curve

(i) If $f(x,y)$ contains only even powers of x i.e.

$f(-x,y) = f(x,y)$ then it is symmetric about y-axis.

(ii) If $f(x,y)$ contains only even powers of y i.e. $f(x,-y) = f(x,y)$ then it is symmetric about x-axis.

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(iii) If $f(x,y) = f(y,x)$, then, the curve is symmetric about $y = x$.

1. <u>Limit of a function</u>:$\rightarrow$ Let $f(x)$ be defined in deleted neighbour-hood of $a \in R$, then, $l \in R$ is said to be limit of $f(x)$ as x approaches a if for given $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } |x-a| < \delta.$$

$$\boxed{\lim_{x \to a} f(x) = l}$$

<u>Left limit</u>:$\rightarrow$ when $x < a$, $x \to a$

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

<u>Right Limit</u>:$\rightarrow$ when $x > a$, $x \to a$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

A limit exists iff LHL = RHL

<u>Indeterminate form</u>:$\rightarrow$ $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

whenever we have $\lim_{x \to a} \frac{f(x)}{g(x)} \left[\text{as } \frac{0}{0} \text{ or } \frac{\infty}{\infty}\right] = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

This rule is applied until we are free from indeterminate form. (This rule is called <u>L'Hospital Rule</u>)

Standard Limits:$\rightarrow$

(i) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n\,a^{n-1}$

(ii) $\lim\limits_{x \to 0} \dfrac{e^{mx} - 1}{x} = e$

(iii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$

(iv) $\lim\limits_{x \to 0} [1 + ax]^{1/x} = e^a$

(v) $\lim\limits_{x \to \infty} \left[1 + \dfrac{a}{x}\right]^x = e^a$

(vi) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ or $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

(vii) $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$

(viii) $\lim\limits_{x \to 0} \dfrac{\sin mx}{x} = m$

(ix) $\lim\limits_{x \to 0} \left[\dfrac{a^x + b^x}{2}\right]^{1/x} = \sqrt{ab}$

(x) $\lim\limits_{x \to 0} [\cos x + a \sin bx]^{1/x} = e^{ab}$

(xi) $\lim\limits_{x \to 0} \left[\dfrac{1 - \cos ax}{x^2}\right] = \dfrac{a^2}{2}$

Questions:→

1. $\lim_{x\to 0} \frac{1-\cos 3x}{x\sin 2x}$.

Sol^n:→ $\lim_{x\to 0} \frac{1-\cos 3x}{x\sin 2x} = \lim_{x\to 0} \frac{3\sin 3x}{\sin 2x + 2x\cos 2x}$

$= \lim_{x\to 0} \frac{9\cos 3x}{2\cos 2x + 2\cos 2x - 4x\sin 2x}$

$= \frac{9}{4}$.

OR

$\lim_{x\to 0} \frac{(1-\cos 3x)/x^2}{(\sin 2x)/x} = \frac{3^2/2}{2} = \frac{9}{4}$.

2. $\lim_{x\to \frac{\pi}{2}} \frac{\sin^2 2x}{(x-\frac{\pi}{2})^2}$ is ______.

(a) 1 (b) 2 (c) 4 (d) -4

Sol^n:→ $\lim_{x\to\frac{\pi}{2}} \frac{[\sin 2(\frac{\pi}{2}-x)]^2}{(\frac{\pi}{2}-x)^2}$

Take, $\frac{\pi}{2} - x = t$, then

$\lim_{t\to 0} \left[\frac{\sin 2t}{t}\right]^2 = (2)^2 = 4$.

3. $\lim_{x\to 0} \frac{\log x}{\cot x}$.

Sol^n:→ $\lim_{x\to 0} \frac{\log x}{\cot x} = \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}^2 x} = -\left[\lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \sin x\right)\right]$

$= -1 \times 0$

$= 0$.

4. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan x}{\log(\cos x)}$

Solⁿ:→ $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan x}{\log(\cos x)} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sec^2 x}{-\sec x \sin x} = -\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1}{\sin x \cos x}$

$= -\infty.$

5. $\lim\limits_{x \to \frac{\pi}{2}} \tan x$

Solⁿ:→ $\cancel{\lim\limits_{x \to \frac{\pi}{2}} \tan x = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin x}{\cos x}}$ limit doesn't exists ($\because$ LHL $\neq$ RHL)

6. $\lim\limits_{x \to 0} \sin x \log x^2$

Solⁿ:→ $\lim\limits_{x \to 0} \sin x \log x^2 = \lim\limits_{x \to 0} \dfrac{\log x^2}{\operatorname{cosec} x} = \lim\limits_{x \to 0} \dfrac{2/x}{-\operatorname{cosec} x \cot x}$

$= \lim\limits_{x \to 0} \left(-2 \times \dfrac{\sin x}{x} \times \tan x\right)$

$= -2 \times 1 \times 0 = 0.$

Note:→ $\log_a 0 = \begin{cases} \infty & ; \ a < 1 \\ -\infty & ; \ a > 1 \end{cases}$

7. $\lim\limits_{x \to 1} (x-1) \tan \dfrac{\pi x}{2}$

Solⁿ:→ $\lim\limits_{x \to 1} (x-1) \tan \dfrac{\pi x}{2} = \lim\limits_{x \to 1} \dfrac{(x-1)}{\cot \frac{\pi x}{2}} = \lim\limits_{x \to 1} \left[\dfrac{1}{-\frac{\pi}{2} \operatorname{cosec}^2\left(\frac{\pi x}{2}\right)}\right]$

$= -\dfrac{2}{\pi}.$

Note:→ (i) $\lim\limits_{x \to 0} x \sin(1/x) = 0$

(ii) $\lim\limits_{x \to \infty} x \sin(1/x) = 1$

(iii) $\lim\limits_{x \to 0} \sin(1/x)$ doesnot exists.

8. $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1+x) \right]$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $-\frac{1}{2}$ (d) $-\frac{1}{3}$

Solⁿ:→ $\lim_{x \to 0} \left[\frac{x - \log(1+x)}{x^2} \right] = \lim_{x \to 0} \left[\frac{1 - \frac{1}{1+x}}{2x} \right]$

$= \lim_{x \to 0} \frac{1/(1+x)^2}{2} = \frac{1}{2}$

9. $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x} \right]$ is

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Solⁿ:→ $\lim_{x \to 0} \left[\frac{\sin^2 x - x^2}{x^2 \sin^2 x} \right] = \lim_{x \to 0} \left[\frac{\sin^2 x - x^2}{x^4 \cdot \cancel{\frac{\sin^2 x}{x^2}}} \right]$ 1

$= \lim_{x \to 0} \left[\frac{\sin^2 x - x^2}{x^4} \right]$

$= \lim_{x \to 0} \left[\frac{\sin 2x - 2x}{4x^3} \right]$

$= \lim_{x \to 0} \frac{2\cos 2x - 2}{12 x^2}$

$= -\frac{2}{12} \lim_{x \to 0} \left(\frac{1 - \cos 2x}{x^2} \right)$

$= -\frac{2}{12} \times \frac{4}{2} = -\frac{1}{3}$

10. $\lim_{x \to 0} x^x$.

Solⁿ:→ Let $y = x^x$

$\Rightarrow \log y = x \log x$

$\Rightarrow \lim_{x \to 0} [\log y] = \lim_{x \to 0} (x \log x) = \log \left(\lim_{x \to 0} x^x \right) = 0$

11. $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$

12. $\lim_{x \to 1} [\log x]^{\log x}$

<u>Note</u>:→ If we have 1^{∞} form, $\lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}$

13. $\lim_{x \to 0} (1 - \sin x)^{1/\sin x}$

<u>Solⁿ</u>:→ $e^{\lim_{x \to 0} \frac{1}{\sin x}[1 - \sin x - 1]} = e^{-1} = \frac{1}{e}$

14. $\lim_{x \to 0} [\cos x]^{1/x^2}$

<u>Solⁿ</u>:→ $e^{\lim_{x \to 0} \frac{1}{x^2}[\cos x - 1]} = e^{-\lim_{x \to 0} \left[\frac{1 - \cos x}{x^2}\right]} = e^{-1/2} = \frac{1}{\sqrt{e}}$

15. $\lim_{x \to 0} \left(\frac{2^x + 8^x}{2}\right)^{1/x} = \sqrt{ab} = \sqrt{2 \times 8} = \sqrt{16} = 4.$

16. $\lim_{x \to 0} \left(\frac{2^x + 4^x + 8^x}{3}\right)^{1/x} = \sqrt[3]{abc} = \sqrt[3]{2 \times 4 \times 8} = 4.$

17. $\lim_{x \to 0} [\cos x + 2 \sin 3x]^{1/x} = e^{ab} = e^{2 \times 3} = e^{6}.$

18. $\lim_{x \to 0} \left[2\cos x + 3 \sin 4x \right]^{1/x}$ 5

$= \lim_{x \to 0} 2^{1/x} \left[\cos x + \frac{3}{2} \sin 4x \right]^{1/x} = \lim_{x \to 0} 2^{1/x} \cdot \lim_{x \to 0} \left[\cos x + \frac{3}{2} \sin 4x \right]^{1/x}$

$=$ doesnot exists ($\because$ LHL $\neq$ RHL for $\lim_{x \to 0} 2^{1/x}$)

19. $\lim_{x \to 0} \frac{|x|}{x} = ?$

Solⁿ:→ LHL $= \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{h \to 0} \frac{|-h|}{-h} = -1$

RHL $= \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{h \to 0} \frac{|h|}{h} = 1$

$\because$ LHL $\neq$ RHL

$\therefore$ limit doesn't exists

20. $\lim_{x \to a} [x]$ doesn't exists when a is ———

(a) Real no. (b) Rational no. (c)✓ Integer (d) all of these

Solⁿ:→ let $a = 2$

LHL $= \lim_{x \to 2^-} [x] = 1$

RHL $= \lim_{x \to 2^+} [x] = 2$

$\because$ LHL $\neq$ RHL $\Rightarrow$ limit doesn't exists.

Continuity of a function:→

(i) Continuity at a point:→ A f^n is said to be continuous at a point $x = a$, if $\lim_{x \to a} f(x) = f(a)$

(ii) Continuity in an interval:→ A f^n $f(x)$ is said to be continuous in $[a, b]$ if it satisfies the following three conditions:-

(a) $f(x)$ is continuous $\forall\ x \in (a, b)$

(b) $\lim_{x \to a^+} f(x) = f(a)$

(c) $\lim_{x \to b^-} f(x) = f(b)$

Eg:- (i) If $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$ check its continuity at $x = 2$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \left(\frac{x^2-4}{x-2}\right) = \lim_{x \to 2} \left(\frac{2x}{1}\right) = 4$$

But at $x = 2$, $f(x) = 2$

$\therefore$ $f(x)$ is not continuous at $x = 2$.

(ii) If $f(x) = \begin{cases} (1+3x)^{1/x} & ; \ x \neq 0 \\ e^3 & ; \ x = 0 \end{cases}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+3x)^{1/x} = e^{\lim_{x \to 0} \frac{1}{x}(1+3x-1)} = e^3$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Questions: $\rightarrow$ 6

1. If $f(x) = \begin{cases} 0 & ; \quad x = 0 \\ \frac{1}{2} - x & ; \quad 0 < x < \frac{1}{2} \\ \frac{1}{2} & ; \quad x = 1/2 \\ \frac{3}{2} - x & ; \quad \frac{1}{2} < x < 1 \\ 1 & ; \quad x \geq 1 \end{cases}$

Then which of the following is true :-

(a) $f(x)$ is right continuous at $x=0$

(b) $f(x)$ is discontinuous at $x = \frac{1}{2}$

(c) $f(x)$ is continuous at $x=1$

(d) b & c

Sol^n: $\rightarrow$ (a) $f(0) = 0$.

2. <u>Differentiation</u>:→ A funⁿ f(x) is said to be differentiable at a pt. $x = c$, if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists & finite. & is represented by $f'(c)$.

<u>Left Hand Derivative</u>:→ $\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

<u>Right Hand Derivative</u>:→ $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Necessary $cond^n$ for a fu^n to be differentiable is LHD = RHD.

<u>Note</u>:→ (i) $f(x) = |x|$ is not differentiable at $x=0$

$$LHD = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|-h| - 0}{-h} = -1$$

$$RHD = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = 1$$

∵ LHD ≠ RHD

⇒ Not differentiable

(ii) $|x-a|$ is not differentiable at $x = a$.

(iii) $|ax+b|$ is not differentiable at $x = -b/a$.

<u>Questions</u>:→

1. $f(x) = |x| + |x+1| + |x-2|$ is differentiable at $x =$ ______.

   (a) 0   (b) 1 ✓   (c) −1   (d) 2

2. Let $f(x) = |x+1|$ be defined in the interval $[0, 4]$ then,

   ✓ (a) f(x) is continuous & differentiable

   (b) f(x) is continuous but non-differentiable

   (c) f(x) is not continuous but differentiable

   (d) f(x) is neither differentiable nor continuous

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3. If $f(x) = |x|^3$ where $x \in R$, then, $f(x)$ at $x=0$ is ____

(a) continuous but not differential

(b) Once differentiable but not twice

✓(c) Twice differentiable but not thrice

(d) Thrice differentiable ~~be~~

Solⁿ:→

$$f(x) = |x|^3 = \begin{cases} x^3 & ; \quad x>0 \\ -x^3 & ; \quad x<0 \\ 0 & ; \quad x=0 \end{cases}$$

$f(0)=0$, LHL $=0$, RHL $=0$ $\Rightarrow$ continuous

$$\text{LHD} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|^3 - 0}{-h} = 0$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|^3 - 0}{h} = 0$$

$\because$ LHD = RHD $\Rightarrow$ differentiable

$$f'(x) = \begin{cases} 3x^2 & , \quad x>0 \\ -3x^2 & , \quad x<0 \\ 0 & , \quad x=0 \end{cases}$$

$$f''(x) = \begin{cases} 6x & , \quad x>0 \\ -6x & , \quad x<0 \\ 0 & , \quad x=0 \end{cases} = 6|x|$$

$$f'''(x) = \begin{cases} 6 & , \quad x>0 \\ -6 & , \quad x<0 \\ \text{not diff.} & \quad x=0 \end{cases}$$

4. If $f(x) = \begin{cases} 2+x & x \geq 0 \\ 2-x & x < 0 \end{cases}$ then $f(x)$ at $x=0$ is ______

(a) Continuous & differentiable

(b) ✓ continuous but not differentiable

(c) Differentiable but not continuous

(d) Neither diff. nor continuous

Solⁿ :→ $f(0) = 2+0 = 2$, LHL $= 2-0 = 2$, RHL $= 2+0 = 2$ ⇒ continuous

$$\left.\begin{aligned} \text{LHD} &= \lim_{h\to 0} \frac{[2-(-h)]-2}{-h} = -1 \\ \text{RHD} &= \lim_{h\to 0} \frac{[2+h-2]}{h} = 1 \end{aligned}\right\} \text{Non-Differentiable.}$$

Note :→ (i) Every differentiable funⁿ is a continuous funⁿ.

(ii) But every continuous funⁿ is not differentiable.

Mean Value Theorem :→

(i) Rolle's Theorem :→ Let $f(x)$ be defined in $[a,b]$ s.t. It satisfies three condⁿ :-

(a) $f(x)$ is continuous funⁿ in $[a,b]$

(b) $f(x)$ is differentiable funⁿ in (a,b)

(c) $f(a) = f(b)$

then, there exists atleast one point $c \in (a,b)$ ~~where~~ s.t.

$f'(c) = 0$

(ii) Lagrange's Mean Value Theorem :→ Let $f(x)$ be defined in $[a,b]$ s.t. it satisfies two condⁿ :-

(a) $f(x)$ is continuous funⁿ in $[a,b]$

(b) $f(x)$ is differential funⁿ in (a,b)

then, $\exists$ atleast one point c in (a,b) s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

<u>Note:</u>→ If $f(x)$ is defined in $[a, a+h]$ s.t.

(a) $f(x)$ is continuous in $[a, a+h]$

(b) $f(x)$ is differentiable in $(a, a+h)$

then $\exists$ $\theta \in (0,1)$ s.t. $\boxed{\theta = \frac{c-a}{a-b}}$

$f(a+h) = f(a) + h f'(a+\theta h)$

<u>Questions:</u>→

1. The mean-value 'c' for the f^n $f(x)=e^x(\sin x - \cos x)$ in $[\frac{\pi}{4}, \frac{5\pi}{4}]$ is ______

(a) 0 (b) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (d)✓ π

<u>Sol^n:</u>→ $f(\frac{\pi}{4}) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = 0$

$f(\frac{5\pi}{4}) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = 0$

$f'(x) = e^x[\cos x + \sin x] + e^x[\sin x - \cos x] = 2e^x \sin x$

$\Rightarrow f'(c) = 2e^c \sin c = 0$

$\Rightarrow c = 0, \pm\pi, \pm 2\pi, \ldots$

2. The mean-value 'c' for the f^n $f(x) = x^3 - 6x^2 + 11x - 6$ in $[0,4]$ is -

(a) $2+\frac{2}{\sqrt{3}}$ (b) $2-\frac{2}{\sqrt{3}}$ (c)✓ $2 \pm \frac{2}{\sqrt{3}}$ (d) None

<u>Sol^n:</u>→ $f(0) = -6$

$f(4) = 64 - 96 + 44 - 6 = 6$

$f'(x) = 3x^2 - 12x + 11$

$\therefore f'(c) = 3c^2 - 12c + 11 = \frac{f(4)-f(0)}{4-0}$

$\Rightarrow 12c^2 - 48c + 44 = 12$

$\Rightarrow c = 2 \pm \frac{2}{\sqrt{3}}$

3. The value of ζ of $f(b)-f(a) = (b-a)f'(\zeta)$ for the fu^n $f(x) = Ax^2+Bx+C$, in $[a,b]$ is ____.

(a) ✓ $\frac{b+a}{2}$ (b) $\frac{b-a}{2}$ (c) $b-a$ (d) $b+a$

Sol^n:→ $f'(x) = 2Ax+B$

$$f'(\zeta) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 2A\zeta + B = \frac{f(b)-f(a)}{b-a} = \frac{(Ab^2+Bb+C)-(Aa^2+Ba+C)}{b-a}$$

$$\Rightarrow 2A\zeta + B = (b+a)A + B$$

$$\Rightarrow \zeta = \frac{b+a}{2}$$

4. The mean-value 'c' for the fu^n $f(x) = 3x^2+5x+11$ in the $\left[\frac{11}{2}, \frac{16}{2}\right]$ is ____.

Sol^n:→ $c = \dfrac{\frac{11}{2}+\frac{16}{2}}{2} = \dfrac{27}{4}$

Note:→ If the fu^n $f(x)$ is polynomial of degree 2 i.e. quadratic then, c will be the average value of the extreme value of the fu^n in given interval i.e. $c = \frac{a+b}{2}$ if $f(x)$ is defined in $[a,b]$

5. The value of $\theta \in (0,1)$ for the fu^n $f(x) = \log_e x$ in $[1,e]$ using an appropriate mean-value theorem is ____.

Sol^n:→ $f'(x) = \frac{1}{x}$ $\Rightarrow$ $f'(c) = \frac{1}{c}$

$f(1) = \log_e 1 = 0$

$f(e) = \log_e e = 1$

∴ Using LMVT, $\frac{1}{c} = \frac{1-0}{e-1}$ $\Rightarrow$ $c = e-1$

$$\therefore \theta = \frac{c-a}{b-a} = \frac{e-1-1}{e-1} = \frac{e-2}{e-1} \in (0,1)$$

6. LMVT cannot be applied for $f(x) = x^{1/3}$ in $[-1, 1]$ because ?

(a) $f(x)$ is not continuous in $[-1, 1]$

(b) ✓ $f(x)$ is not differentiable in $(-1, 1)$

(c) a & b

(d) $f(-1) \neq f(1)$

Sol^n:→ $f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$

∴ $f(x)$ is not differentiable at $x = 0 \in (-1, 1)$

$f(0) = 0$

$LHL = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} (-h)^{1/3} = 0$

$RHL = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h^{1/3} = 0.$

$LHL = RHL = f(0) = 0 \Rightarrow$ continuous.

7. Rolle's Theorem cannot be applied for $f(x) = |x+2|$ in $[-2, 0]$.

(a) $f(x)$ is not continuous in $[-2, 0]$

(b) $f(x)$ is not differentiable in $(-2, 0)$

(c) ✓ $f(-2) \neq f(0)$

(d) b & c

8. If $f'(x) = \frac{1}{5-x^2}$ and $f(0) = 1$ then the lower & upper bounds of $f(1)$ are ______.

Sol^n:→ Let $f(x)$ be defined in $[0, 1]$

By LMVT, $\exists\ c \in (0, 1)$ s.t.

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = f(1) - 1$$

$\text{Min}\{f'(x)\} < f'(c) < \text{Max}\{f'(x)\} \qquad 0 \leq x \leq 1$

$$\frac{1}{5} < f(1) - 1 < \frac{1}{4}$$

$$\Rightarrow \quad 1 + \frac{1}{5} < f(1) < \frac{1}{4} + 1$$

(iii) <u>Cauchy's Mean Value Theorem</u> :⟶ Let $f(x)$ & $g(x)$ be defined in a closed interval $[a, b]$ s.t. they satisfy the condⁿ:-

(a) $f(x)$ & $g(x)$ are continuous in $[a, b]$

(b) $f(x)$ & $g(x)$ are differentiable in (a, b)

(c) $g'(x) \neq 0 \quad \forall \; x \in (a, b)$, then,

$$\exists \; c \in (a, b) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

<u>Questions</u> :⟶

1. The mean-value 'c' for the funⁿ $f(x) = e^{x}$ & $g(x) = e^{-x}$ in $[0, 1]$ is ______.

<u>Solⁿ</u> :⟶ $f'(x) = e^{x} \Rightarrow f'(c) = e^{c}$

$g'(x) = -e^{-x} \Rightarrow g'(c) = -e^{-c}$

$f(1) = e$, $f(0) = 1$

$g(1) = \frac{1}{e}$, $g(0) = 1$

$$\therefore \quad \frac{e^{c}}{-e^{-c}} = \frac{e - 1}{(1/e) - 1} \Rightarrow -e^{2c} = \left[\frac{e-1}{1-e}\right] e$$

$$\Rightarrow + e^{2c} = e \quad \Rightarrow \quad c = \frac{1}{2} \in [0, 1].$$

2. $f(x) = \sin x$ & $g(x) = \cos x$ in $[-\frac{\pi}{2}, 0]$ is ______.

<u>Solⁿ</u> :⟶ $f'(x) = \cos x$, $g'(x) = -\sin x \neq 0 \quad \forall \; x \in (-\frac{\pi}{2}, 0)$

$$\frac{f'(c)}{g'(c)} = \frac{f(0) - f(-\pi/2)}{g(0) - g(-\pi/2)}$$

$$\Rightarrow \frac{\cos c}{-\sin c} = \frac{0-(-1)}{1-0} \Rightarrow -\cot c = 1 \Rightarrow \cot c = -1$$

$$\Rightarrow c = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, 0\right)$$

(iv) Taylor's Theorem:→ OR (Generalised Mean-Value Theorem)

Let $f(x)$ be defined in $[a, a+h]$ s.t.

(a) $f, f', f'', f''', \ldots, f^{n-1}$ are continuous in $[a, a+h]$

(b) $f, f', f'', \ldots, f^{n-1}$ are differentiable in $(a, a+h)$

then $\exists\ \theta \in (0,1)$ s.t.

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where $R_n = \dfrac{h^n}{(n-1)!\, p} (1-\theta)^{n-p} f^n(a+\theta h)$

for $p = n \rightarrow$ ~~Lang~~ Lagrange's form of remainder

$p = 1 \rightarrow$ Cauchy's form of remainder

Case I:- when $p = n$, $R_n = \dfrac{h^n}{n!} f^n(a+\theta h)$

Case II:- when $p = 1$, $R_n = \dfrac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h)$

Taylor's Series:→ As $n \to \infty$, $R_n \to 0$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

(i) $f(x) = f(a) + (x-a) f'(a) + \dfrac{(x-a)^2}{2!} f''(a) + \cdots$ is a Taylor's series expansion of $f(x)$ about $x = a$.

(ii) $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \cdots$ is a Taylor's series expansion of $f(x)$ about $x = 0$. (Maclaurian's Series)

Questions:→

1. The coeff. of x^2 in the Taylor's Series expansion of $\cos^2 x$ about $x = 0$ is ——

(a) 0 (b) 1 (c)✓ −1 (d) 2

Solⁿ:→ coeff. of $x^2 = \dfrac{f''(0)}{2!} = \dfrac{-2}{2} = -1.$

where, $f'(x) = -\sin 2x$

$f''(x) = -2\cos 2x \Rightarrow f''(0) = -2$

2. The coeff. of $(x-2)^4$ in the Taylor's series expansion of e^x about $x = 2$ is ——

(a) $\dfrac{e^2}{2!}$ (b)✓ $\dfrac{e^2}{4!}$ (c) $\dfrac{e^4}{2!}$ (d) $\dfrac{e^4}{4!}$

Solⁿ:→ coeff. of $(x-2)^4 = \dfrac{f''''(2)}{4!} = \dfrac{e^2}{4!}$

3. The coeff. of $(x-\pi)^3$ in the power series expansion of $e^x + \sin x$ in the ascending power of $(x-\pi)$ is ——

(a) $\dfrac{e^\pi}{6}$ (b) $\dfrac{e^\pi+1}{3}$ (c) $\dfrac{e^\pi-1}{3}$ (d)✓ None

Solⁿ:→ coeff. of $(x-3)^3 = \dfrac{f^3(\pi)}{3!} = \dfrac{e^\pi+1}{6}$

where, $f'(x) = e^x + \cos x$

$f''(x) = e^x - \sin x$

$f'''(x) = e^x - \cos x \Rightarrow f^3(\pi) = e^\pi + 1$

4. which of the following funⁿ ~~who~~ would have only odd powers of x in its Taylor's Series expansion about $x = 0$,

(a) $\sin x^2$ (b) $\cos x^2$ (c) $\cos x^3$ (d)✓ $\sin x^3$

$Sol^n:\rightarrow$ $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$$

5. The Taylor's Series expansion of $f(x) = \tan^{-1}x$ about $x=0$ is ______.

$Sol^n:\rightarrow$ $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

$$f(x) = \tan^{-1}x$$

$$f(0) = \tan^{-1}0 = 0$$

$$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow f''(x) = 0$$

$$f'''(x) = -2\left[\frac{(1+x^2)^2 - x\cdot 2(1+x^2)2x}{(1+x^2)^4}\right] = -2\left[\frac{1+x^2-4x^2}{(1+x^2)^3}\right] = -2\left[\frac{1-3x^2}{(1+x^2)^3}\right]$$

$$\Rightarrow f'''(0) = -2$$

$$\therefore \tan^{-1}x = x - 2\frac{x^3}{3!} + \cdots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

6. The power series expansion of $\frac{\sin x}{x-\pi}$ about $x=\pi$ is ______.

$Sol^n:\rightarrow$ Let, $x-\pi = t$, then $f = \frac{\sin(\pi+t)}{t}$ about $t=0$

$$= -\frac{\sin t}{t} \text{ about } t=0$$

$$= -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right]$$

$$= -1 + \frac{t^2}{2!} - \frac{t^4}{5!} + \frac{t^6}{7!} - \cdots$$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{5!} + \frac{(x-\pi)^6}{7!} - \cdots$$

7. For the fu^n e^{-x}, linear approximation around $x = 2$ is ______

(a) $(3-x)\,e^{-2}$ (b) $(1-x)\,e^{-2}$ (c) $[\sqrt{3} + 2\sqrt{2} - (1+\sqrt{2})x]\,e^{-2}$

Sol^n:→ $$f(x) = \underbrace{f(a) + (x-a)\,f'(a)}_{\text{linear approx.}} + \frac{(x-a)^2}{2!}\,f''(a) + \cdots$$

$$\therefore \quad e^{-x} = f(2) + (x-2)\frac{d}{dx}e^{-x}\Big|_{x=2} = e^{-2} + (x-2)(-1)\,e^{-2}$$

$$= e^{-2}\,(3-x)$$

3. Definite Integrals:→

Theorem:→ Let $f(x)$ is a continuous fu^n defined in $[a,b]$ & $F(x)$ be the anti-derivative of $f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Note:→ $$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)\,dx\right] = f(v)\,\frac{dv}{dx} - f(u)\,\frac{du}{dx}$$

Properties of Definite Integrals:→

(i) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(ii) If $c \in (a,b)$ then, $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(iii) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(iv) $$\int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\,dx = \frac{b-a}{2}$$

(v) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ;\ \text{if } f(x) \text{ is even} \\ 0 & ;\ \text{if } f(x) \text{ is odd} \end{cases}$$

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(vi) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$

(vii) $\int_0^{a} x\, f(x)\,dx = \frac{a}{2}\int_0^{a} f(x)\,dx \quad ; \text{ if } f(a-x) = f(x)$

(viii) $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx \quad ; \text{ if } f(x+a) = f(x)$

(ix) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{2}{3} \text{ (or) } \frac{1}{2}\right] k$

where, $k = \begin{cases} 1 & ; \text{ if } n \text{ is odd} \\ \pi/2 & ; \text{ if } n \text{ is even} \end{cases}$

(x) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left\{\frac{[(m-1)(m-3)\cdots 2 \text{ (or) } 1][(n-1)(n-3)\cdots 2 \text{ (or) } 1]}{[(m+n)(m+n-2)\cdots 2 \text{ (or) } 1]}\right\}$

where, $k = \begin{cases} \pi/2 & ; \text{ when } m \,\&\, n \text{ are even} \\ 01 & ; \text{ otherwise} \end{cases}$

Questions:→

1. $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\,dx =$ ______ .

Sol^n:→ Let $f(x) = \tan x \Rightarrow f(0+\frac{\pi}{2}-x) = \cot x$

$\therefore I = \int_0^{\pi/2} \frac{f(x)}{f(x) + f(0+\frac{\pi}{2}-x)} = \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4}.$

2. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}}$ is ______.

Sol^n:→ $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4}$

3. $\displaystyle\int_2^{3} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}\,dx$ is ______.

Sol^n:→ $I = \frac{3-2}{2} = \frac{1}{2}$

4. $\displaystyle\int_0^{\pi} |\cos x|\,dx$ is ______.

(a) 1 (b) 0 (c) ✓ 2 (d) None

Sol^n:→ $\displaystyle I = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx$

$$= \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{\pi}$$

$$= 1-(-1) = 2$$

5. $\displaystyle\int_0^{4} (|x|+|3-x|)\,dx$ is ______.

Sol^n:→ $\displaystyle I = \int_0^4 x\,dx + \int_0^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$

$$= \frac{x^2}{2}\Big|_0^4 + \left[3x-\frac{x^2}{2}\right]_0^3 + \left[\frac{x^2}{2}-3x\right]_3^4$$

$$= 8 + \left[9-\frac{9}{2}\right] + \left[8-12-\frac{9}{2}+9\right]$$

$$= 34-21$$

$$= 13.$$

6. $\int_0^n [x]\,dx$ is ____.

(a) $\frac{n(n+1)}{2}$ (b)✓ $\frac{n(n-1)}{2}$ (c) $\frac{n}{2}$ (d) None.

Solⁿ:→ $I = \int_0^1 0.\,dx + \int_1^2 1.\,dx + \int_2^3 2.\,dx + \cdots + \int_{n-1}^n (n-1)\,dx$

$= 0 + [2-1] + 2[3-2] + \cdots + (n-1)[n-n+1]$

$= 1 + 2 + \cdots + (n-1)$.

$= \frac{n(n-1)}{2}$

7. $\int_0^1 x(1-x)^5\,dx$ is ____.

(a)✓ $\frac{1}{42}$ (b) $\frac{1}{30}$ (c) $\frac{1}{24}$ (d) None

Solⁿ:→ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore \int_0^1 x(1-x)^5\,dx = \int_0^1 (1-x)\,x^5\,dx = \frac{x^6}{6}\Big|_0^1 - \frac{x^7}{7}\Big|_0^1 = \frac{1}{42}$.

8. $\int_0^{\pi/2} \log(\tan x)\,dx$ is ____.

Solⁿ:→ Replace x by $(\frac{\pi}{2}-x)$.

$I = \int_0^{\pi/2} \log \cot x$

$\therefore 2I = \int_0^{\pi/2} [\log \tan x + \log \cot x]\,dx = \int_0^{\pi/2} \log[\tan x \cdot \cot x]\,dx = 0$

$\Rightarrow I = 0$.

9. $\int_0^{\pi/4} \log(1+\tan x)\, dx$ is ______.

(a)✓ $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{4}\log 2$ (c) $\frac{\pi}{2}\log 2$ (d) None

Solⁿ:→ $I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) dx$

$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right) dx$

$= \int_0^{\pi/4} \log\left(2/1+\tan x\right) dx$

$= \int_0^{\pi/4} \left[\log 2 - \log(1+\tan x)\right] dx = \int_0^{\pi/4} \log 2\, dx - \underbrace{\int_0^{\pi/4} \log(1+\tan x)\, dx}_{I}$

$\Rightarrow \quad 2I = \int_0^{\pi/4} \log 2\, dx = \frac{\pi}{4}\log 2$

$\Rightarrow \quad I = \frac{\pi}{8}\log 2.$

10. $\int_{-1}^{1} \frac{|x|}{x}\, dx$ is ______.

Solⁿ:→ $I = 0$ (odd f^n)

11. $\int_{-\pi}^{\pi} \log\frac{(1+\sin x)}{(1-\sin x)}\, dx$ is ______.

Solⁿ:→ $f(-x) = \log\frac{(1-\sin x)}{(1+\sin x)} = -\log\left(\frac{1+\sin x}{1-\sin x}\right) = -f(x) \Rightarrow$ odd f^n

$\therefore I = 0.$

12. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$ is ______.

Solⁿ:→ $\int_0^{a} x f(x)\, dx = \frac{a}{2}\int_0^{a} f(x)\, dx$; if $f(a-x) = f(x)$

$$f(\pi - x) = \frac{\sin x}{1+\cos^2 x} = f(x)$$

$$\therefore\ I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Put, $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\Rightarrow I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = -\frac{\pi}{2}\left[\tan^{-1} t\right]_1^{-1} = -\frac{\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{4}.$$

13. $\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$ is ______.

(a) ✓ $\frac{\pi}{ab}$ (b) πab (c) $\frac{2\pi}{ab}$ (d) None

Solⁿ → $I = \int_0^{\pi} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & ;\ \text{if } f(2a-x) = f(x) \\ 0 & ;\ \text{if } f(2a-x) = -f(x) \end{cases}$$

$f(\pi - x) = f(x)$

$$\therefore\ I = 2\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$$

Let, $\tan x = t$

$\sec^2 x\,dx = dt$

$$\therefore\ I = 2\int_0^{\infty} \frac{dt}{a^2 + (bt)^2} = 2\times\frac{1}{a}\left[\frac{\tan^{-1}(bt/a)}{b}\right]_0^{\infty} = \frac{2}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{ab}$$

14. If $f(t)$ is a continuous fu^n defined in $[0, 1]$ then $\lim_{t\to 0} \frac{1}{t}\int_0^t f(t)\,dt$ ____

(a) 0 (b) ∞ (c) ✓ $f(0)$ (d) $f(1)$

Solⁿ:→ $$\lim_{t\to 0} \frac{\int_0^t f(t)\,dt}{t} = \lim_{t\to 0} \frac{\frac{d}{dt}\left\{\int_0^t f(t)\,dt\right\}}{1}$$

$$= \lim_{t\to 0} \left[f(t)\times 1 - f(0)\times 0\right]$$

$$= f(0)$$

15. $\int_0^{\pi/2} \sin^8 x\,dx$ ____.

Solⁿ:→ $$I = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} =$$

16. $\int_0^{\pi/2} \cos^7 x\,dx$ ____.

Solⁿ:→ $$I = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1 =$$

17. $\int_0^{\pi/2} \sin^5 x\,\cos^9 x\,dx$ ____.

Solⁿ:→ $$I = \frac{(4\times 2)\times(8\times 6\times 4\times 2)}{14\times 12\times 10\times 8\times 6\times 4\times 2}\times 1 =$$

18. $\int_0^{\pi/2} \sin^6 x\,\cos^3 x\,dx$ ____.

Solⁿ:→ $$I = \frac{(5\times 3\times 1)\times(2)}{9\times 7\times 5\times 3\times 1}\times 1 =$$

19. $\int_0^{\pi/2} \sin^5 x\,\cos^8 x\,dx$ ____.

Solⁿ:→ $$I = \frac{(4\times 2)\times(7\times 5\times 3\times 1)}{13\times 11\times 9\times 7\times 5\times 3\times 1}\times 1 =$$

20. $\int_0^{\pi/2} \sin^6 x \cdot \cos^8 x \, dx$ ______.

Sol^n: → $I = \frac{(5\times3\times1)(7\times5\times3\times1)}{14\times12\times10\times8\times6\times4\times2} \times \frac{\pi}{2} =$

21. $\int_{-\pi}^{\pi} \sin^4 x \, dx$ ______.

Sol^n: → $I = 2\int_0^{\pi} \sin^4 x \, dx$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$$

$\because \sin^4(\pi - x) = \sin^4 x$

$\therefore I = 2\times2\times\int_0^{\pi/2} \sin^4 x \, dx = 4\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} =$

22. $\int_0^{\pi} \sin^4 x \cos^3 x \, dx$ ______.

Sol^n: → $f(\pi - x) = \sin^4 x\,(-\cos x)^3 = -f(x)$

$\therefore I = 0.$

23. $\int_0^{\pi} \sin^3 x \cos^4 x \, dx$ ______.

Sol^n: → $f(\pi - x) = \sin^3 x \cos^4 x = f(x)$

$\therefore I = 2\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx = 2\times\frac{(2\times1)(3\times1)}{7\times5\times3\times1}\times 1 =$

24. $\int_{-2\pi}^{2\pi} \sin^4 x \cos^8 x \, dx$ ______

Solⁿ:→ $I = 2\int_0^{2\pi} \sin^4 x \cos^8 x \, dx = 2\times 2\times \int_0^{\pi} \sin^4 x \cos^8 x \, dx$

$$= 2\times 2\times 2\times \int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$$

$$= 8 \times \frac{(3\times 1)\times(7\times 5\times 3\times 1)}{12\times 10\times 8\times 6\times 4\times 2} \times \frac{\pi}{2}$$

Note:→ $\int_a^b \sin^4 x \cos^8 x \, dx = K \int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

where, $K = \frac{b-a}{\pi/2}$

25. $\int_0^{\pi/2} \sqrt{\sin x} \times \cos^3 x \, dx$ ______

Solⁿ:→ Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$\therefore I = \int_0^1 \sqrt{t}\,(1-t^2)\, dt = \left.\frac{t^{3/2}}{3/2}\right|_0^1 - \left.\frac{t^3}{3}\right|_0^1 = \frac{2}{3} - \frac{2}{7} =$

Improper Integrals:→

First Kind:→ $\int_a^b f(x)\,dx$ if $a=-\infty$ (or) $b=\infty$ (or) both

i.e $\int_{-\infty}^{b} f(x)\,dx$, $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$

<u>Second Kind:</u> $\rightarrow$ $\int_a^b f(x)\,dx$ if a & b are finite but $f(x)$ is infinite for some $x \in [a,b]$.

Eg:- $\int_{-1}^{1} \log(1+x)\,dx$, $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$, $\int_0^3 \frac{1}{x^2-5x+4}\,dx$

<u>Convergence of an Improper Integrals:</u> $\rightarrow$

(i) If $\int_a^b f(x)\,dx$ = finite, then, it is a convergent improper integral

(ii) If $\int_a^b f(x)\,dx$ = infinite, then, it is a divergent improper integral.

<u>Questions:</u> $\rightarrow$

1. Find the convergence of following improper integrals

(i) $\int_0^\infty \frac{1}{a^2+x^2}\,dx$ ———.

<u>Solⁿ:</u> $\rightarrow$ $I = \frac{1}{a}\tan^{-1}\frac{x}{a}\Big|_0^\infty = \frac{1}{a}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{2a}$ = finite

which is convergent.

(ii) $\int_0^\infty x\sin x\,dx$ ———.

<u>Solⁿ:</u> $\rightarrow$ $I = x\Big[[-\cos x] - (1)[-\sin x]\Big]_0^\infty$ = infinite

which is divergent improper integral.

(iii) $\int_{-\infty}^{0} e^{ax} \cos px \, dx =$ ______.

Sol^n:→ $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$

$$\therefore \ I = \left[\frac{e^{ax}}{a^2+p^2} [a \cos px + p \sin px] \right]_{-\infty}^{0}$$

$$= \frac{a}{a^2+p^2} - 0 = \text{finite}$$

i.e. convergence

(iv) $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \, dx$

Sol^n:→ $I = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx + \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx$

$$= 2 \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} dx + 0$$

$$= 2 \sin^{-1}x \Big|_0^1 = 2 \cdot \frac{\pi}{2} = \pi = \text{finite}$$

i.e. convergent improper integral.

(v) $\int_{-1}^{1} \frac{1}{x^2} dx$ ______.

Sol^n:→ $I = \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx = \left[\frac{-1}{x} \right]_{-1}^{0} + \left[-\frac{1}{x} \right]_{0}^{1} = \text{infinite}$

i.e. divergent improper integral

(vi) $\int_0^3 \frac{1}{x^2-3x+2}\,dx$

<u>Sol^n</u>: $\rightarrow$ $I = \int_0^3 \frac{1}{(x-1)(x-2)}\,dx = \int_0^1 \frac{dx}{(x-1)(x-2)} + \int_1^2 \frac{dx}{(x-1)(x-2)} + \int_2^3 \frac{dx}{(x-1)(x-}$

$$\int \frac{dx}{(x-1)(x-2)} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1} = \log\left(\frac{x-2}{x-1}\right)$$

$$\therefore\ I = \log\left(\frac{x-2}{x-1}\right)\Big|_0^1 + \log\left(\frac{x-2}{x-1}\right)\Big|_1^2 + \log\left(\frac{x-2}{x-1}\right)\Big|_2^3 = \text{infinite}$$

i.e., divergent improper integral.

Comparison Test : $\rightarrow$

<u>For first kind of improper integrals</u>: $\rightarrow$

(a) Let $0 \leq f(x) \leq g(x)$, then,

(i) $\int_a^b f(x)\,dx$ converges if $\int_a^b g(x)$ is convergent

(ii) $\int_a^b g(x)\,dx$ diverges if $\int_a^b f(x)$ is divergent

(b) <u>Limit form</u>: $\rightarrow$ Let $f(x)$ & $g(x)$ be two positive fu^n s.t.

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$ (non-zero, finite) then $\int_a^b f(x)\,dx$ &

$\int_a^b g(x)\,dx$ both are convergent or divergent together.

<u>Questions</u>: $-$

1. $\int_1^\infty e^{-x^2}\,dx$ ______.

<u>Sol^n</u>: $\rightarrow$ $e^{x^2} \geq e^{x}$ $\forall$ $x \geq 1$

$$\Rightarrow e^{-x^2} \leq e^{-x} \quad \forall \quad x \geq 1$$

$$\int_1^{\infty} e^{-x}\, dx = \left.\frac{e^{-x}}{-1}\right|_1^{\infty} = 1 \Rightarrow \int_1^{\infty} e^{-x}\, dx \text{ is convergent}$$

$$\therefore \int_1^{\infty} e^{-x^2}\, dx \text{ is also convergent.}$$

2. $\int_2^{\infty} \frac{1}{\log x}\, dx$ ________.

Sol^n:→ $\log x < x \quad \forall \; x \geq 2$

$$\Rightarrow \frac{1}{\log x} > \frac{1}{x} \quad \forall \; x \geq 2$$

$$\int_2^{\infty} \frac{1}{x}\, dx = \log x \Big|_2^{\infty} = \text{infinite} \Rightarrow \text{divergent}$$

$$\therefore \int_2^{\infty} \frac{dx}{\log x} \text{ is also divergent.}$$

3. $\int_1^{\infty} \frac{1}{x^2(e^{-x}+1)}\, dx$ ________.

Sol^n:→ $\left.\begin{aligned} & x^2(e^{-x}+1) > x^2(0+1) \quad \forall \; x > 1 \\ & \Rightarrow \frac{1}{x^2(e^{-x}+1)} < \frac{1}{x^2} \quad \forall \; x > 1 \end{aligned}\right\}$ Method- 1

Method- II :- Let $g(x) = x^2$, $\frac{f(x)}{g(x)} = \frac{1}{e^{-x}+1}$

so that, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ thus, $g(x)$ will gives the nature of $f(x)$.

$$\therefore \int_1^{\infty} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^{\infty} = 1 \Rightarrow \text{Convergent}$$

Hence, $f(x)$ is also convergent.

4. $\int_1^{\infty} \frac{x \tan^{-1}x}{\sqrt{4+x^3}}\,dx$ ———.

Sol^n:→ $f(x) = \frac{x\tan^{-1}x}{x\sqrt{x}\sqrt{\frac{4}{x^3}+1}} = \frac{\tan^{-1}x}{\sqrt{x}\sqrt{\frac{4}{x^3}+1}}$

Let $g(x) = \frac{1}{\sqrt{x}}$

$\frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{\sqrt{\frac{4}{x^3}+1}} \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \pi/2$

$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^{\infty} =$ infinite $\Rightarrow$ divergent

$\therefore$ $f(x)$ is also divergent

<u>Comparison Test for the second kind of improper integral</u> :→

(b) <u>Limit Form</u>:→ Let $f(x)$ & $g(x)$ be two +ve fun s.t.

(i) 'a' is a point of discontinuity and $\lim_{x\to a^+} \frac{f(x)}{g(x)} = l_1$

(non-zero & finite).

(ii) 'b' is a point of discontinuity and $\lim_{x\to b^-} \frac{f(x)}{g(x)} = l_2$

(non-zero & finite)

then, $\int_a^b f(x)\,dx$ & $\int_a^b g(x)\,dx$ both converge (or) diverge together.

Questions:$\longrightarrow$

1. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\,dx$ ________.

Sol^n:$\longrightarrow$ $\displaystyle\frac{\sin x}{x} \leq 1 \quad \forall \quad x > 0$

$$\Rightarrow \frac{\sin x}{x\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \forall \quad x > 0$$

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_0^{\pi/2} = 2\times\sqrt{\frac{\pi}{2}} = \text{finite} \Longrightarrow \text{convergent}$$

$$\therefore \int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\,dx \text{ is also convergent}$$

2. $\displaystyle\int_1^{2} \frac{\sqrt{x}}{\log x}\,dx$ ________.

Sol^n:$\longrightarrow$ $\displaystyle\frac{1}{\log x} > \frac{1}{x} \quad \forall \quad x > 1$

$$\Rightarrow \frac{\sqrt{x}}{\log x} > \frac{1}{\sqrt{x}} \quad \forall \quad x > 1$$

$$\int_1^{2} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^{2} = 2\sqrt{2} - 2 = \text{finite} \Rightarrow \text{convergent}$$

$$\therefore \int_1^{2} \frac{\sqrt{x}}{\log x}\,dx \text{ may / may not be convergent.}$$

Thus, first method fails.

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 1$$

Let, $g(x) = \dfrac{1}{x\log x}$

$$\Rightarrow \frac{f(x)}{g(x)} = x\sqrt{x}$$

$\displaystyle\int_1^2 \frac{1}{x\log x}\,dx$; Put $\log x = t$

$$\Rightarrow \frac{1}{x}dx = dt$$

$$\therefore \int_1^2 \frac{1}{x\log x}\,dx = \int_0^{\log 2} \frac{1}{t}\,dt = \log t \Big|_0^{\log 2} = \text{infinite} \Rightarrow \text{divergent}$$

$$\therefore \int_1^2 \frac{\sqrt{x}}{\log x}\,dx \text{ is also divergent}$$

3. which of the following fu^n is strictly bounded :-

(a) x^2 (b) e^x (c) $\frac{1}{x}$ (d)✓ e^{-x^2}

4. which of the following integrals is unbounded :-

(a) $\int_0^{\pi/4} \tan x\,dx$ (b) $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ (c)✓ $\int_0^{1} \frac{1}{1-x}\,dx$ (d) $\int_0^{\infty} x e^{-x}\,dx$

Sol^n:→ (a) $I = \log \sec x \Big|_0^{\pi/4} = \log \sqrt{2}$

(b) $I = \tan^{-1} x \Big|_0^{\infty} = \pi/2$

(c) $I = -\log(1-x) \Big|_0^{1} = \text{infinite}$

(d) $I = x\frac{e^{-x}}{-1}\Big|_0^{\infty} - (1)\frac{e^{-x}}{(-1)^2}\Big|_0^{\infty} = 1$

5. Consider the integrals $I_1 = \int_1^{\infty} \frac{1}{x^2(e^x+1)}\,dx$ & $I_2 = \int_1^{\infty} \frac{x+1}{x\sqrt{x}}\,dx$

then which of the following is true :-

(a) I_1 & I_2 are convergent ~~(b) I_1 is co~~

(b)✓ I_1 is convergent, I_2 is divergent

(c) I_1 is divergent, I_2 is convergent

(d) I_1 & I_2 are divergent

Solⁿ:→ $x^2(e^x+1) > x^2(0+1) \quad \forall \; x \geq 1$

$$\Rightarrow \quad \frac{1}{x^2(e^x+1)} < \frac{1}{x^2} \qquad \forall \; x \geq 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = 1 \quad \Rightarrow \text{convergent}$$

$\therefore$ I_1 is convergent.

$$I_2 = \int_1^\infty \frac{x}{x\sqrt{x}}\,dx + \int_1 \frac{1}{x\sqrt{x}}\,dx = \underbrace{\int_1^\infty \frac{1}{\sqrt{x}}\,dx}_{2\sqrt{x}\big|_1^\infty \Rightarrow \text{Divergent}} + \int_1^\infty \frac{1}{x\sqrt{x}}\,dx$$

$\therefore$ I_2 is divergent.

6. Consider, $I_1 = \int_0^1 \frac{1}{x^{1/3}}\,dx$, $I_2 = \int_{-1}^1 \frac{1}{x}\,dx$, $I_3 = \int_0^1 x\log x\,dx$

then which of the following is convergent:-

(a) I_1 & I_2

(b) I_2 & I_3

(c) I_1 & I_3 ✓

(d) Only I_1

Solⁿ:→ $I_1 = \frac{x^{2/3}}{2/3}\Big|_0^1 = \text{finite}$

$$I_2 = \int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx = \log x\Big|_{-1}^0 + \log x\Big|_0^1 = \text{infinite}$$

$$I_3 = \log x \cdot \frac{x^2}{2}\Big|_0^1 - \int_0^1 \frac{1}{x}\cdot\frac{x^2}{2}\,dx = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1$$

$$= -\frac{1}{4} - \left[\lim_{x\to 0}\frac{x^2}{2}\log x\right]$$

$$\lim_{x\to 0} \frac{\log x}{2/x^2} = \lim_{x\to 0} \frac{1/x}{-4/x^3} = \lim_{x\to 0} -\frac{x^2}{4} = 0$$ 20

$\therefore \quad I_3 = \text{finite}$

<u>Gamma Function</u>:⟶

$$\boxed{n} = \int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n>0)$$

<u>Note</u>:⟶ (i) $\Gamma 1 = 1$

(ii) $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

(iii) $\Gamma (n+1) = n\Gamma n \quad \forall \; n>0$

(iv) $\Gamma (n+1) = n! \quad \forall \; n \in z^+$

(v) $\displaystyle\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma n}{a^n}$

<u>Questions</u>:⟶

1. $\displaystyle\int_0^\infty e^{-x^2} dx = \_\_\_\_\_\_.$

<u>Solⁿ</u>:⟶ Let $x^2 = t \;\Rightarrow 2x\, dx = dt \;\Rightarrow\; dx = \frac{1}{2} t^{-1/2} dt$

$$\therefore \; I = \int_0^\infty e^{-t} \cdot \frac{1}{2} t^{-1/2}\, dt = \frac{1}{2}\int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt = \frac{1}{2}\Gamma \tfrac{1}{2} = \frac{\sqrt{\pi}}{2}.$$

<u>Note</u>:⟶ $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$

2. $\displaystyle\int_0^\infty e^{-2x^2} x^7\, dx = \_\_\_\_\_\_.$

<u>Solⁿ</u>:⟶ Let $2x^2 = t \;\Rightarrow\; 4x\, dx = dt$

$$\therefore \; I = \int_0^\infty e^{-t} \left(\frac{t}{2}\right)^3 \frac{dt}{4} = \frac{1}{32}\int_0^\infty e^{-t} t^3\, dt = \frac{1}{32}\Gamma 4 = \frac{3!}{32} = \frac{6}{32}.$$

3. $\int_0^1 (x\log x)^4\, dx =$ ______.

Sol^n:→ Let, $\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}\,dt$

$$\therefore I = \int_{-\infty}^{\infty} [e^{-t}(-t)]^4 (-e^{-t})\,dt = \int_0^{\infty} e^{-5t}\, t^{5-1}\,dt = \frac{\Gamma 5}{5^5} = \frac{4!}{5^5}$$

4. $\int_0^{\infty} 5^{-4x^2}\, dx =$ ______.

Sol^n:→ Let. $5^{-4x^2} = e^{-t} \Rightarrow -4x^2 \log 5 = -t \Rightarrow x = \frac{1}{2\sqrt{\log 5}}\sqrt{t}$

$$\Rightarrow dx = \frac{1}{2\sqrt{\log 5}}\, \frac{1}{2\sqrt{t}}\,dt$$

$$\therefore I = \int_0^{\infty} e^{-t}\, \frac{1}{2\sqrt{\log 5}}\, \frac{t^{-1/2}}{2}\,dt = \frac{1}{4\sqrt{\log 5}} \int_0^{\infty} e^{-t}\, t^{\frac{1}{2}-1}\,dt$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 5}}$$

Note:→ $\log_a 0 = \begin{cases} \infty & ; \ a<1 \\ -\infty & ; \ a>1 \end{cases}$

Beta function:→

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad (m>0,\ n>0)$$

Note:→ (i) $\beta(m,n) = \beta(n,m)$

(ii) $\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma (m+n)}$

(iii) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

(iv) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2m-1}\theta \; d\theta$

i.e. $\int_0^{\pi/2} \sin^p\theta \; \cos^q\theta \; d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$\left(p > -\frac{1}{2},\; q > -\frac{1}{2}\right)$

Questions:→

1. $\int_0^2 x^7(16-x^4)^{10}\, dx =$ ______.

Soln:→ Let, $x^4 = 16t \Rightarrow 4x^3\,dx = 16\,dt \Rightarrow x^3\,dx = 4\,dt$

$\therefore I = \int_0^1 16t\,(16-16t)^{10}\, 4\,dt = 16^{11}\times 4\int_0^1 t\,(1-t)^{10}\, dt$

$= 4\times 16^{11}\times\beta(2,11)$

$= 4\times 16^{11}\times\frac{\Gamma 2\;\Gamma 11}{\Gamma 13}$

$= 4\times 16^{11}\times\frac{1\times 10!}{12!}$

2. $\int_0^\infty \frac{x^3(1+x^5)}{(1+x)^{13}}\, dx =$ ______.

Soln:→ $\int_0^\infty \frac{x^3}{(1+x)^{13}}\,dx + \int_0^\infty \frac{x^8}{(1+x)^{13}}\,dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+9}}\,dx + \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+4}}\,dx$

$= \beta(4,9) + \beta(9,4)$

$= 2\beta(4,9) = 2\times\frac{\Gamma 4\times\Gamma 9}{\Gamma 13} = 2\times\frac{3!\times 8!}{12!}$

3. $\int_0^\infty \left(\frac{x}{1+x^2}\right)^3 dx =$ ______.

Solⁿ: $\longrightarrow$ Let $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$\therefore I = \int_0^{\pi/2} \left(\frac{\tan\theta}{\sec^2\theta}\right)^3 \sec^2\theta \, d\theta = \int_0^{\pi/2} (\tan^3\theta / \sec^4\theta) \, d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{3+1}{2}, \frac{1+1}{2}\right) = \frac{1}{2} \beta(2,1)$$

$$= \frac{1}{2} \times \frac{\sqrt{2}\,\sqrt{1}}{\sqrt{3}} = \frac{1}{2} \times \frac{1 \times 1}{2!} = \frac{1}{4}$$

4. Partial Differentiation: $\longrightarrow$

Let $z = f(x,y)$ then

$$z_x = \frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$z_y = \frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

Similarly, $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ and so on.

Homogeneous function: $\longrightarrow$

Eg:- (i) $x^2 + xy + y^2$; $n = 2$

(ii) $2x^3 + xy^2 + z^3$; $n = 3$

(iii) $\frac{xy^2 - y^3}{2x + 3y}$; $n = 3 - 1 = 2$

If $f(kx, ky) = k^n f(x,y)$ then $f(x,y)$ is a homogeneous funⁿ with degree 'n'.

Note :→ (i) If $f(x,y)$ is a homogeneous fu^n with degree 'n', then,

$$f(x,y) = \begin{cases} x^n \phi(y/x) \\ y^n \psi(x/y) \end{cases}$$

Euler's Theorem :→ If $f(x,y)$ is a homogeneous fu^n with degree 'n', then,

(a) $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = nf$

(b) $x^2 \dfrac{\partial^2 f}{\partial x^2} + 2xy \dfrac{\partial^2 f}{\partial x \partial y} + y^2 \dfrac{\partial^2 f}{\partial y^2} = n(n-1) f$

Note :→ If $u(x,y) = f(x,y) + g(x,y) + h(x,y)$, where f, g & h are homogeneous fu^n with degree m, n & p respectively, then,

(a) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = mf + ng + ph$

(b) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g + p(p-1)h$.

Note :→ If $f(u)$ is a homogeneous fu^n in two variables x & y with degree 'n', then,

(a) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = n \dfrac{f(u)}{f'(u)} = F(u)$

(b) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = F(u) [F'(u) - 1]$

Total differentiation :→ If $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$, then, the total derivative of 'z' w.r.t. 't' is

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Total differential coefficient,

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Note: $\rightarrow$ (i) If $f(x,y) = c$ is an implicit funⁿ, then,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

(ii) If $z = f(x,y)$ where $x = \phi(u,v)$ & $y = \psi(u,v)$, then,

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

& $$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Questions: $\rightarrow$

1. If $z = e^x \sin y$, where $x = \log t$ & $y = t^2$, then, $\frac{dz}{dt} = ?$

Solⁿ: $\rightarrow$ $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$= e^x \sin y \times \frac{1}{t} + e^x \cos y \times 2t$$

$$= \frac{e^x \sin y}{t} + 2te^x \cos y$$

$$= \frac{e^x}{t}(\sin y + 2t^2 \cos y)$$

$$= \sin y + 2y \cos y \qquad \{\because t = e^x \;\&\; t^2 = y\}$$

2. The Total derivative of $x^2 y$ w.r.t. x, where, x & y are connected by the relation $x^2 + xy + y^2 = 1$ is ——.

Solⁿ: $\rightarrow$ Let, $u = x^2 y$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= 2xy + x^2 \frac{dy}{dx}$$

Now, $f(x,y) = x^2+xy+y^2 = 1$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x+y}{x+2y}$$

$$\therefore \frac{du}{dx} = 2xy + x^2\left(-\frac{2x+y}{x+2y}\right) = 2xy - x^2\left(\frac{2x+y}{x+2y}\right).$$

3. If $u = f(x+cy) + g(x-cy)$, then, $\frac{u_{xx}}{u_{yy}} =$ ——.

(a) ✓ c^{-2} (b) c^2 (c) $-c^{-2}$ (d) $-c^2$

Solⁿ:→ Let, $r = x+cy$, $s = x-cy$

$$u = f(r) + g(s)$$

$$u_x = f'(r)\frac{\partial r}{\partial x} + g'(s)\frac{\partial s}{\partial x} = f'(r) + g'(s)$$

$$u_{xx} = f''(r) + g''(s)$$

$$u_y = f'(r)\frac{\partial r}{\partial y} + g'(s)\frac{\partial s}{\partial y} = c\,f'(r) - c\,g'(s)$$

$$u_{yy} = c^2 f''(r) + c^2 g''(s)$$

$$\therefore \frac{u_{xx}}{u_{yy}} = \frac{1}{c^2} = c^{-2}$$

4. If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, then, $6u_x + 4u_y =$ ——.

(a) $3u_z$ (b) $4u_z$ (c) ✓ $-3u_z$ (d) $-4u_z$

Solⁿ:→ $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

$$u = f(r,s,t)$$

$$u_x = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= f_r(2) + f_s(0) + f_t(-2)$$

$$\therefore 6u_x = 12 f_r - 12 f_t$$

$$u_y = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= f_r(-3) + f_s(3) + f_t(0)$$

$$\therefore 4u_y = -12f_r + 12f_s$$

$$\therefore 6u_x + 4u_y = 12f_s - 12f_t$$

$$u_z = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= f_r(0) + f_s(-4) + f_t(4)$$

$$= -4f_s + 4f_t$$

5. If $u = \frac{y}{z} + \frac{z}{x}$, then, $xu_x + yu_y + zu_z =$ ______.

(a) $\frac{xy}{z^2}$ (b) $\frac{yz}{x^2}$ (c) $\frac{xz}{y^2}$ (d) ✓ 0

Solⁿ:→ $xu_x + yu_y + zu_z = 0\cdot u = 0 \quad (\because n=0)$

6. If $\mu = \frac{x^2 y}{x^{5/2}+y^{5/2}}$, then, $x^2\mu_{xx} + 2xy\,\mu_{xy} + y^2\mu_{yy} =$ ______.

(a) $\frac{3}{4}\mu$ (b) ✓ $-\frac{1}{4}\mu$ (c) $-\frac{3}{4}\mu$ (d) $\frac{1}{4}\mu$

Solⁿ:→ $n = 3 - \frac{5}{2} = \frac{1}{2}$

$$\therefore x^2\mu_{xx} + 2xy\,\mu_{xy} + y^2\mu_{yy} = n(n-1)\mu = -\frac{1}{4}\mu.$$

7. If $u = \operatorname{cosec}^{-1}\left[\frac{x^{1/4}-y^{1/4}}{x^{1/5}+y^{1/5}}\right]$, then, $xu_x + yu_y =$ ______.

(a) $-\frac{1}{20}u$ (b) $-\frac{1}{20}\cot u$ (c) $\frac{1}{20}\tan u$ (d) ✓ $-\frac{1}{20}\tan u$

Solⁿ:→ $\operatorname{cosec} u = \frac{x^{1/4}-y^{1/4}}{x^{1/5}+y^{1/5}}$

$$\Rightarrow n = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

$\therefore\ x u_x + y u_y = n\dfrac{f(u)}{f'(u)} = \dfrac{1}{20} \times \dfrac{\operatorname{cosec} u}{-\operatorname{cosec} u \cot u} = -\dfrac{1}{20}\tan u.$

8. If $u = \log\left(\frac{x^2}{y}\right)$, then, $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} =$ ______.

(a) u (b) 0 (c) -1 (d) 1

Solⁿ:→ $e^u = \left(\frac{x^2}{y}\right) \Rightarrow n = 2-1 = 1$

$$x u_x + y u_y = n\frac{f(u)}{f'(u)} = \frac{e^u}{e^u} = 1 = F(u)$$

$$\therefore\ x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = F(u)\left[F'(u) - 1\right] = -1.$$

9. If $z = x^n f(y/x) + y^{-n} g(x/y)$, then, $xz_x + yz_y + x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy}$ is ______.

(a) $n(n-1)z$ (b) $n^2 z$ (c) $n(n+1)z$ (d) nz

Solⁿ:→ The given funⁿ is the sum of two homogeneous funⁿ having degree n & $-n$ respectively.

$$\therefore\ xz_x + yz_y + x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = (nf - ng) + [n(n-1)f] + [(-n)(-n-1)g]$$

$$= n^2 z$$

10. If $u = \sin^{-1}(x/y) + \cos^{-1}(y/x)$, then, $\frac{u_x}{u_y} =$ ______.

(a) $-\frac{y}{x}$ (b) $-\frac{x}{y}$ (c) $\frac{y}{x}$ (d) $\frac{x}{y}$

Solⁿ:→ $x u_x + y u_y = 0 \cdot u = 0 \quad (\because n = 0)$

$$\Rightarrow \frac{u_x}{u_y} = -\frac{y}{x}.$$

Maxima & Minima:→

For function of Single Variable:→

$f(x) \to \max \to x = c$ if $f(x) \leq f(c)$ $\forall x$

$f(x) \to \min \to x = c$ if $f(x) \geq f(c)$ $\forall x$

Method:→ (i) find $f'(x)$

(ii) Equate $f'(x) = 0$ for obtaining the stationary points

(iii) At each stationary pt. find $f''(x)$

(a) If $f''(x_0) > 0$ then $f(x)$ has minima at $x = x_0$.

(b) If $f''(x_0) < 0$ then $f(x)$ has maxima at $x = x_0$.

(c) If $f''(x_0) = 0$ then $f(x)$ has no extreme at $x = x_0$

and it is called critical point.

Questions:→

1. The funⁿ $f(x) = 2x^3 - 3x^2 - 36x + 10$ has a minimum value at $x =$ ______.

(a) 2 (b) ✓ 3 (c) −2 (d) −3.

Solⁿ:→ $f'(x) = 6x^2 - 6x - 36 = 0$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow x = -2, 3$

$f''(x) = 12x - 6$

$f''(x)\big|_{x=-2} = 12(-2) - 6 = -24 - 6 < 0 \Rightarrow$ maxima at $x = -2$.

$f''(x)\big|_{x=3} = 12(3) - 6 > 0 \Rightarrow$ minima at $x = 3$.

2. The maximum value of fu^n $\frac{e^{\sin x}}{e^{\cos x}}$ is ______. where, $x \in R$

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(a) $e^{1/\sqrt{2}}$ (b) $e^{-\sqrt{2}}$ (c)✓ $e^{\sqrt{2}}$ (d) $e^{-1/\sqrt{2}}$

Sol^n:-- $f(x) = \frac{e^{\sin x}}{e^{\cos x}} = e^{(\sin x - \cos x)}$

f(x) will have max. value when $(\sin x - \cos x)$ will be max.

let, $g(x) = \sin x - \cos x$

$$g'(x) = \cos x + \sin x = 0 \Rightarrow x = -\pi/4 , \frac{3\pi}{4}$$

$$g''(x) = -\sin x + \cos x$$

$$\therefore g''(x)\Big|_{x=-\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} > 0 \Rightarrow \text{minima at } \frac{-\pi}{4}$$

$$g''(x)\Big|_{x=\frac{3\pi}{4}} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0 \Rightarrow \text{maxima at } \frac{3\pi}{4}$$

$$\therefore f\left(\frac{3\pi}{4}\right) = e^{\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)} = e^{\sqrt{2}}$$

3. For the fu^n $f(x) = x^x$, minimum appears at $x =$ ______.

(a) e (b)✓ $\frac{1}{e}$ (c) $e+1$ (d) $e-1$

Sol^n:→ let, $y = x^x \Rightarrow \log y = x \log x$

$$\Rightarrow \frac{1}{y} y' = x \cdot \frac{1}{x} + \log x = 1 + \log x$$

$$\Rightarrow y' = y(1 + \log x) = 0$$

$$\Rightarrow x = \frac{1}{e}$$

4. Consider $f(x) = (x^2 - 4)^2$ where $x \in R$, then, f(x) has

(a) Only one minima (b)✓ Only two minima

(c) three minima (d) three maxima

$\underline{Sol^n}$:$\rightarrow$ $f(x) = (x^2-4)^2$

$f'(x) = 2\cdot 2x(x^2-4) = 0 \Rightarrow x = \pm 2, 0$

$f''(x) = 3x^2-4$

$f''(0) = -4 < 0 \Rightarrow$ maxima at $x=0$

$f''(2) = 3\times 4 - 4 > 0 \Rightarrow$ minima at $x = 2$

$f''(-2) = 3\times 4 - 4 > 0 \Rightarrow$ minima at $x=-2$

5. If $f(x) = a\log x + bx^2 - x$ has an extreme value at $x=-1, 2$ then a & b is ______.

(a) $2, \frac{1}{2}$ (b) $2, -\frac{1}{2}$ (c)✓ $-2, \frac{1}{2}$ (d) $-2, -\frac{1}{2}$

$\underline{Sol^n}$:$\rightarrow$ $f'(x) = \frac{a}{x} + 2bx - 1 = 0.$

$\Rightarrow a + 2bx^2 - x = 0$

$\Rightarrow 2bx^2 - x + a = 0$

$\Rightarrow x^2 - \frac{1}{2b}x + \frac{a}{2b} = 0.$

$\therefore \frac{1}{2b} = -1+2 = 1 \Rightarrow b = \frac{1}{2}$

$\frac{a}{2b} = -2 \Rightarrow a = -2$

6. The maximum value of $f(x) = x^2 - x - 2$ in $[-4, 4]$ is ______.

(a)✓ 18 (b) 10 (c) -2.25 (d) indeterminate

$\underline{Sol^n}$:$\rightarrow$ $f'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$f''(x) = 2 > 0 \Rightarrow$ minima at $x = \frac{1}{2}$

$f(-4) = 18$

$f(4) = 10$

Maxima & Minima for fu^n of two variables :→

Let $z = f(x, y)$,

consider. $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Method :→ (i) Find p, q, r, s & t

(ii) Equate p & q to zero for obtaining the stationary points

(iii) At each stationary point find r, s & t.

(a) If $rt - s^2 > 0$, $r > 0$ then the fu^n $f(x,y)$ has a minima at that stationary point.

(b) If $rt - s^2 > 0$, $r < 0$ then $f(x,y)$ has a maxima at that stationary point.

(c) If $rt - s^2 < 0$, then $f(x,y)$ has no extreme at that stationary point & it is known as Saddle point

Questions :→

1. The fu^n $f(x,y) = x^2 + y^2 + 6x = 0$ has

(a) min. at $(-3, 0)$

(b) max. at $(-3, 0)$

(c) $(-3, 0)$ is a saddle point

(d) none

Sol^n :→

2. The fu^n $f(x,y) = x^3 - 3x^2 + 4y^2 + 6$ has a minimum value at $x =$ ______.

(a) $(0,0)$ (b) $(2,0)$ ✓ (c) $(2,1)$ (d) $(-2,0)$

Sol^n:→ $p = \frac{\partial f}{\partial x} = 3x^2 - 6x = 0 \Rightarrow x = 0, 2.$

$q = \frac{\partial f}{\partial y} = 8y = 0 \Rightarrow y = 0$

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = 8$

At $(0,0)$	& $(2,0)$
$r = -6 < 0$	$r = 6 > 0$
$s = 0$	$s = 0$
$t = 8$	$t = 8$
$rt - s^2 < 0$	$rt - s^2 > 0$
Saddle point	minima at $(2,0)$

3. The fu^n $f(x,y) = x^3 + y^3 - 3axy$ has

(a) max. at (a,a)

(b) max. at (a,a) if $a < 0$ ✓

(c) min. at (a,a)

(d) max. at (a,a) if $a > 0$

Sol^n:→ $p = 3x^2 - 3ay = 0 \Rightarrow x^2 = ay$

$q = 3y^2 - 3ax = 0 \Rightarrow y^2 = ax$

$\left.\begin{matrix} x^2 = ay \\ y^2 = ax \end{matrix}\right\}$ solving

4. A rectangular box open at the top is to have a volume of $32\ ft^3$, then, the dimensions of the box requiring the least material for its construction is ______.

(a) 8, 2, 2 (b) 4, 4, 2 (c) 16, 1, 2 (d) 8, 8, $\frac{1}{2}$

Solⁿ:→ Let the dimension of the box is $x, y, z \Rightarrow xyz = 32$

$$\therefore \quad S = xy + 2yz + 2xz$$

i.e. $f(x,y) = xy + 2y \cdot \frac{32}{xy} + 2x \cdot \frac{32}{xy}$

$$= xy + \frac{64}{x} + \frac{64}{y}$$

$$\left.\begin{aligned} p &= y - \frac{64}{x^2} = 0 \\ q &= x - \frac{64}{y^2} = 0 \end{aligned}\right\} \text{solving } x = 4,\ y = 4$$

5. The distance between the origin and a point nearest to it on the surface $z^2 = 1 + xy$ is ______.

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 1 (d) None

Solⁿ:→ Let $P(x, y, z)$ be on the surface $z^2 = 1 + xy$

$$\therefore D = \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2+xy+1}$$

Consider, $f = x^2+y^2+xy+1$

$$\left.\begin{aligned} p &= 2x + y = 0 \\ q &= 2y + x = 0 \end{aligned}\right\} \text{solving } x = 0,\ y = 0$$

$$\left.\begin{aligned} r &= 2 \\ s &= 1 \\ t &= 2 \end{aligned}\right\} \begin{aligned} &rt - s^2 = 4 - 1 = 3 > 0 \quad \text{also } r > 0 \\ &\text{i.e. minima at } x = 0, y = 0 \end{aligned}$$

at $x = 0, y = 0$ we have $z^2 = 1 + 0 = 1 \Rightarrow z = \pm 1$

$$\therefore D = \sqrt{1} = 1.$$

Constrained Maxima & Minima :→

Lagrange's method of undetermined multipliers :→

Let $f(x,y,z)$ & $\phi(x,y,z) = c$, then

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z) = 0$$

$$\left.\begin{aligned} F_x = 0 \quad \text{i.e.} \quad \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \\ F_y = 0 \quad \text{i.e.} \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \\ F_z = 0 \quad \text{i.e.} \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \text{Lagrange's Eq}^n$$

$\lambda \rightarrow$ Lagrange's Multiplier

Questions :→

1. The volume of the greatest parallelopiped that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is ——.

(a) ✓ $\frac{8abc}{3\sqrt{3}}$ (b) $\frac{4abc}{3\sqrt{3}}$ (c) $\frac{abc}{\sqrt{3}}$ (d) $\frac{abc}{3\sqrt{3}}$

Sol^n :→ Let $2x, 2y, 2z$ be dimension of parallelopiped.

$$\text{volume} = 8xyz = f(x,y,z)$$

$$\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\therefore F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0$$

$$F_x = 8yz + \frac{2\lambda x}{a^2} = 0 \Rightarrow -\frac{\lambda}{4} = \frac{a^2yz}{x}$$

$$F_y = 8xz + \frac{2\lambda y}{b^2} = 0 \Rightarrow -\frac{\lambda}{4} = \frac{b^2xz}{y}$$

$$F_z = 8xy + \frac{2\lambda z}{c^2} = 0 \Rightarrow -\frac{\lambda}{4} = \frac{c^2xy}{z}$$

$$\therefore \frac{a^2yz}{x} = \frac{b^2xz}{y} \ \& \ \frac{b^2xz}{y} = \frac{c^2xy}{z} \ \& \ \frac{a^2yz}{x} = \frac{c^2xy}{z}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} \ \& \ \frac{y^2}{b^2} = \frac{z^2}{c^2} \ \& \ \frac{z^2}{c^2} = \frac{x^2}{a^2}$$

$$\Rightarrow \frac{3x^2}{a^2} = 1 \Rightarrow x = \frac{a}{\sqrt{3}} \ \& \ y = \frac{b}{\sqrt{3}}, \ z = \frac{c}{\sqrt{3}}$$

$\therefore$ volume, $V = 8xyz = \frac{8abc}{3\sqrt{3}}$.

2. The min value of $x^2+y^2+z^2$ where $x+y+z=1$ is ____.

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$ (c) $\frac{1}{27}$ (d) 1

Sol^n:→ $f = (x^2+y^2+z^2)$ & $\phi = x+y+z-1$

$\therefore F = f + \lambda\phi$

$\Rightarrow F_x = 2x + \lambda \Rightarrow -\lambda = 2x$

$F_y = 2y + \lambda \Rightarrow -\lambda = 2y$

$F_z = 2z + \lambda \Rightarrow -\lambda = 2z$

$\Rightarrow x = y = z$

$\therefore x = \frac{1}{3}, y = \frac{1}{3}, z = \frac{1}{3}$

$\therefore f_{min} = x^2+x^2+x^2 = 3x^2 = \frac{1}{3}$

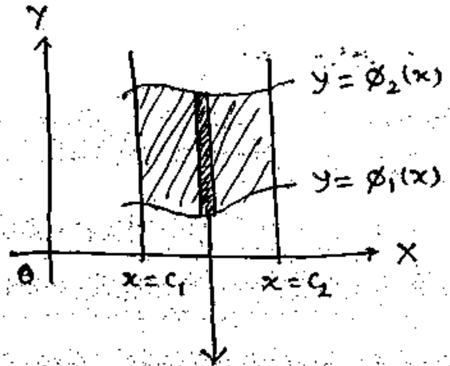
5. Multiple Integrals:→

Double Integral:→ Let $f(x,y)$ be defined at each point in the given region R, then its double integral is, $\iint_R f(x,y)\,dx\,dy$, where, continuity of $f(x,y)$ in the region R ensures the existence of the integral.

Methods of Evaluation:→

Case I:→ Let the limits of integration be $y = \phi_1(x)$ to $y = \phi_2(x)$ & $x = c_1$ to c_2.

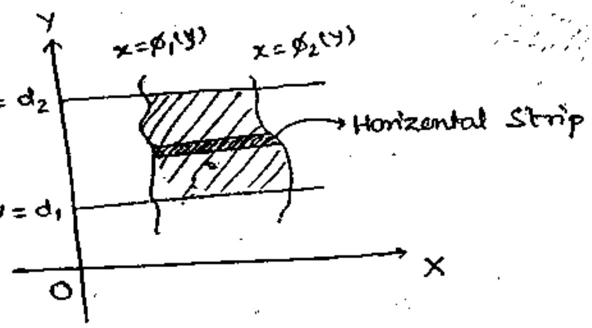
$$\iint_R f(x,y)\,dx\,dy = \int_{x=c_1}^{c_2} \int_{y=\phi_1(x)}^{\phi_2(x)} \{f(x,y)\,dy\}\,dx$$



Vertical strip slided over c_1 to c_2 to get area

Case 2:— when the limits of integration are $x=\phi_1(y)$ & $x=\phi_2(y)$ and $y=d_1$ to $y=d_2$,

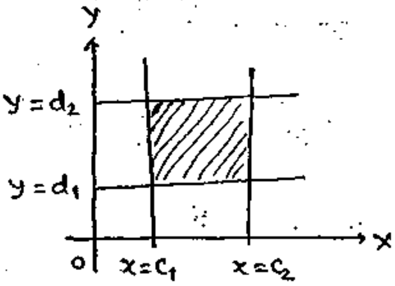
$$\iint_R f(x,y)\,dx\,dy = \int_{y=d_1}^{d_2}\int_{x=\phi_1(y)}^{\phi_2(y)} \{f(x,y)\,dx\}\,dy$$



Case 3:— when the limits are $x=c_1$ to $x=c_2$ & $y=d_1$ ~~to~~ $y=d_2$,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=c_1}^{c_2}\left[\int_{y=d_1}^{d_2} f(x,y)\,dy\right]dx$$

$$= \int_{y=d_1}^{d_2}\left[\int_{x=c_1}^{c_2} f(x,y)\,dx\right]dy.$$



Questions:→

1. Evaluate the following

(i) $\displaystyle\int_0^1\int_0^2 (xy+x^3)\,dx\,dy.$ (ii) $\displaystyle\int_0^4\left[\int_0^{x^2} e^{y/x}\,dy\right]dx.$

Solⁿ:→ (i) $\displaystyle\int_0^1 \left(\frac{x^2}{2}y+\frac{x^4}{4}\right)_0^2 dy = \int_0^1 (2y+4)\,dy = [y^2+4y]_0^1 = 5.$

(ii) $\displaystyle\int_0^4 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx = \int_0^4 (x e^x - x)\,dx = \left[xe^x - e^x - \frac{x^2}{2}\right]_0^4$

$= [4e^4 - e^4 - 8] - [-1] = 3e^4 - 7.$

2. The value of $\iint_R xy\,dx\,dy$, where, R is a region in the ~~I~~ +ve quad-rant at the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is ——.

(a)✓ $\dfrac{a^2b^2}{8}$ (b) $\dfrac{ab}{8}$ (c) $\dfrac{a^3b^3}{8}$ (d) $\dfrac{a^2b^2}{4}$

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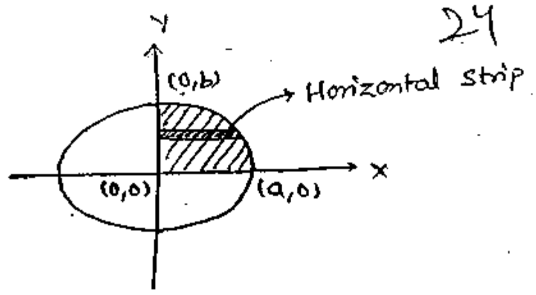
Sol^n:→ Consider the horizontal strip,

$x=0$ to $x=\frac{a}{b}\sqrt{b^2-y^2}$

$y=0$ to $y=b$



$$\therefore \iint_R xy\,dx\,dy = \int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\;dx\,dy$$

$$= \int_{y=0}^{b}\left[\frac{x^2y}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \int_0^b \frac{\frac{a^2}{b^2}(b^2-y^2)\,y}{2}\,dy$$

$$= \left[\frac{a^2}{2}\cdot\frac{y^2}{2} - \frac{a^2}{2b^2}\cdot\frac{y^4}{4}\right]_0^b = \frac{a^2b^2}{4} - \frac{a^2b^2}{8} = \frac{a^2b^2}{8}$$

3. The value of $\iint_R y\,dx\,dy$, where, R is a region $y=x^2$, $x+y=2$ & $x=0$ is ______.

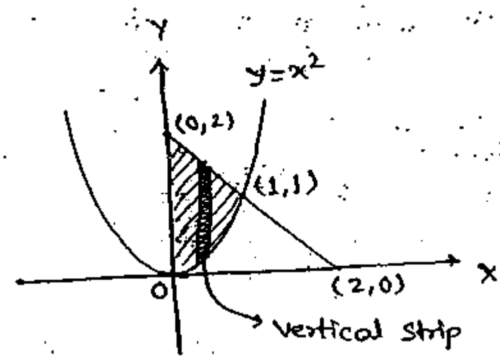
Sol^n:→ $x+x^2=2$

$\Rightarrow x^2+x-2=0$

$\Rightarrow (x-1)(x+2)=0$

$\Rightarrow x=1, \cancel{-2}$

$\therefore$ pt. of intersection is $(1,1)$



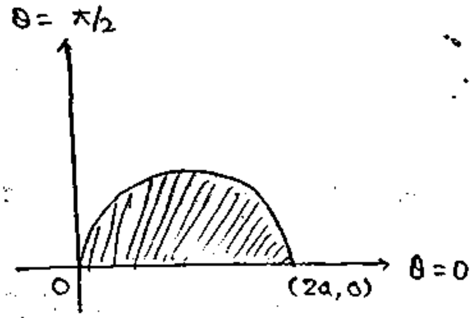
Consider the vertical strip,

$y=x^2$ to $y=2-x$

$x=0$ to 1

$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{2-x} y\,dy\,dx = \int_0^1\left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx = \frac{16}{15}.$$

4. The value of $\iint_R r^2 \sin\theta \, dr \, d\theta$, where, R is the region, bounded by the semicircle $r = 2a\cos\theta$ above the initial line is ______.



Solⁿ:→ $r = 0$ to $2a\cos\theta$

$\theta = 0$ to $\pi/2$

$$\therefore \iint_R r^2 \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \int_{r=0}^{2a\cos\theta} r^2 \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

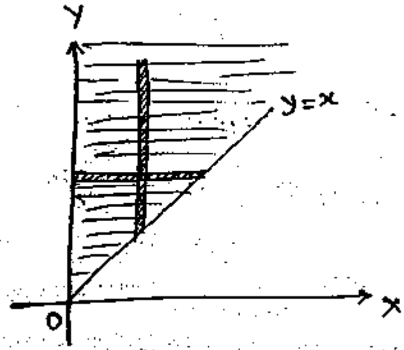
$$= \int_0^{\pi/2} \frac{8a^3\cos^3\theta \sin\theta}{3} \, d\theta$$

$$= -\frac{8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{2a^3}{3}.$$

Change of order of Integration:→

Questions:→

1. The value of $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$ is ______.



Solⁿ:→ Given limits are $y = x$ to $y = \infty$

$x = 0$ to $x = \infty$

Horizontal Strip:–

$x = 0$, $x = y$

$y = 0$, $y = \infty$

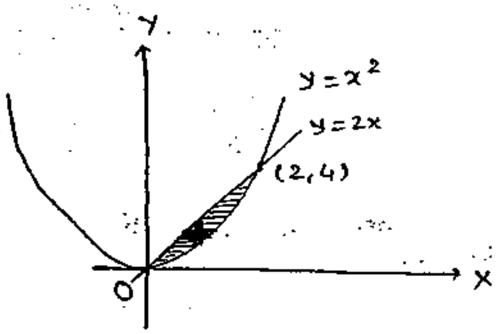
$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx = \int_{y=0}^\infty \left[\int_{x=0}^{y} \frac{e^{-y}}{y} \, dx\right] dy = \int_0^\infty \left[\frac{e^{-y} x}{y}\right]_0^y dy = \int_0^\infty e^{-y} \, dy$$

$$= 1.$$

2. By reversing the order of integration $\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$ may be represented as, 30

(a) $\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$ (b) $\int_0^2 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy$ (c) ✓ $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$

(d) $\int_{x^2}^{2x} \int_0^2 f(x,y)\,dx\,dy$.



Sol^n:→ Given limits are

$x=0$ to $x=2$

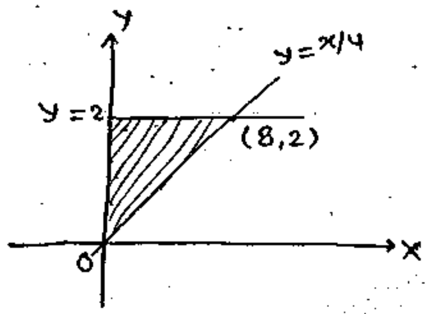
$y=x^2$ to $y=2x$

Horizontal Strip:-

$y=0$ to $y=4$

$x=\frac{y}{2}$ to $x=\sqrt{y}$

3. By changing the order of integration $\int_0^8 \int_{x/4}^{2} f(x,y)\,dy\,dx$ leads to $\int_r^s \int_p^q f(x,y)\,dx\,dy$ then q is ______.



Sol^n:→ Given limits are

$y=\frac{x}{4}$ to $y=2$

$x=0$ to $x=8$

By changing limits,

$y=0$ to $y=2$

$x=0$ to $x=4y$

$\therefore\ q=4y$.

Triple Integrals :→ Let $f(x,y,z)$ be defined at each point in the region 'R' of space then its triple integral is $\iiint_R f(x,y,z)\,dx\,dy\,dz$.

Let $z = \phi_1(x,y)$ to $z = \phi_2(x,y)$

$y = \psi_1(x)$ to $y = \psi_2(x)$

$x = c_1$ to $x = c_2$, then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{x=c_1}^{c_2} \int_{y=\psi_1(x)}^{\psi_2(x)} \int_{z=\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Questions :→

1. Evaluate $\displaystyle\int_0^2 \int_0^x \int_0^{\sqrt{x+y}} z \; dz\,dy\,dx$.

Sol^n :→ $$I = \int_0^2 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\,dx = \int_0^2 \left[\frac{(x+y)^2}{4}\right]_0^x dx = 2.$$

2. The value of $\iiint_R y\,dx\,dy\,dz$; where R is the region bounded by the plane $x=0$, $y=0$, $z=0$ & $x+y+z=1$ is ——.

Sol^n :→ $$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\;dz\,dy\,dx = \int_0^1 \int_0^{1-x} y\,(1-x-y)\,dy\,dx$$

$$\cancel{= \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx}$$

$$\cancel{= \int_0^1 \left[1-x - x + x^2 - \frac{(1-x)^2}{2}\right] dx}$$

$$= \int_0^1 \left[(1-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\right] dx = \left.\frac{(1-x)^4}{-24}\right|_0^1 = \frac{1}{24}$$

Change of variables :$\longrightarrow$ In a double integral, if $x = f(u,v)$ & $y = g(u,v)$

then, $$\iint_R \phi(x,y) = \iint_R \phi(f,g)\ |J|\ du\,dv = \iint_R \phi(u,v)\ |J|\ du\,dv$$

where, $|J| \rightarrow$ Jacobian of transformation used to transform one system to another.

$$|J| = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Cartesian form $\longrightarrow$ Polar form :$\longrightarrow$
(x,y) $\qquad$ (r,θ)

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\Rightarrow \iint_R \phi(x,y)\ dx\,dy = \iint_R \psi(r,\theta)\ r\ dr\,d\theta$$

In a triple integral, if $x = f(u,v,w)$, $y = g(u,v,w)$ & $z = h(u,v,w)$

then, $$\iiint_R f(x,y,z)\ dx\,dy\,dz = \iiint_R \psi(u,v,w)\ |J|\ du\,dv\,dw$$

where, $$|J| = J\left(\frac{x,y,z}{u,v,w}\right) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Cartesian to Cylindrical Polar Form :→

(x,y,z) $\qquad$ (r,θ,z)

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \iiint_R \phi(x,y,z)\,dx\,dy\,dz = \iiint_R \psi(r,\theta,z)\, r\,dr\,d\theta\,dz$$

Cartesian to Spherical Polar form:→

(x,y,z) $\qquad$ (r,θ,ϕ)

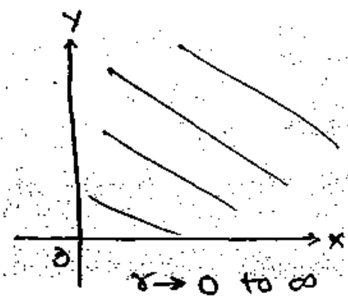
$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

$$|J| = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\therefore \iiint_R \phi(x,y,z)\,dx\,dy\,dz = \iiint_R \psi(r,\theta,\phi)\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

Questions:→

1. Evaluate $\displaystyle\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$.



$r \to 0$ to ∞

$\theta \to 0$ to $\frac{\pi}{2}$

Sol^n:→ Let $x = r\cos\theta$, $y = r\sin\theta$, then $|J| = r$

& $x^2+y^2 = r^2$

$$\therefore \int_0^\infty\int_0^{\pi/2} e^{-r^2}\cdot r\,dr\,d\theta$$

Let $r^2 = t \Rightarrow r\,dr = \frac{dt}{2}$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta = \int_0^{\pi/2}\left[\frac{e^{-t}}{2}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}.$$

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2. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$.

Sol^n:→ $z=0$ to $z=\sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2 \Rightarrow x^2+y^2+z^2=1$

Using spherical coordinates,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$x^2+y^2+z^2 = r^2$

$\Rightarrow r \to 0$ to 1

$\theta \to 0$ to $\pi/2$

$\phi \to 0$ to $\pi/2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{1-r^2}} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \sin\theta\left[\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right] dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\sin^{-1}r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1}(r)\right)\right]_0^1 d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\frac{\pi}{4}\right] d\theta\,d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2} d\phi = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

3. By a change of variable $x(u,v) = uv$ & $y(u,v) = \frac{v}{u}$ in double integr the integrand $f(x,y)$ changes to $f(uv, \frac{v}{u})\,\phi(u,v)$ then $\phi(u,v)$ is ___

(a) $2\frac{v}{u}$ (b) $2uv$ (c) v^2 (d) $=1$

Sol^n:→ $\phi(u,v) = |J| = \begin{vmatrix} v & u \\ -\frac{v}{u^2} & \frac{1}{u}\end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}.$

Lengths, Areas & Volumes:→

(i) Length of an arc of a curve $y = f(x)$ between the lines $x = x_1$ & $x = x_2$

is, $l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

(ii) length of an arc of a curve $x = f(t)$ & $y = g(t)$ between $t = t_1$ to $t = t_2$

is, $l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

(iii) Area of the region bounded by the curve $y = f(x)$ & $y = g(x)$ between

$x = x_1$ & $x = x_2$ is,

$$A = \int_{x_1}^{x_2} [g(x) - f(x)]\, dx \quad \text{(or)} \quad \int_{x_1}^{x_2} \int_{f(x)}^{g(x)} dy\, dx$$

(iv) The volume of solid generated by revolving $y = f(x)$ between $x = x_1$

& $x = x_2$ about x-axis is,

$$V = \int_{x_1}^{x_2} \pi y^2\, dx$$

about y-axis is, $V = \int_{y_1}^{y_2} \pi x^2\, dy$

In polar form:— (i) about initial line (i.e. $\theta = 0^\circ$)

$$V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta\, d\theta$$

(ii) about line $\theta = \frac{\pi}{2}$

$$V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta\, d\theta$$

Questions:→

1. The length of the curve $y = \frac{2}{3}x^{3/2}$ between $x = 0$ & 1 is ____.

(a) 0.27 (b) 0.67 (c) 1 (d) 1.22

Sol^n:→ $\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2} x^{\frac{3}{2}-1} = x^{1/2}$

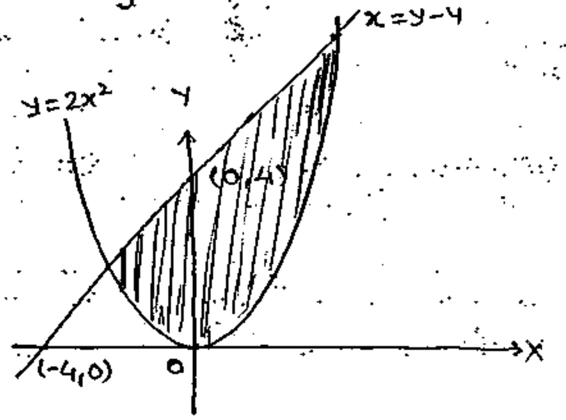
$$\therefore \quad l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+x}\, dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = 1.22$$

2. The length of the curve, $x = \cos^3\theta$, $y = \sin^3\theta$ between $\theta = 0$ & $\pi/2$ is –

Sol^n:→ $$l = \int_0^{\pi/2} \sqrt{(3\cos^2\theta(-\sin\theta))^2 + (3\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= \int_0^{\pi/2} (3\sin\theta\cos\theta)\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \frac{3}{2}.$$

3. The area bounded by the ~~pra~~ parabola $y = 2x^2$ & st. line $x = y - 4$ is ____.



Sol^n:→ Pt. of intersection:–

$y = 2(y-4)^2$

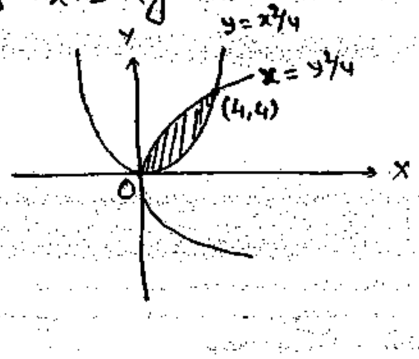
$\Rightarrow y = 2(y^2 + 16 - 8y)$

$\Rightarrow 2y^2 - 17y + 32 = 0$

$$\Rightarrow y = \frac{17 \pm \sqrt{289 - 256}}{4} = \frac{17 \pm \sqrt{33}}{4}$$

$\cancel{A = \int_{y=\frac{17-\sqrt{33}}{4}}^{\frac{17+\sqrt{33}}{4}} [(x+4) - 2x^2]}$ $\quad \therefore A = \int_{\frac{17-\sqrt{33}}{4}}^{\frac{17+\sqrt{33}}{4}} \left[(y-4) - \sqrt{\frac{y}{2}}\right] dy$

4. The area between the curves $y^2 = 4x$ & $x^2 = 4y$ is, ____.

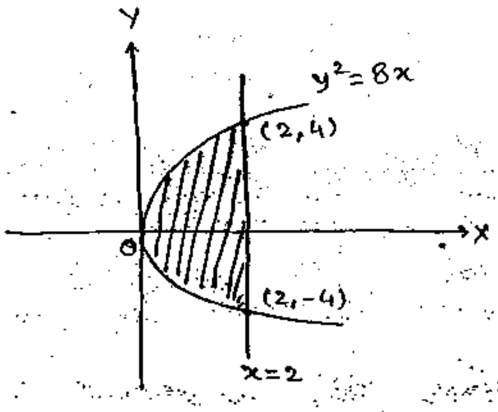


Sol^n:→ $$A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \frac{16}{3}.$$

5. The volume generated by revolving the area bounded by parabola $y^2 = 8x$ and st. line $x = 2$ about y-axis is ______.

(a) ✓ $128\frac{\pi}{5}$ (b) $\frac{5\pi}{128}$ (c) $\frac{127\pi}{5}$ (d) none

Sol^n:→ $V = \int_{y_1}^{y_2} \pi x^2 \, dy$



$= \int_{-4}^{4} \pi \frac{y^4}{64} \, dy$

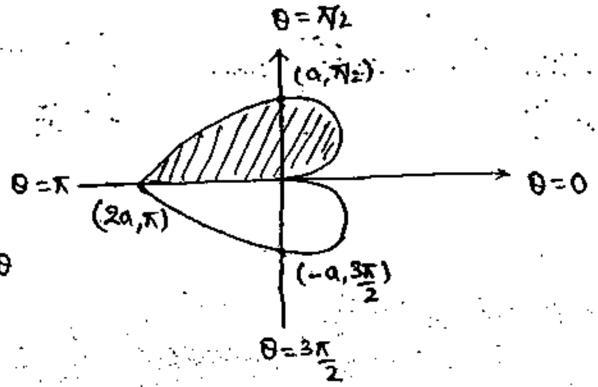
$= \frac{\pi}{64}\left[\frac{y^5}{5}\right]_{-4}^{4} = \frac{32\pi}{5}$

Total volume $= \int_{-4}^{4} \pi x^2 \, dy = \int_{-4}^{4} \pi (2)^2 dy = 4\pi [y]_{-4}^{4} = 32\pi$

∴ Remaining volume $= 32\pi - \frac{32\pi}{5} = \frac{128\pi}{5}$.

6. The volume of solid generated by revolving the cardiac $r = a(1-\cos\theta)$ about the initial line is ______.

Sol^n:→ $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta \, d\theta$



$= \int_{0}^{\pi} \frac{2\pi}{3} a^3 (1-\cos\theta)^3 \sin\theta \, d\theta$

Let $1-\cos\theta = t \Rightarrow \sin\theta \, d\theta = dt$

as $\theta = 0$, $t = 0$

$\theta = \pi$, $t = 2$

$\therefore V = \int_{0}^{2} \frac{2\pi}{3} a^3 t^3 \, dt = \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 = \frac{8\pi a^3}{3}.$

Vector Calculus:→

Position Vector:→ The position vector of the point $P(x,y,z)$ in the space is

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

In parametric form,

$$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

(i) $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos(\vec{a}\cdot\vec{b}) = a_1b_1 + a_2b_2 + a_3b_3$

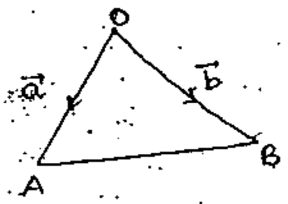
(ii) $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin(\vec{a}\cdot\vec{b})\,\hat{n}$; where $\vec{n}$ is vector of unit length perpendicu-lar to the plane contains $\vec{a}$ & $\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note:→ (I) Area of ΔOAB

$= \frac{1}{2}|\vec{OA}\times\vec{OB}|$

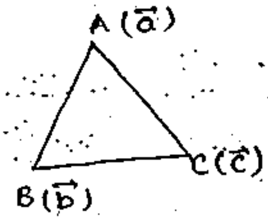
$= \frac{1}{2}|\vec{a}\times\vec{b}|$



(ii) Area of ΔABC

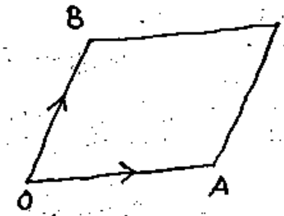
$= \frac{1}{2}|\vec{AB}\times\vec{AC}|$

$= \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|$



(iii) Area of parallelogram

$= |\vec{a}\times\vec{b}|$



Scalar Triple Products:→

$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note:$\to$ (i) Volume of parallelopiped, $V = \left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

(ii) Volume of tetrahadron, $V = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

where, $\vec{a}, \vec{b}$ & $\vec{c}$ are edge vectors of ~~parallelopiep~~ parallelopipe.

Vector Triple Product :$\to$

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{b}\cdot\vec{a})\vec{c}$$

6. Vector Differentiation :$\to$ Let $\vec{r}(t) = \vec{f}(t)$ be the vector can of the curve

than $\dfrac{d\vec{r}}{dt} = \lim\limits_{\Delta t\to 0} \dfrac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$.

If t is a time variable then $\dfrac{d\vec{r}}{dt}$ represents a velocity vector.

Note:$\to$ (i) $\dfrac{d\vec{r}}{dt}$ is a vector in the direction of tangent to the curve at that point.

(ii) $\vec{F}(t)$ is constant in magnitude then, $\vec{F}\cdot\dfrac{d\vec{F}}{dt} = 0$

(iii) $\vec{F}(t)$ vector with fixed direction then, $\vec{F}\times\dfrac{d\vec{F}}{dt} = 0$

Properties:$\to$ Let $\vec{a}(t)$ & $\vec{b}(t)$ be the vector fun of the scalar variable 't' and ϕ be a scalar fun then,

(i) $\dfrac{d}{dt}(\vec{a}\pm\vec{b}) = \dfrac{d\vec{a}}{dt} \pm \dfrac{d\vec{b}}{dt}$

(ii) $\dfrac{d}{dt}(\vec{a}\cdot\vec{b}) = \dfrac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\dfrac{d\vec{b}}{dt}$

(iii) $\dfrac{d}{dt}(\vec{a}\times\vec{b}) = \left(\dfrac{d\vec{a}}{dt}\times\vec{b}\right) + \left(\vec{a}\times\dfrac{d\vec{b}}{dt}\right)$

(iv) $\dfrac{d}{dt}(\phi\vec{a}) = \dfrac{d\phi}{dt}\vec{a} + \phi\dfrac{d\vec{a}}{dt}$

Point Function: → If the value of function depends on position of point, then it is said to be point $f u^n$. 35

Scalar Point Function: → If to each point $P(x,y,z)$ in region R of space ∃ a unit scalar associated with it, then, $\phi(x,y,z)$ is a scalar-point function.

The set of all points in the region R of space together with the scalar values forms a scalar field.

Eg: → The temp. $T(x,y,z)$ at any point on a body is a scalar point function and the medium it self is a scalar field.

Vector Point Function: → The velocity at any time 't' of a particle in a fluid flow is a vector point $f u^n$.

Vector Differential Operator: → (del) $\vec{\nabla}$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Level Surface: → Let $\phi(x,y,z)$ be a scalar field in the region 'R', then the set of points satisfying $\phi(x,y,z) = c$, where, 'c' is an arbitrary const, constitutes a family of surfaces called level surfaces.

Gradient of a Scalar Function: → Let $\phi(x,y,z)$ be a differentiable scalar pt. $f u^n$ then gradient of scalar is denoted by grad ϕ

(or) $$\vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

→ vector normal to surface ϕ

$\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ → unit vector normal to surface ϕ.

~~(i)~~

Questions:→

1. $\vec{\nabla} r$ is ______ if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol^n:→ $r = \sqrt{x^2+y^2+z^2} \Rightarrow \vec{\nabla} r = \frac{\vec{r}}{|\vec{r}|}$

Note:→ $\vec{\nabla} f(r) = f'(r)\,\frac{\vec{r}}{r}$

Eg:- $\vec{\nabla}(\log r) = \frac{1}{r}\cdot\frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$

$\vec{\nabla}(\sin\log r) = \frac{\cos\log r}{r}\cdot\frac{\vec{r}}{r} = \cos\log r\cdot\frac{\vec{r}}{r^2}$

2. The unit vector normal to the surface $y^2 = 8x$ at $(1, 2)$ is ______.

Sol^n:→ Let $\phi = y^2 - 8x$

$\vec{\nabla}\phi = -8\hat{i} + 2y\hat{j}$

$\vec{\nabla}\phi\big|_{(1,2)} = -8\hat{i} + 4\hat{j}$

$\therefore\ \hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{-8\hat{i}+4\hat{j}}{\sqrt{64+16}} = \frac{-8\hat{i}+4\hat{j}}{\sqrt{80}}$.

3. A sphere of unit radius is centred at origin the unit normal at a point $P(x, y, z)$ to the surface of the sphere is the vector

(a) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}, \frac{z}{\sqrt{3}}\right)$ (c) ✓ (x, y, z) (d) $\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}, \frac{z}{\sqrt{2}}\right)$

Sol^n:→ $x^2+y^2+z^2 = 1$

Let $\phi = x^2+y^2+z^2$

$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore\ \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\hat{i}+y\hat{j}+z\hat{k}$.

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Directional Derivative: → The directional derivative of a differentiable scal fun $\phi(x,y,z)$ in the direction of $\vec{a}$ is given by,

$$D.D. = \vec{\nabla}\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

Let, $\vec{a} = \hat{i}$, then,

$$D.D. = \vec{\nabla}\phi \cdot \frac{\hat{i}}{|\hat{i}|} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\hat{i}$$

$$= \frac{\partial\phi}{\partial x}.$$

Let. $\frac{\vec{a}}{|\vec{a}|} = \hat{b}$ then, $D.D. = \vec{\nabla}\phi\cdot\hat{b} = |\vec{\nabla}\phi||\hat{b}|\cos\theta = |\vec{\nabla}\phi|\cos\theta$

The max. value of $\cos\theta = 1$ i.e. when $\theta = 0$ i.e. when $\hat{b}$ is paral to $\vec{\nabla}\phi$. Therefore the value of the directional derivative is maximum in the direction of normal to the surface ϕ and the maximum valu of directional derivative is $|\vec{\nabla}\phi|$.

Angle Between Surfaces: → Angle between the normal to the surfaces at the pt of intersection is taken as the angle between the surface. Let θ be the angle b/w the surfaces $\phi_1(x,y,z) = c_1$ & $\phi_2(x,y,z) = c_2$ th

$$\cos\theta = \frac{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1|\,|\vec{\nabla}\phi_2|}$$

Questions: →

1. The directional derivative of $f(x,y) = x^2 - y^2$ at $(1,2)$ in the direction of $4\hat{i}+3\hat{j}$ is

(a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) ✓ $-\frac{4}{5}$ (d) $-\frac{3}{5}$

Soln: → $D.D. = \vec{\nabla}f \cdot \frac{\vec{a}}{|\vec{a}|} = (2x\hat{i} - 2y\hat{j})\frac{(4\hat{i}+3\hat{j})}{5} = \left.\frac{8x-6y}{5}\right|_{(1,2)} = \frac{8-12}{5} = -$

2. The directional derivative $\phi = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in direction $\overrightarrow{PQ}$ where $Q = (5,0,4)$ ______.

Solⁿ:→ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5,0,4) - (1,2,3) = (4,-2,1)$

$$\therefore \ D.D. = \vec{\nabla}\phi \cdot \frac{(\overrightarrow{PQ})}{|\overrightarrow{PQ}|} = (2x\hat{i} - 2y\hat{j} + 4z\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+1}}$$

$$= \left.\frac{8x + 4y + 4z}{\sqrt{21}}\right|_{(1,2,3)} = \frac{28}{\sqrt{21}}.$$

3. The directional derivative $f = xy^2 + yz^2 + zx^2$ along to the tangent to the curve $x = t$, $y = t^2$ & $z = t^3$ at $(1,1,1)$ is ______.

Solⁿ:→ Vector eqⁿ of the curve is given by

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$
$$= t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

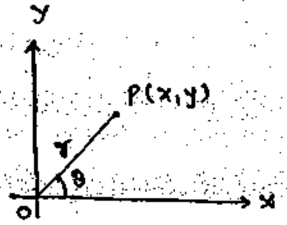
$$\left.\frac{d\vec{r}}{dt}\right|_{(1,1,1)} = \left.\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right|_{t=1} = \hat{i} + 2\hat{j} + 3\hat{k} = \vec{a}$$

$$\therefore \ D.D. = \vec{\nabla}f \cdot \frac{\vec{a}}{|\vec{a}|} = \left[(y^2 + 2zx)\hat{i} + (2xy + z^2)\hat{j} + (2yz + x^2)\hat{k}\right] \cdot \frac{(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+9}}$$

$$= \frac{18}{\sqrt{14}}.$$

4. The directional derivative of $\phi = \frac{y}{x^2+y^2}$ at $(0,1)$ along the line which makes angle 30° with positive x-axis is ______.

Solⁿ:→ $\vec{r} = x\hat{i} + y\hat{j}$



$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$\Rightarrow \dfrac{\vec{r}}{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Note:→ The vector eqⁿ of a st. line which makes an angle θ with +ve x-axis is $\cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$$\therefore \frac{\vec{r}}{r} = \cos 30^\circ\,\hat{i} + \sin 30^\circ\,\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$$\therefore D.D. = \vec{\nabla}\phi \cdot \frac{\vec{r}}{r} = -\frac{1}{2}.$$

5. The greatest rate of increase of $\phi = x^2yz$ at $(2,-1,2)$ is ______.

Solⁿ:→ $\vec{\nabla}\phi = 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$

$$\therefore \vec{\nabla}\phi\Big|_{(2,-1,2)} = -8\hat{i} + 8\hat{j} - 4\hat{k}$$

Greatest rate of increase (or) max. value of D.D. $= |\vec{\nabla}\phi|$

$$= \sqrt{64+64+16}$$

$$= \sqrt{144} = 12$$

6. The angle b/w the surfaces $x^2+y^2+z^2=9$ & $x^2+y^2-z=3$ at $(2,-1,2)$ is ______.

Solⁿ:→ Let, $\phi_1 = x^2+y^2+z^2 \Rightarrow \vec{\nabla}\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$$\Rightarrow \vec{\nabla}\phi_1\Big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

Let, $\phi_2 = x^2+y^2-z \Rightarrow \vec{\nabla}\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$$\Rightarrow \vec{\nabla}\phi_2\Big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1|\,|\vec{\nabla}\phi_2|} = \frac{16+4-4}{\sqrt{16+4+16}\,\sqrt{16+4+1}} = \frac{16}{\sqrt{36}\,\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

7. If $\vec{a}$ & $\vec{b}$ are two arbitrary vectors with magnitudes a & b respectively then, $|\vec{a}\times\vec{b}|^2 =$ ______.

Solⁿ:→ $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$

$$|\vec{a}\times\vec{b}|^2 = a^2b^2\sin^2\theta = a^2b^2(1-\cos^2\theta) = a^2b^2\left(1 - \frac{(\vec{a}\cdot\vec{b})^2}{a^2b^2}\right)$$

$$= a^2b^2 - (\vec{a}\cdot\vec{b})^2$$

Divergence of a Vector Function: → Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be the differential vector point function then

$$\text{div } \vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note: → If $\vec{\nabla}\cdot\vec{F} = 0$ then $\vec{F}$ is called solenoidal vector.

Curl of a Vector Function: →

$$\text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note: → (i) If $\vec{\nabla}\times\vec{F} = \vec{0}$ then $\vec{F}$ is called Irrotational vector.

(ii) If $\vec{v} \rightarrow$ velocity vector

$\vec{\omega} \rightarrow$ angular velocity

then, $\vec{v} = \vec{\omega}\times\vec{r}$

$$\text{curl } \vec{v} = \vec{\nabla}\times(\vec{\omega}\times\vec{r}) = 2\vec{\omega}$$

$$\Rightarrow \boxed{\vec{\omega} = \frac{1}{2}\text{curl } \vec{v}}$$

Scalar Potential function: → For every rotation-al vector, $\exists$ a scalar function ϕ s.t. $\vec{F} = \vec{\nabla}\phi$, then ϕ is said to be scalar potential fu^n.

Note: → (i) $\text{curl}(\text{grad } \phi) = \vec{0}$

(ii) $\text{div}(\text{curl } \vec{F}) = 0$

(iii) $\text{div}(\text{grad } \phi) = \nabla(\nabla\phi) = \nabla^2\phi$

where, $\nabla^2 \rightarrow$ Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Questions:→

1. The value of $\vec{\nabla}.(r^n \vec{r}) =$ ______, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(a) $(n+3)r^n$ ✓ (b) $(n-2)r^n$ (c) nr^{n-3} (d) $(n+2)r^{n-1}$

and hence which of the following is solenoidal

(a) $r^3\vec{r}$ (b) $r\vec{r}$ (c) $\frac{\vec{r}}{r^3}$ ✓ (d) $\frac{\vec{r}}{r^2}$

Solⁿ:→ $r^n \vec{r} = \underbrace{r^n x}_{f_1}\hat{i} + \underbrace{r^n y}_{f_2}\hat{j} + \underbrace{r^n z}_{f_3}\hat{k}$

$$\vec{\nabla}.(r^n\vec{r}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}(r^n x) = r^n + x\, n\, r^{n-1}\frac{\partial r}{\partial x}$$

$$= r^n + n x r^{n-1}\frac{x}{r}$$

$$= r^n + n r^{n-2} x^2$$

Similarly, $\frac{\partial f_2}{\partial y} = r^n + n r^{n-2} y^2$; $\frac{\partial f_3}{\partial z} = r^n + n r^{n-2} z^2$

$$\therefore \vec{\nabla}.(r^n\vec{r}) = 3r^n + nr^{n-2}(x^2+y^2+z^2) = 3r^n + nr^{n-2}.r^2$$

$$= 3r^n + nr^n = (n+3)r^n$$

Now, for solenoidal $\vec{\nabla}.(r^n\vec{r}) = 0$

$$\Rightarrow (n+3)r^n = 0 \Rightarrow n = -3$$

$$\therefore \vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right) = 0.$$

2. If $\vec{F} = (3x^2+2y)\hat{i} - 4xz\hat{j} + 3xy^2\hat{k}$ represents a velocity vector then corresponding angular velocity at $(2, 2, -1)$ is ______.

Solⁿ:→ curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2+2y & -4xz & 3xy^2 \end{vmatrix}_{(2,2,-1)} = 32\hat{i} - 12\hat{j} + 2\hat{k}$

$$\therefore \vec{\omega} = \frac{1}{2}\,\text{curl}\,\vec{F} = 16\hat{i} - 6\hat{j} + \hat{k}$$

3. If $\phi(x,y) = ax^2y - y^3$ & $\nabla^2\phi = 0$ then, $a =$ ______.

(a) 2 (b)✓ 3 (c) −2 (d) −3

Solⁿ:→ $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\Rightarrow 2ay - 6y = 0 \Rightarrow a = 3.$$

4. If $\vec{F} = (5x+7z^2)\hat{i} + (4x^2+\lambda y)\hat{j} + (7z-2xy)\hat{k}$ is solenoidal then $\lambda =$ ___.

Solⁿ:→ $\vec{\nabla}\cdot\vec{F} = 0$

$$\Rightarrow 5+\lambda+7 = 0 \Rightarrow \lambda = -12.$$

5. If $\vec{F} = 5x^2z\,\hat{i} - 7xy^2\hat{j} + (12x+7z)\hat{k}$ then $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})$ at $(5,3,-2)$ is $\underline{0}$.

6. If $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational vector funⁿ then $a, b, c =$ ______.

Solⁿ:→
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \hat{i}(c+1) + \hat{j}(4-a) + \hat{k}(b-2)$$

$$= 0$$

$\therefore\ c = -1,\ a = 4,\ b = 2.$

7. Vector Integration:→

Line Integral:→ An integral evaluated over a curve is called line integral.

Let, $\vec{F}(x,y,z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a differentiable point ~~vector~~ function defined at each point on the curve 'c' then its line integral is

$$\int_c \vec{F}\cdot d\vec{r} = \int_c (f_1\,dx + f_2\,dy + f_3\,dz)$$

Note:→ If c is a closed curve then line integral of $\bar{F}$ over 'c' is called circulation of $\bar{F}$ i.e. $\oint_c \bar{F}\cdot d\bar{r}$.

Work done by a Force:→ The total work done by a force $\vec{F}$ in moving a particle along a curve 'c' is $\int_c \vec{F}\cdot d\vec{r}$

Note:→ If $\bar{F}$ is irrotational, then, the line integral of $\bar{F}$ is independent of the path.

i.e. when $\bar{F}$ is irrotational we have $\bar{F} = \vec{\nabla}\phi$, where ϕ is a scalar potential fun then,

$$\int_a^b \bar{F}\cdot d\bar{r} = \phi_b - \phi_a .$$

Questions:→

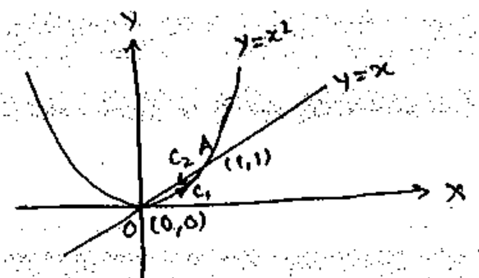
1. The value of $\int_c \bar{F}\cdot d\bar{r}$, where, $\bar{F} = 3xy\,\hat{i} - y^2\hat{j}$ & 'c' is the curve $y = 2x^2$ joining pts (0,0) & (1,2) is ______.

Soln:→ $\int_c \bar{F}\cdot d\bar{r} = \int_c (F_1\,dx + F_2\,dy + F_3\,dz) = \int_c (3xy\,dx - y^2\,dy)$

$$= \int_0^1 [3x(2x^2)\,dx - 4x^4 . 4x\,dx] \qquad \left\{\begin{aligned} &\Rightarrow y = 2x^2 \\ &\Rightarrow dy = 4x\,dx \end{aligned}\right\}$$

$$= \frac{6}{4} - \frac{16}{6} =$$

2. The value of $\int_c \vec{F}\cdot d\bar{r}$, where, $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$ & 'c' is the curve bounded by $y = x$ & $y = x^2$ is ______.



Soln:→ $\int_c \vec{F}\cdot d\bar{r} = \int_{C_1}(\,) + \int_{C_2}(\,)$

along C_1, $y = x^2 \Rightarrow dy = 2x\,dx$

$\therefore \int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} (3xy\,dx - y^2\,dy)$

$$= \int_0^1 [3x(x^2)\,dx - x^4 \cdot 2x\,dx] = \frac{5}{12}$$

Now, along C_2. $y = x \Rightarrow dy = dx$

$$\therefore \int_{C_2} \bar{F} \cdot d\bar{r} = \int_1^0 [3x(x)\,dx - x^2\,dx] = -\frac{2}{3}$$

$$\therefore \int \bar{F} \cdot d\bar{r} = \frac{5}{12} - \frac{2}{3} = -\frac{3}{12} = -\frac{1}{4}.$$

3. The value of $\int_C [(3x+4y)\,dx + (2x-3y)\,dy]$ where 'C' is the circle of radius 2, with centre at origin in X-Y plane is ___.

(a) 4π (b) 8π (c) -4π ✓(d) -8π.

Solⁿ → Here, $C \rightarrow x^2 + y^2 = 2^2$

whenever the curve is circle we will go for polar form.

let, $x = 2\cos\theta$; $y = 2\sin\theta$

$\Rightarrow dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$

$$\therefore \int_0^{2\pi} [(3\times 2\cos\theta + 4\times 2\sin\theta)(-2\sin\theta\,d\theta) + (2\times 2\cos\theta - 3\times 2\sin\theta)(2\cos\theta\,d\theta)]$$

$$= \int_0^{2\pi} [-12\sin\theta\cos\theta\,d\theta - 16\sin^2\theta\,d\theta + 8\cos^2\theta\,d\theta - 12\sin\theta\cos\theta\,d\theta]$$

$$= \int_0^{2\pi} [-24\sin\theta\cos\theta\,d\theta -$$

4. The value of $\int_C \bar{F}\cdot d\bar{r}$, where, $\bar{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ along the line joining

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(i) (0,0,1) & (0,1,1)

(ii) (0,1,1) & (2,1,1)

is ________.

Sol^n:→ (i) Along that st. line $x=0,\ z=1 \Rightarrow dx=0\ ,\ dz=0$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C F_2\, dy = \int_0^1 xz\, dy = 0.$$

(ii) $y=1,\ z=1 \Rightarrow dy=0\ ,\ dz=0$

$$\int_C \vec{F}\cdot d\bar{r} = \int_C F_1\, dx = \int_0^2 (2y+3)\, dx = \int_0^2 5\, dx = 5x\Big|_0^2 = 10,$$

5. The total work done by a force $\bar{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ in moving a particle along a st. line joining the pts. (0,0,0) & (1,2 is ________.

Sol^n:→ $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \Rightarrow \frac{x-0}{1-0} = \frac{y-0}{2-0} = \frac{z-0}{3-0} = t$$

$$\Rightarrow x=t,\ y=2t,\ z=3t$$

$$dx=dt,\ dy=2dt\ \ dz=3dt.$$

$$\int_C \vec{F}\cdot d\bar{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_0^1 \left[(3t^2+12t)\,dt + (-14\times 2t\times 3t)(2\,dt) + (20\times t\times 9t^2)(3dt)\right]$$

$$= \frac{540}{4}.$$

6. The line integral $\int \vec{F} \cdot d\vec{r}$ of vector f^{n} $\vec{F} = 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$ from the origin $(0,0,0)$ to the pt. $(1,1,1)$ is ______.

(a) 0 (b) 1 (c) -1

(d) cannot be determined without specifying the path.

Sol^n:→ $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix} = \hat{i}(x^2-x^2) - \hat{j}(2xy-2xy) + \hat{k}(2xz - 2xz)$$

$$= \vec{0}.$$

$\Rightarrow F$ is irrotational.

$$\Rightarrow \vec{F} = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xyz, \quad \frac{\partial\phi}{\partial y} = x^2z, \quad \frac{\partial\phi}{\partial z} = x^2y$$

The total differentiation of ϕ,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= 2xyz\,dx + x^2z\,dy + x^2y\,dz$$

$$= d(x^2yz)$$

$$\Rightarrow \phi = x^2yz$$

$$\therefore \int_a^b \vec{F}\cdot d\vec{r} = \phi_b - \phi_a = \phi_{(1,1,1)} - \phi_{(0,0,0)} = 1.$$

Green's Theorem in a Plane:→ Let $M(x,y)$ & $N(x,y)$ be continuous f^{n} having continuous 1st order partial derivative defined in the closed region R bounded by the closed curve 'C' then,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Questions:→

1. Evaluate $\oint_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where, c is a curve bounded by $x = 0$, $y = 0$ & $x + y = 1$.

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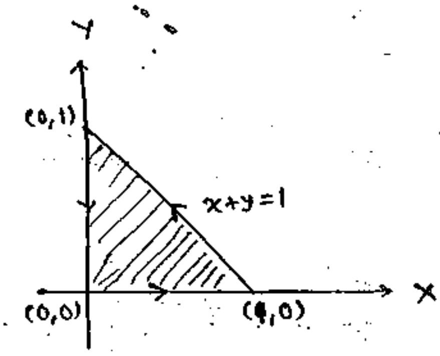


Solⁿ:→ $M = 3x^2 - 8y^2$

$$\Rightarrow \frac{\partial M}{\partial y} = -16y$$

$$N = 4y - 6xy$$

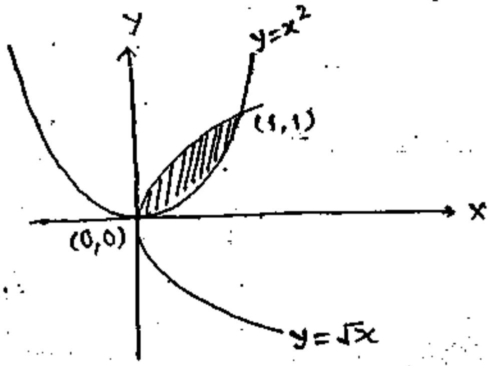
$$\Rightarrow \frac{\partial N}{\partial x} = -6y$$

$$\therefore \frac{\partial M}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

$$\therefore \oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} 10y\;dx\,dy = 5\int_0^1 (1-x)^2\,dx = \frac{5}{3}.$$

2. Evaluate $\oint_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where, c is a curve bounded by $y = \sqrt{x}$ & $y = x^2$.

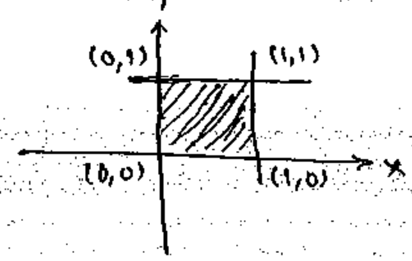


Solⁿ:→ $\cancel{M = (3x^2 - 8y^2)}$

$$\oint_c M\,dx + N\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\;dx\,dy$$

$$= 5\int_0^1 (x - x^4)\;dx = \frac{3}{2}.$$

3. Evaluate $\oint_c xy\,dy - y^2\,dx$, where, c is a square cut from the first quadrant ~~from~~ by the lines $x = 1$ & $y = 1$.



Solⁿ:→ $M = -y^2 \Rightarrow \frac{\partial M}{\partial y} = -2y$

$N = xy \Rightarrow \frac{\partial N}{\partial x} = y$

$\therefore$ $\cancel{\frac{\partial M}{\partial x}}$ $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3y$

$$\therefore \oint_C M\,dx + N\,dy = \int_0^1\int_0^1 3y\,dx\,dy = \frac{3}{2}.$$

Surface Integral: $\longrightarrow$ Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be differentiable vector point fun^n defined over the surface S then, its surface integral is $\int_S \vec{F}\cdot d\vec{S} = \int_S \vec{F}\cdot\vec{n}\,ds$.

where, $\vec{n}$ — unit outward drawn normal to the surface.

In cartesian form,

$$\int_S \vec{F}\cdot d\vec{S} = \int_S \vec{F}\cdot\vec{n}\,ds = \int (F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy)$$

Methods of Evaluation: $\longrightarrow$

(i). If R_1 is the projection of 'S' on to x-y plane then,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \iint_{R_1} \vec{F}\cdot\vec{n}\,\frac{dx\,dy}{|\vec{n}\cdot\hat{k}|}.$$

(ii) If $R_2 \rightarrow$ y-z plane then,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \iint_{R_2} \vec{F}\cdot\vec{n}\,\frac{dy\,dz}{|\vec{n}\cdot\hat{i}|}$$

(iii) If R_3 — x-z plane then,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \iint_{R_3} \vec{F}\cdot\vec{n}\,\frac{dx\,dz}{|\vec{n}\cdot\hat{j}|}$$

Questions:→

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1. The value of $\int_S \vec{F}\cdot\vec{n}\, ds$, where, $\vec{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the 1st octent between $z=0$ & $z=5$.

Sol^n:→ Let $\phi = x^2+y^2$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j}$$

$$\vec{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$$

$$\therefore \vec{n} = \frac{x\hat{i}+y\hat{j}}{4}$$

$$\vec{F}\cdot\vec{n} = \frac{xz}{4} + \frac{xy}{4} = \frac{x}{4}(y+z)$$

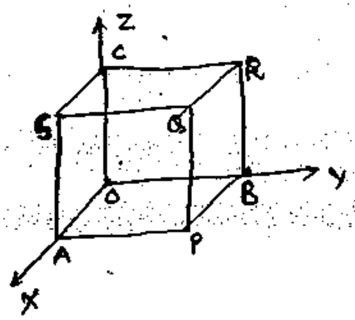
Let, $R \to$ Y-Z plane.

$$\int_S \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\,\frac{dy\,dz}{|\vec{n}\cdot\hat{i}|} = \iint_R \frac{x}{4}(y+z)\,\frac{dy\,dz}{(x/4)}$$

$$= \int_{z=0}^{5}\int_{y=0}^{4} (y+z)\, dy\, dz$$

$$= \int_0^5 \left(\frac{y^2}{2} + yz\right)_0^4 dz = 90.$$

2. The value of $\int_S \vec{F}\cdot\vec{n}\, ds$, where, $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ & S is surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ & $z=1$, is ______



Sol^n:→ Method I:—

$$\int \vec{F}\cdot\vec{n}\, ds = \int_{S_1} + \int_{S_2} + \cdots + \int_{S_6}$$

Over S_1:- In X-Y plane (OAPR).

$$z=0,\ \vec{n} = -\hat{k}$$

$$\vec{F}\cdot\vec{n} = -yz = 0$$

$$\therefore \int_{S_1} \vec{F}\cdot\vec{n}\, ds = 0$$

Over S_2:- parallel to xy plane (SQRC)

$$z=1,\ \vec{n}=\hat{k}\ ,\quad \vec{F}\cdot\vec{n} = yz = y$$

$$\therefore \int_{S_2} \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\ \frac{dx\,dy}{|\vec{n}\cdot\hat{k}|} = \int_0^1\int_0^1 y\, dx\,dy = \frac{1}{2}.$$

Over S_3:- in y-z plane (OBRC)

$$x=0,\ \vec{n}=-\hat{i}\ ,\quad \vec{F}\cdot\vec{n} = -4xz = 0$$

$$\therefore \int_{S_3} \vec{F}\cdot\vec{n}\, ds = 0$$

Over S_4:- parallel to y-z plane (AP

$$x=1,\ \vec{n}=\hat{i}\ ,\quad \vec{F}\cdot\vec{n} = 4xz = 4z$$

$$\therefore \int_{S_4} \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\ \frac{dy\,dz}{|\vec{n}\cdot\hat{i}|} = \int_0^1\int_0^1 4z\, dy\,dz = 2$$

Over S_5:- In x-z plane (OCSA)

$$y=0,\ \vec{n}=-\hat{j}\ ,\quad \vec{F}\cdot\vec{n} = y^2 = 0$$

$$\therefore \int_{S_5} \vec{F}\cdot\vec{n}\, ds = 0$$

Over S_6:- parallel to x-z plane (BRQP)

$$y=1,\ \vec{n}=\hat{j}\ ,\quad \vec{F}\cdot\vec{n} = -y^2 = -1$$

$$\therefore \int_{S_6} \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\ \frac{dx\,dz}{|\vec{n}\cdot\hat{j}|} = \int_0^1\int_0^1 (-1)\, dx\,dz = -1$$

~~Method 2~~ $$\therefore \int_S \vec{F}\cdot\vec{n}\, ds = 0+\frac{1}{2}+0+2+0-1 = \frac{3}{2}.$$

Method 2:-

$$\int_S \vec{F}\cdot\vec{n}\, ds = \int_V \vec{\nabla}\cdot\vec{F}\ dv \qquad \text{(Divergence Theorem)}$$

$$\vec{\nabla}\cdot\vec{F} = 4z-2y+y = 4z-y$$

$$\therefore \int_S \vec{F}\cdot\vec{n}\, ds = \int_0^1\int_0^1\int_0^1 (4z-y)\, dz\,dy\,dx = \frac{3}{2}.$$

Volume Integral: → Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be the differentiable vector point funⁿ defined in volume V, then, its volume integral is $\int_V \vec{F}\cdot dV$

Similarly, if $\phi(x,y,z)$ is scalar point funⁿ, then, $\int_V \phi\, dV$

Questions: →

1. Evaluate $\int_V \nabla\cdot\vec{F}\, dV$ where, $\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ and V is the region bounded by the planes $x=0,\ y=0,\ z=0$ & $2x+2y+z=4$.

Solⁿ: → $\vec{\nabla}\cdot\vec{F} = 4x - 2x = 2x$

$$\int_V 2x\, dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\, dz\, dy\, dx = \frac{8}{3}$$

2. The volume of an object expressed in spherical coordinate. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} r^2 \sin\theta\, dr\, d\phi\, d\theta$, then V is ______.

(a) $\frac{\pi}{3}$ ✓ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

Solⁿ: → $V = \int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left.\frac{r^2}{2}\right|_0^1 d\phi\, d\theta = \frac{1}{2}\int_0^{2\pi} (-\cos\phi)_0^{\pi/2}\, d\theta = \frac{\pi}{3}$

Gauss-Divergence Theorem: → Let ~~[illegible]~~ S be a closed surface enclosing a volume V & $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be the differentiable vector point funⁿ defined over S, then,

$$\int_S \vec{F}\cdot d\vec{s} = \int_V \operatorname{div}\vec{F}\, dV$$

Questions: →

1. Evaluate $\int_S \vec{r}\cdot\vec{n}\, ds$, where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & S is a closed surface enclosing a volume V is ____.

(a) V (b) 2V (c) 3V ✓ (d) 4V.

2. The value of $\int_S (x\,dy\,dz + y\,dx\,dz + z\,dx\,dy)$, where S is a surface,

(i) cylinder $x^2+z^2=16$ & $y=0$ to $y=3$,

(ii) is a sphere $x^2+y^2+z^2=9$,

is ________.

Sol^n:→ $\bar{F} = x\hat{i}+y\hat{j}+z\hat{k}$ $\left(\because \int_S \bar{r}\cdot\bar{n}\,ds = \int_S \bar{F}\cdot d\bar{S} = \int_S (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\,d\bar{S}\right)$

$$\int_V \bar{r}\cdot\bar{n}\,ds = 3V$$

(i) $\int_V \bar{r}\cdot\bar{n}\,ds = 3\pi r^2 h = 3\pi\times 4^2\times 3 =$

(ii) $\int_V \bar{r}\cdot\bar{n}\,ds = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\times 3^3 =$

3. Evaluate $\int_S (x^2+2y^2+3z^2)\,ds$, where, S is the surface $x^2+y^2+z^2=1$.

Sol^n:→ $\bar{F}\cdot\bar{n} = x^2+2y^2+3z^2$ --- (1)

$\phi = x^2+y^2+z^2$

$$\bar{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = x\hat{i}+y\hat{j}+z\hat{k}$$

Let, $F_1\hat{i}+F_2\hat{j}+F_3\hat{k} = \bar{F}$, then,

$$\bar{F}\cdot\vec{n} = F_1x+F_2y+F_3z = x^2+2y^2+3z^2$$

$$\Rightarrow F_1 = x,\ F_2 = 2y,\ F_3 = 3z$$

$$\therefore \bar{F} = x\hat{i}+2y\hat{j}+3z\hat{k}$$

$$\therefore \int_S \bar{F}\,ds = \int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_V (1+2+3)\,dV = 6V = 6\times\frac{4}{3}\pi\times 1 = 8\pi.$$

4. The value of $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\vec{F} = 4x^2\hat{i}-3y\hat{j}+8xz\hat{k}$ & S is a surface $0\le x\le 1$, $0\le y\le 2$, & $0\le z\le 3$, is ________.

Sol^n:→ $\vec{\nabla}\cdot\bar{F} = 8x-3+8x = 16x-3$

$$\therefore \int_S \bar{F}\cdot\vec{n}\,ds = \int_V \vec{\nabla}\cdot\bar{F}\,dV = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3}(16x-3)\,dx\,dy\,dz = 30.$$

5. Evaluate $\int_S \vec{F}\cdot\vec{n}\,ds$, where, $F = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, taken over the regio bounded by $x^2+y^2=4$ & $z=0$ to $z=3$.

Sol^n:→ $\vec{\nabla}\cdot\vec{F} = 4 - 4y + 2z$

$$\therefore \int_S \vec{F}\cdot\vec{n}\,ds = \int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dx\,dy\,dz$$

$$= 2\int_0^3\int_{-2}^{2} (4+2z)\sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_0^3\int_0^2 (4+2z)\sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_0^3 (4+2z)\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 dz$$

$$= 4\pi\int_0^3 (4+2z)\,dz = 84\pi$$

Note:→ we can also solve the above problem in polar form.

6. Evaluate $\int_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,ds$, where $\vec{F} = 4x^2z\,\hat{i} - (yz-7)\hat{j} + xy^2z\,\hat{k}$, and S is the surface bounded by $y^2+z^2=25$ & $x=0$ to $x=2$.

Sol^n:→ $\int_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}(\vec{\nabla}\times\vec{F})\,dV = 0.$

Stoke's Theorem:→ Let S be an open surface bounded by a closed curve 'C' & $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a differentiable vector fun^n defined over 'S', then $\oint_C \vec{F}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \int_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,ds$

i.e. $\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,ds$

Questions: → 1. The value of $\oint_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$, & c is a curve $x^2+y^2 = 4$ in XY plane is ______.

(a) 0 (b) $\frac{1}{2}$ (c) 2 (d) 3

Sol^n: →

$$\vec{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z) = \vec{0}$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S (\vec{\nabla}\times\bar{F})\cdot\vec{n}\, ds = \int_S \vec{0}\cdot\vec{n}\, ds = 0.$$

2. The value of $\int_C \bar{F}\cdot d\bar{x}$, where, $\vec{F} = -y^3\hat{i} + x^3\hat{j}$ & 'c' is the circular disc $x^2+y^2 \le 1$, $z=0$ is ______.

Sol^n: →

$$\vec{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(3x^2+3y^2)$$

$$= 3(x^2+y^2)\,\hat{k}$$

$$\therefore \vec{\nabla}\times\bar{F}\cdot\vec{n} = 3(x^2+y^2) \qquad \{\because \vec{n} = \hat{k}\}$$

$$\therefore \int_C \bar{F}\cdot d\bar{x} = \int_S (\vec{\nabla}\times\bar{F})\cdot\vec{n}\, ds$$

$$= \int_S 3(x^2+y^2)\, ds$$

Let, $R \to xy$ plane

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \iint_R 3(x^2+y^2)\frac{dx\,dy}{|\vec{n}\cdot\hat{k}|} = \iint_R 3(x^2+y^2)\, dx\, dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$x^2+y^2 = r^2, \quad |J| = r$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi}\int_0^1 3r^2\, r\, dr\, d\theta = \frac{3\pi}{2}.$$

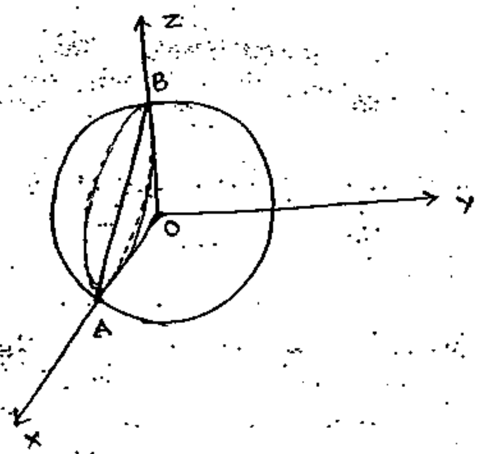
3. Evaluate $\oint_C (y\,dx + z\,dy + x\,dz)$, where C is the curve of intersection of $x^2+y^2+z^2 = a^2$ & $x+z = a$.

Solⁿ: → $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1)$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

The intersection of a sphere $x^2+y^2+z^2 = a^2$ with the plane $x+z = a$ is a circle in the plane $x+z=a$, with AB as diameter, where A $(a,0,0)$ & B $(0,0,a)$



$AB = \sqrt{a^2+a^2} = a\sqrt{2}$

$\Rightarrow$ radius $= \dfrac{a}{\sqrt{2}}$

Let, $\phi = x+z$

$\vec{\nabla}\phi = \hat{i}+\hat{k}$

$\vec{n} = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \dfrac{\hat{i}+\hat{k}}{\sqrt{2}}$

$(\vec{\nabla}\times\vec{F})\cdot\vec{n} = (-\hat{i}-\hat{j}-\hat{k})\cdot\dfrac{(\hat{i}+\hat{k})}{\sqrt{2}} = -\sqrt{2}$

$$\oint_C (y\,dx + z\,dy + x\,dz) = \int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\; dS = \int_S -\sqrt{2}\; dS = -\sqrt{2}\, S$$

$$= -\sqrt{2}\,\pi\times\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}.$$

8. <u>Fourier Series</u> :→ Let f(x) be a periodic fu^n defined in $(c, c+2l)$ with period $2l$, then, the fourier series of f(x) is given by,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

where, a_0, a_n & b_n are fourier coeff. given by,

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

<u>Note</u>:→ $[-l, l]$, $[0, 2l]$, $[-\pi, \pi]$ (or) $[0, 2\pi]$

<u>Dirchilet's Conditions</u> :→ A fu^n is said to satisfy dirchilet's $cond^n$ if

(i) f(x) and its integrals are finite & single valued.

(ii) f(x) has finite no. of finite discontinuities.

(iii) f(x) has finite no. of maxima & minima.

<u>Note</u>:→ If f(x) ~~is~~ satisfied Dirichlet's $cond^n$ then the fourier series is convergent.

<u>Convergence</u>:→ (i) If f(x) is continuous at $x = c \in (a, b)$ then fourier series of f(x) at $x = c$ converges to f(c).

(ii) If f(x) is discontinuous at $x = c \in (a, b)$ then fourier series of f(x) at $x = c$ converges to $\frac{1}{2}\left[\lim_{x\to c^-} F(x) + \lim_{x\to c^+} F(x)\right]$

(iii) fourier Series of f(x) at the end ~~pt~~ pts i.e $x = a$ (or) b converges to $\frac{1}{2}\left[\lim_{x\to a^+} F(x) + \lim_{x\to b^-} F(x)\right]$

Fourier Series of Even & Odd Function in $[-l, l]$ (or) $[-\pi, \pi]$:→

(i) Fourier Series of an even function :→

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where, $a_0 = \frac{2}{l} \int_0^l f(x)\, dx$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

(ii) Fourier Series of an Odd function :→

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where, $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$.

Half - Range Series :→

(i) Half - Range cosine series in $[0, l]$:→

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where, $a_0 = \frac{2}{l} \int_0^l f(x)\, dx$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

(ii) Half - Range sine series in $[0, l]$:→

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where, $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

Questions :→

1. The coeff. of $\sin x$ in the fourier series expansion of $f(x) = x^2$ in $(-\pi, \pi)$ is ______.

(a) $\sum \frac{(-1)^n}{n^2}$ (b) $\sum \frac{1}{n^2}$ (c) $\frac{\pi^2}{6}$ (d) ✓ 0

Sol^n :→ Even fu^n hence, coeff. of $\sin x = 0$.

2. If $f(x) = \begin{cases} 0 \ ; & -2 < x < 0 \\ 1 \ ; & 0 < x < 2 \end{cases}$

then the term independent of x in the fourier series of $f(x)$ is —

(a) 0 (b) 1 (c) ✓ $\frac{1}{2}$ (d) 2

Sol^n :→ Here, $(-2, 2) \Rightarrow l = 2$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx = \frac{1}{2} \int_{0}^{2} 1 \cdot dx = 1$$

$\therefore$ Independent term $= \frac{a_0}{2} = \frac{1}{2}$

3. The fu^n $f(x) = \begin{cases} -x+1 \ ; & -\pi \le x \le 0 \\ x+1 \ ; & 0 \le x \le \pi \end{cases}$

then $f(x)$ has following terms in its expansion

(a) ✓ cosine (b) sine (c) both (d) cannot be determined.

Sol^n :→ $f(-x) = \begin{cases} x+1 \ ; & -\pi \le -x \le 0 \\ -x+1 \ ; & 0 \le -x \le \pi \end{cases}$

$$= \begin{cases} x+1 \ ; & \pi \ge x \ge 0 \\ -x+1 \ ; & 0 \ge x \ge -\pi \end{cases}$$

$$= f(x)$$

$\Rightarrow$ Even fu^n.

4. If $f(x) = \begin{cases} 0 & ; & -2 < x < 0 \\ 1 & ; & 0 < x < 2 \end{cases}$ 47

then the coeff. of $\cos \frac{n\pi x}{2}$ is ____.

(a) ✓ 0 (b) $\frac{1}{n}$ (c) $\frac{1}{n^2}$ (d) $-\frac{1}{n}$

Soln:→ $(-2, 2) \Rightarrow l = 2$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{2} \int_0^2 \cos \frac{n\pi x}{2} \, dx$$

$$= \frac{1}{2} \left[\frac{\sin \frac{n\pi x}{2}}{n\pi/2} \right]_0^2$$

$$= \frac{1}{2} \times \frac{2}{n\pi} [0 - 0] = 0$$

5. If $f(x) = x^2$ in $[-\pi, \pi]$ has its fourier expansion as $f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \infty\right]$ then the value of

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ is ____.

(a) ✓ $\frac{\pi^2}{6}$ (b) $\frac{\pi^2}{12}$ (c) π^2 (d) None

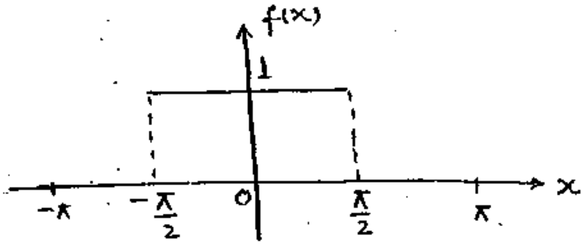
Soln:→ At $x = \pi$, we have

$$\frac{\pi^2}{3} - 4\left[\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots\right] = \frac{1}{2}\left[\lim_{x \to \pi^+} f(x) + \lim_{x \to \pi^-} f(x)\right]$$

$$= \frac{1}{2}[\pi^2 + \pi^2] = \pi^2$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{4}\left[\pi^2 - \frac{\pi^2}{3}\right] = \frac{\pi^2}{6}$$

6. A fu^n with period 2π is shown below



then fourier series is ______ .

(a) $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos nx$

(b) $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \sin nx$

✓(c) $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos nx$

(d) $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \sin nx$

Sol^n:$\rightarrow$

$$f(x) = \begin{cases} 0 \; ; & -\pi < x < -\pi/2 \\ 1 \; ; & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 \; ; & \frac{\pi}{2} < x < \pi \end{cases}$$

$f(-x) = f(x) \Rightarrow$ Even $fu^n \Rightarrow$ cosine series

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi/2} 1 \cdot dx = 1$$

$$\therefore \frac{a_0}{2} = 1$$

7. In $[0, \pi]$ the constant term in the cosine series of $f(x) = x^2 + 2x$ is

(a) $\pi\left(\frac{\pi}{3} - 1\right)$ ✓(b) $\pi\left(\frac{\pi}{3} + 1\right)$ (c) $\pi\left(\frac{\pi}{2} + 1\right)$ (d) None

Sol^n:$\rightarrow$ $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} [x^2 + 2x]\, dx = \frac{2}{\pi}\left[\frac{\pi^3}{3} + \pi^2\right]$

$$= \cancel{\frac{2\pi^2}{3} + \pi}$$

$$= \cancel{\pi\left(\frac{2\pi}{3} + 1\right)}$$

$$\therefore \frac{a_0}{2} = \frac{1}{\pi}\left(\frac{\pi^3}{3} + \pi^2\right) = \pi\left(\frac{\pi}{3} + 1\right).$$

8. If $f(x) = x$ is expressed on a half range cosine series in $[0,2]$ then the coeff. of $\cos \pi x$ is ____. 48

(a) $\frac{4}{\pi^2}$ (b) $\frac{2}{\pi^2}$ (c) ✓ 0 (d) None.

9. In the interval $[0,\pi]$ if a const. c is expressed as a half range sine series then coeff. of $\sin 5x$ is ____.

(a) $\frac{2c}{5\pi}$ (b) 0 (c) $\frac{4c}{5\pi}$ (d) $\frac{c}{5\pi}$

Solⁿ:→ $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, $f(x) = c$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} c \sin 5x \, dx$$

$$= \frac{2c}{\pi} \left[-\frac{\cos 5x}{5} \right]_0^{\pi}$$

$$= \frac{4c}{5\pi}$$

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Diffrencial Equations

* Differential Equations *

eqn - $x+y=2$ diff. eqn - $\frac{dy}{dx} x+y=2$

soln $\Rightarrow x=1$ & $y=1$ soln $\Rightarrow y=f(x)$

Differential eqn

Ordinary diff. eqn

single variable

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+3y=0$$

partial diff. eqn

more than one var

$$\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial^2 u}{\partial y^2}\right)^3=0$$

order = 2
degree = 3

order = highest derivative

degree = highest power of highest degree derivative should free from radicals & fractions.

order = 2; degree = 1

order - The highest derivative involved in diff. eqn.

Radicals = $\left(\frac{dy}{dx}\right)^{1/2, 1/3, 1/4 \dots}$

fraction - derivative should not in denomenator

① $\left(\frac{d^2y}{dx^2}\right)^2=\left(x+\frac{dy}{dx}\right)^{3/2}$

$\left(\frac{d^2y}{dx^2}\right)^4=\left(x+\frac{dy}{dx}\right)^3$

$\therefore O=2 \quad D=4$

② $\frac{dy}{dx}=x+\frac{e^y}{\frac{dy}{dx}}$

$\left(\frac{dy}{dx}\right)^2=x\frac{dy}{dx}+e^y$

$O=1, \quad D=2$

* Differential Equations *

Eqn - $x+y=2$ diff eqn $\frac{dy}{dx} x + y = 2$

soln $\Rightarrow$ $x=1$ & $y=1$ soln $\Rightarrow$ $y = f(x)$

Differential eqn

Ordinary diff. eqn

single variable

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 3y = 0$$

partial diff. eqn

more than one var

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = 0$$

order = highest derivative

degree = highest power of highest degree derivative should free from radicals & fractions.

order = 2 ; degree = 1

order = 2
degree = 3

order - The highest derivative involved in diff. eqn.

Radical - $\left(\frac{dy}{dx}\right)^{1/2, 1/3, 1/4 \ldots}$

fraction - derivative should not in denomenator

1) $\left(\frac{d^2y}{dx^2}\right)^2 = \left(x + \frac{dy}{dx}\right)^{3/2}$

$\left(\frac{d^2y}{dx^2}\right)^4 = \left(x + \frac{dy}{dx}\right)^3$

$\therefore O = 2 \quad D = 4$

2) $\frac{dy}{dx} = x + \frac{e^y}{\frac{dy}{dx}}$

$\left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx} + e^y$

$O = 1, \; D = 2$

) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + 3y = 0$

$O = 3$, $D = 2$

(4) $dy = (t + \sin t)\,dt$

$\frac{dy}{dt} = t + \sin t$

$O = 1$, $D = 1$

) $\left(\frac{\partial^2 u}{dx^2}\right)^3 = c^2\left(\frac{\partial^3 y}{\partial v^3}\right)^2$

$O = 3$, $D = 2$

Formation of differential eqns -

$F(x, y, a, b) = 0$ — (1)

Diff. (1) w.r.t. x

$F_1(x, y, \frac{dy}{dx}, a, b) = 0$ — (2)

Diff. (2) w.r.t. x

$F_2(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b) = 0$ — (3)

Eliminate a, b $\Rightarrow$ $\phi(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$

Note:

The no. of arbitrary constants eliminated should be equal to the order of resulting ordinary diff. eqns

(1) The diff eqn of family of [illegible] passing through origin is

(a) [illegible] (b) [illegible]

(c) $xdx + ydy = 0$ (d) none

$y = mx$

$$\frac{dy}{dx} = m$$

$$\therefore \quad y = \frac{dy}{dx}\, x$$

$$dy\,x = dx\cdot y$$

$$\therefore \quad xdy - ydx = 0$$

(2) Find the diff eqn of family of circle centred at (h, k) & radius 'a' where h, k are arbitrary constants.

$$(x-h)^2 + (y-k)^2 = a^2 \quad \text{——— (1)}$$

diff w.r.t. x

$$2(x-h) + 2(y-k)y' = 0 \qquad x-h = -(y-k)y' \quad \text{——— (2)}$$

diff w.r.t. x

[illegible]

[illegible] y'' [illegible] y''

$$x-h = \frac{(1+y'^2)\cdot y'}{y''} \quad \text{--- (4)}$$

Put (4) in (1)

$$\frac{(1+y'^2)^2}{y''^2}y'^2 + \frac{(1+y'^2)^2}{y''^2} = a^2$$

$$(1+y'^2)^2(y'^2+1) = a^2y''^2$$

$$(1+y'^2)^3 = a^2y''^2$$

(3) $y = c(x-c)^2$ --- (1)

Diff. w.r.t. x

$y' = 2c(x-c)$ ← (2)

$$\frac{(1)}{(2)} \Rightarrow \frac{y}{y'} = \frac{\cancel{c}(x-c)^{\cancel{2}}}{2\cancel{c}\cancel{(x-c)}} = \frac{x-c}{2} \quad \therefore x-c = \frac{2y}{y'}$$

$$c = x - \frac{2y}{y'} = \frac{xy'-2y}{y'}$$

$$y = \frac{xy'-2y}{y'}\left(x - \frac{xy'-2y}{y'}\right)^2 = \left(\frac{xy'-2y}{y'}\right)\frac{4y^2}{y'^2}$$

$$y'^3 = (xy'-2y)4y$$

Note:- If the given eqn is of the form $y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + \cdots$

Then D.E is

$(D-a)(D-b)(D-c)y = 0$

$D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ...

e.g.

① $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{2x}$

→

$(D+1)(D-3)(D-2)y = 0$

$(D^2-2D-3)(D-2)y = 0$

$(D^3-4D^2+D+6)y = 0$

$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

② $i = c_1 e^{2t} + c_2 e^{-4t}$

→ $(D-2)(D+4)i = 0$

$(D^2+2D-8)i = 0$

$\frac{d^2i}{dt^2} + 2\frac{di}{dt} - 8i = 0$

Note:-

If the given eqn is of the form

$y = Af(x) + Bg(x)$ Then the D.E. is obtained by simplifying the following determinant.

$$\begin{vmatrix} y & f(x) & g(x) \\ y' & f' & g' \\ y'' & f'' & g'' \end{vmatrix} = 0$$

e.g. ① $y = Ae^{2x} + Bx$

$$\begin{vmatrix} y & e^{2x} & x \\ y' & 2e^{2x} & 1 \\ y'' & 4e^{2x} & 0 \end{vmatrix} = 0$$

$\therefore y(-4e^{2x}) - e^{2x}(-y'') + x(4y'e^{2x} - 4y''e^{2x}) = 0$

$\therefore y''(1-2x) + 4xy' - 4y = 0$

② $y = e^{x}(A\cos x + B\sin x)$ —①

$y' = e^{x}(A\cos x + B\sin x) + e^{x}(-A\sin x + B\cos x)$ —②

$y' = y + e^{x}(-A\sin x + B\cos x)$

$y'' = y' + e^{x}(-A\sin x + B\cos x) + e^{x}(-A\cos x - B\sin x)$

$y'' = y' + (y' - y) - y$

$\therefore y'' - 2y' + 2y = 0$

$$\boxed{\frac{d^2y}{dt^2} = g}$$

$y(0)$
$y'(0) = 0$

$\frac{dy}{dt} = gt + c_1$; $y' = 0$, $t = 0$; $0 = 0 + c_1$; $c_1 = 0$

$y = \frac{gt^2}{2} + c_1 t + c_2$; $y = 0$, $t = 0$; $0 = 0 + c_2$

General sol^n

∂ no. of arbitrary const

$$\boxed{y = \frac{gt^2}{2}}$$ P. sol^n

$$\boxed{\frac{d^2y}{dt^2} = g}$$

(a) $y = \frac{1}{2}gt^2 + c_1 e^{t} + c_2$

(b) $y = \frac{1}{2}gt^2 + c_1 t^3 + c_2$

(c) $y = \frac{1}{2}gt^2 + c_1 t + c$

(d) None

First order first degree diff. eq^ns.

Notation - $\boxed{\frac{dy}{dx} = F(x,y)}$ or $\boxed{M\,dx + N\,dy = 0}$ $M(x,y)$ & $N(x,y)$

① Variable separable method.

$\frac{dy}{dx} = F(x,y)$

$f(x)\,dx = g(y)\,dy$

$\int f(x)\,dx = \int g(y)\,dy + c$

① $\log\left(\frac{dy}{dx}\right) = 2x-y$

$\rightarrow$

~~log dy − log dx = 2x−y~~

$\frac{dy}{dx} = e^{2x-y} = \cancel{e^{2x} \times e^{-y}}$

$\int e^{y} dy = \int e^{2x} dx$

$$\boxed{e^{y} = \frac{e^{2x}}{2} + c}$$

② $\frac{dy}{dx} = \frac{y^2}{1-xy}$

$(1-xy)\,dy = y^2\,dx$

$dy = y^2 dx + xy\,dy$

$= y\,(\underbrace{y\,dx + x\,dy})$

$dy = y\,d(xy)$

$\int \frac{dy}{y} = \int d(xy)$

$$\boxed{\log y = xy + c}$$

③ $\frac{dx}{dy} = e^{x-2y} + e^{x}.y$

$\frac{dx}{dy} = e^{x}(e^{-2y} + y)$

$\int \frac{dx}{e^{x}} = \int (e^{-2y} + y)\,dy$

$-e^{-x} = \frac{e^{-2y}}{-2} + \frac{y^2}{2} + c$

④ $\frac{dy}{dx} = 1+y^2$

$\int \frac{dy}{1+y^2} = \int dx$

$\tan^{-1} y = x + c$

$$\boxed{y = \tan(x+c)}$$

⑤ $\frac{dy}{dx} = -\frac{x}{y}$ at $x=1, y=\sqrt{3}$

$\int y\,dy = \int -x\,dx$

$y^2 = -x^2 + 2c$

$3 - -1 = 2c$

$2c = 4$

$\boxed{c = 2}$

$\boxed{x^2 + y^2 = 4}$

(6) Find the curve passing through the point (0,1) and satisfing $\sin\left(\frac{dy}{dx}\right) = b$

→

$$\sin\left(\frac{dy}{dx}\right) = b$$

$$\frac{dy}{dx} = \sin^{-1} b$$

$$\int dy = \int \sin^{-1} b \, dx$$

$$y = x \sin^{-1} b + c$$

$$\boxed{c = 1}$$

$$\boxed{y = x \sin^{-1} b + 1}$$

(7) $\frac{dx}{dt} + 3x = 0$

(a) $x = 3e^{3t}$ (b) $x = 3t^3$

(c) $x = 2e^{-2t}$ ✓(d) $x = 3e^{-3t}$

→

$$\frac{dx}{dt} = -3x$$

$$\frac{dx}{3x} = -3dt$$

$$\log x = -3t + \log c$$

$$x = ce^{-3t}$$

$$\therefore \boxed{x = 3e^{-3t}}$$

(8) $\frac{dy}{dx} = e^{x+y}$ with $y(1) = 1$

then find y when $x = -1$

→

$$\int e^{-y} dy = \int e^{x} dx$$

$$-e^{-y} = e^{x} + c$$

$$-e^{-1} = e^{1} + c$$

$$-\frac{1}{e} - e = c$$

$$\therefore c = \frac{-1 - e^2}{e}$$

$$-e^{-y} = e^{x} - e^{-1} - e$$

$x = -1$

$$-e^{-y} = \cancel{e^{-1}} - \cancel{e^{-1}} - e$$

$$e^{-y} = e$$

$$\therefore -y = 1$$

$$\therefore \boxed{y = -1}$$

⑨ $\frac{dy}{dx} = 3x^2 - 2x$ at $x=1, y=1$
then find $y(3) = ?$

→ $\int dy = \int (3x^2 - 2x)\, dx$

$y = x^3 - x^2 + c$

$\boxed{c=1}$

$y = x^3 - x^2 + 1$

$y(3) = 27 - 9 + 1$

$\boxed{y(3) = 19}$

⑩ $\frac{dy}{dx} = y^2 \sin x$ with $y(\pi/2) = 1$

ⓐ $y \sin x = 1$ ✓ⓑ $y \cos x = 1$

ⓒ $y = \sin x$ ⓓ $x = \sin y$

⑫ If the equation contains the terms like $\cos(xy)$, $\sin(x+y)$, $(ax+by+c)^2$, etc. can be reduced to variable separable form with a substitution $xy = v$, $x+y = v$, $ax+by+c = v$ resp.

① $\frac{dy}{dx} = (4x + y + 1)^2$

$4x + y + 1 = v$

$4 + \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{dv}{dx} - 4$

$\therefore \frac{dv}{dx} - 4 = v^2$

$\int \frac{dv}{(4+v^2)} = \int dx$

$\frac{1}{2} \tan^{-1} \frac{v}{2} = x + c$

$\tan^{-1} \frac{v}{2} = 2x + 2c$ $\therefore \tan(2x+2c) = \frac{v}{2}$

$\boxed{= \frac{4x+y+1}{2}}$

Homogeneous diff method -

$$\frac{dy}{dx} = F(x,y)$$

Homo. f^n of degree n

num. degree = den. degree

$$F(kx, ky) = k^n F(x,y)$$

$\frac{dy}{dx} = F(x,y)$ is said to be homogeneous diff^{a} eqn if $f(x,y)$ should be a homogeneous function of degree zero.

Note - Eq^n $Mdx + Ndy = 0$ is said to homogeneous diff. eqn if all the terms of M & N should be of same degree.

$$x^2+y^2 \Rightarrow (kx)^2 + (ky)^2 \quad \therefore \text{ degree} = 2.$$

Substitution $\boxed{y = vx}$ or $\boxed{x = vy}$ reduces homogeneous eqn to variable separation form.

$$\frac{dy}{dx} = \frac{x^2y - y^3}{xy^2} = \frac{\cancel{x^3}\left(y/x - (y/x)^3\right)}{\cancel{x^3}(y/x)^2}$$

Every homogeneous f^n of degree zero can be written as f^n of y/x or x/y & substitution reduces it to variable separation form.

$$x\frac{dy}{dx} = y\{\log y - \log x + 1\}$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\log\left(\frac{y}{x}\right) + 1\right]$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v\left[\log v + 1\right]$$

$$x\frac{dv}{dx} = v\log v$$

$$\int \frac{1}{v\log v}\,dv = \int \frac{dx}{x}$$

$$\log v = t$$

$$\frac{1}{v}\,dv = dt$$

$$\int \frac{1}{t}\,dt = \int \frac{dx}{x}$$

$$\log t = \log x + \log c$$

$$t = cx$$

$$\log v = cx$$

$$\therefore v = e^{cx}$$

$$\therefore \frac{y}{x} = e^{cx}$$

$$\therefore \boxed{y = xe^{cx}}$$

Non homogeneous diff. eqn method -

$$\boxed{\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}}$$

case 1 :- $\left[\frac{a_1}{a_2} = \frac{b_1}{b_2}\right]$

There exists a subst. which reduces given eq^n to variable separable form.

case II :- $\left[\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right]$

$x = X + h \quad , \quad y = Y + K$

$dx = dX \quad , \quad dy = dY$

$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+K) + c_1}{a_2(X+h) + b_2(Y+K) + c_2}$$

$$= \frac{a_1X + b_1Y + (a_1h + b_1K + c_1)}{a_2X + b_2Y + (a_2h + b_2K + c_2)}$$

choose h, K so that

$$\left.\begin{aligned} a_1h + b_1K + c_1 &= 0 \\ a_2h + b_2K + c_2 &= 0 \end{aligned}\right\}$$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y} \quad \text{then } Y = VX$$

(i) $(2x+2y-1)dx = (x+y+1)\,dy$

$$\frac{dy}{dx} = \frac{2x+2y-1}{x+y+1} \qquad \left(\frac{a_1}{a_2} = \frac{b_1}{b_2}\right)$$

$$= \frac{2(x+y)-1}{(x+y)+1}$$

$x+y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = \frac{2v-1}{v+1}$$

$$\frac{dv}{dx} = \frac{2v-1+v+1}{v+1} = \frac{3v}{v+1}.$$

$$\int \frac{(v+1)}{v}\,dv = \int 3\,dx$$

$$v + \log v = 3x + c$$

$$x+y + \log(x+y) = 3x + c$$

$$y - 2x + \log(x+y) = c$$

⑦ Which of following subst. reduces diff. eqn $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ to homogeneous form

$\longrightarrow$

$x = X+h$, $y = Y+k$

$k + \cancel{h} - 2 = 0$

$k - \cancel{h} - 4 = 0$

$2k - 6 = 0$

$k = 3$, $h = -1$

$\boxed{x = X-1}$, $\boxed{y = Y+3}$

$\frac{dY}{dX} = \frac{Y+X}{Y-X}$ Homo. form

Linked que.

⑧ (a) $x = X+h$, $y = Y+k$ reduces $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ to homo. form then find h, k.

Ans. $h = -1$, $k = 3$

(b) $\frac{d^2y}{dx^2} + (h+3)\frac{dy}{dx} + (k-1)y = 0$

* Exact differential eqn

$\boxed{Mdx + Ndy = 0}$

A differential eqn $Mdx + Ndy = 0$ is said to be an exact differential eq [illegible]

$d[f(x,y)] = Mdx + Ndy$

e.g. $y^2 dx + 2xy\, dy = 0$

$d[xy^2] = y^2 dx + 2xy\, dy$

The D.E $Mdx + Ndy = 0$ is exact $\Longleftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$Mdx + Ndy = 0$

$\Big\downarrow$ exact i.e. $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$

$\int M dx + \int$ (terms of N without x) dy

take 'y' as constant

Q. $py\,dx$ [illegible] $+\cos^2 x)$ [illegible] eq^n [illegible]

$\frac{\partial M}{\partial y} = p$ $\quad \frac{\partial N}{\partial x} = 2\cos x \sin x$

$p = $ [illegible]

$P\,dx - (1+\sin^2 y + \cos^2 x)\,dy = 0$ is exact then find P.

→ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \frac{\partial N}{\partial x} = \sin 2x$

$\therefore \int \frac{\partial M}{\partial y} = \int \sin 2x$

$\therefore \boxed{P = y \sin 2x}$ ✓

$P = y\sin 2x + \cos x$ ✓ ans

3) $(3a^2x^2 + by\cos x)\,dx + (2\sin x - 4ay^3)\,dy = 0$ is exact then

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(a) Exactness depends on both a and b

(b) Exactness depends only on a

(c) Exactness depends only on b

(d) Exactness not depends on both a & b.

④ $(ax+hy+g)dx+(hx+by+f)dy=0$ — non homogeneous

$$\frac{\partial M}{\partial y}=h \quad , \quad \frac{\partial N}{\partial x}=h$$

$$\int (ax+hy+g)\,dx+\int (by+f)\,dy=c$$

$$\frac{ax^2}{2}+hxy+gx+\frac{by^2}{2}+fy=c$$

⑤ $\left[y(1+1/x)+\cos y\right]dx+\left[x+\log x-x\sin y\right]dy=0$

→ $$\frac{\partial M}{\partial y}=-\sin y+\left(1+\frac{1}{x}\right) \qquad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$$

$$\int\left(y+\frac{y}{x}+\cos y\right)dx+\int 0\,dx=c$$

$$\therefore\ yx+y\log x+x\cos y=c$$

⑥ $y\,dx-x\,dy=0$ — ①

$$\frac{\partial M}{\partial y}=1 \quad , \quad \frac{\partial N}{\partial x}=-1$$

multiply ① with $1/y^2$

$$\frac{y}{y^2}dx-\frac{x}{y^2}dy=0$$

$$\frac{1}{y}dx-\frac{x}{y^2}dy=0$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

Integrating factor -

A non exact eqⁿ is converted to the exact by multiplying it with a function $f(x,y)$. Then $f(x,y)$ is called integrating factor.

$$y\,dx - x\,dy = 0$$

$$\Big\downarrow \text{I.F.}$$

$$\frac{1}{y^2}, \frac{1}{x^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$$

A constant multiple of an integrating factor is also an integrating factor.

$$\boxed{M\,dx + N\,dy = 0}$$

$$\Big\downarrow \text{Non exact eqn (i.e. } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x})$$

i) All terms of M and N should be of same degree.

$$\boxed{IF = \frac{1}{Mx+Ny}} \quad (Mx+Ny \neq 0)$$

ii) The eqn is of the form

$$y\,f(x,y)\,dx + x\,g(x,y)\,dy = 0$$

$$\boxed{IF = \frac{1}{Mx-Ny}} \quad (Mx-Ny \neq 0)$$

(ii) $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ only fn. of x

$$\boxed{IF = e^{\int f(x)dx}}$$

(iii) $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ only or constant

$$\boxed{IF = e^{\int g(y)dy}}$$

1) Inspection method

Note:

A non-exact homogeneous differential eqn $Mdx + Ndy$ is converted to exact by multiplying with $\left(\frac{1}{Mx+Ny}\right)$

Q. Find integrated factor of

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

→

$\dfrac{\partial M}{\partial y} = 4y^3 + 2 \qquad \dfrac{\partial N}{\partial x} = y^3 - 4$

$\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \dfrac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \dfrac{-3(y^3+2)}{y(y^3+2)} = \dfrac{-3}{y}$

$IF = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = y^{-3}$

$f(x)$	$e^{\int f(x)dx}$
$\frac{1}{x}$	x
$\frac{2}{x}$	x^2
$\frac{3}{x}$	x^3
$\frac{-1}{x}$	$x^{-1} = 1/x$
$-2/x$	$x^{-2} = 1/x^2$

① $y(1-xy)dx - x(1+xy)dy = 0$ — ①

→ $Mx - Ny = xy - x^2y^2 + xy + x^2y^2 = 2xy$

$IF = \frac{1}{2xy}$ or $IF = \frac{1}{xy}$

Multiply ① with $1/xy$

$$\frac{y(1-xy)}{xy}dx - \frac{x(1+xy)}{xy}dy = 0$$

$$\int\left(\frac{1}{x} - y\right)dx - \int\frac{1}{y}dy = c$$

$$\log x - xy - \log y = c$$

$$\boxed{\log x/y - xy = c}$$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = [illegible]$

[illegible]

$I.F = 1/x^2$

$x^2 y\,dx - (x^3+y^3)\,dy = 0$

$\rightarrow \quad Mx + Ny = x^3 y - (x^3+y^3)y = -y^4$

$I.F = 1/y^4$

$$\frac{x^2 y}{-y^4}\,dx + \frac{x^3+y^3}{y^4}\,dy = 0$$

$$-\int \frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

$$-\frac{x^3}{3y^3} + \log y = c$$

* $y\,dx - x\,dy + (1+x)dx + x^2\cos y\,dy = 0$

multiply with $1/x^2$

$$\frac{y\,dx - x\,dy}{x^2} + \frac{(1+x)}{x^2}dx + \frac{x^2\cos y}{x^2}\,dy = 0$$

$$\int d\left[\frac{-y}{x}\right] + \int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx + \int \cos y\,dy = 0$$

$$-\frac{y}{x} - \frac{1}{x} + \log x + \sin y = c$$

* $y\,dx - x\,dy + (y^2+y)x\,dx + xye^x\,dx = 0$

multiply with $1/xy$

$$\frac{y\,dx - x\,dy}{xy} + \frac{(y^2+y)\,x\,dx}{xy} + \frac{xy\,e^x\,dx}{xy} = 0$$

$$\int d\left[\log(x/y)\right] + \int (y+1)dy + \int e^x dx = 0$$

$$\log(x/y) + \frac{y^2}{2} + y + e^x = c$$

* $y\,dx + x\,dy + xe^{-x}dx = 0$

$$\int d(xy) + \int xe^{-x}dx = 0$$

$$xy - e^{-x}(x+1) = c$$

* $y\,dx + x\,dy + x^2 y e^x dx = 0$

multiply with $1/xy$.

$$\frac{y\,dx + x\,dy}{xy} + \frac{x^2 y e^x dx}{xy} = 0$$

$$\int d(\log(xy)) + \int x e^x dx = 0$$

$$\log(xy) + e^x(x-1) = c$$

* Linear differential eqns -

The differential eqn is said to be linear if it satisfies the following two conditions:

i) The dependent variable and all it's derivatives should be of 1st degree only i.e.

$(y')^1, \left(\frac{dy}{dx}\right)^1, \left(\frac{d^2y}{dx^2}\right)^1,$

ii) There is no product of the dependent variable and any of it's higher derivative.

$\left(\frac{y\,dy}{dx}\right), \frac{dy}{dx}\frac{d^2y}{dx^2},$

* $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + xy = x^3$

$\Longrightarrow$ Linear

The eqn containing fn of dependent variable is not linear in that variable.

e.g.

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + xe^y = x^3$$

$\Longrightarrow$ Non-linear

* Linear in y –

$$\boxed{\frac{dy}{dx} + Py = Q}$$

soln ↓ P and Q are fn of 'x' or constant.

$$\boxed{y \cdot e^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx + c}$$

$$\boxed{IF = e^{\int P\,dx}}$$

* Linear in x –

$$\boxed{\frac{dx}{dy} + Px = Q}$$

soln ↓ P and Q are fn of 'y' or constant

$$x \cdot e^{\int P\,dy} = \int Q \cdot e^{\int P\,dy}\,dy + c$$

$x \frac{dy}{dx} + y = x^4$ with $y(1) = 6/5$

$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

$P = 1/x$, $Q = x^3$

$$e^{\int \frac{1}{x} dx} = x$$

$$\therefore y \cdot x = \int x^3 \cdot x \, dx + c = \int x^4 dx + c$$

$$yx = \frac{x^5}{5} + c$$

at $y(1) = 6/5$

$$\frac{6}{5} = \frac{1}{5} + c$$

$$\boxed{c = 1}$$

$$\boxed{y = \frac{x^4}{5} + \frac{1}{x}}$$

* $x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1$

$\longrightarrow$

$$\frac{dy}{dx} + \left(\frac{x\sin x + \cos x}{x\cos x}\right)y = 1$$

$$\frac{dy}{dx} + (\tan x + 1/x)y = \frac{1}{x\cos x}$$

$$e^{\int p\,dx} = e^{\int(\tan x + 1/x)dx} = e^{\log\sec x + \log x}$$

$$= e^{\log(\sec x \cdot x)}$$

$$\therefore e^{\int p\,dx} = x\sec x.$$

$$y \cdot x\sec x = \int \frac{1}{x\cos x} \times x\sec x\,dx + c$$

$$\therefore xy\sec x = \tan x + c$$

* $x^2 \frac{dy}{dx} + 2xy = \frac{2\log x}{x}$ with $y(1)=0$ then find y when $x=e$

$\longrightarrow$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{2\log x}{x^3}$$

$$e^{\int p\,dx} = e^{\int 2/x\,dx} = x^2$$

$$y \cdot x^2 = \int \frac{2\log x}{x^{\cancel{3}}} \times \cancel{x^2}\,dx + c$$

$\cancel{x}$ = $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$x^2y = \int 2t\,dt + c$

$x^2y = t^2 + c$

$x^2y = (\log x)^2 + c$

$\Rightarrow \quad c = 0$

$x^2y = (\log x)^2$

when $x = e$ then

$e^2y = 1$

$\therefore \boxed{y = e^{-2}}$

* $(x + 2y^3)\frac{dy}{dx} = y$ with $x(1) = 0$

→

$\cancel{\frac{x\,dy}{dx} + \frac{2y^3\,dy}{dx} = y}$ $\qquad \frac{x+2y^3}{y} = \frac{dx}{dx}$

$\frac{dx}{dy} - \frac{x}{y} = 2y^2$

$e^{\int P\,dy} = e^{\int -1/y\,dy} = 1/y$

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y}\, dy + c$$

$$\frac{x}{y} = y^2 + c$$

$x(1) = 0$

$\Longrightarrow$

$c = -1$

$$\frac{x}{y} = y^2 - 1$$

$$\boxed{x = y^3 - y}$$

* $\frac{dy}{dx} + y = e^x$ with $y(0) = 1$ find $y(1) = ?$

$\longrightarrow$

$$e^{\int p\,dx} = e^{\int 1\,dx} = e^x$$

$$y \cdot e^x = \int e^x \cdot e^x\, dx + c$$

$$y e^x = \frac{e^{2x}}{2} + c$$

$y(0) = 1$

$\Rightarrow c = \frac{1}{2}$

$$\therefore y e^x = \frac{e^{2x}}{2} + \frac{1}{2}$$

Ans $y(1) =$

$$\therefore y = \underline{e^x + e^{-x}} \quad \boxed{y = \underline{e^1 + e^{-1}}}$$

* Bernoullis differential eqn :-

$$\boxed{\frac{dy}{dx} + Py = Q\, y^{n}} \rightarrow \text{Linear if } n=0$$

P and Q are f^n of x (or) constants

sol^n

$$\boxed{y^{1-n}\, e^{\int(1-n)P\,dx} = \int (1-n)\, Q\, e^{\int(1-n)P\,dx}\, dx + c}$$

$$\boxed{y^{1-n} = v} \quad \text{substitution}$$

The substitution $y^{1-n} = v$ reduces Bernoullis eqn to linear eqn

GATE

Q. ... n $y^{1-n} = v$ reduces the non-linear eqn to which of the following ...near form.

$$\frac{dv}{dx} \; - \; \frac{1}{x} v$$

$$e^{\int \frac{1}{x} dx} = e^{\log x}$$

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

$$\boxed{\frac{dv}{dx} + (1-n)Pv = (1-n)Q}$$

Q. Which of the following substitution reduces the non-linear eqn $x\frac{dx}{dy} + x^2y^2 = x^3y^3$ to linear form?

(a) $y^{-1} = v$ (b) $y^{-2} = v$ ✓ (c) $x^{-1} = v$ (d) $x^{-2} = v$

→

$$\frac{dx}{dy} + xy^2 = x^2y^3$$

$$\therefore x^{1-2} = v$$

$$\therefore \boxed{x^{-1} = v}$$

Q. $\frac{dy}{dx} - y\tan x = -y^2\sec x$ with $y(0) = 1$

→

$$e^{\int(1-2)(-\tan x)dx} = e^{\log\sec x} = \sec x$$

$$y^{1-2}\sec x = \int(1-2)(-\sec x)(\sec x)dx + c$$

$$y^{-1}\sec x = \tan x + c$$

$$y(0) = 1 \Rightarrow c = 1$$

$$\boxed{y = \frac{\sec x}{1 + \tan x}}$$

* Equations of the form :-

$$\boxed{f'(y)\frac{dy}{dx} + P f(y) = Q} \longrightarrow \text{Non-linear}$$

$$f(y) = v \implies f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

$$\boxed{\frac{dv}{dx} + Pv = Q} \longrightarrow \text{Linear}$$

* What substitution reduces the non-linear eq^{ns} $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$. Also find I.F. of linear form.

$\longrightarrow$

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{x}(\log z)^{-1} = \frac{1}{x^2}$$

$$\boxed{(\log z)^{-1} = v}$$

$$\frac{-1}{z}(\log z)^{-2}\frac{dz}{dx} = \frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2}$$

$$\frac{dv}{dx} - \frac{1}{x}v = \frac{-1}{x^2}$$

$$e^{\int -\frac{1}{x}dx} = e^{\log x^{-1}} = 1/x$$

* $x\frac{dy}{dx} + y\log y = xye^{x}$

$\longrightarrow$

$$\frac{x}{y}\frac{dy}{dx} + \log y = xe^{x}$$

$$\log y = v$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$$

$$x\frac{dv}{dx} + v = xe^{x}$$

$$\frac{dv}{dx} + \frac{1}{x}v = e^{x}$$

$$e^{\int \frac{1}{x}dx} = x$$

$$v\cdot x = \int e^{x}\cdot x\,dx + c$$

$$vx = e^{x}\cdot(x-1) + c$$

$$x\log y = e^{x}(x-1) + c$$

* Clairaut's eqⁿ

$$\boxed{y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)}$$

or

$$\boxed{y = px + f(p)} \quad \text{where } p = \frac{dy}{dx}$$

soln

Directly replace $\frac{dy}{dx}$ by 'c'

$$\boxed{y = cx + f(c)}$$

* $\left(y - x\frac{dy}{dx}\right)\left(\frac{dy}{dx} - 1\right) = \frac{dy}{dx}$

→

$$y - xy' = \frac{y'}{y'-1}$$

$$y = \frac{y'}{y'-1} + xy'$$

$$\boxed{y = cx + \frac{c}{c-1}}$$

* $p = \sin(y - xp)$ where $p = \frac{dy}{dx}$

$\longrightarrow$

$$y - xp = \sin^{-1} p$$

$$y = xp + \sin^{-1} p$$

$$\boxed{y = cx + \sin^{-1} c}$$

* Higher order linear differential eqns with constant coefficients :-

$$\boxed{\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} \frac{dy}{dx} + k_n y = X}$$

$k_1, k_2, \cdots, k_n$ are constants

X - f^n of 'x' or constant.

$$\boxed{D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^3 = \frac{d^3}{dx^3}} \Longrightarrow \text{e.g. } D e^{2x} = 2e^{2x}$$

$$\boxed{\frac{1}{D} = \int, \quad \frac{1}{D^2} = \iint \Longrightarrow \text{e.g. } \frac{1}{D} \cos 2x = \frac{\sin 2x}{2}}$$

$$(D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n) y = X$$

$$\boxed{F(D)\, y = X}$$

$F(D) = D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n$

f^n of differential operator

① The complete soln of F(D)y = x is

~~eqn~~ y = Complementary fn + Particular integral

$= CF + PI$

② If $X = 0$ then $F(D)y = 0$ is called homogeneous linear differential eqn

③ If $X \neq 0$ the $F(D)y = x$ is called non homogeneous linear differential eqn

④ The solution of homogeneous linear differential eqn $F(D)y = 0$ is called complementary fn.

The no of arbitrary constants in the complementary fn should be equal to the order of given differential eqn.

⑤ The particular integral of $F(D)y = X$ is

$$PI = \left[\frac{1}{F(D)}\right] X$$

. PI will not contain any arbitrary constants.

⑥ If $X = 0$ then complete soln of the given eqn is only complementary function.

⑦ By assuming 'D' as an algebric ~~eqn~~ quantity $F(D) = 0$ becomes an algebric eqn

And is called Auxillary eqn

* Procedure to find CF –

By solving an A.E. we get the roots based on nature of this roots we write the CF as follows.

Nature of roots	CF
Real and distinct $D = m_1, m_2, m_3$	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2 Real & repeated $D = m_1, m_1$	$CF = (c_1 + c_2 x) e^{m_1 x}$
③ Complex & distinct $D = a \pm ib$	$CF = e^{ax}[c_1 \cos bx + c_2 \sin bx]$
④ Complex & repeated $D = a \pm ib, a \pm ib$	$CF = e^{ax}[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx]$
⑤ Surds (Real no.) $a \pm \sqrt{b}$	$CF = c_1 e^{(a+\sqrt{b})x} + c_2 e^{(a-\sqrt{b})x}$ OR $CF = e^{ax}[c_1 \cosh\sqrt{b}\,x + c_2 \sinh\sqrt{b}\,x]$

Roots $\qquad$ CF

$D = 1, -\frac{1}{2}, 2$ $\qquad$ $CF = C_1e^{x} + C_2e^{-x/2} + C_3e^{2x}$

$D = 2, -2, -2$ $\qquad$ $CF = C_1e^{2x} + (C_2 + C_3x)e^{-2x}$

$D = 2 \pm 3i, \frac{1}{3}$ $\qquad$ $CF = e^{2x}[C_1\cos 3x + C_2\sin 3x] + C_3e^{x/3}$

$D = \pm 2i, \pm 4$ $\qquad$ $CF = C_1e^{4x} + C_2e^{-4x} + [C_3\cos 2x + C_4\sin 2x]$

* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$\longrightarrow$

$D^2 - 5D + 6 = 0$

$\therefore D = +3, D = +2$

$CF = C_1e^{+3x} + C_2e^{+2x}$

* $\frac{d^4y}{dx^4} - 16y = 0$

$\longrightarrow$

$(D^4 - 16)y = 0$

$D^4 - 16 = 0$

$(D^2-4)(D^2+4) = 0$

$D = 2, -2, \pm 2i$

$CF = C_1e^{2x} + C_2e^{-2x} + [C_3\cos 2x + C_4\sin$

* $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + (p^2+q^2)y = 0$

$\longrightarrow$

$$D^2 + 2pD + (p^2+q^2) = 0$$

$$D_1, D_2 = \frac{-2p \pm \sqrt{4p^2 - 4p^2 - 4q^2}}{2} = \frac{-2p \pm \sqrt{-4q^2}}{2}$$

$$= \frac{-2p \pm 2qi}{2}$$

$$D_1, D_2 = -p \pm qi$$

$$CF = e^{-px}\left[c_1 \cos qx + c_2 \sin qx\right]$$

Note!

If $y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots$ is the complete solution of homogeneous differential eqn $F(D)y = 0$ then each one of $y_1, y_2, y_3, \cdots$ is linearly independent solution of the homogeneous of linear differential eqn $F(D)y = 0$

Que: y_1, y_2 are linearly independent solns of the corresponding ~~linearly~~ ~~indepe~~ homogeneous eqn of $F(D)y = \cancel{X}$ ~~then~~ $y_1 - y_2$ is a soln of which of the following eqns

(a) $F(D) = 0$ (b) $X = 0$

(c) $F(D)y = X$ ✓ (d) $F(D)y = 0$

are linearly independent

$CF = c_1 \cos 3x + c_2 \sin 3x$

$D = \pm 3i$

$D^2 + 9 = 0$

$(D^2+9)y = 0$

* e^{3x}, e^{-3x} are linearly independent sol^n of –

(a) $\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 9y = 0$ (b) $\frac{d^3y}{dx^3} + \frac{2d^2y}{dx^2} + \frac{3dy}{dx} + y = 0$

✓ (c) $\frac{d^4y}{dx^4} - 81y = 0$ (d) None

$$\boxed{\text{Order} \geq \text{No. of indep. sol}^{ns}}$$

* $y''' + y'' + y' + y = 0$

⟶

$D^3 + D^2 + D + 1 = 0$

$D^2(D+1) + (D+1) = 0 \quad \therefore (D^2+1)(D+1) = 0$

$\cancel{D^2 = 0}, \; D^2 = -1, \; D = -1$

$\therefore D = \pm i$

$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

* $\frac{d^4y}{dx^4} + a^4 y = 0$

$D^4 + a^4 = 0$

$(D^2 + a^2)^2 - 2D^2a^2 = 0$

$(D^2 + a^2)^2 - (\sqrt{2}Da)^2 = 0$

$(D^2+a^2+\sqrt{2}Da)(D^2+a^2-\sqrt{2}Da)=0$

$D^2+\sqrt{2}aD+a^2=0$ & $D^2-\sqrt{2}aD+a^2=0$

$$\frac{-\sqrt{2}a \pm \sqrt{2a^2-4a^2}}{2}$$

$$\frac{-\sqrt{2}a \pm \sqrt{-2a^2}}{2}$$

$$\frac{-\sqrt{2}a \pm \sqrt{2}a\,i}{2}$$

$$-\frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i \qquad \frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i$$

$$y = e^{-\frac{ax}{\sqrt{2}}}\left[c_1\cos\frac{a}{\sqrt{2}}x + c_2\sin\frac{a}{\sqrt{2}}x\right] + e^{\frac{ax}{\sqrt{2}}}\left[c_3\cos\frac{a}{\sqrt{2}}x + c_4\sin\frac{a}{\sqrt{2}}x\right]$$

* $\frac{d^2f}{dn^2} + 4\frac{df}{dn} + 4f = 0$

(a) $f_1 = e^{-2n}$, $f_2 = e^{2n}$ ✓(c) $f_1 = ne^{-2n}$, $f_2 = e^{-2n}$

(b) $f_1 = e^{-2n}$, $f_2 = ne^{2n}$ (d) $f_1 = ne^{-2n}$, $f_2 = e^{2n}$

* The D.E. $\frac{d^2y}{dx^2} + 2\cos x\frac{dy}{dx} - 3y = x^2$ is

(a) Homogeneous linear DE ✗(b) Non homogeneous linear eq with constant coeff

✗(c) Non linear D.E. order 2 ✓(d) Linear D.E.

The sol^n of

* $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ where $R^2C = 4L$

(a) $e^{Rt/2R}$ (b) $te^{Rt/L}$

(c) $e^{-Rt/L}$ ✓(d) $te^{-Rt/2L}$

→ $D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$

$$D_1, D_2 = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2C - 4L}{L^2C}}}{2}$$

$$= \frac{-\frac{R}{L} \pm 0}{2}$$

$$D_1, D_2 = \frac{-R}{2L}, \frac{-R}{2L}$$

$$\therefore y = (C_1 + C_2t)e^{-\frac{R}{2L}t}$$

$$\therefore y = C_1e^{-Rt/2L} + C_2te^{-\frac{Rt}{2L}}$$

* $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 13y = 0$

with $y(0) = 0$ & $y'(0) = 1$

→

$D^2 + 4D + 13 = 0$

$D_1, D_2 = \frac{-4 \pm 6i}{2}$

$= -2 \pm 3i$

$y = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$

at $x = 0$, $y = 0 \Rightarrow c_1 = 0$

$y = c_2 e^{-2x} \sin 3x$

$y' = c_2 [-2e^{-2x} \sin 3x + \cos 3x \, e^{-2x}]$

$y'(0) = 1$

$\therefore c_2 = 1/3$

$$\boxed{y = \frac{e^{-2x}}{3} \sin 3x}$$

* $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0$ with $y(0)=3$ & $y'(0)=1$

→

$9D^2 - 6D + 1 = 0$

$\therefore D = 1/3, 1/3$

$y = (c_1 + c_2 x)e^{x/3}$

$y(0) = 3 \Rightarrow \boxed{3 = c_1}$

$\therefore y = 3 + c_2 x e^{x/3}$

$y' = c_2\left[\frac{x}{3}e^{x/3} + e^{x/3}\right]$

$y'(0) = 1$

$\boxed{0 = c_2}$

$\therefore \boxed{y = (3 + 0x)e^{x/3}}$

$\therefore \boxed{y = 3e^{x/3}}$

* $\frac{d^2y}{dx^2}$ ...

(a) ... (b) ...

(c) $y = \sum_n c_n \sin\frac{n\pi x}{l}$ (d) $y = \sum_n ...$

* Direct put cond^{ns} in options i.e. $y(0) = 0$ *

* $\frac{d^2n}{dx^2} - \frac{n}{L^2} = 0$ where L is a constant & $n(0) = k$ & $n(\infty) = 0$

(a) $n = ke^{x/L}$ (b) $n = k^2 e^{-x/L}$

(c) $n = ke^{-x/\sqrt{L}}$ (d) $n = ke^{-x/L}$

Here check both cond^n then c & d satisfies so from roots of A.E (d) is ans.

$D = \pm 1/L$

$y = c_1 e^{x/L} + c_2 e^{-x/L}$

* For what value of λ the D.E $\frac{d^2y}{dx^2} + \lambda y = 0$ with $y(0) = 0$ and $y(\pi) = 0$ will have non-trivial sol^n → Non-zero

(a) ... (b) ...

(c) $\lambda = 1, 3, 5$ (d) ...

$y(0) = 0 \Rightarrow y = c_2 \sin\sqrt{\lambda}x$

$y(\pi) = 0$ $0 = c_2 \sin\sqrt{\lambda}\pi$

$\sin\sqrt{\lambda}\pi = 0$ $= \sin n\pi$

$D = \pm\sqrt{\lambda}\, i$ $y = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\sqrt{\lambda} = n$

$\lambda = n^2 = 1, 4, 9, \ldots$

Linked

1) * The complete sol^n of the differential eqn $\frac{d^2y}{dx^2} + \frac{p\,dy}{dx} + qy = 0$ is $y = c_1 e^{-x} + c_2 e^{-3x}$ then values of p & q are –

(a) $p = 3$, $q = 3$ ✓(b) $p = 4$, $q = 3$

(c) $p = 4$, $q = 4$ (d) $p = 3$, $q = 4$

→ $(D+1)(D+3)y = 0$

$(D^2 + 4D + 3)y = 0$

$D^2y + 4Dy + 3y = 0$

$p = 4$, $q = 3$

(b) * The sol^n of $\frac{d^2y}{dx^2} + \frac{p\,dy}{dx} + (q+1)y = 0$ is

→ (a) xe^{2x} (b) e^{2x} ✓(c) xe^{-2x} (d) x^2e^{-2x}

→

$y'' + 4y' + 4y = 0$

$\therefore D^2 + 4D + 4 = 0$

$(D+2)^2 = 0$

$D = -2, -2$

$y = (c_1 + c_2 x)e^{-2x} = xe^{-2x}$ if $c_1 = 0$, $c_2 = 1$

Particular integrals :-

$$F(D)y = X$$

$$PI = \left[\frac{1}{F(D)}\right] X$$

- I: e^{ax} or constant
- II: $\sin ax$ or $\cos bx$
- III: x^m or poly. in x
- IV: $e^{ax} V$, where V: $\sin ax$, $\cos bx$, x^m, poly. in x

Case I - $X = e^{ax}$ or constant

$$PI = \left[\frac{1}{F(D)}\right] e^{ax} = \frac{1}{F(a)} e^{ax} \quad (\text{where } F(a) \neq 0)$$

Replace 'D' by 'a' in f^n $f(D)$

if $f(a) = 0$ then

$$PI = x\left[\frac{1}{F'(D)}\right] e^{ax}$$

Replace 'D' by 'a' in $F'(D)$

$$PI = x\left[\frac{1}{F'(a)}\right] e^{ax} \quad (F'(a) \neq 0)$$

if $F'(a)=0 \Rightarrow PI = x^2\left(\frac{1}{F''(a)}\right)e^{ax}$ $(F''(a)\neq 0)$

* P.I. of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 7y = e^{-2x}$

⟶

$$PI = \frac{1}{D^2+2D+7}e^{-2x} = \frac{1}{(-2)^2+2(-2)+7}e^{-2x}$$

$$\boxed{PI = \frac{e^{-2x}}{7}}$$

* PI of $\frac{d^2y}{dx^2} - 9y = e^{3x}+3$

⟶

$$PI = \frac{1}{D^2-9}\left(e^{3x}+3e^{0x}\right)$$

$$= x\left(\frac{1}{2D}\right)e^{3x} + 3\frac{1}{0-9}e^{0x}$$

$$= \frac{xe^{3x}}{6} + \frac{3}{-9}$$

$$\therefore PI = \frac{xe^{3x}}{6} - \frac{1}{3}.$$

v PI of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$

$\rightarrow$ $PI = \left[\frac{1}{D^2+4D+4}\right] e^{-2x} = x\left(\frac{1}{2D+4}\right) e^{-2x} = x^2\left(\frac{1}{2}\right)$

$\therefore PI = \frac{x^2}{2} e^{-2x}$

* 'a' is the root of the A.E. of $F(D)y = e^{ax}$ then PI is (k constant)

(a) ke^{-ax} (b) ke^{ax} (c) kxe^{-ax} (d) kxe^{ax}

* Case II –

$X = \sin ax$ (or) $\cos bx$ ($\sin(ax+b)$ or $\cos(cx+d)$)

$PI = \left[\frac{1}{F(D)}\right] \sin ax = \left[\frac{1}{F(-a^2)}\right] \sin ax$ ($F(-a^2) \neq$

Replace 'D^2' by '$-a^2$' in $F(D)$

if $F(-a^2) = 0$ –

$PI = x\left[\frac{1}{F'(D)}\right] \sin ax = \frac{x}{F'(-a^2)} \sin ax$

Replace 'D^2' by '$-a^2$' in $F'(D)$

* PI of $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + 3y = \cos 3x$

→

$$PI = \frac{1}{D^4+D^2+3}\cos 3x = \frac{1}{(-9)^2+(-9)+3}\cos 3x$$

$$PI = \frac{\cos 3x}{75}$$

* PI of $\frac{d^2y}{dx^2} + 4y = \cos(2x+4) + e^{-2x}$

→

$$PI = \frac{1}{D^2+4}\left(\cos(2x+4) + e^{-2x}\right)$$

$$= x\left(\frac{1}{2D}\right)\cos(2x+4) + \frac{1}{(-2)^2+4}e^{-2x}$$

$$= \frac{x}{2}\,\frac{\sin(2x+4)}{2} + \frac{e^{-2x}}{8}$$

$$\therefore PI = \frac{x\sin(2x+4)}{4} + \frac{e^{-2x}}{8}$$

* PI of $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin x$

→

$$PI = \frac{1}{D^3+D^2+2D+2}\sin x$$

$(-1)D + (-1) + 2D + 2$

$\cos x - \sin x$

$PI = \dfrac{\sin x - \cos x}{8}$

* case III $X = x^m$ or poly. in x (m is +ve integer)

$$PI = \left[\frac{1}{F(D)}\right] x^m = \frac{1}{*[1 + \phi(D)]} x^m$$

$$= \frac{1}{*} [1 + \phi(D)]^{-1} x^m$$

$*$ is taking least ~~power~~ common in $F(D)$

$$PI = \left(\frac{1}{D^2 + 2D + 2}\right)(x^2 + \ldots) = \frac{1}{2\left(1 + \frac{D^2 + 2D}{2}\right)}$$

$$\frac{1}{2}\left[1 + \left(\frac{D^2 + 2D}{2}\right)\right]^{-1}$$

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$$= \frac{1}{2}\left[1-\left(\frac{D^2+2D}{2}\right)+\left(\frac{D^2+2D}{2}\right)^2 - \cdots\right](x^2+2)$$

$$= \frac{1}{2}\left[x^2+2-1-2x+2\right]$$

$$\therefore PI = \frac{x^2-2x+3}{2}$$

* PI of $\frac{d^3y}{dx^3}+\frac{3d^2y}{dx^2} = x^3+3x$

$\rightarrow$

$$PI = \frac{1}{D^3+3D^2}(x^3+3x) = \frac{1}{3D^2\left[1+\frac{D^3}{3D^2}\right]}(x^3+3x)$$

$$= \frac{1}{3D^2}\left[1+\frac{D}{3}\right]^{-1}(x^3+3x)$$

$$= \frac{1}{3D^2}\left[1-\frac{D}{3}+\frac{D^2}{9}-D^3/27 \cdots\right](x^3+3x)$$

$$= \frac{1}{3D^2}\left[x^3+3x-\frac{3x^2}{3}-1+\frac{6x}{9}-\frac{6}{27}\right]$$

$$= \frac{1}{3D^2}\left[x^3-x^2+\frac{33x}{9}-\frac{33}{27}\right] = \frac{1}{3D^2}\left[x^3-x^2+\frac{11}{3}x-\frac{11}{9}\right]$$

$$= \frac{1}{3}\left[\frac{x^5}{20}-\frac{x^4}{12}+\frac{11x^3}{18}-\frac{11x^2}{18}\right]$$

* Case IV :- $X = e^{ax} \cdot V$

$$PI = \left[\frac{1}{F(D)}\right]\left(e^{ax} \cdot V\right) = e^{ax}\left[\frac{1}{F(D+a)}\right]V$$

Replace 'D' by 'D+a' in F(D)

$$\boxed{\frac{1}{F(D+a)} V}$$

→ II. if $V = \sin bx$ or $\cos bx$

→ III. if $V = x^m$ or poly. in x

DE

(i) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 4y = 0$ — ordinary DE

since only one independent variabl

(ii) $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = 0$ — partial DE

since 2 independent variables.

i) <u>order</u> :- The highest derivative involved in the DE is called order of DE.

(ii) <u>Degree</u> :- The highest power of highest derivative is called degree of DE (provided the derivatives of the dependent variable should free from radicals & fractions).

⊛ $\left(\frac{d^2y}{dx^2}\right)^2 = \left[x + 4\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

$\left(\frac{d^2y}{dx^2}\right)^4 = \left[x + 4\left(\frac{dy}{dx}\right)^2\right]^3$

order — 2 degree — 4.

⊛ $\frac{dy}{dx} = x + \frac{2}{\frac{dy}{dx}}$

$\left(\frac{dy}{dx}\right)^2 = x\left(\frac{dy}{dx}\right) + 2$ order = 1
degree = 2.

⊛ $dr = (\theta + \cos\theta)\, d\theta$

$\frac{dr}{d\theta} = \theta + \cos\theta$ order = 1
degree = 1.

⊛ $\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 + 3y = 0$ order = 3
degree = 1.

⊛ $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial y^2}\right)^4$

order = 2

$\Rightarrow$ Formation of ordinary DE :-

$$F(x, y, a, b) = 0 \quad \text{—①}$$

diff ① w.r.t x.

$$F_1\left(x, y, \frac{dy}{dx}, a, b\right) = 0 \quad \text{—②}$$

diff ② w.r.t x

$$F_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b\right) = 0 \quad \text{—③}$$

eliminating a & b from above 3 eq

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

The no. of arbitrary constants eliminated should be equal to the order of resulting ordinary DE.

* The DE of family of straight line passing through the origin is

(a) $y\,dy + x\,dx = 0$ b) none

c) $y\,dx - x\,dy = 0$ d) $y\,dy - x\,dx = 0$

$$y = mx$$

$$\frac{dy}{dx} = m \Rightarrow y = \frac{dy}{dx} \cdot x$$

$$y\,dx - x\,dy = 0$$

* $y = c(x-c)^2$ c is arbitrary const

$$y' = \frac{dy}{dx} = 2c(x-c)$$

$$\frac{y}{y'} = \frac{x-c}{2} \Rightarrow x - c = \frac{2y}{y'}$$

$$c = x - \frac{2y}{y'}$$

✱ DE of family of circles

Note : If the given eq^n of the form $y = Af(x) + Bg(x)$ then the resulting DE is

$$\begin{vmatrix} y & f & g \\ y' & f' & g' \\ y'' & f'' & g'' \end{vmatrix} = 0.$$

✱ $y = Ae^{2x} + Bx$.

$$\begin{vmatrix} y & e^{2x} & x \\ y' & 2e^{2x} & 1 \\ y'' & 4e^{2x} & 0 \end{vmatrix} = 0$$

$$y(-4) - 1(0 - y'') + x(4y' - 2y'') = 0$$

$$y''(1 - 2x) + 4xy' - 4y = 0$$

✱ $y = e^x(A\cos x + B\sin x)$ — ①

$$y' = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

$$y' = e^x(-A\sin x + B\cos x) + y.$$

$$y'' = y' + e^x(-A\cos x - B\sin x) +$$
$$e^x(-A\sin x + B\cos x).$$

$$y'' = y' - y + y' - y$$

$$y'' - 2y' + 2y = 0$$

Note :- If the given eq of the form

$y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + \dots$ then

$$\boxed{\begin{array}{c}(D-a)(D-b)(D-c)\,y = 0 \\ D = \frac{d}{dx} \qquad D^2 = \frac{d^2}{dx^2}\end{array}}$$

$$y = Ae^{2x} + Be^{-3x}$$

$$(D-2)(D+3)y = 0$$

$$(D^2 + D - 6)y = 0$$

$$y'' + y' - 6y = 0$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

$$(D+1)(D+2)(D+3)\,y = 0$$

$$(D^2 + 3D + 2)(D+3)\,y = 0$$

$$(D^3 + \underset{1+2+3}{\underset{\downarrow}{6D^2}} + \underset{1\cdot 2+2\cdot 3+3\cdot 1}{\underset{\downarrow}{11D}} + \underset{1\cdot 2\cdot 3}{\underset{\downarrow}{6}})\,y = 0$$

$$y''' + 6y'' + 11y' + 6y = 0$$

> Solution of DE :-

$$\frac{d^2y}{dt^2} = g \qquad y(0) = 0 \quad y'(0) = 0$$

$$\frac{dy}{dt} = gt + c_1$$

$$y = \frac{1}{2}gt^2 + c_1 t + c_2$$

$$y(0) = 0 \Rightarrow c_2 = 0$$

$$y'(0) = 0 \Rightarrow c_1 = 0$$

$$y = \frac{1}{2}gt^2$$

$\Rightarrow$ 1st order 1st degree DE :-

$$\frac{dy}{dx} = F(x, y)$$

(or)

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

Variable - seperable method :-

(*) $\frac{dy}{dx} = e^{x-y} + x^3 e^{-y}$

$= e^{-y}(e^x + x^3)$

$\int e^y\,dy = \int (e^x + x^3)\,dx$

$e^y = e^x + \frac{x^4}{4} + c$

(*) $\log\left(\frac{dy}{dx}\right) = 2x + 3y$

$\frac{dy}{dx} = e^{2x} \cdot e^{3y}$

$\int e^{-3y}\,dy = \int e^{2x}\,dx$

$\frac{-1}{3} e^{-3y} = \frac{e^{2x}}{2} + c$

(*) $\frac{dy}{dx} = \frac{y^2}{1-xy}$

$(1-xy)\,dy = y^2\,dx$

$dy = y^2\,dx + xy\,dy$

$dy = y(y\,dx + x\,dy)$

$\int \frac{1}{y}\,dy = \int d(xy)$

$\log y = xy + c$

(*) $\frac{dy}{dx} = \frac{-x}{y}$ at $x = 1$, $y = \sqrt{3}$

$\int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = \frac{-x^2}{2} + c \Rightarrow c = \frac{3}{2} + \frac{1}{2} = 2$

$y^2 = -x^2 + 4$

(*) $\frac{dy}{dx} = y^2 \sin x$, $y(2\pi) = 1$

a) $y \sin x = 1$ $\quad$ b) $y \cos x = 1$

$$\int \frac{1}{y^2}\,dy = \int dx \sin x$$

$$\frac{-1}{y} = -\cos x + c \quad \Rightarrow c = 0$$

$$y\cos x = 1.$$

$$\frac{dy}{dx} = 3x^2 - 2x$$ passes through (1,1) then find mag of y when $x = 3$

$$\int dy = \int 3x^2 - 2x\,dx$$

$$y = x^3 - x^2 + c$$

$(1,1) \Rightarrow c = 1.$

$$y = x^3 - x^2 + 1.$$

when $x = 3 \Rightarrow y = 19.$

Bio transformation of an organic compound having conc. 'x' can be modelled using DE $\frac{dx}{dt} + kx^2 = 0$. At $t = 0$ the conc is 'a' then the soln is

$$\frac{dx}{dt} = -kx^2$$

$$\int \frac{1}{x^2}\,dx = -k\int dt$$

$$\frac{-1}{x} = -kt + c \quad \Rightarrow c = -1/a$$

$$\frac{dx(t)}{dt} + 3x(t) = 0$$

$$\int \frac{1}{x}\,dx = \int -3\,dt$$

$$\log x = -3t + \log c$$

$$x = e^{-3t}c$$

$$\frac{dy}{dx} = 1 + y^2$$

$$\int \frac{1}{}\,dy = \int dx$$

$$\tan^{-1} y = x + c$$

$$y = \tan(x+c).$$

⊛ Find the curve passing through the point $(0, 1)$ & satisfying $\sin\left(\frac{dy}{dx}\right) = b$.

$$\frac{dy}{dx} = \sin^{-1} b$$

$$\int dy = \int \sin^{-1} b \, dx$$

$$y = (\sin^{-1} b)x + c$$

$(0, 1)$ $\quad c = 1$

$$y = (\sin^{-1} b)x + 1.$$

⊛ $\frac{dy}{dx} = e^{x+y}$ given that for $x = 1, y = 1$; find y when $x = -1$.

$$\int e^{-y} \, dy = \int e^{x} dx$$

$$-e^{-y} = e^{x} + c \Rightarrow c = -e^{-1} - e.$$

$$\Rightarrow -e^{-y} = e^{x} - (e^{-1} + e)$$

at $x = 1 \Rightarrow -e^{-y} = \cancel{e^{-1}} - \cancel{e^{-1}} - e$

$$y = -1.$$

⊛ $\frac{dy}{dx} = (4x + y + 1)^2$

$$4x + y + 1 = v.$$

$$4 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 4$$

$$\frac{dv}{dx} = v^2 + 4$$

$$\int \frac{1}{v^2 + 2^2} \, dv = \int dx$$

$$\frac{1}{2} \tan^{-1}\left(\frac{v}{2}\right) = x + c.$$

$$\frac{4x + y + 1}{2} = \tan(2x + c).$$

$$4x + y + 1 = 2\tan(2x + c).$$

omogenous differential eq^n :-

$$\frac{dy}{dx} = F(x,y)$$

is said to be homogenous DE if $F(x,y)$
uld be a homogenous fn of degree '0'.

e : $M dx + N dy = 0$ is said to be homogenous if all the terms of M & N should be same degree.

for variable - seperable sub $y = vx$ (or) $x = vy$.

$$x\frac{dy}{dx} = y\{\log y - \log x + 1\}$$

$$\frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right) + 1\right\}$$

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$v + x\frac{dv}{dx} = v\{\log v + 1\}.$$

$$x\frac{dv}{dx} = v\log v$$

$$\int\frac{1}{v\log v}\,dv = \int\frac{1}{x}\,dx$$

$$\log(\log v) = \log x + \log c.$$

$$\log v = xc \Rightarrow v = e^{cx} \Rightarrow y = xe^{cx}.$$

$$(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$$

Non - homogenous DE :-

An eq^n of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

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case (i) :- $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

In this case there exists a substitution which reduces the given eqn to variable - seperable form.

case (ii) :- $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

sub, $x = X + h$, $y = Y + k$

$dx = dX$, $dy = dY$

$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

choose h, k so that $a_1h + b_1k + c = 0$, $a_2h + b_2k + c_2 =$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}.$$

⊛ $(2x + 2y - 1)\, dx = (x + y + 1)\, dy$

$$\frac{dy}{dx} = \frac{2x + 2y - 1}{x + y + 1} \quad \left(\frac{a_1}{a_2} = \frac{b_1}{b_2}\right)$$

$$\frac{dv}{dx} - 1 = \frac{2v - 1}{v + 1} \qquad x + y = v$$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{3v}{v + 1}.$$

$$\int \frac{v+1}{v}\, dv = 3\int dx \Rightarrow v + \log v = 3x + c.$$

$$x + y + \log(x+y) = 3x + c \Rightarrow y - 2x + \log(x+y) = c$$

⊛ which of the following sub reduces the non-homogenoos eqn $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ to hom. form.

Sol $a_1 = 1$ $b_1 = 1$ $a_1h + b_1k + c_1 = 0$

$K + h - 2 = 0 \qquad K = 3 \qquad h = -1$

$K - h - 4 = 0$

$x = (X + 1) \qquad y = (Y + 3)$

hom form $\Rightarrow \dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$

$\Rightarrow$ Exact differential eqⁿ :-

An eqⁿ $M\,dx + N\,dy = 0$ is said to be an exact DE if

$\exists\ f(x,y)$ such that $d[f(x,y)] = M\,dx + N\,dy$.

condition to be exact $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

if this satisfies then find soln.

$$M\,dx + N\,dy = 0$$

$$\underset{\text{Keep 'y' const}}{\int M\,dx} + \int \left(\begin{matrix}\text{terms of } N \\ \text{without } x\end{matrix}\right) dy = c$$

$$\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + \left\{x + \log x - x\sin y\right\} dy = 0$$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y.$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

exact//

soln.

$$\int_{y \to \text{const}} \left(y\left(1 + 1/x\right) + \cos y\right) dx + \int 0 \cdot dy = c$$

$$= y\left(x + \log x\right) + x\cos y + 0 = c.$$

be

(*) $y\sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0.$

M $\qquad N$

$$\frac{\partial M}{\partial y} = \sin 2x \qquad \frac{\partial N}{\partial x} = 2\sin x \cos x$$
$$= \sin 2x$$

eq^n is exact.

$$\int y\left(\frac{-\cos 2x}{2}\right) + \int (-1 - y^2)\, dy = c.$$

$$\frac{-y\cos 2x}{2} - y - \frac{y^3}{3} = c.$$

(*) The eq $py\,dx + (1 + \sin^2 y + \cos^2 x)\,dy = 0$ is exact then

a) $p = \cos 2x$ $\qquad$ b) $p = \sin 2x$

c) $p = -\cos 2x$ $\qquad$ d) ✓ $p = -\sin 2x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$p = -\sin 2x.$$

(*) $p\,dx + (1 + \cos^2 x + \sin^2 y)\, dy = 0$ is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(p) = 2\cos x(-\sin x).$$
$$= -\sin 2x.$$

$$p = \int -\sin 2x\, dy$$

$$P = -y\sin 2x$$

(*) The DE

$$(3a^2x^2 + by\cos x)\,dx + (2\sin x - 4ay^3)\,dy = 0$$

is exact then

a) exactness depends on both a & b

b) " not

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$+b\cos x = 2\cos x$$

$b = +2$ not depends on 'a' only depends on b.

3) $y\,dx - x\,dy = 0$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$$

not exact.

to make it exact ; multiply with $1/y^2$

$$\frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} = \frac{\partial N}{\partial x} \quad \text{(exact)}$$

$\therefore$ $1/y^2$ is called integrating factor.

it is form II

⇒ Integrating factor :- A non-exact eq^n is converted to exact by multiplying a fn $f(x,y)$ then $f(x,y)$ is called integrating factor.

The integrating factors of $y\,dx - x\,dy = 0$ are

$$\frac{1}{y^2},\ \frac{1}{x^2},\ \frac{1}{xy},\ \frac{1}{x^2+y^2} \ldots$$

$\boxed{M\,dx + N\,dy = 0}$ non-exact eq $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

I. All the terms of M & N should be of same degree, then

II. eq^n is of the form $y f(xy)\,dx + x\,g(xy)\,dy = 0$

IV. $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ only

$\int f(x)\,dx$

IV $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \text{ only (or) constant}$$

$$IF = e^{\int g(y)\, dy}$$

V Inspection method.

(*) $y(xy-2)\,dx + x(x^2y^2 + 2xy + 1)\,dy = 0$ — ①

$(xy^2 - 2y)\,dx + (x^3y^2 + 2x^2y + x)\,dy = 0$

$$\frac{\partial M}{\partial y} = 2xy - 2$$

$$\frac{\partial N}{\partial x} = 3x^2y^2 + 4xy + 1$$

from ① it is in form $y f(xy)\,dx + x\,g(xy)\,dy = 0$

it is form II.

so $IF = \dfrac{1}{Mx - Ny}$

$f(xy) = x^2y^2,\ xy \ldots$
$f(x,y) = x^2y^2 + 2x \ldots$

$$IF = \frac{1}{x^2y^2 - 2xy - (x^3y^3 + 2x^2y^2 + xy)}$$

$$= \frac{-1}{x^3y^3 + 3x^2y^2 + 3xy}$$

(*) The non-exact homogenous DE $M\,dx + N\,dy = 0$ is converted to exact by multiplying with which of following fns.

Ans: $\dfrac{1}{Mx + Ny}$.

given homogenous so degree of M & N will be same so use II.

(*) $x^2y\,dx - (x^3 + y^3)\,dy = 0$.

$f(x)$ only

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$$

all the M & N have same degree

$x)\,dx$

$$\frac{x^2 y}{-y^4}\,dx - \frac{(x^3+y^3)}{-y^4}\,dy = 0 \quad \rightarrow \text{exact}$$

$\int_{y\,\text{const}} M\,dx + \int \text{terms no } x$

$$\frac{-x^3}{3y^3} + \log y = c \quad \text{is the soln.}$$

⊛ $$x(x-2y)\,dy + (x^2+y^2+1)\,dx = 0$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2x - 2y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 2x + 2y}{x(x-2y)}$$

$$= \frac{-2(x-2y)}{x(x-2y)}$$

$$= \frac{-2}{x} \quad f(x)$$

$$IF = e^{-\int \frac{2}{x}} = e^{-2\log x} = \frac{1}{x^2}$$

$$\frac{x(x-2y)}{x^2}\,dy + \frac{(x^2+y^2+1)}{x^2}\,dx = 0$$

soln:- $$x - \frac{y^2}{x} - \frac{1}{x} + \int 1\,dy$$

$$x - \frac{(1+y^2)}{x} + y = c$$

⊛ $f(x)$	$e^{\int f(x)\,dx}$
$\frac{1}{x}$	x
$\frac{2}{x}$	x^2
$\frac{3}{x}$	x^3
$\frac{-1}{x}$	$x^{-1} = 1/x$
$\frac{-2}{x}$	$1/x^2$
	$1/x^3$

$$y(1-xy)\,dx - x(1+xy)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 1-2xy \qquad \frac{\partial N}{\partial x} = -1-2xy.$$

case (ii)

$$Mx - Ny = xy - x^2y^2 + xy + x^2y^2$$

$$= 2xy.$$

$$IF = \frac{1}{2xy}.$$

$$\frac{y(1-xy)}{\cancel{2}xy} - \frac{x(1+xy)}{\cancel{2}xy} = 0$$ (don't cancel variables — cancel constants)

$$\frac{1-xy}{x} - \frac{1+xy}{y} = 0$$

soln is $\log x - xy - \int \frac{1}{y}\,dy$

$$\log x - xy - \log y = c.$$

By multiplying with which of the following functions the eq $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$ is converted to exact

$$\frac{\partial M}{\partial y} = 4y^3+2 \qquad \frac{\partial N}{\partial x} = y^3-4$$

case (4)

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y}$$

$$= \frac{-3(y^3+2)}{y(y^3+2)}$$

$$= \frac{-3}{y}.$$

$$IF = 1/y^3 \quad \text{(a)} \quad e^{\int -3/y} = e^{-3\log y} = \frac{1}{y^3}$$

The IF of

$$y(x^2y^2+xy+1)\,dx + x(x^2y^2-xy+1)\,dy = 0$$

case ② since $x f(xy) + y g(xy) = 0$.

$$Mx - Ny = x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy$$

$$= 2x^2y^2$$

$$IF = \frac{1}{2x^2y^2} \text{ (or) } \frac{1}{x^2y^2}$$ (const doesnot make differen)

⊛ $y\,dx - x\,dy + (1+x^2)\,dx + x^2\sin y\,dy = 0$.

$\cancel{(x^2 + y + 1)\,dx + (-x + x^2\sin y)\,dy = 0}$

when there is $y\,dx - x\,dy$ is a part of DE then simply multiply with $1/x^2$, $1/y^2$, $\frac{1}{xy}$, $\frac{1}{x^2+y^2}$ & then integrate.

By multiplying with $1/x^2$ we can easily integrate it is not possible with other terms. Depending on the DE multiply with IF.

Note:

$$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$$

$$\frac{y\,dx - x\,dy}{x^2} = -d\left(\frac{x}{y}\right)$$

$$\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right]$$

$$\frac{y\,dx - x\,dy}{x^2+y^2} = d\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$\int y e^{y} = e^{y}(y-1)$

$\int y e^{-y} = -e^{-y}(y+1)$

sol multiply with $1/x^2$

$$\frac{y\,dx - x\,dy}{x^2} + \left(\frac{1+x^2}{x^2}\right)dx + \frac{\cancel{x^2}\sin y\,dy}{\cancel{x^2}} = 0$$

$$= \frac{-x}{y} - \frac{1}{x} + x + (-\cos y) = c$$

- xy

(*) $\underbrace{y\,dx - x\,dy} + y(1+x^2)\,dx + xy^2 e^y\,dy = 0$

multiply with $1/xy$

$$\frac{y\,dx - x\,dy}{xy} + \left(\frac{1+x^2}{x}\right) dx + \underline{ye^y}\,dy = 0$$

snot
ffence

$$\int d\left[\log\left(\frac{x}{y}\right)\right] + \int\left(\frac{1}{x} + x\right)dx + \int y e^y\,dy = 0$$

$$\log\left(\frac{x}{y}\right) + \log x + \frac{x^2}{2} + e^y(y-1) = c$$

(*) $\boxed{y\,dx + x\,dy = d(xy)}$

$$\boxed{\frac{y\,dx + x\,dy}{xy} = d\left[\log(xy)\right]}$$

(*) $y\,dx + x\,dy + xy^2 e^{-y}\,dy = 0$

multiply with $1/xy$

$$\int \frac{y\,dx + x\,dy}{xy} + \int y e^{-y}\,dy = 0$$

$$\log(xy) - e^{-y}(y+1) = c$$

$e^y(y-1)$

$e^{-y}(y+1)$

⇒ Linear Dequations :-

A DE is said to be linear if the dependent variable & derivatives should be of 1st degree only & there should be no product of them.

(should) $(y)^1$, $\left(\frac{dy}{dx}\right)^1$, $\left(\frac{d^2y}{dx^2}\right)^1$ ..... degree = 1.

(should not) $y\frac{dy}{dx}$, $\frac{d^2y}{dx^2}\,\frac{d^4y}{dx^4}$ (there should not be multiplication)

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right) + e^y = 0$$ multiplication came

is not a linear eq^n becoz $\left(e^y = 1 + y + y^2 + y^3 \ldots\right)$

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 + 4y = 0$$ is non-linear eq.

$\frac{dy}{dx} \cdot \frac{dy}{dx}$ multiplication came.

⇒ Linear in y :-

$$\boxed{\frac{dy}{dx} + Py = Q}$$

P & Q are fns of x (or) constants.

↓ soln.

IF

$$\boxed{y\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C}$$

⇒ Linear in x :-

$$\boxed{\frac{dx}{dy} + Px = Q}$$

P & Q are fns of y (or) constants

↓ soln is.

IF

$$\boxed{x\, e^{\int p dy} = \int Q\, e^{\int p dy} + C}$$

) $x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1.$

$$\frac{dy}{dx} + y\tan x + \frac{y}{x} = \frac{1}{x\cos x}$$

$$\frac{dy}{dx} + \underbrace{y\left(\tan x + \frac{1}{x}\right)}_{P} = \underbrace{\frac{1}{x\cos x}}_{Q}$$

$$IF = e^{\int\left(\tan x + \frac{1}{x}\right)dx} = e^{\log\sec x + \log x}$$

lication came (y^2+y^3x)

ar eq.

of x

fns stands

soln :- $y(x\sec x) = \int \frac{1}{\cancel{x}\cos x} \cdot \cancel{x}\sec x \, dx + c$

$= \int \sec^2 x \, dx$

$xy\sec x = \tan x + c$

(*) $(x + 2y^3)\frac{dy}{dx} = y$ with $x(1) = 0$

linear in x.

$x + 2y^3 = y\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{x + 2y^3}{y}$

$\frac{dx}{dy} - \frac{x}{y} = 2y^2$ $\quad P = \frac{-1}{y}$

IF $= e^{-\int 1/y} = 1/y$

soln: $x \cdot \frac{1}{y} = \int 2y^{\cancel{2}} \cdot \frac{1}{\cancel{y}} + c$

$\frac{x}{y} = y^2 + c$ $\quad$ at $y = 1$, $x = 0$

$\frac{0}{1} = 1 + c$ $\quad c = -1$

$x/y = y^2 - 1 \cdot y$

(*) $x^2\frac{dy}{dx} + 2xy = \frac{2\log x}{x}$; $y(1) = 0$ find y

when $x = e$.

$\underline{g}$ linear in y:

$\frac{dy}{dx} + \frac{2y}{x} = \frac{2\log x}{x^3}$

IF $= e^{\int \frac{2}{x}} = x^2$.

soln $yx^2 = \int \frac{2\log x}{x^{\cancel{3}}} \cdot \cancel{x^2}$

$\int f(x)^n \cdot f'(x) = f(x)^{n+1}$

at $x = 1$; $y = 0$ $\Rightarrow C = 0$

$$x^2 y = (\log x)^2 + 0$$

at $x = e$ $\quad e^2 y = (\log e)^2$

$$e^2 y = 1 \Rightarrow y = 1/e^2$$

(*) $x\frac{dy}{dx} + y = x^4$; $y(1) = \frac{6}{5}$

$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

$$IF = e^{\int 1/x} = x$$

soln $\quad yx = \int x^3 \cdot x + c$

$$yx = \frac{x^5}{5} + c$$

at $x = 1$ $y = 6/5$ $\Rightarrow c = 1$

$$xy = \frac{x^5}{5} + 1$$

$$y = \frac{x^4}{5} + \frac{1}{x}$$

$\Rightarrow \frac{dy}{dx} + Py = Qy^n$ → is not a linear eqn

sub $y^{1-n} = v$ reduces Bernoulli eq to linear eqn

soln.

$$y^{1-n} e^{\int (1-n) P\, dx} = \int (1-n) Q\, e^{\int (1-n) dx}\, dx + c.$$

(If y^n is there is q^n ~~then~~ multiply $(1-n)$ to every term in soln)

(*) The sub $y^{1-n} = v$ reduces the non-linear eq

$\frac{dy}{dx} + Py = Qy^n$ to which of following linear

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{y}{y^n} = Q.$$

$$y^{1-n} = v \Rightarrow (1-n)y^{1-n-1}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$$

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

$$\boxed{\frac{dv}{dx} + Pv(1-n) = Q(1-n)}$$

* which of the following sub reduces the non-linear eq^n to linear form

$$y\frac{dy}{dx} + x^2y^3 = x^3y^3$$

$$\frac{dy}{dx} + x^2y^2 = x^3y^2$$

$$y^{1-n} = v \Rightarrow y^{1-2} = v \Rightarrow y^{-1} = v$$

* $$\frac{dy}{dx} - y\tan x = -y^2\sec x$$

$$e^{\int(1-2)(-\tan x)} = e^{\log \sec x} = \sec x$$

$$y^{1-2}\sec x = \int(1-2)(-\sec x)\sec x\, dx$$

$$\frac{\sec x}{y} = \tan x + c$$

ear eq

in soln)

eq

$\Rightarrow$ Equation of the form

inear

$$\boxed{f'(y)\frac{dy}{dx} + P f(y) = Q}$$

$$f(y) = v \Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + Pv = Q$$

⊛ The sub $(\log z)^{-1} = v$ reduces the non-linear eq

$$\frac{dz}{dx} + \frac{z\log z}{x} = \frac{z(\log z)^2}{x^2}$$

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{(\log z)^{-1}}{x} = \frac{1}{x^2}$$

$$(\log z)^{-1} = v \Rightarrow -\frac{(\log z)^{-2}}{z}\frac{dz}{dx} = \frac{dv}{dx}$$

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} = -\frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

$$\frac{dv}{dx} - \frac{v}{x^2} = \frac{-1}{x^2}$$

$$x\frac{dy}{dx} + y\log y = xye^x$$

$$\frac{1}{y}\frac{dy}{dx} + \frac{\log y}{x} = e^x$$

$$\log y = v \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + \frac{v}{x} = e^x$$

$$IF = e^{\int\frac{1}{x}} = x.$$

$$v\,x = \int e^x . x\, dx$$

$$x \log y = e^{x}(x-1) + c.$$

$\Rightarrow$ Clairaut's eq^n :-

An eq of the form

—Linear

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

(or)

$$y = px + f(p)$$

replace $\frac{dy}{dx} \to c$.

$\boxed{y = cx + f(c)}$ is the fn of soln.

(*) $p = \sin(y - xp)$

$$y - xp = \sin^{-1} p$$

$$y = xp + \sin^{-1} p$$

$$y = cx + \sin^{-1} c \quad (c \text{ is a const})$$

(*) $(x-a)y'^2 + (x-y)y' - y = 0$

$$xy'^2 - ay'^2 + xy' - yy' - y = 0$$

$$xy'(y'+1) - ay'^2 = y(y'+1)$$

$$xy' - \frac{ay'^2}{(y'+1)} = y$$

$$y = cx - \frac{ac^2}{1+c^2}$$

(*) $\left(y - x\frac{dy}{dx}\right)\left(\frac{dy}{dx} - 1\right) = \frac{dy}{dx}$

$$(y - xy')(y' - 1) = y'$$

⇒ Higher order linear eq^n with constant coefficients

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

$k_1, k_2 \ldots k_{n-1}$ are constants

X is fn of x.

$\boxed{D = \frac{d}{dx}}$ — differential operator.

$\boxed{\frac{1}{D} = \int}$ — inverse differential operator.

$$\left(D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n\right) y = X$$

$$F(D)\, y = X.$$

$\boxed{F(D) = D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n = 0}$

is called auxillary eq^n.

* The complete solution of $F(D)\, y = X$ is

y = complementary fn (CF) + particular integral (PI)

* If $X = 0$ then $F(D)y = 0$ is the called the homogenous linear DE.

* If $X \neq 0$ then $F(D)\,y = X$ is called non-homogenous linear DE

* The soln of homogenous linear DE i.e. $(F(D)\,y = 0)$ is called CF.

The no. of arbitrary constants in the CF should be equal to the higher order of given DE.

fficients

PI will not contain any arbitrary constants.

* If $X = 0$ the complete soln of $F(D)y = X$ is is only CF.

⇒ Procedure to find the CF :-

By assuming 'D' as an algebraic quantity. $F(D) = 0$ becomes an algebraic eqn & by solving it we get roots for these. Based on the nature of these roots we write CF as follows.

Nature of roots	CF
1. Real & Distinct $D = m_1, m_2, m_3$	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2. Real & repeated $D = m_1, m_1, m_3$	$CF = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x}$
3. complex & distinct $D = a \pm ib, m_3$	$e^{ax}[c_1 \cos bx + c_2 \sin bx] + c_3 e^{m_3 x}$
4. complex & repeated $D = a \pm ib, a \pm ib, m_5$	$e^{ax}[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx] + c_5 e^{m_5 x}$

Eg: roots.

$D = 1, -1/2, 1/2 \quad CF = c_1 e^{x} + c_2 e^{-1/2 x} + c_3 e^{x/2}$

$D = -2, -2, 1 \quad CF = (c_1 + c_2 x) e^{-2x} + c_3 e^{x}$

$D = 3 \pm 4i, -1 \quad CF = e^{3x}[c_1 \cos 4x + c_2 \sin 4x] + c_3 e^{-x}$

$D = \pm 2i, \pm 3 \quad CF = [c_1 \cos 2x + c_2 \sin 2x]$

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genous

hould

DE

6

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$D^2 - 5D + 6 = 0 \qquad D = 3, 2$$

$$CF = c_1 e^{3x} + c_2 e^{2x}.$$

$$\frac{d^4y}{dx^4} - 81y = 0$$

$$D^4 - 81 = 0$$

$$(D^2 - 9)(D^2 + 9) = 0$$

$$D = \pm 3, \pm i3$$

$$y = c_1 e^{3x} + c_2 e^{-3x} + (c_3 \cos 3x + c_4 \sin 3x)$$

in option it ~~will~~ maybe
$e^{-3x} + \sin 3x$
choosing different values of c_1, c_2, c_3, c_4 ...

$$y'' + 2py' + (p^2+q^2)y = 0$$

$$D^2 + 2pD + (p^2+q^2) = 0$$

$$D = \frac{-2p \pm \sqrt{4p^2 - 4(p^2+q^2)}}{2}$$

$$= -p \pm \sqrt{p^2 - p^2 - q^2}$$

$$= -p \pm iq$$

$$y = e^{-xp}(c_1 \cos qx + c_2 \sin qx).$$

$$y''' - 4y'' + 5y' - 2y = 0$$

$$D^3 - 4D^2 + 5D - 2 = 0$$

$D = 1$ is root

$$\begin{array}{c|cccc} 1 & 1 & -4 & 5 & -2 \\ & 0 & 1 & -3 & 2 \end{array}$$

$D^2 - 3D + 2 = 0$

$D = +1, +2$

$D^2 - 2D - D + 2 = 0$

$D(D-2) - 1(D-2) = 0$

$(D-1)(D-2) = 0$

roots are $1, +1, +2$

$y = \cancel{c_1 e^x + c_2 e^x + c_3 e^{2x}}$

$y = (c_1 + c_2 x) e^x + c_3 e^{2x}$

maybe

values

$3x)$

<u>Note</u> :- If $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots$ is a soln of $F(D) y = 0$ then each one $y_1, y_2, y_3 \cdots$ is linearly independent soln of $F(D) y = 0$.

⊛ $\dfrac{d^2 f}{dt^2} - 4 \dfrac{df}{dt} + 4f = 0$

a) $f_1 = e^{2t}, f_2 = e^{-2t}$ ~~b)~~ $f_1 = e^{2t}, f_2 = te^{2t}$

c) $f_1 = e^{-2t}, f_2 = te^{-2t}$ d) $f_1 = e^{-2t}, f_2 = te^{2t}$

$D^2 - 4D + 4 = 0$

$(D-2)^2 = 0$

$D = 2, 2$

$y = (c_1 + c_2 t) e^{2t}$

$c_1 e^{2t} + c_2 t e^{2t}$

⊛ The DE $(D^2 + 4\cos x\, D + 3) y = x^2$ is b

a) homogenous DE

b) non-homogenous DE with const coef

~~c)~~ Linear DE

d) Non-Linear DE

⊛ y_1, y_2 are linearly independent solns of $F(D) y = 0$ then which of the following is the soln of the same eqn.

⊛ y_1, y_2 are linearly independent soln of the corresponding homogenous eqn of $F(D)y = x$ then $y_1 + y_2$ is the soln of which of following eqn

a) $F(D)y = x$ b) $F(D)y = 0$

c) $x = 0$ ~~d)~~ $F(D)y = 0$

⊛ e^{2x}, e^{-3x} are the ~~so~~ linearly independent solns

$$D = 2, -3$$

$$(D-2)(D+3)y = 0$$

$$(D^2+D-6)y = 0$$

$$y'' + y' - 6y = 0$$

⊛ e^{2x}, xe^{2x}

$$= (c_1 + c_2 x)e^{2x}$$

$$D = 2, 2$$

$$(D-2)^2 y = 0$$

$$(D^2 - 4D + 4)y = 0 \Rightarrow y'' - 4y' + 4y = 0$$

⊛ e^{2x}, e^{-2x} are soln of which ...

(see option sub $D = 2$ & -2 which satisfies the eq) It may be

$$D = 2, -2$$

$$(D^2 - 4)y = 0 \Rightarrow y'' - 4y = 0 \qquad \frac{d^4y}{dx^4} - 16y = 0$$

⊛ $\sin 3x, \cos 3x$ are LI soln of which of the following DE

$$y = c_1 \cos 3x + c_2 \sin 3x$$

the

then

eqn

solns

such

S.y. =0

the

$y'' + 4y' + 13y = 0$; $y(0) = 0$, $y'(0) = 1$.

$(D^2 + 4D + 13) = 0$.

$$D = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{-4 \pm i6}{2} = -2 \pm 3i$$

$$y = e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right).$$

at $x = 0$; $y = 0$.

$$0 = c_1$$

$$y = e^{-2x} c_2 \sin 3x.$$

$$y' = c_2\left[e^{-2x}(3\cos 3x) + (-2)e^{-2x}\sin 3x\right]$$

at $x = 0$ $y' = 1$

$$1 = c_2[3] \Rightarrow c_2 = 1/3.$$

$$y = \frac{e^{-2x}}{3}\sin 3x.$$

which of the following is not a soln of

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0.$$

where $R^2 C = 4L$ & R, L, C be the constants

a) $i = e^{-Rt/2L}$ b) $i = te^{-Rt/2L}$

c) $i = te^{Rt/2L}$ d) $i = e^{-Rt/2L} + te^{-Rt/2L}$

$$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$$

$$D = \frac{-R/L \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$D = \frac{-R}{2L}, \frac{-R}{2L}$$

$$y = (c_1 + c_2 t) e^{-R/2L}$$

$$= e^{-R/2L} + t e^{-R/2L}$$

⊛ $\frac{d^2y}{dx^2} + \omega^2 y = 0$; $y(0) = 0$, $y(l) = 0$

a) $y = \sum_n C_n \cos \frac{n\pi x}{l}$ b) $y = \sum_n C_n e^{\frac{n\pi x}{l}}$

~~c)~~ $y = \sum_n C_n \sin \frac{n\pi x}{l}$ d) $y = \sum_n C_n x^{\frac{n\pi}{l}}$

Sol sub $y = 0$ at $x = 0$ & $x = l \Rightarrow y = 0$.

⊛ $9y'' - 6y' + y = 0$; $y(0) = 3$, $y'(0) = 1$

$(9D^2 - 6D + 1) y = 0$

$$D = \frac{+6 \sqrt{36 - 36}}{18}$$

$$D = \frac{1}{3}, 1/3$$

$$y = (c_1 + c_2 x) e^{x/3}$$

$y(0) = 3 \Rightarrow 3 = c_1$

$y'(0) = 1 \Rightarrow y' = c_2 e^{x/3} + (c_1 + c_2 x) e^{x/3} \frac{1}{3}$

$\Rightarrow 1 = c_2 + 3c_1$

$\Rightarrow c_2 = 0$

$$y = (3 \quad) e^{x/3}$$

⊛ For what value of 'λ' the DE $(D^2 + \lambda) y = 0$ will have non-trivial soln. $y(0) = 0$, $y(\pi) = 0$

$$y = (c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x)$$

$$y(0) = 0 \Rightarrow 0 = c_1$$

$$y(\pi) = 0 \Rightarrow 0 = c_2 \sin \pi\sqrt{\lambda}$$

$$\sin \pi\sqrt{\lambda} = 0$$

$$\pi\lambda = n^2\pi$$

$$\lambda = 1, 4, 9, 16 \ldots$$

Q

(a) The complete soln of DE $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$ is $y = c_1 e^{-x} + c_2 e^{-3x}$. p, q are

$$D = -1, -3$$

$$(D+1)(D+3)y = 0$$

$$(D^2 + 4D + 3)y = 0$$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dy} + 3y = 0 \qquad p = 4, \ q = 3.$$

(b) The soln of $y'' + py' + (q+1)y = 0$ is

$$y'' + 4y' + 4y = 0$$

$$(D+2)^2 y = 0$$

$$D = -2, -2 \Rightarrow y = (c_1 + c_2 x)e^{-2x}.$$

$$\frac{d^2n}{dx^2} + \frac{-n}{L^2} = 0; \quad n(0) = K, \ n(\infty) = 0.$$

$$\left(D^2 - \frac{1}{L^2}\right)n = 0.$$

$$D = \pm\frac{1}{L}$$

$$y = c_1 e^{x/L} + c_2 e^{-x/L}.$$

sub. conditions:

⊛ **Particular integral :-**

$$F(D)\,y = X$$

$$PI = \left[\frac{1}{F(D)}\right] X$$

X may be

I. e^{ax} or constant

II. $\sin ax$ or $\cos ax$

III. x^m

IV. $e^{ax}.V$

V can be

- $\sin bx$ or $\cos bx$
- x^m

case (I) : $X = e^{ax}$ or constant

$$PI = \left[\frac{1}{F(D)}\right] e^{ax}$$

replace 'D' by 'a' in $F(D)$

$$\boxed{PI = \left[\frac{1}{F(a)}\right] e^{ax}} \qquad (F(a) \neq 0)$$

if $F(a) = 0$:-

$$PI = x\left[\frac{1}{F'(D)}\right] e^{ax}$$

replace $D \to a$ in $F'(D)$

$$\boxed{PI = x\left[\frac{1}{F'(a)}\right] e^{ax}}$$

if $F'(a) = 0$

$$PI = x^2\left[\frac{1}{F''(a)}\right] e^{ax}$$

⊛ Find PI of $y'' + 3y' - 2y = e^{2x} + 3$

$$PI = \left[\frac{1}{D^2+3D-2}\right]\left(e^{2x}+3\right) \qquad 3 = 3e^{0x}$$

$$= \frac{1}{2^2+3(2)-2}e^{2x} + \frac{3}{0^2+3(0)-2}e^{0x}.$$

$$= \frac{e^{2x}}{8} - \frac{3}{2}.$$

he

⊛ PI of $y'' + 5y' + 6y = e^{-3x}$

$$PI = \left[\frac{1}{D^2+5D+6}\right]e^{-3x} \qquad (\text{Dr is } 0).$$

so differentiate

$$= x\left(\frac{1}{2D+5}\right)e^{-3x}$$

$$= -xe^{-3x}$$

⊛ PI of $y'' + 4y' + 4y = e^{2x}$.

$$PI = \left[\frac{1}{D^2+4D+4}\right]e^{-2x} \qquad (\text{Dr is } 0)$$

$$= \left(\frac{x}{2D+4}\right)e^{-2x} \qquad (\text{Dr is } 0)$$

$$= \frac{x^2}{2}e^{-2x}$$

Case(2) :- $X = \sin ax$ (or) $\cos ax$

(or) $\sin(ax+b)$ (or) $\cos(ax+b)$

$$PI = \left[\frac{1}{F(D)}\right]\sin ax$$

replace D^2 by $-a^2$ in $F(D)$.

$$= \frac{1}{\quad}\sin ax.$$

if $F(-a^2) = 0$

then $PI = x\left[\frac{1}{F'(D)}\right]\sin ax$

$$PI = x\,\frac{1}{F'(-a^2)}\sin ax$$

⊛ $\frac{d^2y}{dx^2} + 5y = \cos(3x+2)$

$$PI = \left[\frac{1}{D^2+5}\right]\cos(3x+2)$$

$$= \frac{1}{-9+5}\cos(3x+2)$$

$$= \frac{-\cos(3x+2)}{4}$$

⊛ $y'' + 4y = \sin 2x$

$$PI = \left[\frac{1}{D^2+4}\right]\sin 2x \qquad (Dr = 0)$$

$$= \frac{x}{2D}\sin 2x = \frac{x}{2}\cdot\left(\frac{1}{D}\sin 2x\right)$$

Don't do integ on x only do on $\sin 2x$.

$$= \frac{x}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{-x\cos 2x}{4}$$

⊛ $(D^3 + D^2 + 2D + 2)y = \cos x$

$$PI = \left[\frac{1}{D^3+D^2+2D+2}\right]\cos x$$

replace $D^2 = -a^2 = -1$

$$= \frac{1}{-D-1+2D+2}\cos x \qquad (D^3 = D\cdot D^2)$$

$$= \frac{D-1}{(D^2} \cos x)$$

$$= \frac{D-1}{-1-1} \cos x = \frac{(D-1)}{-2} \cos x$$

$$= \frac{1}{2}\left(\cos x + D(\cos x)\right)$$

$$= \frac{1}{2}\left[\cos x - \sin x\right]$$

case :(3) :- $x = x^m$ or poly in x (m is +ve).

$$PI = \left[\frac{1}{F(D)}\right] x^m$$

$$= \frac{1}{*\left[1+\phi(D)\right]} x^m$$

(* is the least power in F(D).)

$$= \frac{1}{*}\left[1+\phi(D)\right]^{-1} x^m$$

Ex $\frac{d^2y}{dx^2} + 2y = x^3$

$$PI = \left(\frac{1}{D^2+2}\right) x^3$$

$$= \frac{1}{2}\left(\frac{1}{1+0.5D^2}\right) x^3$$

$$= \frac{1}{2}\left[1+\frac{D^2}{2}\right]^{-1} x^3$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{D^4}{4}\cdots\right] x^3$$

$$= \frac{1}{2}\left[x^3 - 3x + 0 \cdots\right] = \frac{x^3-3x}{2}$$

Ex $D^2 + D = x^2 + 2$

$$PI = \left(\frac{1}{D^2+D}\right)(x^2+2)$$

on x

sin $2x$

$= \frac{1}{D}\left(1+\frac{D}{D}\right. \quad \left. +2\right)$

$= \frac{1}{D}(1+D)^{-1}(x^2+$

$= \frac{1}{D}(1-D+D^2)(x^2+2)$

$= \frac{1}{D}(x^2+2-2x+2)$

$= \frac{x^3}{3}+4x-x^2$

Case (4) :- $X = e^{ax}.V$

$(V = \cos bx$ (or) $\sin bx)$ (or) $(V = x^m$ (or) poly in $x)$

$PI = \left[\frac{1}{F(D)}\right]e^{ax}V.$

replace $D \to$ '$D+a$' in $F(D)$.

$= e^{ax}\left[\frac{1}{F(D+a)}\right]V.$

$\left[\frac{1}{F(D+a)}\right]V$ → case (2) if $V = \sin bx$ (or) $\cos bx$
→ case (3). if $V = x^m$.

(*) $\frac{d^2y}{dx^2} + 3y = e^x \sin 2x.$

$PI = \left[\frac{1}{D^2+3}\right]e^x \sin 2x$

$= e^x\left[\frac{1}{(D+1)^2+3}\right] \sin 2x$

$= e^x\left[\frac{1}{D^2+2D+4}\right]\sin 2x$

$$= \frac{e^x}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{e^x \cos 2x}{4}$$

(*) $\frac{d^2y}{dx^2} - 4y = \cos^2 x$

$F(D) = D^2 - 4 = 0 \qquad D = \pm 2$

$CF = c_1 e^{2x} + c_2 e^{-2x}$

poly in x

$$PI = \left(\frac{1}{D^2-4}\right)\cos^2 x \quad = \frac{1}{2}\left[\frac{1}{D^2-4}\right]\left(\overset{e^{0x}}{1} + \cos 2x\right)$$

$$= \frac{1}{2}\left[\frac{1}{-4}\right] + \frac{1}{2}\left[\frac{1}{-4-4}\right]\cos 2x$$

$$PI = \frac{-1}{8} - \frac{1}{16}\cos 2x$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} - \frac{1}{16}\cos 2x$$

$\frac{x}{2}\cos bx$

(*) The PI of $\frac{d^2y}{dx^2} + a^2 y = \sin ax$ is

$$PI = \left(\frac{1}{D^2+a^2}\right)\sin ax \quad = \frac{x}{2D}\sin ax$$

$$= \frac{-x}{2a}\cos ax$$

(*) The PI of $\frac{d^2y}{dx^2} + 4y = \sin 2x + \cos 3x$ is

$(Ax\cos 2x + B \cancel{\sin}\,\underset{\cos}{} 3x) \qquad A, B = \_\_$

$$(PI)_1 = \frac{x}{2D}\sin 2x \qquad (PI)_2 = \frac{1}{-5}\cos 3x$$

$$= -\frac{x}{4}\cos 2x$$

$$(D-2)^3 y = x e^{2x}$$

$$D = 2, 2, 2$$

$$CF = (c_1 + c_2 x + c_3 x^2) e^{2x}$$

$$PI = \frac{1}{(D-2)^3} x e^{2x}$$

$$= e^{2x} \frac{1}{((D+2)-2)^3} x$$

$$= e^{2x} \frac{1}{D^3} x$$

$$= e^{2x} \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{x^4}{4}$$

$$y = (c_1 + c_2 x + c_3 x^2) e^{2x} + \frac{x^4 e^{2x}}{24}$$

$$y'' - 2y' + y = x - 1$$

$$(D^2 - 2D + 1) = 0$$

$$(D-1)^2 = 0 \qquad D = 1, 1$$

$$CF = (c_1 + c_2 x) e^x$$

$$PI = \frac{1}{(D^2 - 2D + 1)} (x-1)$$

$$= (1 + D^2 - 2D)^{-1} (x-1)$$

$$= (1 - (D^2 - 2D)) (x-1)$$

$$PI = (x-1) + 2 = x + 1$$

$$\frac{d^2 y}{dx^2} = e^x \cos x$$

$$PI = \frac{1}{D^2} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2} \cos x$$

$$= e^x \left(\frac{1}{D^2+2D+1}\right) \cos x = \frac{e^x}{2} \sin x$$

⊛ $\frac{d^2y}{dx^2} + y = x$; at $x = 0$, $y = 1$

& $x = \pi/2$, $y = \pi/2$

$$D^2 + 1 = 0$$

$$D = \pm i$$

$$CF = (c_1 \cos x + c_2 \sin x)$$

$$PI = \frac{1}{D^2+1} x$$

$$= (1 + D^2)^{-1} x$$

$$= (1 - D^2) x \Rightarrow x - 0 = x$$

$$y = c_1 \cos x + c_2 \sin x + x$$

$x = 0, y = 1 \Rightarrow 1 = c_1$

$x = \pi/2, y = \pi/2 \Rightarrow \frac{\pi}{2} = c_2 + \pi/2 \Rightarrow c_2 = 0$

$$y = \cos x + x$$

⊛ $y'' - 3y' + 2y = \cosh x$

$$= \frac{e^x + e^{-x}}{2}$$

$$D^2 - 3D + 2 = 0 \Rightarrow D^2 - 2D - D + 2 = 0$$

$$D = 1, 2$$

$$CF = c_1 e^x + c_2 e^{2x}$$

$$PI = \frac{1}{2}\left(\frac{1}{\quad}\right) e^x + \frac{1}{2}\left(\frac{1}{\quad}\right) e^{-x}$$

$$y'' - 3y' + 2y = e^x \; ; \quad y = 3 \;\; \& \;\; \frac{dy}{dx} = 3 \text{ at } x = 0$$

$$D^2 - 3D + 2 = 0$$

$$D = 1, 2$$

$$CF = c_1 e^x + c_2 e^{2x}$$

$$PI = \left[\frac{1}{D^2 - 3D + 2}\right] e^x$$

$$= -x e^x .$$

$$y = c_1 e^x + c_2 e^{2x} - x e^x$$

$$y' = c_1 e^x + 2 c_2 e^{2x} - e^x(x+1)$$

$x = 0, \; y = 3 \;\Rightarrow\; 3 = c_1 + c_2$

$x = 0, \; y' = 3 \;\Rightarrow\; 3 = c_1 + 2c_2 - 1$

$$c_2 - 1 = 0$$

$$c_2 = 1 \qquad c_1 = 2$$

$\Rightarrow$ <u>Cauchy's homogenous D.eqⁿ</u> :-

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} x \frac{dy}{dx} + k_n y = X$$

$k_1, k_2 \dots$ are constants.

$x = e^t$ $\quad X$ is function of x.

it reduces to linear DE with constant coef.

$$x \frac{dy}{dx} = D_1 y = \frac{dy}{dt}$$

$$x^2 \frac{d^2 y}{dx^2} = D_1 (D_1 - 1) y$$

$$x^3 \frac{d^3 y}{dx^3} = D_1 (D_1 - 1)(D_1 - 2) y$$

$$\cancel{(D_1^2 + D_1 + 1) y = 4\cos t}$$

$$\cancel{PI = \left[\frac{1}{D_1^2 + D_1 + 1}\right] 4\cos t}$$

$$\cancel{= 4\left(\frac{1}{D_1}\cos t\right)}$$

$$\cancel{= 4\sin t}$$

$$(D_1(D_1 - 1) + D_1 + 1) y = 4\cos t$$

$$(D_1^2 + 1) y = 4\cos t$$

$$PI = 4\left(\frac{1}{D_1^2 + 1}\right) \cancel{4}\cos t$$

$$= 2t \sin t$$

$$= 2\log(1+x)\sin[\log(1+x)]$$

$\Rightarrow$ Method of variation of parameters :-

This is a method to find the PI

of

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R}$$

1. $CF = c_1 y_1 + c_2 y_2$

2. $PI = Ay_1 + By_2$

$$A = -\int \frac{R y_2}{\omega}\, dx \quad , \quad B = \int \frac{R y_1}{\omega}\, dx$$

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

3. $y = CF + PI$

Note :- If P & Q are functions of x then we should know two independent soln y_1, y_2 of the corresponding homogenous eq^n $y'' + Py' + Qy = 0$ so that we can write C_F $(CF = c_1 y_1 + c_2 y_2)$.

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$$

$$D^2 - 6D + 9 = 0$$

$$D = 3, 3$$

$$CF = c_1 e^{3x} + c_2 x e^{3x}$$

$$y_1 = e^{3x} \quad y_2 = x e^{3x}$$

$$w = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix}$$

$$= e^{6x} + 3xe^{6x} - 3xe^{6x}$$

$$w = e^{6x}$$

$$A = -\int \frac{\frac{\cancel{e^{3x}}}{x^2} \cdot x \cancel{e^{3x}}}{\cancel{e^{6x}}} dx = -\int \frac{1}{x} = -\log x$$

$$B = +\int \frac{\cancel{e^{3x}}}{x^2} \cdot \frac{\cancel{e^{3x}}}{\cancel{e^{6x}}} dx = \frac{-1}{x}$$

$$PI = -\log x \; e^{3x} - \frac{1}{\cancel{x}} \cancel{x} e^{3x}$$

$$= -e^{3x}(\log x + 1)$$

The PI of $\frac{d^2y}{dx^2} + 4y = \sec^2 2x$ is $Ay_1 + By_2$ where $y_1 = \cos 2x$, $y_2 = \sin 2x$ find A & B

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$$

Sub. $x = e^t \Rightarrow \log x = t$

$$(D_1(D_1-1) - 3D_1 + 4)y = 0$$

$$(D_1^2 - 4D_1 + 4)y = 0$$

$$D_1 = 2, 2$$

$$y = (c_1 + c_2 t)e^{2t}$$

$$y = (c_1 + c_2 \log x)\, x^2$$

Find PI of $x^2y'' - xy' + 3y = x^2 \log x$

Sub. $x = e^t \Rightarrow \log x = t$

$$(D_1(D_1-1) - D_1 + 3)y = e^{2t}\cdot t$$

$$(D_1^2 - 2D_1 + 3)y = te^{2t}$$

$$D_1 = -1, 3$$

$$PI = \left[\frac{1}{D_1^2 - 2D_1 + 3}\right] t e^{2t}$$

$$= e^{2t}\left[\frac{1}{(D_1+2)^2 - 2(D_1+2) + 3}\right] t$$

$$= e^{2t}\left[\frac{1}{D_1^2 + 2D_1 + 3}\right] t$$

$$= \frac{e^{2t}}{3}\left[1 - \left(\frac{D_1^2}{3} + \frac{2D_1}{3}\right)\right] t$$

$$= \frac{e^{2t}}{3}\left[t - \frac{2}{3}\right].$$

$$PI = \frac{x^2}{3}\left[\log x - \frac{2}{3}\right]$$

Legenders eq^n :-

$\Longrightarrow (a+bx)^n \dfrac{d^n y}{dx^n} + K_1 (a+bx)^{n-1} \dfrac{d^{n-1} y}{dx^{n-1}} + \cdots k_{n-1} (a+bx) \dfrac{dy}{dx}$

$+ k_n y = X.$

$(a+bx) = e^t \downarrow$

Linear DE with const coef.

$(a+bx) \dfrac{dy}{dx} = b D_1 y$ $\boxed{t = \log(a+bx)}$

$(a+bx)^2 \dfrac{d^2 y}{dx^2} = b^2 D_1 (D_1 - 1) y$

$(a+bx)^3 \dfrac{d^3 y}{dx^3} = b^3 D_1 (D_1 - 1)(D_1 - 2) y.$

$D_1 = \dfrac{d}{dt}, \quad D_1^2 = \dfrac{d}{dt^2}, \ldots$

$(1+2x)^2 y'' - 6(1+2x) y' + 16 y = 0$

$4 D_1 (D_1 - 1) - 6(2) D_1 + 16 = 0$

$4 D_1^2 - 8 D_1 + 16 = 0$ $(1+2x) = e^t.$

$D_1^2 - x D_1 + 4 = 0$

$D_1 = 2, 2$

$CF = (C_1 + C_2 t) e^{2t}$

$PI = \cancel{\left[\dfrac{1}{D_1^2 - 2D_1 + 4}\right]}$

$CF = \left[C_1 + C_2 \log(1+2x)\right] (1+2x)^2$

PI of $(1+x^2)^2 y'' + (1+x) y' + y = 4 \cos(\log(1+x))$

$(1+x) = e^t$

1) Res. of $f(z) = \frac{1}{(z^2+1)^2}$ @ $z = i$ is

$f(z) = \frac{1}{(z+i)^2(z-i)^2} \rightarrow \lim_{z\to i} \frac{d}{dz}\left\{ \cancel{(z-i)^2} \times \frac{1}{(z+i)^2\cancel{(z-i)^2}} \right\}$.

$\lim_{z\to i} \frac{-2}{(z+i)^3} \Rightarrow \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{1}{4i}$

*) $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ Calculate res. at ech of the poles.

$z = 1, \; z = -2$

Res at $z = 1 \rightarrow \lim_{z\to 1} \frac{d}{dz}\left\{ \cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z+2)} \right]$

$\Rightarrow \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{z+2}\right] \Rightarrow \lim_{z\to 1} \left\{ \frac{2z}{z+2} - \frac{z^2}{(z+2)^2} \right]$

$= \frac{2}{3} - \frac{1}{9} = \frac{6-1}{9} = \frac{5}{9}$.

Res at $z = -2 \rightarrow \lim_{z\to -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2\cancel{(z+2)}} \Rightarrow \frac{4}{9}$.

1) Res. of $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles

a) $1/2, -1/2, 1$ b) $1/2\; 1/2, -1$ c) $1/2, 1, -3/2$ d) $1/2, -1, 3/2$

At $z \Rightarrow 0$ $\lim_{z\to 0} \frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}$.

At $z \Rightarrow 1$ $\lim_{z\to 1} \frac{1-2z}{z(z-2)} = \frac{-1}{-1} = 1$

At $z \Rightarrow 2$ $\lim_{z\to 2} \frac{1-2z}{z(z-1)} = \frac{-3}{2}$.

(*) $f(z) = \dfrac{1}{(z+2)^2(z-2)^2}$ at $z=2$ is

$$\lim_{z\to 2} \frac{d}{dz}\left\{ \cancel{(z-2)^2} \times \frac{1}{(z+2)^2\cancel{(z-2)^2}} \right\} \Rightarrow \lim_{z\to 2} \frac{-2}{(z+2)^3} = \frac{-2}{64}$$

$$= \frac{-1}{32}$$

<u>Cauchy's Residue Theorem.</u>

$f(z)$ is analytic in a closed curve c except at a finite no. of points, then $\int_c f(z)\,dz = 2\pi i\,\{$ sum of residues $f(z)$ at each of poles

(*) $\oint_c \dfrac{1}{(z+2)^2(z-2)^2}\,dz \qquad |z| = 5$

$$= 2\pi i\,[\operatorname{Res}(2) + \operatorname{Res}(-2)]$$

$$= 2\pi i\left[\lim_{z\to 2}\frac{d}{dz}\frac{1}{(z+2)^2} + \lim_{z\to -2}\frac{d}{dz}\frac{1}{(z-2)^2}\right]$$

$$= 2\pi i\left[\lim_{z\to 2}\frac{-2}{(z+2)^3} + \lim_{z\to -2}\frac{-2}{(z-2)^3}\right] = 2\pi i\left[\frac{-2}{64} + \frac{2}{64}\right]$$

$$= 0$$

(*) $\oint_c \dfrac{e^z}{z^2+1}\,dz$ where c is the region $|z| = 2$

$$\oint_c \frac{e^z}{(z-i)(z+i)} = 2\pi i\,\{\operatorname{Res}(i) + \operatorname{Res}(-i)\}$$

$$= 2\pi i\left\{\lim_{z\to i}\frac{e^z}{z+i} + \lim_{z\to -i}\frac{e^z}{z-i}\right\}$$

$$= 2\pi i\left\{\frac{e^i}{2i} + \frac{e^{-i}}{-2i}\right] = \pi\,[e^i - e^{-i}]$$

(*) $\int_c \dfrac{z^2+1}{z(2z+1)}\,dz \qquad |z| = 1$

$$\frac{1}{2}\int_c \frac{z^2+1}{z(z+\frac{1}{2})} = \frac{2\pi i}{2}\,\{\operatorname{Res}(0) + \operatorname{Res}(-\tfrac{1}{2})\}$$

$$= \pi i\left\{\lim_{z\to 0}\frac{z^2+1}{z+\frac{1}{2}} + \lim_{z\to -\frac{1}{2}}\frac{z^2+1}{z}\right\}$$

$$\frac{\frac{5}{4}}{-\frac{1}{2}} = -\frac{5}{2}$$

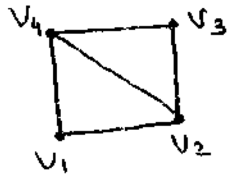
$$= \pi i\left\{2 + -\frac{5}{2}\right] = \frac{-\pi i}{2}$$

Graph Theory

Graph :- A graph G is defined by $G = (V, E)$ where

V = set of all vertices in G

E = set of all edges in G



$|V|$ = no. of ~~e~~ Vertices in G = order of G

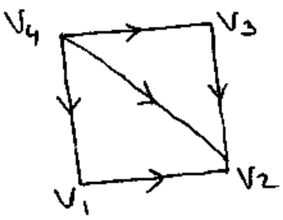
$|E|$ = no. of edges in G = size of Graph

Nondirected (Undirected) graph

- In a nondirected graph, an edge is represented by set of two two vertex

$\{V_i, V_j\}$ = An edge betn V_i & V_j

Directed Graph :- (Diagraph)



- In directed graph, an edge is represented by an ordered pair of two vertex.

(V_i, V_j) = An edge from V_i to V_j

Connected Graph :-

- A graph is said to be connected if there exist a path betn every pair of vertices.
- The graph which is not connected will have 2 or more connected components.

- In a graph two vertices are said to be adjacent if there exist a ~~path~~ edge betn the two vertices.
- Two edges are said to be adjacent if there is a common vertex for the two edges.

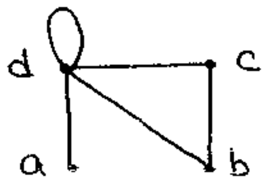
NULL GRAPH :-

- The graph with no edges is called Null graph / Empty graph

TRIVIAL GRAPH :-

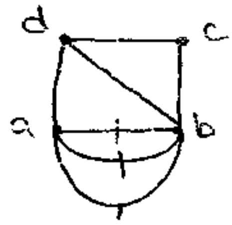
- The graph with one vertex and no edges.

Loop :- An edge is drawn from a vertex to itself.

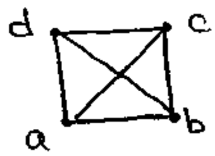


Parallel edges (Multiple edges) :-

- In a graph if a pair of vertex is allow to join by more than 1 edge then those edges are called parallel edges and resulting graph is called a Multigraph.



Simple Graph :- The graph with no loops and no parallel edges is called Simple Graph.



- Maximum no. of edges possible in a simple graph with n vertices $= \frac{n(n-1)}{2}$

$$^6C_6 + {}^6C_5 + {}^6C_4 + {}^6C_3 + {}^6C_2 + {}^6C_1 + {}^6C_0 = 2^6$$

- No. of simple graphs possible with n vertices $= 2^{n(n-1)/2}$

Ex: Max. no. of ~~edges~~ simple graphs possible ~~in a simple graph~~ with 5 vertices and 3 edges is —

$\rightarrow$ Max. no. edges $= C(5,2) = 10$ $\left\{ \because \frac{n(n-1)}{2} = \frac{10 \times 2}{2} = 10 \right\}$

Req. no. Graphs $= C(10,3)$

$= 120$

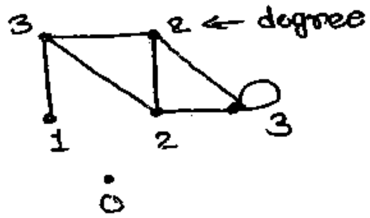
* No. of simple graphs possible with n vertices and m edges

$$= {}^{\frac{n(n-1)}{2}}C_m$$

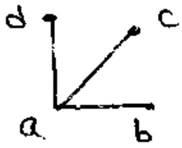
$$= C\left(\frac{n(n-1)}{2}, m\right)$$

* Degree of a vertex v = deg(v) = no. of edges incident with v

- In nondirected graph the loop at vertex is counted as two edges by writing degree of that vertex



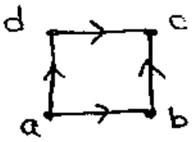
* In a simple graph with n vertices, Degree of any vertex $v = \deg(v) \leq (n-1) \ \forall \ v \in G$



simple graph

* In directed graph, indegree of a vertex is

$\deg^{+}(v)$ = no. of edges incident to the vertex

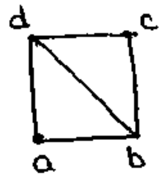


$\deg^{+}(d) = 1$

outdegree = $\deg^{-}(v)$

= no. of edges incident from the vertex

$\deg^{-}(v) = \deg^{-}(d) = 1$



* $\delta(G)$ = Minimum of the degrees of all vertices in G.

$\delta(G) = \{2, 3, 2, 3\}$

$= 2$

* $\Delta(G)$ = Max. of the degrees of all vertrices in G

$\Delta(G) = 3$

* Degree Sequence :-

- If the degrees of all vertices in G are arranged in descending order then the seq. to obtained is called Degree seq. of the G.

D.S. = $\{3, 3, 2, 2\}$

* Regular Graph :-

- If all the ventrces have same degree then the graph is called as Regular Graph.
- If the degree of each vertex is k then the graph is called as k-Regular graph.
- Every polygon is two Regular Graph.

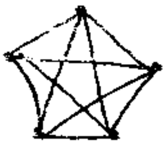
2-regular 3-regular

<u>Complete Graph</u> :- In a simple graph with n mutually adjacent vertices is called complete graph

— denoted by 'K_n'.

Ex :

K_2 K_3 K_4 K_4



K_5

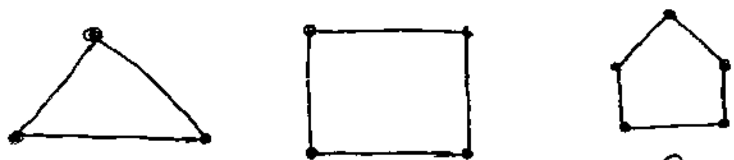
- Every complete graph is a regular graph but every regular graph need not be complete.

- A complete graph is a simple graph with max. no. of edges.

- No. of edges in $K_n = \frac{n(n-1)}{2}$

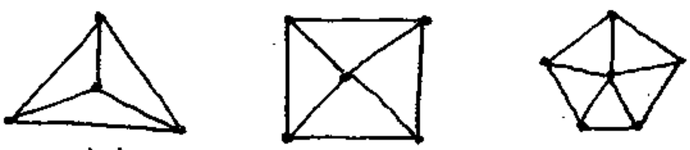
- Degree of each vertex $= (n-1)$

- <u>Cycle Graph</u> :- A simple graph G with n vertices $(n \geq 3)$ and n edges, is called a cycle graph, if all the edges from cycle of length n.



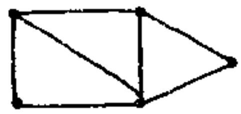
C_3 C_4 C_5

<u>Wheel Graph</u> :- A wheel graph with n vertices $(n \geq 4)$ can be formed from a cycle graph C_{n-1}, by adding a new vertex (hub) which is adjacent to all vertices of C_{n-1}



— ~~Number of edges in $W_n = 2(n-1)$~~

Cyclic Graph :- A simple graph with atleast one cycle is called a cyclic graph.



Acyclic Graph :- A simple graph with no cycles is called Acyclic graph.

Tree : A connected graph with no cycle is called a tree

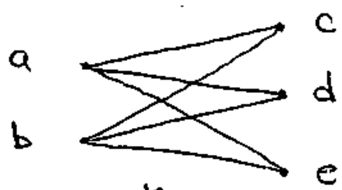
* A tree with n vertices has $(n-1)$ edges.
* Every tree has atleast two vertrices of degree 1.

Bipartite Graph :- A simple graph $G = (V, E)$ with vertex partition $\{V_1, V_2\}$ is said to be a Biparotite graph, if every ~~vertex in~~ edge of G join a vertex in V_1 to a vertex in V_2.

$V = \{a, b, c, d, e\}$

Complete Bipartite Graph :- A bipartite graph $G = \{V, E\}$ with vertex partition $\{V_1, V_2\}$ is said to be complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2.

— If $|V_1| = m$ and $|V_2| = n$ then a complete bipartite graph from V_1 to V_2 is denoted by $K_{m,n}$



$K_{2,3}$

Note :- In general, A complete biparotite graph is not a complete graph.

— $K_{m,n}$ is a complete graph $\Leftrightarrow m = n = 1$

$K_{1,1}$ & K_2 (complete graph as well as complete bipartite graph)

— $K_{m,n}$ has $(m+n)$ vertices and mn edges.

— Max. no. of edges possible in bipartite graph with n vertices is

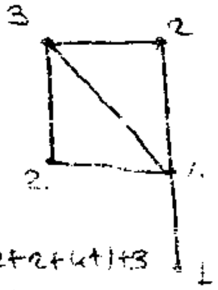
$n = 9$

$K_{5,4} \rightarrow 20$ $\qquad \left\lfloor \frac{81}{4} \right\rfloor$

$K_{6,3} \rightarrow 18$ $\qquad = \underline{20}$

Sum of degrees theorem :-

Let $G = (V, E)$ be a non directed graph with $V = \{v_1, v_2, \ldots v_n\}$ then $\sum_{i=1}^{n} \deg(v_i) = 2 \cdot |E|$



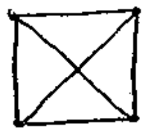
$\deg(v) = 2 \times 6$
$= 12$

$= 2+2+4+1+3$
$= 12$

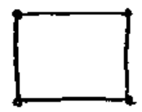
Corollary 1 : Let $G = (V, E)$ be a directed graph with $V = \{v_1, v_2 \ldots v_n\}$ then

$$\sum_{i=1}^{n} \deg^{+}(v_i) = \sum_{i=1}^{n} \deg^{-}(v_i) = |E|$$

Corollary 2 : In a nondirected graph, no. of vertices with odd degree is always even no. of vertices.



no. of vertices with odd degree = 4 (Even)



no. of vertices with odd degree = 0 (even)

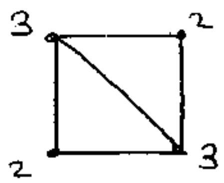
Corollary 3 : If G is a nondirected graph with degree of each vertex k, then $k \cdot |V| = 2|E|$

Corollary 4 : If G is a non directional graph with degree of each vertex atleast k ($\geq k$) then

$k \cdot |V| \leq 2|E|$

$\therefore 2 \times 4 \leq 2 \times 5$

$8 \leq 10$

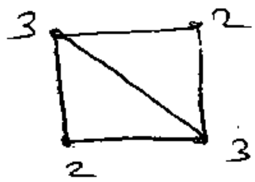


$k = 2$

Corollary 5 : If G is a non directed graph with degree of each vertex atmost k ($\leq k$) then

$k \cdot |V| \geq 2|E|$

$3 \cdot 4 \geq 2 \times 5$



$k = 3$

Note :- for a nondirected graph G

$$\delta(G)\cdot|V| \le 2|E| \le \Delta(G)|V|$$

↑ Min·degree ↑ Max·degree.

Ex: A simple non directed graph G contains 21 edges, 3 vertices of degree 4, and other vertices are of degree 2. then $|V| = ?$

→ $\sum_{i=1}^{n} \deg(v_i) = 2|E| = 2\times 21 = 42$

$$\Longrightarrow 3\times 4 + (x-3)\times 2 = 42$$

$$2x-6 = 30$$

$$2x = 36$$

$$\boxed{x = 18}$$

∴ no. of vertices $= |V| = 18$

Ex: A simple non directed graph G has 34 edges and degrees of each vertex is 4 then find the $|V| = ?$

→ $\sum_{i=1}^{n} \deg(v_i) = 2\times 34$

$$4\times|V| = 68$$

$$|V| = \frac{68}{4}$$

$$\boxed{|V| = 17}$$

Ex: A simple non directed graph G has 24 edges and degree of each vertex is k then $k>1$? which of the following is possible no. of vertices ?

ⓐ 20 ⓑ 15 ⓒ 10 ⓓ 8

→ By ~~cello~~ corollary 3

$$k\cdot|V| = 2|E|$$

$$|V| = \frac{2\times 24}{k} \quad (k = 1, 2, 3, 4, 6)$$

$$= \frac{48}{} = \frac{48}{6} = 8 \quad (48, 24, 16, 12, 8) \text{ in options}$$

Ex : Maximum no. of vertices possible in a simple non directed graph with 35 edges and degree of each vertex is atleast 3 is —

→ By cor. 4

$k \cdot |V| \leq 2|E|$

$\Rightarrow 3 \cdot |V| \leq 2 \cdot 35$

$|V| \leq 23.33$

$|V| \leq 23$

Ex : Min. no. of vertices of edges necessary in a simple non-directed graph, with 25 vertices and degree of each vertex is atleast 4 is —

→ By cor. 4

$k \cdot |V| \leq 2|E|$

$4|25| \leq 2|E|$

$100 \leq 2|E|$

$50 \leq |E|$

$\therefore \boxed{E = 50}$

For $k = 5$ (atleast 5)

$k|V| \leq 2|E|$

$5|25| \leq 2|E|$

$|E| \geq 62.5$

$\boxed{|E| \geq 63}$

Ex : Min. no. of vertices necessary in a simple non directed graph with 13 edges and degree of each vertex is atmost 3 is —

≤ 3

→ By cor. 5

$k \cdot |V| \geq 2|E|$

$\Rightarrow 3 \cdot |V| \geq 2|13|$

$3 \cdot |V| \geq 26$

$|V| \geq \frac{26}{3} \Rightarrow |V| \geq 8.66$

$\boxed{|V| \geq 9}$

Ex: Which of the following degree sequences represent a simple non directed graph?

a) $\{2, \underset{\uparrow}{3}, \underset{\uparrow}{3}, 4, 4, \underset{\uparrow}{5}\}$ $\longrightarrow$ sum of degrees = 21 $\neq$ Even

(OR)

no. of vertices with odd degrees = 3 (not even)

$\therefore$ not represent a simple non-directed graph.

b) $\{2, 3, 4, 4, 5\}$ $\rightarrow$ sum of degrees = 18 = even

and no. of vertices with odd degrees = 2 = Even

But it having 5 vertices and degree of Each vertex must be ≤ 4.

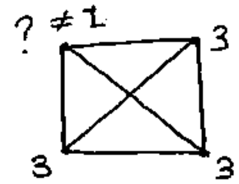
$\therefore$ It is not simple non directed graph.

c) $\{3, 3, 3, 1\}$ $\longrightarrow$ sum of degrees = 10 = Even

and no. of vertices with odd degree = 4 = even

But if 3 vertices having 3 degree then remaining vertex cannot have degree 1

$\therefore$ It is not simple graph.



d) $\{1, 3, 3, 4, 5, 6, 6\}$ $\rightarrow$ sum of degree = 28

and no. of vertices with odd degree = 4

But it cannot represent simple non directed graph because in simple graph with 7 vertex it have two vertices with degree 6 then vertex with degree 1 is not possible.

e) $\{0, 1, 2, ---\ n-1\}$

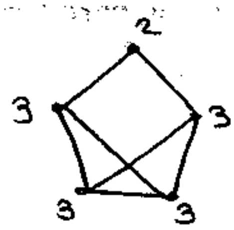
$\underbrace{v_1, v_2, v_3 ---\ v_n}$

Simple graph with n vertices can not represent simple graph because a simple graph with n vertices we have vertex with degree $n-1$ then vertex with degree 0 is not possible.

* In a simple graph with ~~$(n \geq 2)$~~ n vertices at least two vertices should have same degree.

f) $\{2, 3, 3, 3, 3\}$
   a b c d e

→ Simple Graph.



Havel - Hakimi result

Consider the degree sequences S_1 and S_2, and assume that S_1 is in descending order.

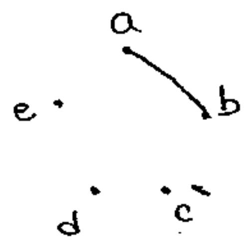
S_1) $\{s, t_1, t_2, \ldots t_s, d_1, d_2, \ldots, d_n\}$

S_2) $\{t_1 - 1, t_2 - 1, \ldots, t_s - 1, d_1, d_2, \ldots d_n\}$

S_1 is graphic $\Leftrightarrow$ S_2 is graphic

Which of the following degree sequences represent a simple non directed graph?

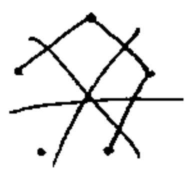
S_1) $\{6, \underline{6, 6, 6, 4, 3, 3}, 0\}$
$\{5, \underline{5, 5, 3, 2, 2}, 0\}$
$\{4, \underline{4, 2, 1, 1}, 0\}$
$\{3, \underline{1, 0, 0}, 0\}$
a b c d e



The last seq. can not be represented by simple graph.

∴ Given seq. also can not be represented by a simple graph.

S_2) $\{6, \underline{5, 5, 4, 3, 3, 2}, 2, 2\}$
$\{4, \underline{4, 3, 2, 2}, 1, 2, 2\}$
$\{3, \underline{2, 1, 1}, 1, 2, 2\}$
$\{1, 0, 0, 1, 2, 2\}$



$= \{4, 4, 3, 2, 2, 1, 2, 2\}$
$= \{4, \underline{4, 3, 2, 2}, 2, 2, 1\}$) Reordering the Elements.
$= \{3, 2, 1, 1, 2, 2, 1\}$
$= \{3, \underline{2, 2, 2}, 1, 1, 1\}$
$\{1, 1, 1, 1, 1, 1\}$ →

$= \{1,1,1,1,0\}$

$= \{0,1,1,0\}$

$= \{1,1,0,0\}$

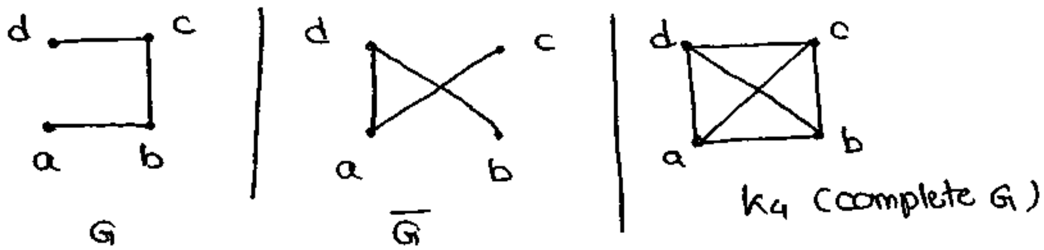
$= \{0,0,0\}$

$\therefore$ Given Graph is simple.

<u>Complement of a graph</u> :-

Let G be a simple non directed graph with n vertices. complement of G, denoted by $\overline{G}$ is a simple non directed graph with same vertices as that of G, and an edge $\{u,v\} \in \overline{G}$ iff $\{u,v\} \notin G$

Ex :



$G \cup \overline{G} = K_n$ — where n = no. of vertices.

$*$ $|E(G)| + |E(\overline{G})| = |E(K_n)|$ — where $n = |V(G)|$

Ex : A simple graph G has 10 vertices and 21 edges find no. of edges in $\overline{G}$ $(|E(\overline{G})|)$.

$\rightarrow$ We have

$$|E(G)| + |E(\overline{G})| = |E(K_n)|$$

$$21 + |E(\overline{G})| = \frac{n(n-1)}{2} = \frac{10 \times 9}{2} = 45$$

$$|E(\overline{G})| = 45 - 21$$

$$|E(\overline{G})| = 24$$

Ex : A simple graph G has 30 edges and $\overline{G}$ has ~~25~~ 36 edges then no. of vertices in G = ?

$\rightarrow$ $|E(G)| + |E(\overline{G})| = |E(K_n)|$

$$30 + 36 = \frac{n(n-1)}{2}$$

$*$ $n(n-1) = (12)(12-1)$

$\therefore \boxed{n = 12}$

Isomorphic Graphs :- Two Graphs G_1 & G_2 are said to be isomorphic, if there exists a function $f : V(G_1) \to V(G_2)$ such that i) f is a bijection and

ii) f preserves adjacency

i.e. if $\{u, v\} \in G_1$ then $\{f(u), f(v)\} \in G_2$ then $G_1 \equiv G_2$

Note :- If $G_1 \equiv G_2$, then

i) $|V(G_1)| = |V(G_2)|$

ii) $|E(G_1)| = |E(G_2)|$

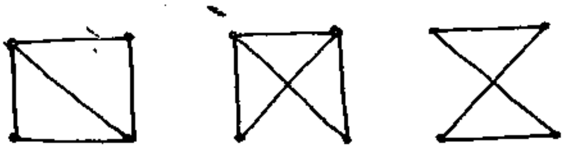
iii) Degree sequences of G_1 & G_2 are same.

iv) If $\{v_1, v_2, \ldots v_k\}$ form a cycle in G_1, then $\{f(v_1)_1, f(v_2)_2, f(v_3)_2 \ldots f(v_k)_1\}$ should form a cycle in G_2.

Result : $(G_1 \equiv G_2)$ iff $\overline{G_1} \equiv \overline{G_2}$

- $(G_1 \equiv G_2)$ if the adjacency matrices of G_1 and G_2 are same
- $(G_1 \equiv G_2)$ iff the corresponding subgraphs (obtained by deleting a vertex in G_1 and it's image in G_2) are isomorphic.

Ex : Which of the following graphs are isomorphic ?



G_1 $\quad$ G_2 $\quad$ G_3

→ Graph G_3 has only 4 edges $\therefore$ G_3 is not isomorphic to G_1 / G_2.

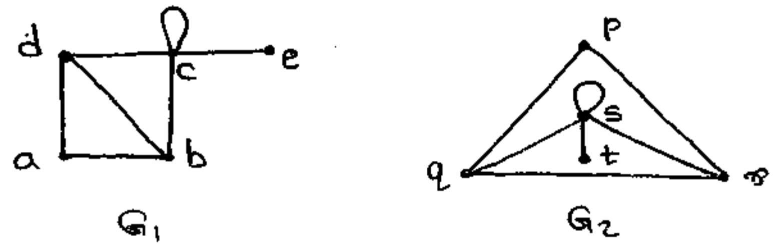
Now for G_1 & G_2

$\overline{G_1}$ $\quad$ $\overline{G_2}$

$\overline{G_1} \equiv \overline{G_2}$

$\therefore G_1 \equiv G_2$

Ex : ~~Find whether the following graphs are isomorphic ?~~

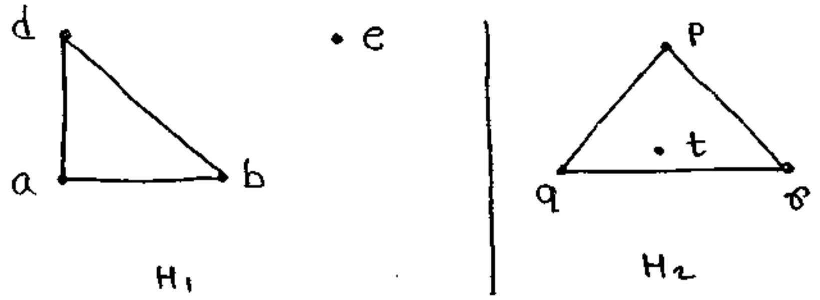


→ If $G_1 \equiv G_2$,

Image of $c = s$

Further, c and s have similar neighbours.

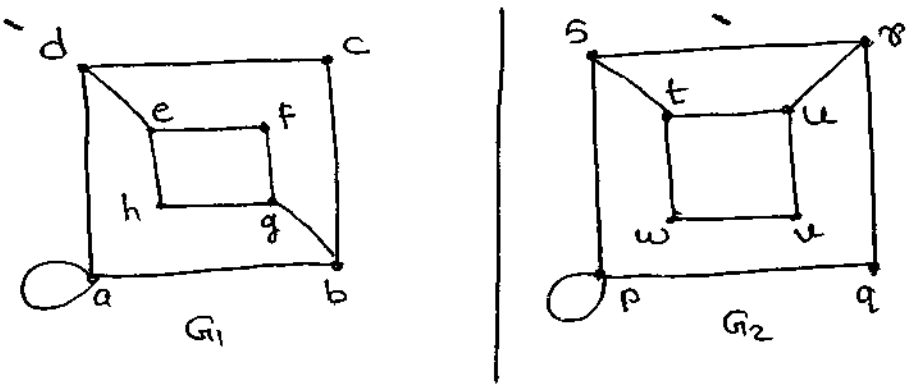
Deleting c and s from G_1 & G_2 we get,



Here, $H_1 \equiv H_2$

$\therefore G_1 \equiv G_2$

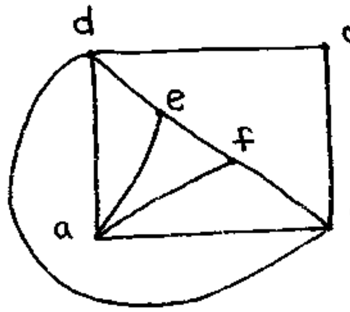
Ex: Find wheather the following graphs are isomorphic ?

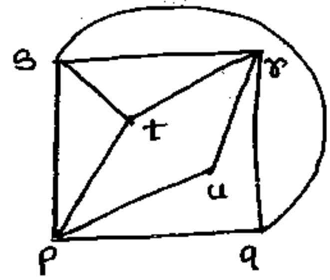


→ In G_2 all vertices form the cycle C_8 where as in G_1 there is no such cycle

$\therefore G_1 \not\equiv G_2$

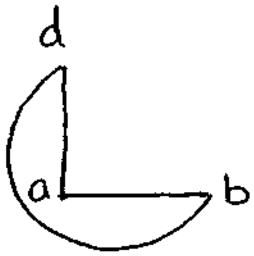
Ex. Find the another following graphs are isomorphic ?





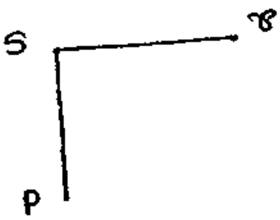
G_1

→ consider vertices with degree 4



G_1

(Form cycle)



G_2

(Not cycle)

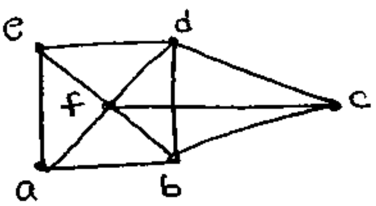
Comparing vertices of degree 4 we have in G_1 vertices form a cycle but in G_2 not.

$\therefore G_1 \not\equiv G_2$

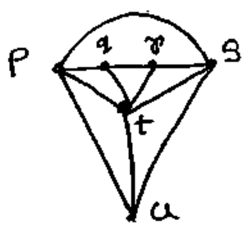
Comparing vertices of degree 3 we have in G_1 two vertices of degree 3 are adjacent whereas in G_2 are not.

$\therefore G_1 \not\equiv G_2$

Ex: Find wheather the following graphs are isomorphic ?



G_1

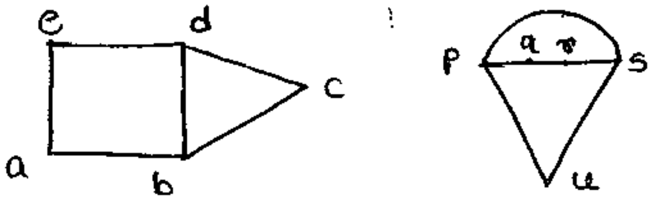


G_2

→ $G_1 \equiv G_2$ since Image of $f = t$

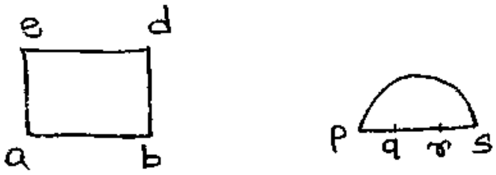
The vertices f & t has similar neighbours

Deleting f & t from corresponding graphs we get following,



$G_{11} \equiv G_{21}$

Image of $c = u$ $\therefore$ Deleting c & u we get



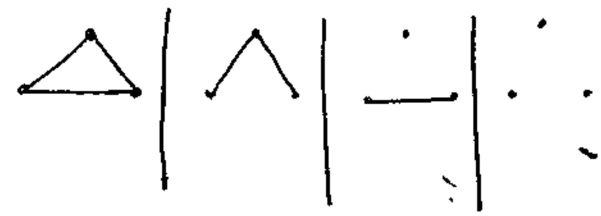
$G_{111} \equiv G_{211}$

$\therefore$ $G_1 \equiv G_2$

Ex : How many simple non isomorphic graphs (pairwise) are possible with 3 vertices?

a) 2 b) 3 c) 4 d) 6

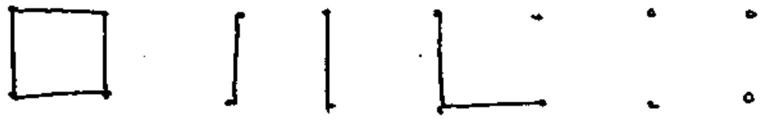
$\rightarrow$



Ex : How many simple non isomorphic graphs (pairwise) are possible with 4 vertices and 2 edges?

a) 2 b) 3 c) 4 d) 6

$\rightarrow$



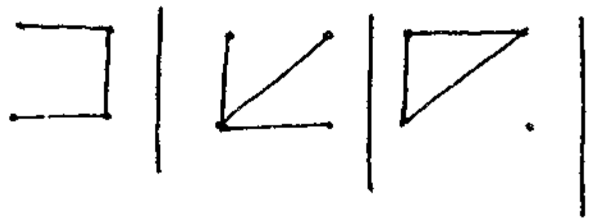
(2-edges) (2-edges)

Two edges may be adjacent or non adjacent.

Ex : —"— with 4 vertices and 3 edges ?

a) 2 b) 3 c) 4 d) 6

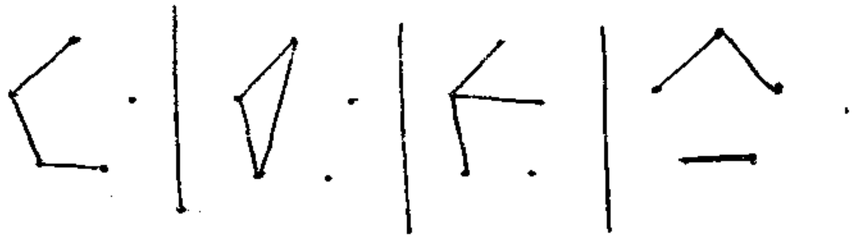
→



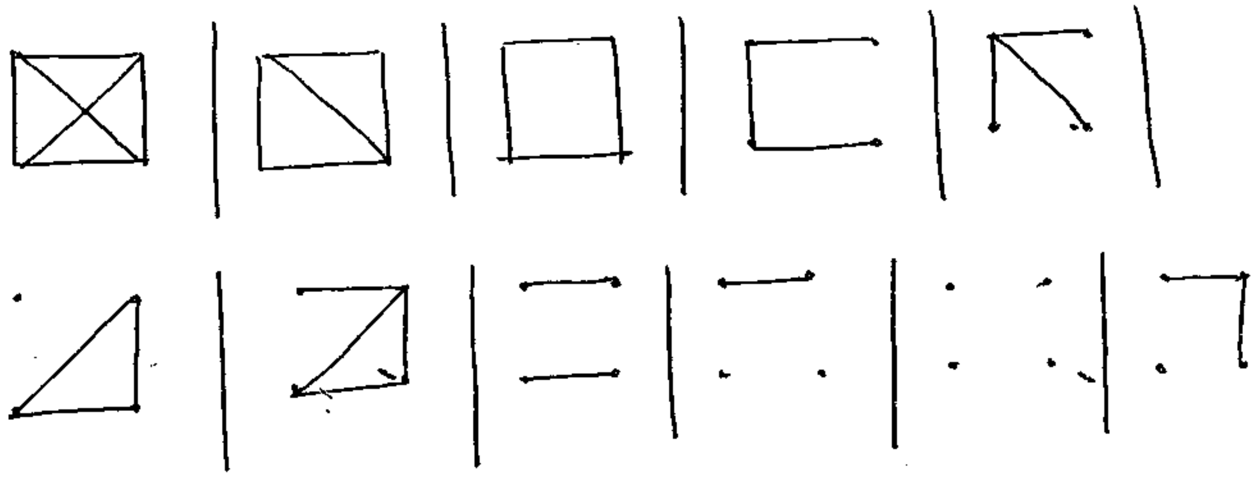
Ex : —"— with 5 vertices and 3 edges ?

a) 2 b) 3 c) 4 d) 6

→



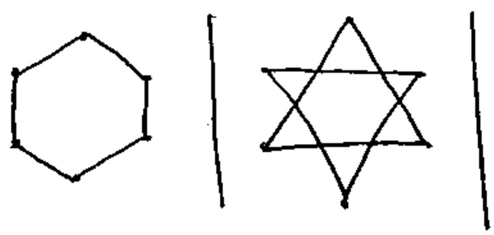
Ex : —"— 4 vertices ?



⇒ Ans = 11

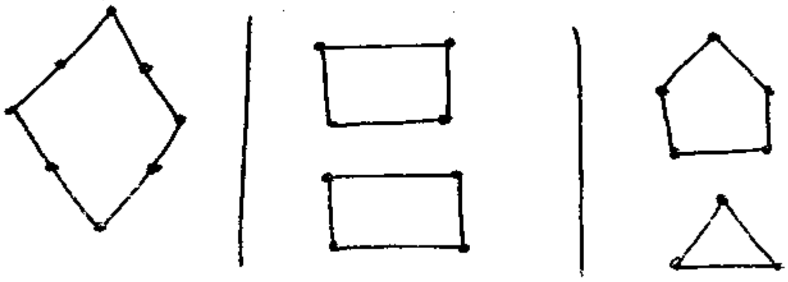
Ex : How many simple non isomorphic graphs (pairwise) are possible with 6 vertices , 6 edges and degree of each vertex is 2 ?

→



Ans : 2

Ex : ——||—— with 8 vertices , 8 edges and degree is 2 ?

→ 

Ans = 3

Ex . ——||—— with 10 vertices , 10 edges and degree is 2 ?

→

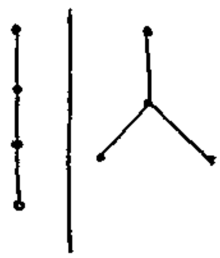
C_{10}	C_5	C_6	C_7	C_4
	C_5	C_4	C_3	C_3
				C_3

Ans = 5

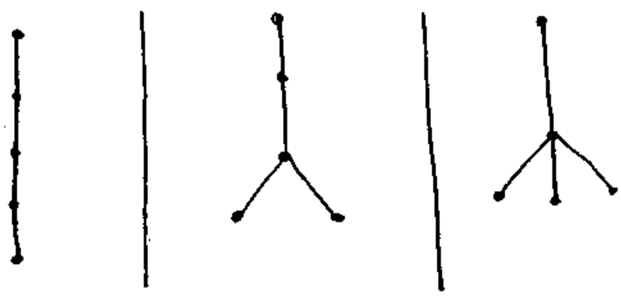
Ex : How many simple non isomorphic trees (pairwise) are possible with 3 vertices ?

→  Ans = 1

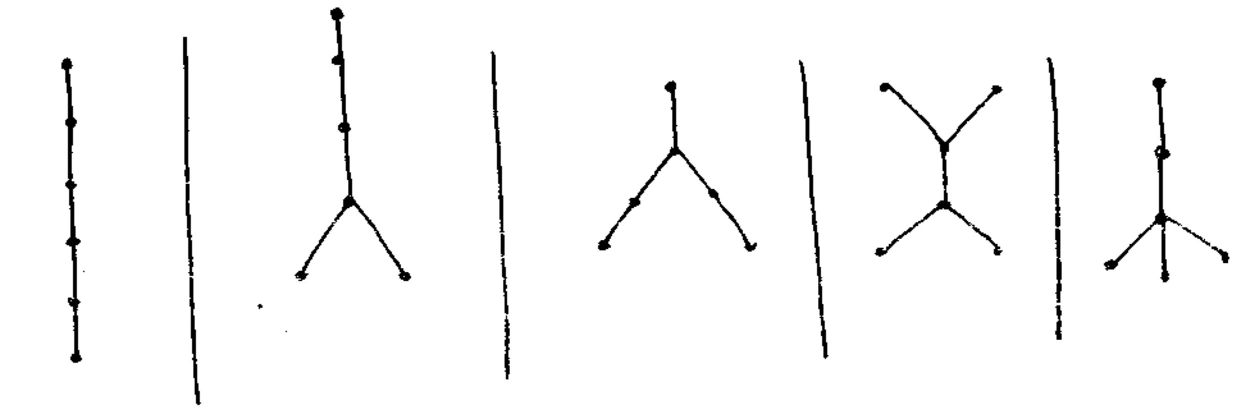
Ex : ——||—— with 4 vertices ?

→  Ans : 2

Ex : ——||—— with 5 vertices ?

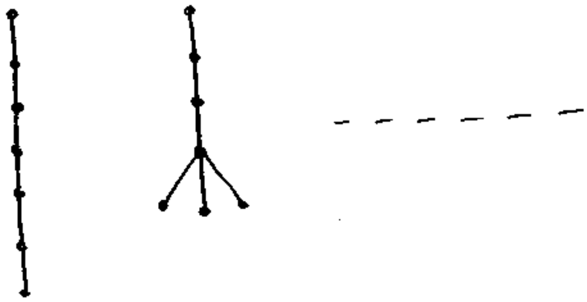
→  Ans = 3

Ex : —— 11 —— with 6 vertices ?

→ 

Ans = 6

Ex : —— 11 —— with 7 vertices ? Ans = 12

→ 

* If a simple graph G is isomorphic to $\overline{G}$ then

i) $|E(G)| = \dfrac{n(n-1)}{4}$ where n = no. of vertices in a graph

$$|E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2}$$

$$\therefore\ 2|E(G)| = \frac{n(n-1)}{2} \qquad (\because |E(G)| = |E(\overline{G})|)$$

$$\therefore\ |E(G)| = \frac{n(n-1)}{4}$$

ii) $|V(G)| = 4k$ or $(4k+1)$ $\qquad k = 1, 2, 3 ----$

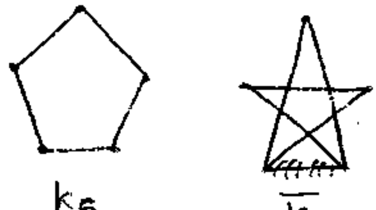
Ex: If $C_n \equiv \overline{C_n}$ then $n = ?$

$\rightarrow$ By the prev. result no. of edges in $C_n = \dfrac{n(n-1)}{4}$

$$|E(C_n)| = \frac{n(n-1)}{4}$$

$$n = \frac{n(n-1)}{4}$$

$$\boxed{n = 5}$$



K_5 $\overline{K_5}$

Ex: If $G \equiv \overline{G}$ then which of the following is not true?

a) $|V(G)| = 20$ b) $|V(G)| = 21$

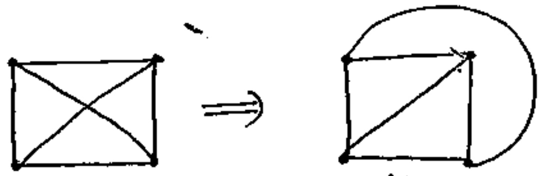
c) $|V(G)| = 25$ d) $|V(G)| = 26$

$\Rightarrow$ $|V| = 4k$ or $4k+1$

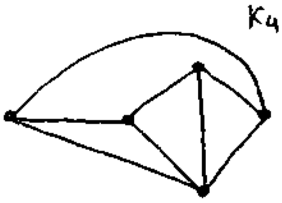
Planer Graphs

A graph G is said to be planer, if it can be drawn on a plane so that no two edges cross each other (at a non vertex point)

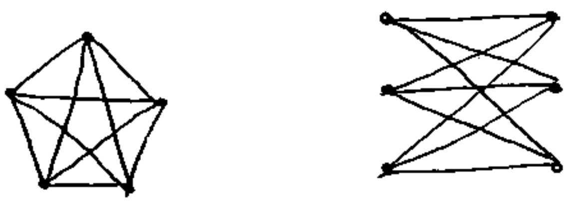
Ex:



K_4 $\Longrightarrow$ K_4 (Planer graph)

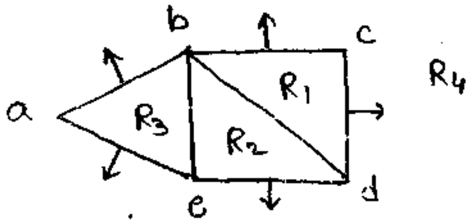


(Planer graph)



K_5 $K_{3,3}$ (non planer graphs)

Regions : Every planar graph divide the plane into connected areas called Regions of the plane.

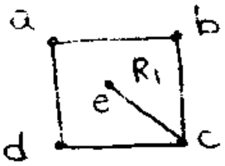


$\deg(R_1) = \deg(R_2) = \deg(R_3) = 3$

$\deg(R_4) = 5$

Degree of an interior region $r = \deg(r)$
= no. of edges enclosing the region r

Degree of exterior region $r = \deg(r)$
= no. of edges exposed to the region r.



$\deg(R_1) = 6$

Properties of Planar Graph :-

1) $\sum_{i=1}^{n} \deg(v_i) = 2|E|$

2) **sum of degrees of regions theorem :-**

→ If P is a planar graph with n regions

$$\sum_{i=1}^{n} \deg(r_i) = 2|E|$$

2.1) In a planar graph if degree of each region is K then
$K \cdot |R| = 2|E|$

2.2) In a planar graph if degree of each region is $\geq K$ (atleast K)
then $K \cdot |R| \leq 2 \cdot |E|$

2.3) In a simple planar graph degree of each region ≥ 3 then

$3 \cdot |R| \leq 2 \cdot |E|$

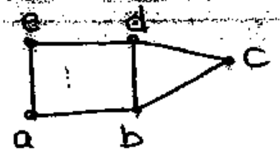
2.4) In a planar graph degree of each region $\leq K$ (atmost K)
then $K \cdot |R| \geq 2|E|$

3) **Euler's formula :-**

If G is a connected planar graph, then

$|V| + |R| = |E| + 2$

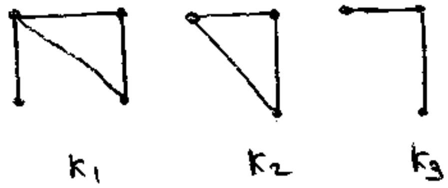
$5+3 = 6+2$



— If G is planar graph with k components

$|V| + |R| = |E| + (K+1)$

$10+3 = 9 + (3+1)$



4) If G is a simple connected planar graph, then

i) $|E| \leq [3|V| - 6]$

ii) $|R| \leq \{2|V| - 4\}$

iii) There exists atleast one vertex $v \in G$ such that $\deg(v) \leq 5$

5) <u>Polyhedral graph</u> : A planar graph in which every interior region is polygon is called polyhedral graph.

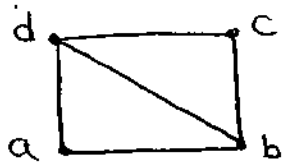
— In a polyhedral graph degree of every vertex i.e. $\deg(V) \geq 3$ $\forall\, v \in G$

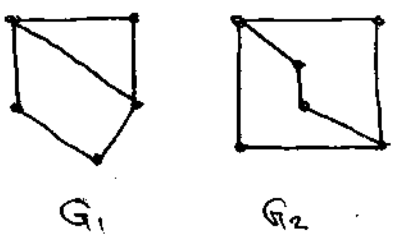
— For a polyhedral graph the following inequality must be holds :-

i) $3|V| \leq 2|E|$ and ii) $3|R| \leq 2|E|$

6) <u>Kuratowski's theorem</u> :- A graph G is non planar iff G has a subgraph is homeomorphic to K_5 or $K_{3,3}$.

<u>Homeomorphic</u> : Two graph G_1 & G_2 are said to be homeomorphic to each other if each of these graphs can be obtained from a graph G by dividing some edges of G with more vertices.





Note :- 1) Any graph with 4 or fewer vertices is planar.

2) Any graph with 8 or fewer edges is planar.

3) A non planar graph minimum no. of vertices is K_5

4) A non planar graph with min. no. of edges is $K_{3,3}$

5) A complete graph K_n is planar iff $n \le 4$

6) $K_{m,n}$ is planar. iff ($m \le 2$ or $n \le 2$)

- If two graphs are isomorphic then they are homomorphic.

Ex. Let G be a connected planar graph with 25 vertices and 60 edges. No. of regions in G is __

→ By Euler's Formula,

$$|V| + |R| = |E| + 2$$

$$\Rightarrow 25 + |R| = 60 + 2$$

$$\Rightarrow |R| = 37$$

Ex: Let G be a planar graph with 10 vertices, 3 components and 9 edges then $|R| = ?$

→

$$|V| + |R| = |E| + (K+1)$$

$$\Rightarrow 10 + |R| = 9 + (3+1)$$

$$\Rightarrow |R| = 3$$

Ex: Let G be a connected planar graph with 20 vertices, degree of each vertex is 3 then $|R| = ?$

→ By sum of vertices of degrees theorm

$$\Sigma \deg(v_i) = 2|E|$$

$$20 \times 3 = 2|E|$$

$$\boxed{|E| = 30}$$

$$|V| + |R| = |E| + 2$$

$$|R| = 32 - 20$$

$$\boxed{|R| = 12}$$

Ex : Let G be a connected planar graph with 35 regions, degree of each region is 6 then |V| = ?

→
$$\sum_{i=1}^{35} \deg(r_i) = 2|E|$$
$$35 \times 6 = 2|E|$$
$$|E| = 105$$
$$\therefore |V| + |R| = |E| + 2$$
$$|V| = 107 - 35$$
$$\boxed{|V| = 72}$$

Ex : Let G be a connected planar graph with 12 vertices & 30 edges and degree of each region is k Then k = ?

→
$$|V| + |R| = |E| + 2$$
$$12 + |R| = 32$$
$$|R| = 20$$

By sum of degrees of regions theorm
$$20 \cdot k = 2|E|$$
$$k = \frac{60}{20}$$
$$\boxed{k = 3}$$

Ex : Max. no. of regions possible in a simple planar graph with 10 edges is_

→ By theorm 2.3
$$3|R| \leq 2|E|$$
$$3|R| \leq 20$$
$$|R| \leq \frac{20}{3}$$
$$|R| \leq 6.66$$
$$\Rightarrow |R| \leq 6$$

Ex : Min no. of edges necessary in a simple planar graph with 15 regions is —

→ By theorm. 2.3

$$3|R| \leq 2|E|$$

$$3|15| \leq 2|E|$$

$$|E| \geq 22.5$$

$$\Rightarrow |E| \geq 23$$

Ex : Max. no. of edges possible in a simple connected planar graph with 8 vertices is —

→ By Theorm 4.1

$$|E| \leq \{3|V| - 6\}$$

$$3 \cdot 8 - 6$$

$$|E| \leq 18$$

Ex : Max. no of regions possible in simple planar connected graph with 13 vertices is —

→ By theorm 4.2

$$|R| \leq \{2|V| - 4\}$$

$$= \{2 \times 13 - 4$$

$$\Longrightarrow |R| \leq 22$$

Ex : If G is simple connected planar graph, The $\delta(G)$ cannot be equal to —

Min. degree of all vertices

→ a) 3 b) 4 c) 5 d) 6

By theorm 4.3

$V \in G$ such that

$\deg(V) \leq 5$

$\Rightarrow \delta(G) \leq 5$

$\Rightarrow \delta(G) \neq 6$

Ex: which of the following is /are true ?

S1) A polyhedral graph with 30 edges and 11 regions does not exists.

→ By Euler's formula

$$|V| + |R| = |E| + 2$$

$$|V| + 11 = 30 + 2$$

$$|V| = 21$$

from polyhedral graph

$3|V| \leq 2|E|$ and $3|R| \leq 2|E|$

$\Rightarrow$ $3 \times 21 \leq 2(30)$ $3 \times 11 \leq 2 \times 30$

$63 \leq 60$ $33 \leq 60$

$\therefore$ True.

S2) A polyhedral graph with 7 edges does not exist

→ $3|V| \leq 2|E|$ and $3|R| \leq 2|E|$

$|V| \leq 2 \times 7/3$ $|R| \leq 4.66$

$|V| \leq 4.66$ $|R| \leq 4$

$|V| \leq 4$

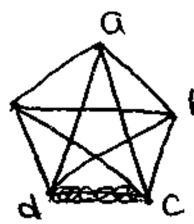
$\therefore$ By Euler's formula $\Rightarrow$ $|V| + |R| = |E| + 2$

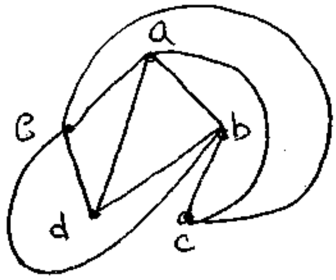
$4 + 4 = 7 + 2$

$8 \neq 9$ $(8 \ngtr 9)$

$\therefore$ True

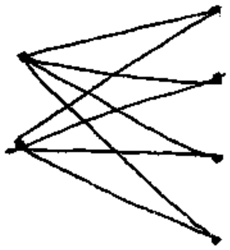
Ex: Which of the following is not a planar graph ?

a) 

$\Longrightarrow$ 

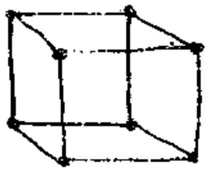
Planar Graph

b)

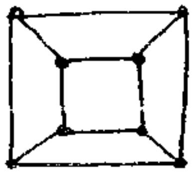


Planar

c)

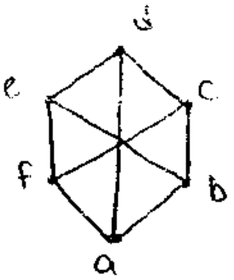


$\equiv$



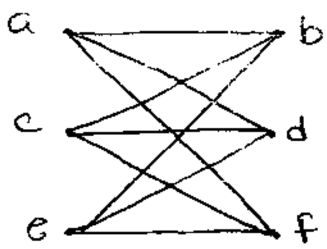
planar

d)



non planar

$\equiv$



$K_{3,3}$

Ex: G is a non planar graph with minimum no. of vertices then

a) G has 6 vertices and 8 edges
b) G has 5 —"— and 9 —"—
c) G has 6 —"— and 9 —"—
d) G has 5 —"— and 10 —"— (K_5)

Ex: Let G be a non planar graph with minimum no. of edges then —

a) G has 6 vertices and 8 edges
b) G has 5 —"— and 9 —"—
c) G has 6 —"— and 9 —"— ($K_{3,3}$)
d) G has 5 —"— and 10 —"—

Graph Coloring :-

Vertex coloring :- An assignment of colors to the vertices of a graph G, so that no two adjacent vertices have same color is called vertex coloring.

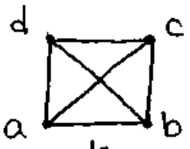
Chromatic Number : Min. no. of colors required for vertex coloring of a graph G is called as chromatic number and denoted by

$G = \chi(G)$

* $\chi(G) = 1$ iff G is a null graph.
* If G is not a null graph, Then $\chi(G) \geq 2$
* A graph G is said to be n-colorable, if there exist a vertex coloring that uses atmost n colors i.e. $\chi(G) \leq n$
* Four color theorm
  — Every planar graph G is 4-colorable i.e. $\chi(G) \leq 4$
* Welch - Powel's algorithm
  1) Arrange vertices of G in the descending order of their degrees.
  2) If 2 or more vertices of same degree, arrange those vertices in Alphabetical or numerical order. ~~so that~~
  3) Assign colors to the vertices in that order so that no 2 adjacent vertices have same color.

Ex : Chromatic number of the complete graph K_n is —

a) n b) $n-1$ c) $\lfloor n/2 \rfloor$ d) $\lceil n/2 \rceil$

→ 

K_4

In K_n each vertex is adjacent to remaining $(n-1)$ vertices

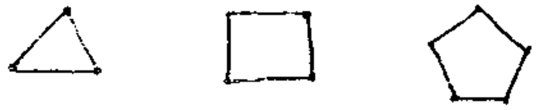
∴ Each vertex requires a new color.

$\therefore \chi(K_n) = n$

Ex : chromatic no. of the graph C_n is ___
cycle

a) 2 b) 3 ~~c)~~ $n - 2\lfloor n/2 \rfloor + 2$ d) $n - 2\lceil n/2 \rceil + 1$

→



If n is odd then 3
If n is even then 2
} verify option for even & odd

Ex : Chromatic no. of the wheel graph W_n $(n \geq 4)$ is ___

→

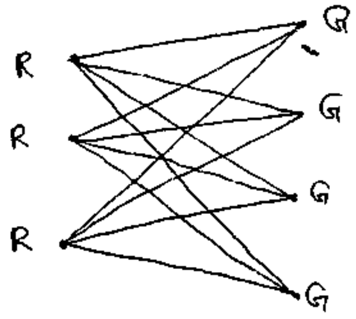
W_7 W_4

If n is even 4
If n is odd 3

Ex : chromatic no. of the complete bipartite graph $K_{m,n}$ is ___

~~a)~~ 2 b) 4 c) min (m,n) d) max of {m,n}

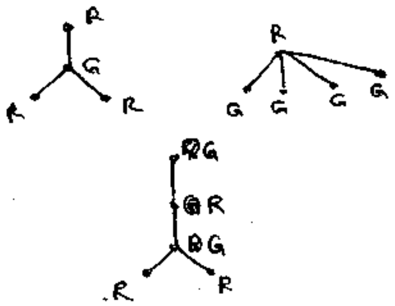
→



Ex : chromatic no. of a tree T with n vertices $(n \geq 2)$ is ___

~~a)~~ 2 b) 4 c) $\lfloor n/2 \rfloor$ d) $\lceil n/2 \rceil$

→



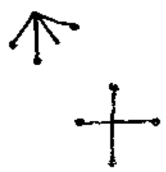
Every tree is bipartite graph

$\therefore \chi(n) = 2$

Ex : chromatic no. of the star graph with n vertices $(n \geq 2)$ is —

a) 2 b) 4 c) $\lfloor n/2 \rfloor$ d) $\lceil n/2 \rceil$

→

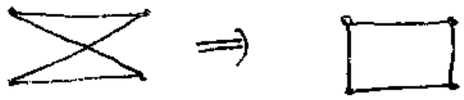


Star graph is special case of bipartite graph.

$\therefore \chi(n) = 2$ $K_{1,n-1}$

Ex : In a graph G if all the cycles are of even length then $\chi(n) = ?$

→ when cycle of even length then it is bipartite graph.

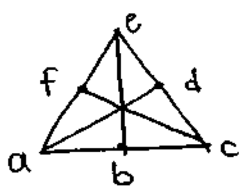


$K_{2,2}$ $\Rightarrow$ C_4

$\chi(n) = 2$

Ex : chromatic no. of the graph shown below is —

a) 2 b) 3 c) 4 d) 5



vertex	a	b	c	d	e	f
color	C_1	C_2	C_1	C_2	C_1	C_2

$\chi(G) \leq 2$ — ①

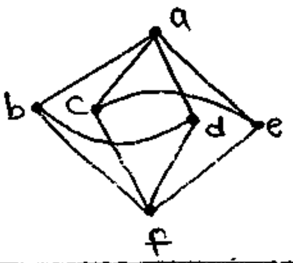
As G is not null graph

$\therefore \chi(G) \geq 2$

from ① & ②

$\chi(G) = 2$

Ex : chromatic no. of the graph shown below is



a) 2 ✓b) 3 c) 4 d) 5

vertex	a	b	c	d	e	f
color	c_1	c_2	c_2	c_3	c_3	c_1

$\chi(G) \leq 3$ —— ①

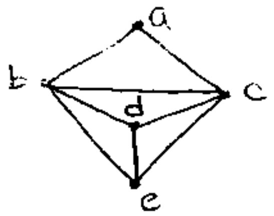
further, we have 3 mutually adjacent vertices $\{a, b, d\}$

$\therefore \chi(G) \geq 3$ —— ②

From ① & ②

$\chi(G) = 3$

Ex : chromatic no. of the graph given below is ——



a) 2

b) 3

✓c) 4

d) 5

vertex	a	b	c	d	e
color	c_1	c_2	c_3	c_1	c_4

Here graph is planar therefore by 4 color theorem chromatic no. $\chi(G) \leq 4$ —— ①

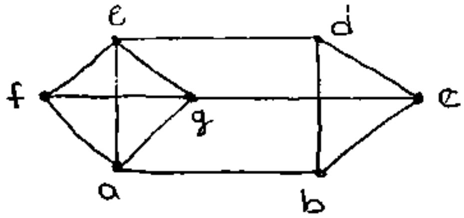
Further we have 4 mutually adjacent vertices $\{b, c, d, e\}$

$\therefore \chi(G) \geq 4$ —— ②

from ① & ②

$\chi(G) = 4$

Ex :



(Planar Graph)

vertex	a	b	c	d	e	F	g
color	c_1	c_2	c_1	c_4	c_2	c_3	c_4

a) 2

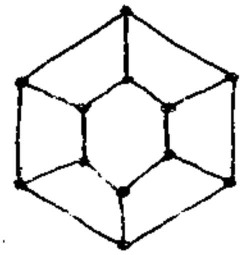
b) 3

✓c) 4

d) 5

Ex : chromatic no. of the graph shown below is —

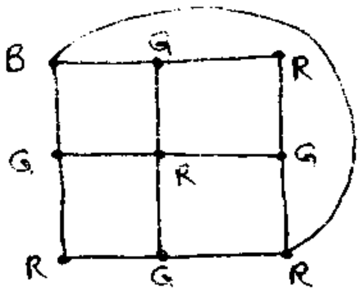
~~a)~~ 2 b) 3 c) 4 d) 5



In the graph all cycles are of Even vertices

$\therefore \chi(G) = 2$

Ex :



a) 2

~~b)~~ 3

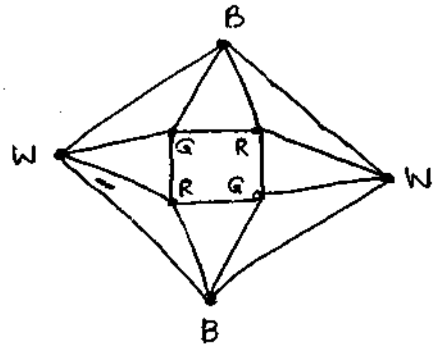
c) 4

d) 5

The graph we have a cycle of odd length

$\therefore \chi(G) \geq 3$

Ex : chromatic no. of graph shown Below is —



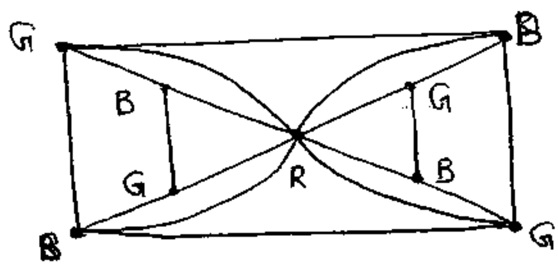
a) 2

b) 3

~~c)~~ 4

d) 5

Ex : chromatic no. of the graph shown below is —



a) 2

~~b)~~ 3

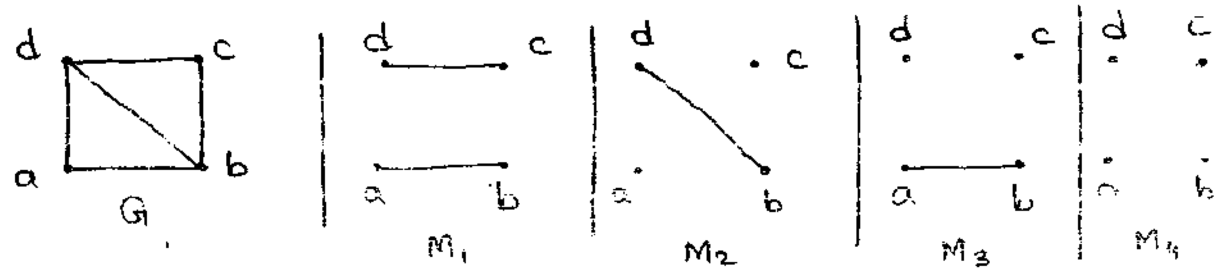
c) 4

d) 5

Matching and Covering

Matching : Let G be a graph. A subgraph M of G is called a matching of G, if every vertex of G is incident with atmost one vertex in M.

i.e. $deg(v) \leq 1 \quad \forall\, v \in G$



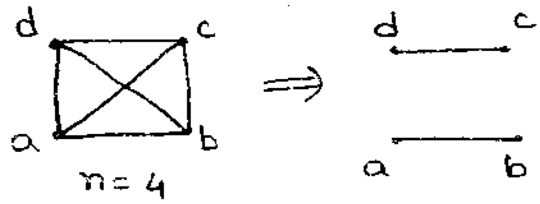
- In a matching, no two edges are adjacent
  $deg(v) \leq 1$
- **Maximal Matching** : A matching M of a graph G is said to be maximal, if no other edges of G can be added to M.
  - In above example for the graph G M_1 & M_2 are maximal matching of G.
- **Maximum Matching** : A matching of a graph with maximum no. of (largest maximal matching) edges is called a maximum matching of G.

  For a graph given in above example M_1 is maximum matching
- **Matching Number** : The no. of edges in a maximum matching of G.

  Matching no. of $M_1 = 2$
- **Perfect Matching** : A matching of a graph in which every vertex is matched is called perfect match.
- **Note** :- Every Perfect Matching is Maximum Matching / Largest maximal matching. (not vice versa)
  - Converse of above need not be true.
  - If Graph G has a perfect match then no. of vertices in G is even. (but not vice versa)

Ex : Matching number of K_n ~~(n is even)~~ is ___

a) $\lfloor n/2 \rfloor$ b) $\lceil n/2 \rceil$ c) $\lfloor \frac{n+1}{2} \rfloor$ d) $\lceil \frac{n+1}{2} \rceil$

→ 

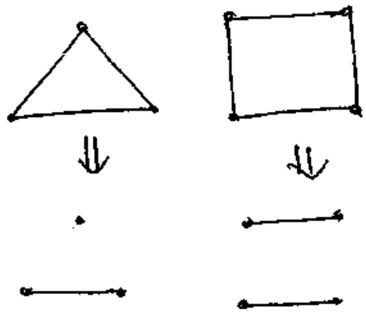
$n = 4$

$= n/2$ if n is even

$= \frac{n-1}{2}$, if n is odd

Ex : What is matching no. C_n $(n \geq 3)$ is ___

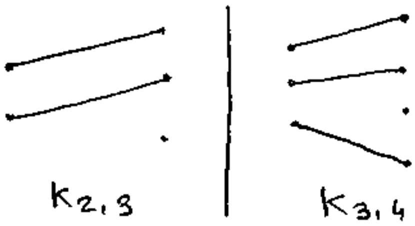
a) $\lfloor n/2 \rfloor$ b) $\lceil n/2 \rceil$ c) $\lfloor \frac{n+1}{2} \rfloor$ d) $\lceil \frac{n+1}{2} \rceil$

→ 

Ex : what is matching no. of $K_{m,n}$ is —

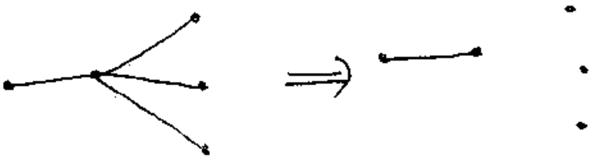
a) Max of $\{m,n\}$ b) min of $\{m,n\}$

c) $|m-n|$ d) G.C.D of $\{m,n\}$

→ 

$K_{2,3}$ | $K_{3,4}$

ex : what is the matching no. of star graph with n vertices $(n \geq 2)$ is

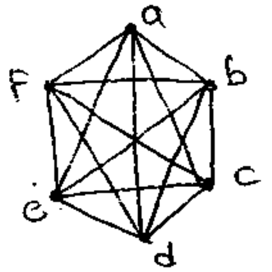


star graph is $K_{1,n-1}$ Bipartite graph

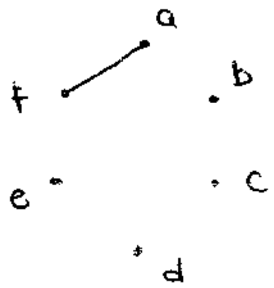
Ans : 1

* K_n has a perfect matching iff n is even.

Ex: How many no. of perfect matching in K_{2n} is ___



$a \to 5$ matches
$b \to 3$ —"—
$e \to 1$ —"—



$\Rightarrow$ 15 matching no.

Generally, $V_1 \to (2n-1)$
$V_2 \to (2n-3)$
$V_3 \to (2n-5)$
⋮

$\therefore$ total perfect matching no. $= (2n-1)(2n-3)(2n-5)\cdots$

$$= \frac{\lfloor 2n}{2n(2n-2)(2n-4)\cdots}$$

$$= \frac{\lfloor 2n}{2^n \lfloor n}$$

Ex: How many perfect matching in K_8 is —

$\to$ $2n = 8$

$$\therefore \text{no.} = \frac{8!}{2^4\, 4!} = \frac{8 \times 7 \times \cancel{6}^{3} \times 5}{\cancel{16}\,\cancel{2}} = 105$$

$K_{m,n}$ has a perfect matching iff $m = n$.

Ex: No. of perfect matching in $K_{n,n}$ $(n \geq 1)$ is —

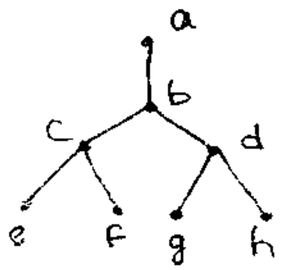
n way $\leftarrow V_1$ $\quad u_1$
$(n-1) \leftarrow V_2$ $\quad u_2$
$(n-2) \leftarrow V_3$ $\quad u_3$
V_4 $\quad u_4$
⋮
$1 \leftarrow V_n$ $\quad u_n$

$= n(n-1)(n-2)\cdots 1$

$= \lfloor n$

* A tree can have atmost 1 perfect matching

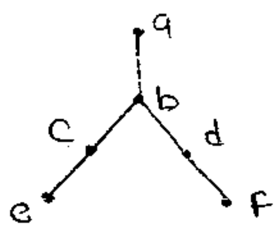
Ex : No. of perfect matching in the tree shown below is ___



(a) 0
b) 1
c) 2
d) 3

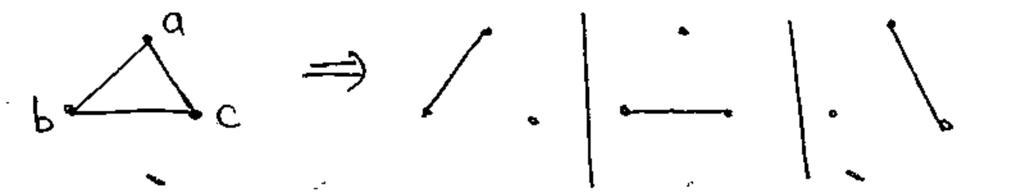
In a perfect matching degree of every vertex is 1 therefore we cannot delete any edge passing through leaf nodes.

Ex : No of perfect matching in the tree given below is ___



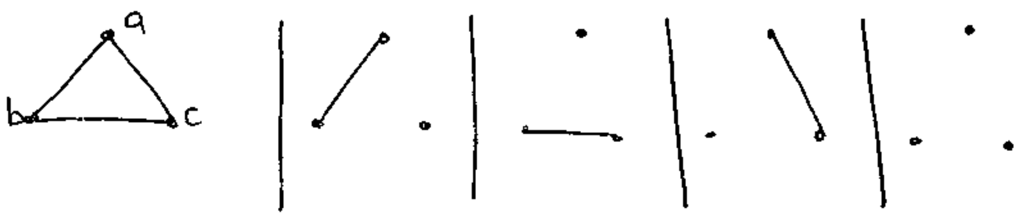
a) 0
~~b)~~ 1
c) 2
d) 3

Ex : No. of maximal matching in the graph shown below is ___



Ans = 3

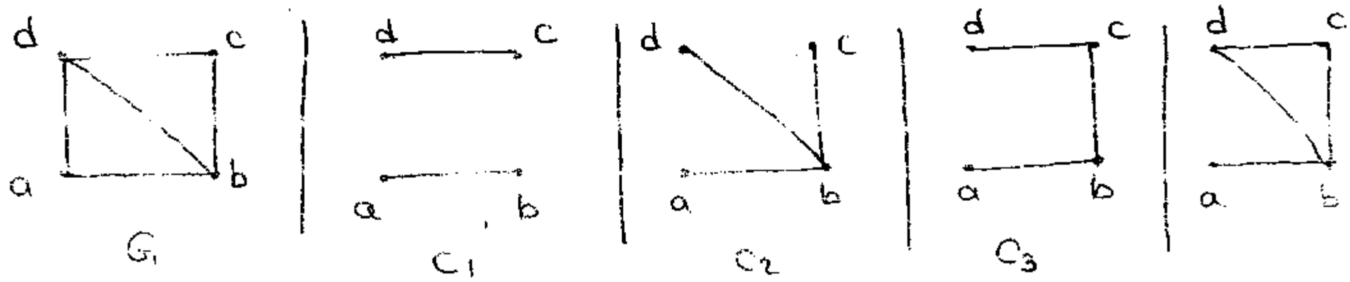
Ex : No. of matching in the graph shown below is 4



Coverings :-

Line Covering :- Let $G = (V, E)$ be a graph. A subset C of E is called a line covering of G, if every vertex of G is incident with atleast one edge in C.

i.e. $\deg(v) \geq 1 \quad \forall v \in G$



— line covering of a graph G does not exists iff G has an isolated vertex.

Minimal line covering : A line covering is said to be minimal, if no edge can be deleted from the line covering, without destroying it's ability to cover the graph.

For the graph given in the above example C_1 & C_2 are minimal line covering.

Minimum Line covering : (smallest minimal line covering)

A line covering with minimum no. of edges is called a minimum line covering.

* The no. of edges in minimum line covering is called line covering number of a graph. $G = \alpha_1$

The graph is above example has $\alpha_1 = 2$.

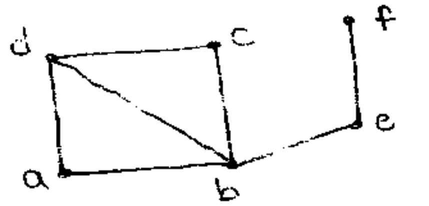
(C_1 graph is minimum line covering.)

* line covering of graph with n vertices contains atleast $\lceil \frac{n}{2} \rceil$ edges.
* No minimal line covering can contain a cycle.
* In a line covering if there is no path of length 3 or more, then C is minimal.

* In the line covering if there are no path of length 3 or more then all components of C is star graph. Then from star graph no edge can be deleted.

<u>Independent line set</u> :

Let $G = (V, E)$ be a graph. A subset L of E is called an independent if no two edges in L are adjacent.



$L_1 = \{(b,d)\}$

$L_2 = \{(b,d), (e,f)\}$

$L_3 = \{(a,d), (b,c), (e,f)\}$

$L_4 = \{(a,b), (e,f)\}$

<u>Maximal independent line set</u> : An independent line set L of a graph G is said to be maximal, if no other edges of G can be added to L.

Ex : $L_2 = \{(b,d), (e,f)\}$

$L_3 = \{(a,d), (b,c), (e,f)\}$ are maximal independent line set because no other edges can be added to it.

<u>Maximum independent line set</u> :- (Largest Maximal independent line set) An independent line set L of a graph G, with maximum no. of edges is called Maximum independent line set.

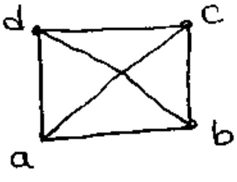
* No. of edges in Maximum independent line set is called line independent number of G denoted by β_1 (say)

* line independent no. = Matching no. of G.

For the graph in above example L_3 is a Maximum independent line set and $\beta_1 = 3$.

* for any graph G, $\boxed{\alpha_1 + \beta_1 = |V|}$

Ex : For the graph shown below which of the following is a minimal line covering ?



(a) $\{(a,b), (a,c), (b,d)\}$ (c) $\{(a,b), (a,c), (a,d)\}$

(b) $\{(a,b), (b,d), (d,c)\}$ (d) $\{(a,b), (c,d), (a,c)\}$

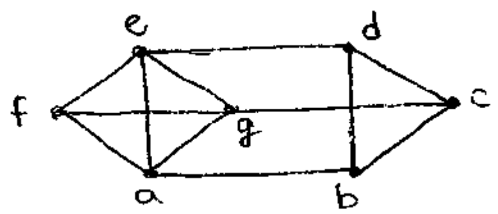
Ex : No. of edges in the line covering of the graph cannot exceed (with n vertices)

a) $\lceil \frac{n}{2} \rceil$ b) $\lfloor n/2 \rfloor$ c) $n-1$ d) $n-2$

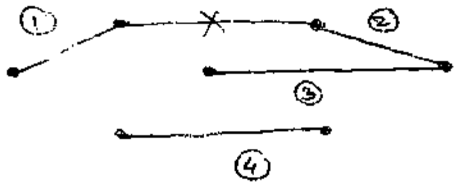
Ex : For the graph shown below

line covering no. $= \alpha_1 = 4$

line independent no. $= \beta_1 = 3$

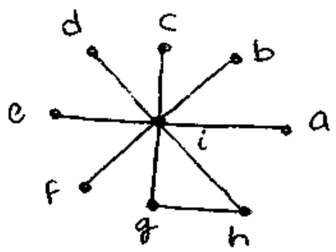


$$\beta_1 = |V| - \alpha_1$$
$$= 7 - 4$$
$$= 3$$



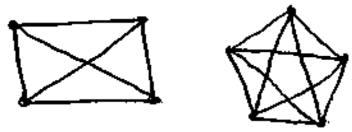
Ex : For the graph shown below

$$\alpha_1 = 7$$
$$\alpha_1 + \beta_1 = |V|$$
$$\beta_1 = 9 - 7$$
$$\beta_1 = 2$$



Ex : For the complete graph K_n find α_1, β_1

$$\alpha_1 = \lceil n/2 \rceil$$
$$\beta_1 = \lfloor n/2 \rfloor$$
$$\alpha_1 + \beta_1 = |V| = n$$



Ex : For the cycle graph C_n $(n \geq 3)$ α_1 & $\beta_1 = ?$

$$\alpha_1 = \lceil n/2 \rceil$$
$$\beta_1 = \lfloor n/2 \rfloor$$
$$\alpha_1 + \beta_1 = n$$

Ex : wheel graph W_n $(n \geq 4)$

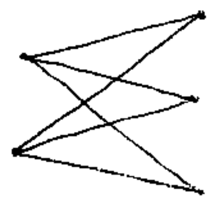
$$\alpha_1 = \lceil n/2 \rceil$$
$$\beta_1 = \lfloor n/2 \rfloor$$

Ex: For complete bipartite graph $K_{m,n}$

$\alpha_1 = \max(m,n)$

$\beta_1 = \min(m,n)$

$\alpha_1 + \beta_1 = m+n$



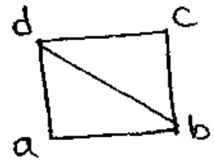
Ex: For the star graph with n vertices $(n \geq 2)$

$\alpha_1 = n-1$

$\beta_1 = 1$

$\alpha_1 + \beta_1 = n$

Vertex Covering :- Let $G=(V,E)$ be graph. A subset k of V is called a vertex covering of G, if every edge of G is incident with a vertex in k.



$k_1 = \{b, d\}$

$k_2 = \{a, b, c\}$

$k_3 = \{b, c, d\}$

Minimal Vertex Covering :- vertex covering k of a graph G is said to be minimal if no vertex can be deleted from k.

k_1 & k_2 are minimal vertex covering.

Minimum vertex covering :- A vertex covering of a graph G with minimum number of vertices is called as Minimum vertex covering. (also called as smallest minimal vertex covering)

— No. of vertices in a minimum vertex covering is called vertex covering no. of Graph G. denoted by α_2

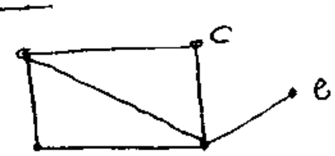
In above graph G of k_1 $\alpha_2 = 2$

Independent Vertex set :-

Let $G=(V,E)$ be a graph. A subset S of V is called an independent set if no two vertices in S are adjacent.

$S_1 = \{b\}$ $S_4 = \{a, c\}$

$S_2 = \{d, e\}$



Maximal indep. vertex set :- An independent vertex set is said to be maximumal, if no other vertex of G can be added to the set.

Ex : $S_1 = \{b\}$

$S_2 = \{d, e\}$

$S_3 = \{a, c, e\}$

Maximum indep. vertex set (Largest maximal indep. vertex set) :-

An indep. vertex set of graph G with maximum no. of vertices is called Maximum indep. vertex set.

* The no. of vertices in Maximum indep. vertex set is called as ~~indep~~ vertex indep. number of G denoted by β_2

Ex : $S_3 = \{a, c, e\}$

$\therefore \beta_2 = 3$

* For any graph $\alpha_2 + \beta_2 = |V|$
* For any graph if S is independent set of G then $V - S$ = A vertex covering of G.

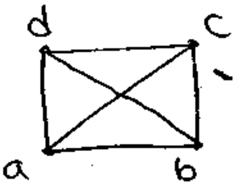
Ex : For a complete graph K_n, vertex covering no $= \alpha_2 = ?$

vertex ~~e~~ indep. no $= \beta_2 = ?$

$\rightarrow$ $\alpha_2 + \beta_2 = |V| = n$

$\cancel{\alpha_2 = 1}$ $\beta_2 = 1$

(because in K_n every vertex adjacent to each other)



$S_1 = \{a\}$

$\therefore$ $\cancel{\beta_2 = n-1}$ $\alpha_2 = |V| - \beta_2$

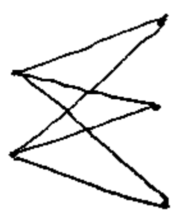
$\alpha_2 = n - 1$

Ex : For the complete bipartite graph $K_{m,n}$

$\alpha_2 = \min(m, n)$

$\beta_2 = \max(m, n)$

$\alpha_2 + \beta_2 = m + n$



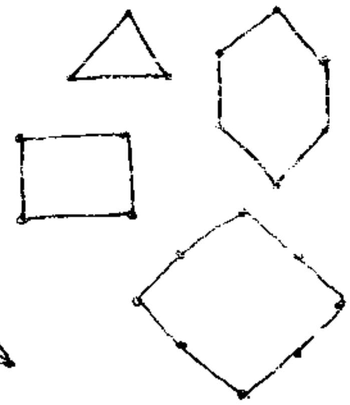
Ex: for the ~~star~~ graph with n vertices $(n \geq 2)$

$\longrightarrow$ $\alpha_2 = 1$

$\beta_2 = n-1$

$\alpha_2 + \beta_2 = n$

Ex: For the cycle graph C_n $(n \geq 3)$



$\alpha_2 = \lceil n/2 \rceil$

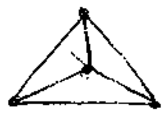
$\beta_2 = \lfloor n/2 \rfloor$

Ex: wheel graph W_n $(n \geq 4)$

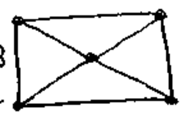
$\longrightarrow$ $\alpha_2 = \left\lceil \frac{n+1}{2} \right\rceil$

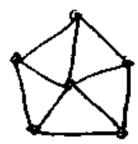
$\beta_2 = \left\lfloor \frac{n-1}{2} \right\rfloor$

$\alpha_2 = 3$
$\beta_2 = 1$



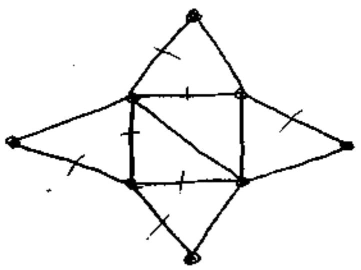
$\alpha_2 = 3$
$\beta_2 = 2$





$\alpha_2 = 4$
$\beta_2 = 2$

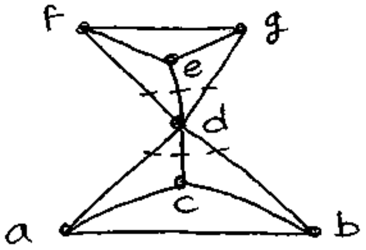
Ex: For the graph shown below



$\beta_2 = 4$

$\alpha_2 = 4$

Ex:

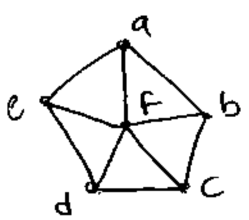


which is not true?

a) $\alpha_2 = 5$ c) $\{a, e\}$ is a max. indep. set

b) $\beta_2 = 2$ d) $\{a, c, e, f\}$ is vertex covering

Ex:



which of the following is not true?

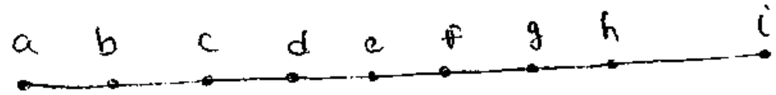
a) $\alpha_2 = 4$ b) $\beta = 2$

c) $\{b, d\}$ is max. indep. vertex set

d) $\{a, c, f\}$ is a vertex covering.

Ex : The no. of vertices in the smallest maximal indep. vertex set in the chain of 9 nodes :

(a) 3 b) 4 c) 5 d) 2



$S_1 = \{a, c, e, g, i\}$
$S_2 = \{b, d, f, h\}$ } Largest Maximal
$S_3 = \{b, e, h\} \longrightarrow$ smallest

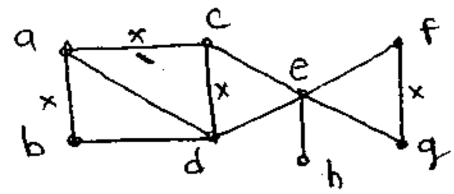
Spanning Trees

Let G be a connected graph. A subgraph H of G is called a spanning tree of G, if i) H is a tree
ii) H contains all vertices of G

Circuit rank :- Let G be connected graph with n vertices and m edges. Any spanning tree of G contains (n-1) edges

– The no. edges we have to delete from G to get spanning tree of G is equal to $\boxed{m-(n-1)}$ is called circuit rank of G.

Ex :



Edges

Circuit rank (G) = $m-(n-1)$
$= 11-(8-1)$
$= 4$

Ex : Let G be a connected graph with 6 vertices and degree of each vertex is 3 circuit rank of G = ?

$\rightarrow$

$$\Sigma \deg(v_i) = 2|E|$$
$$18 = 2|E|$$
$$|E| = 9$$

Circuit rank (G) = $9-(n-1)$
$= 9-(6-1)$
$= 4$

Ex : Let G be connected graph with 5 vertices, max. no. of edges and all cycles in G are of even length Find $C(G) = ?$

→ If all cycles of even length then G is bipartite graph with max. no. edges and 5 vertices.

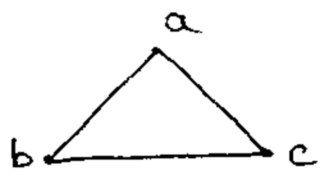
$\therefore G = K_{3,2}$ or $K_{2,3}$

$\therefore$ No. edges $= 6$

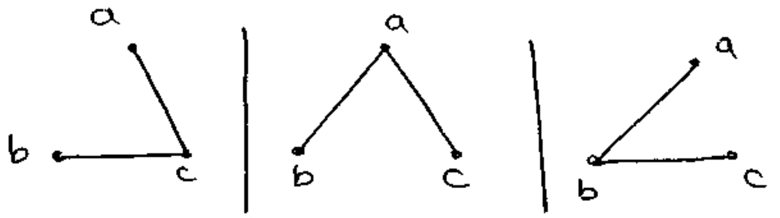
$C(G) = 6 - (5-1)$

$= 2$

Ex. Number of spanning trees in the graph shown below is —

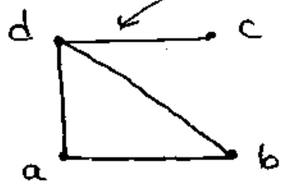


a) 3
b) 4
c) 6
d) 8



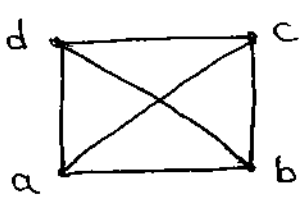
Ex : No. of spanning trees —

can not delete



a) 3 c) 6
b) 4 d) 8

Ex : No. of spanning trees —



a) 6 c) 12
b) 8 d) 16

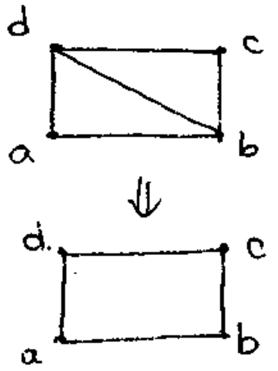
No. of spanning trees in complete graph $= n^{n-2}$ (caley's Formula)

$= 4^{4-2}$

$= 16$

Circuit rank $= m - (n-1) = 6 - (4-1) = 3$

Ex : No. of spanning trees —



a) 6
b) 8
c) 10
d) 4

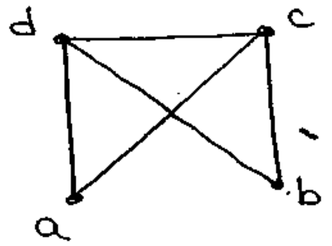
Kirchoff's Theorm :-

Let A be the adjacency matrix of a connected graph G. Let M be the matrix obtained from A, by changing all 1's into -1 and replacing each zero element in the principle diagonal of A with the degree of corresponding vertex.

— cofactor of any element of M is equal to the no. of spanning trees in G.

Ex : No. of spanning trees in the graph shown below :-

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



$$M = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

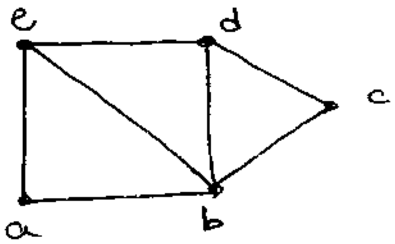
$$\text{cofactor of } M_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 2(9 \oplus -1) + 1(-2) - 1(1+3)$$

$$= 16 - 2 - 4 - 2$$

$$= 8$$

Ex : No of spanning trees in the graph shown below :-

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



$$M = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

$$\text{cofactor of } M_{51} = (-1)^{5+1} \begin{vmatrix} -1 & 0 & 0 & -1 \\ 4 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \end{vmatrix}$$

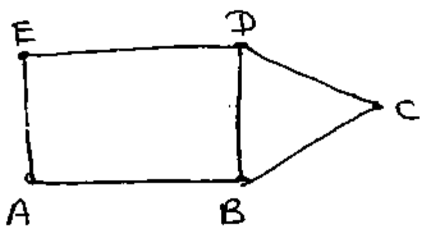
$C_4 - C_1$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 4 & -1 & -1 & -5 \\ -1 & 2 & -1 & 1 \\ -1 & -1 & 3 & 0 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} -1 & -1 & -5 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{vmatrix} = 21$$

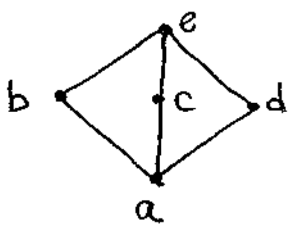
Ex: No. of spanning trees —

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Ans = 11

Ex :

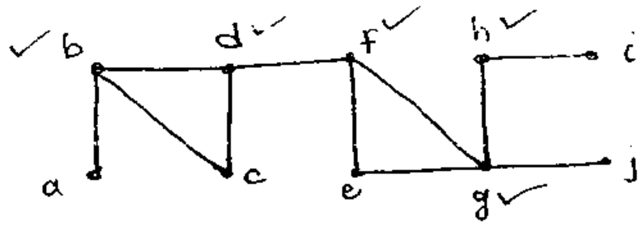


Ans :- 12

1W BFS, DFS, Minimal spanning trees, Kruskal's algo, Prim's Algorithm, Traversability theorm, Euler circuit, Hamiltonian graph.

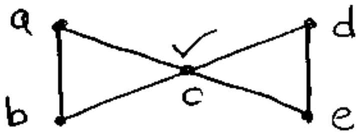
<u>Connectivity</u> :- A graph is not connected will have two or more connected components.

<u>Cut Vertex</u> :- Let G be a connected graph. A vertex $V \in G$ is (Articulation point) called cut vertex of G if $(G-V)$ results in a disconnected graph.



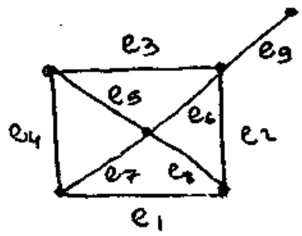
<u>Cut edge</u> :- Let G be a connected graph and an edge $E \in G$ (Bridge) is called a cut edge of G if $(G-E)$ results in a disconnected graph.

- In a connected graph G if $E \in G$ is cut edge iff E is not part of any cycle in G.
- whenever cut edge exists cut vertex also exists and not vice versa. Because atleast one vertex of cut edge is a cut vertex.
- If cut vertex exists then cut edge may or may not be exists. Ex: vertex C,



<u>Cut set</u> :- Let $G = (V, E)$ be a connected graph. A subset E' of E is called a cut set of G, ~~iff~~ ~~(G-E') results in a disconnected graph~~ If deletion of all the edges in E' makes G disconnected and deletion of no proper subset of E' from G, can make G disconnected.

Ex : For the graph shown below which of the following are cut set ?



✓ (a) $\{e_1, e_3, e_5, e_7\}$

× (b) $\{e_2, e_5, e_6, e_8\}$

× (c) $\{e_2, e_6, e_3, e_9\}$
↓
No need of e_9
because using e_2, e_6, e_3 it is disconnected graph.

✓ (d) $\{e_9\}$

✓ (e) $\{e_1, e_4, e_7\}$

Edge connectivity :- In a connected graph G, the min. no. of edges whose deletion makes G disconnected is called 'Edge connectivity of G'.
It is denoted by $\lambda(G)$.

- If G has a cut edge then edge connectivity $\lambda(G) = 1$
- The no. of edges in a smallest cut set of G is said to be edge connectivity of G.

Vertex Connectivity :- In a connected graph G, the min no of vertices whose deletion makes graph disconnected or reduces G into a trivial graph is called vertex connectivity of a connected graph G.
It is denoted by $K(G)$

- If G has a cut vertex then $K(G) = 1$
- For any connected graph G,

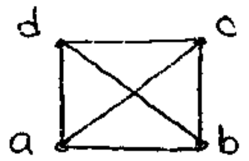
$$\boxed{K(G) \leq \lambda(G) \leq \delta(G)}$$

$\delta(G)$ = min. of all the degrees of all vertices.

Ex: for complete graph K_n

vertex connectivity of $K_n = n-1$

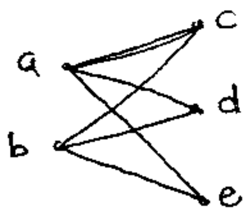
Edge connectivity of $K_n = n-1$



Ex: for a complete bipartite graph $K_{m,n}$

vertex connectivity $= \min(m,n)$

edge connectivity $= \min(m,n)$



$\delta(G) = 2$

$k(G) \leq \lambda(G) \leq 2$

Ex: For a cycle graph C_n $(n \geq 3)$

Edge connectivity $= 2$

vertex connectivity $= 2$

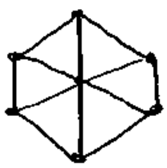
$\delta(G) = 2$

$k(G) \leq \lambda(G) \leq 2$

Ex: for the wheel graph W_n $(n \geq 4)$

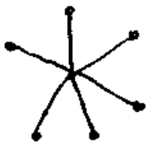
Edge connectivity $= 3$

vertex connectivity $= 3$



$k(G) \leq \lambda(G) \leq \delta(G) = 3$

Ex: for the star graph $(n \geq 2)$ $(K_{1,n-1})$



edge connectivity $= 1$

vertex connectivity $= 1$

$\delta(G) = 1$

Ex: For a tree with n vertices $(n \geq 2)$,

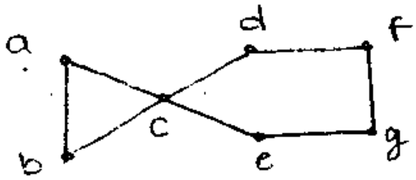
vertex connectivity = 1
Edge connectivity = 1

$\delta(G) = 1$

Ex: For the graph shown below

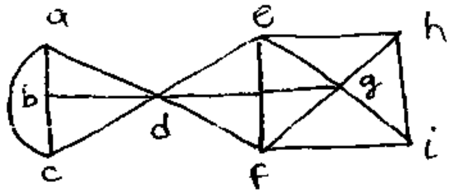
$\delta(G) = 2$

vertex connectivity = 1
edge connectivity = 2



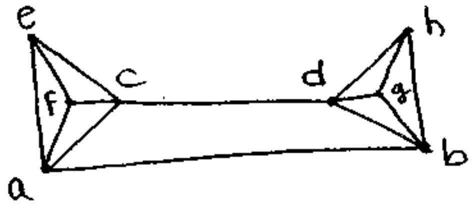
Ex:

$\delta(G) = 3$



vertex conn. = 1
Edge conn. = 3

Ex:



$\delta(G) = 3$
vertex conn. = 2
edge conn. = 2

* A simple graph G with n-vertices is necessarily connected if number of edges in G

$$\boxed{|E(G)| > \left\{\frac{(n-1)(n-2)}{2}\right\}}$$

Ex: Which of the following Graphs are necessarily connected?

(A) A Graph with 6 V and 10 E $(10 \not> 10)$

(B) 7 vertices and 14 edges $(14 \not> 15)$

(C) 8 vertices, 22 edges $(22 > 21)$

(D) 9 vertices, 28 edges $(28 \not> 28)$

Ex: what is the min. no. of edges needed to guarantee the connectivity in a simple graph with 10 vertices?

(a) 37 b) 38 c) 34 d) 28

$|E(G)| > \frac{(n-1)(n-2)}{2} \Rightarrow 9 \times 4 = 36$

* A simple graph with n-vertices and k-components has atleast (n - k) edges.

$$i.e. \; |E(G)| \geq (n-k)$$

* A simple graph with n-vertices and k-components then the max. no. of edges of G is always at most $\frac{(n-k)(n-k+1)}{2}$ edges.

$$|E(G)| \leq \frac{(n-k)(n-k+1)}{2}$$

Ex: Min. no. of edges necessary in a simple graph with 10 v and 3 components.

a) 25 b) 28 c) 30 d) 7 ✓

→ $|E(G)| \geq (n-k)$

$\geq (10-3)$

$|E(G)| \geq 7$

Ex: Max. no. of edges necessary in a simple graph with 10 v and 3 components

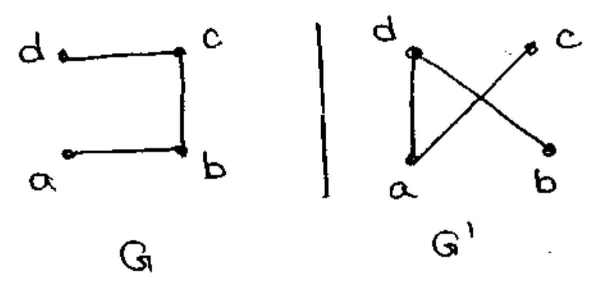
a) 25 b) 28 ✓ c) 30 d) 37

$$|E(G)| \leq \left[\frac{(10-3)(10-3+1)}{2} \right]$$

$$|E(G)| \leq 28$$

Ex: Which of the following is/are true?

(a) If a simple graph G is connected then $\overline{G}$ is not connected.

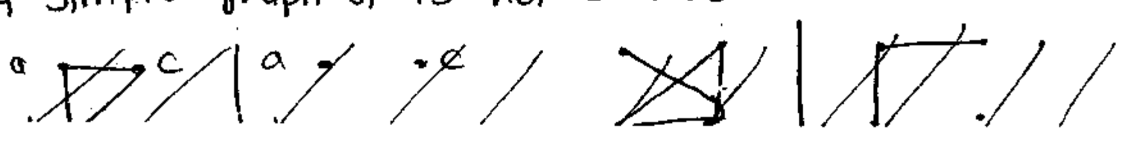


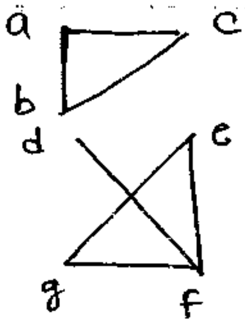
Here G & G' both connected.

∴ FALSE (Need not be always true)

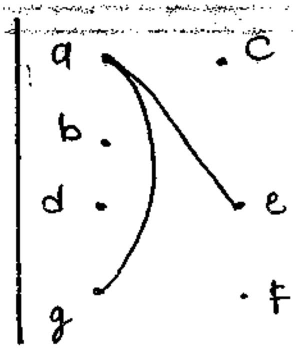
TRUE

(b) A simple graph G is not connected then $\overline{G}$ is connected





G



G'

TRUE

(c) In a complete graph G with n-vertices, if $\delta(G) \geq \left(\frac{n-1}{2}\right)$ then G is connected.

→ Suppose given stat. is False

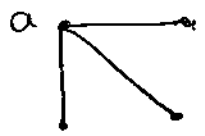
i.e. $\delta(G) \geq \frac{(n-1)}{2}$ and G is not connected

G has atleast two components G_1 & G_2

Let $v \in G_1$

$$\deg(v) \geq \left(\frac{n-1}{2}\right)$$

deg(a) = 3



$$\Rightarrow |V(G_1)| \geq \frac{n-1}{2} + 1$$

$$|V(G_1)| \geq \frac{n+1}{2}$$

Similarly, no. of vertices in G_2 is also $\geq \frac{n+1}{2}$

Now, No. of vertices in G $\geq |V(G_1)| + |V(G_2)|$

$$\geq \frac{n+1}{2} + \frac{n+1}{2}$$

$$|V(G)| \geq n+1$$

∴ It is contradiction.

∴ Given stat. is true.

TRUE

(d) If a simple graph G has exactly two vertices of odd degree, then there exists a path betn the 2 vertices.

→ The given stat. is true if G is connected graph.

If a graph is nonconnected then by sum of degree theorm the two vertices of odd degree belong to same components x: The G be a graph with n vertices and k components. If we delete a vertex in G, then the number of components in G, should lie betn

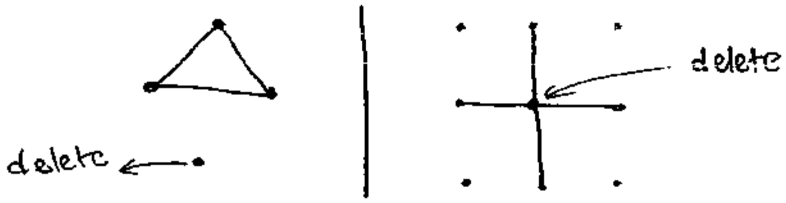
a) $n-k$ and $n-1$ b) $k-1$ and $n-k$

c) k and $n-k$ ✓d) $k-1$ and $n-1$

i) If the vertex we are deleting is the component of itself then the no. of components become $k-1$

ii) If given graph is star graph with n vertices then by deleting cut vertex of star graph we get $n-1$ components

∴ max. no. component $= n-1$



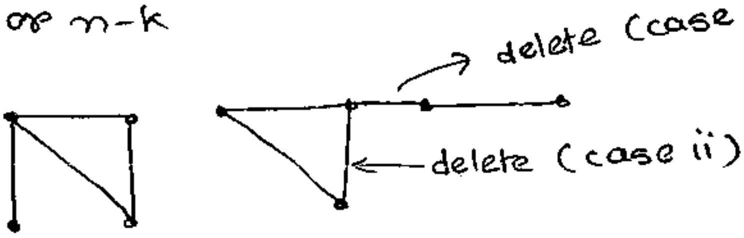
(i) (ii)

Ex: Let G be a graph with n vertices and k-components. If we delete an edge in G, then the no. of components in G are ___

a) $k-1$ or k ✓b) k or $k+1$ c) $k-1$ or $k+1$

d) k or $n-k$

→



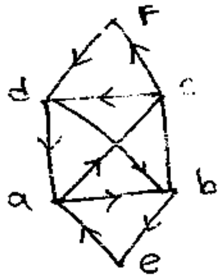
i) If edge we are deleting is cut edge then no. of components become $k+1$.

ii) If deleting edge is not cut edge then no. of components remains same i.e. k

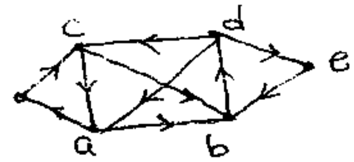
Traversability :-

A graph G is said to be traversal, if there exist a path which contain each edge of G exactly once and each vertex of G atleast once such a path is called Euler path.

Euler Circuit :- An Euler path in which the starting vertex is same as ending vertex of path is called Euler circuit.



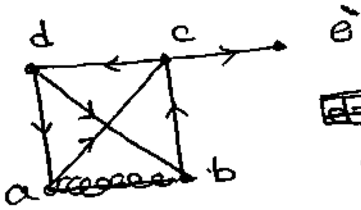
a - b - c - d - b - e - a - c - f - d - a



a - b - d - c - a - f - c - b - d - e - b - d - a

Theorm : A graph G is traversable, if no. of vertices with odd degree is exactly two or ~~more~~ zero.

- If graph G has exactly 2 vertices of odd degree then euler path exists but euler circuit does not exists.
- If graph G has no vertices of odd degree then Euler circuit also exists.
- If graph has exactly two vertices of odd degree then the euler path begins with one odd vertex and ends with other odd vertex.

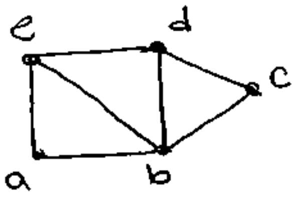


d - a - c - d - b - c - e

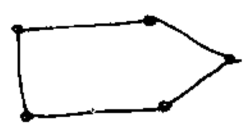
Hamiltonian graph :-

In a graph G, if there exists a cycle which contains each vertex of G exactly once, then the cycle is called Hamitonian cycle and the graph is called hamiltonian graph.

Ex :



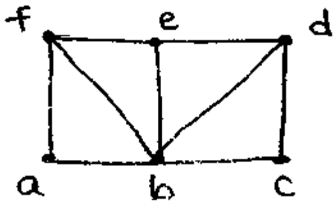
Ex :



- In Euler circuit every edge is present exactly once but in hamiltonian cycle some edges of the graph may be skipped.

* Hamiltonian graphs and traversable graph both are connected graph.

Ex : for the graph shown below which of the following is true?



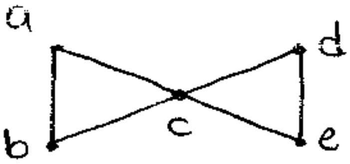
1) Euler path exists
2) Euler circuit exists
3) Hamiltonian cycle exists
4) Hamiltonian path exists

[illegible] vertices of odd degree

[illegible] not traversible

[illegible] Euler path & circuit not exists

which is/are true?



in the graph no. of vertices with odd degree is zero

It is traversible and Euler circuit also exists.

i) Euler path (or) Euler circuit

ii) Ham. cycle (or) Ham. path

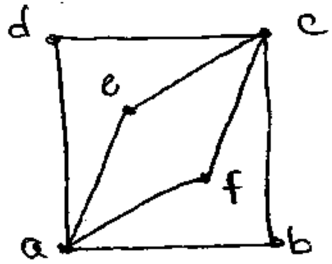
* In hamiltonian cycle degree of each vertex is 2.
Therefore to construct hamiltonian in G we have to delete 2 edges of vertex c, then we are left with 5 vertices & 4 edges ∴ a cycle graph with 5 vertices req. 5 edges.

Ham. path exists a-b-c-d-e which contains all 5 edges (Ham. circuit not exists)

Q. which is/are true?

The graph has no vertices with odd degree

∴ It is traversable and Euler circuit also exists



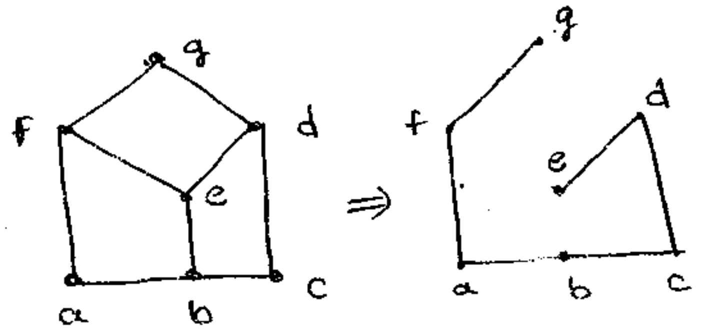
To construct ham. cycle for the graph we have to delete two edges at each of the vertices a & c

Then we are left with 6-vertices and 4-edges, therefore neither ham. cycle nor path exists.

Ex:

Odd degrees of vertices present

∴ No Euler path and Circuit present.



To construct ham. cycle for the graph we have to delete one edge at each of the vertices b, d, f then left with 6 edges & 7 vertices

∴ ham.cycle is not possible. but ham. path exists.

Set theory & Algebra

* sets
* Relations
* functions
* Groups
* lattice
* Boolean Algebra

Set :- A well defined unordered collection of distinct elements.

$$A = \{ x \mid x \text{ is an integer and } 1 \leq x \leq 10 \}$$

$$= \{ 1, 2, 3, \ldots 10 \}$$

Sets are denoted by upper case letters and elements denoted by lower case letters.

Null set :- A set with no elements.

It is denoted by ϕ.

Ex: $A = \{ x \mid x$ is a prime no. and $8 < x < 10 \}$

$A = \{\ \}$

$A = \phi$

Subset : If A and B are sets, such that every element of A is also an element of B then A is a subset of B.

Denoted by $A \subseteq B$

For any set A, $A \subseteq A$ and $\phi \subseteq A$

Proper subset :- Any subset of A, which is not a trivial subset of A is called proper subset of A.

Ex: $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$ then $A \subset B$

* If $A \subseteq B$ and $B \subseteq A$, then $A = B$

Power Set :- Let A be a set then power set of A denoted by $P(A)$ = set of all subset of A

Let $A = \{1, 2, 3\}$

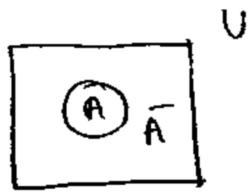
then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

Note : If $|A| = n$ then no. of elements in $P(A) = 2^n$

Universal set : set of all objects under discussion (U)

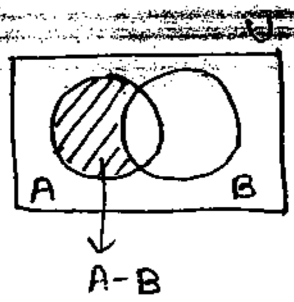
Complement of set :- A universal set U and a set A contained in U, then all elements that belong to U but not A is called complement of A (in U)

It is denoted by $\bar{A}$.



Set difference :- (Relative complements)

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$



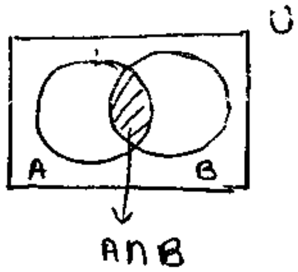
If $A = \{1, 2, 3, 4\}$

$B = \{2, 4, 5, 6\}$

$A - B = \{1, 3\}$

Set intersection :

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

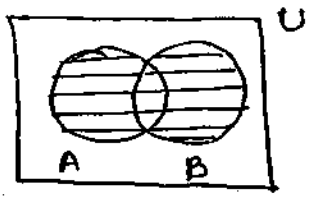


$A \cap B \subseteq A$ and

$A \cap B \subseteq B$

Set Union :

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



$A \subseteq (A \cup B)$ and

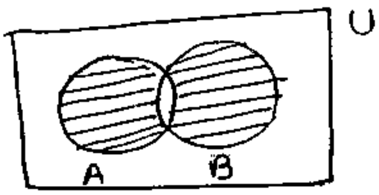
$B \subseteq (A \cup B)$

Symmetric Difference :- (Boolean sum)

$$A \triangle B = \{x \mid x \in A \text{ or } x \in B \text{ but } x \notin (A \cap B)\}$$

$A \triangle B = (A \cup B) - (A \cap B)$

$A \oplus B = (A - B) \cup (B - A)$



$A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3, 4, 5, 6\}$

$A \triangle B = \{5, 6\}$

The laws of set theory :

1. **commutative laws :**

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
$$A \oplus B = B \oplus A$$
$$A - B \neq B - A$$

2. **Associative laws :**

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$
$$A - B \neq B - A$$

3. **Distributive law :**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. **DeMorgan laws :-**

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$(\overline{A \cap B}) = \overline{A} \cup \overline{B}$$
$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$

5 **Idempotent laws :-**

$$A \cup A = A$$
$$A \cap A = A$$

6. **Identities :-** $A \cup \phi = A$, $A \cup U = U$, $A \cup \overline{A} = U$

$A \cap U = A$ $\quad A \cap \overline{A} = \emptyset$, $A - B = A \cap \overline{B}$

7. **Absorption law :-** $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

8. **Modular laws :-**

$$A \cup (B \cap C) = (A \cup B) \cap C \text{ iff } A \subseteq C$$
$$\Rightarrow (A \cup B) \cap (A \cup C) = (A \cup B) \cap C$$
$$A \cup C = C$$
$$A \subseteq C$$
$$A \cap (B \cup C) = (A \cap B) \cup C \text{ iff } C \subseteq A$$

Note : * $P(A \cap B) = P(A) \cap P(B)$

* $P(A \cup B) \neq P(A) \cup P(B)$
* $A - (A - B) = A \cap B$
* $A - (A \cap B) = A - B$

Ex: which of the following is not true ?

a) If $A \subseteq \phi$, then $A = \phi$ —— true

b) $B \cup (A \cap B) = B$ —— true

$A \cap B \subseteq B$

c) $(A \cap B) \cup (A \cap B^c) = A$ | $(A \cdot B) + (A \cdot \overline{B})$
$= A \cdot (B + \overline{B}) = A$

$L.H.S = A \cap (B \cup B^c)$
$= A \cap U$
$= A = R.H.S$ —— True

d) $(A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c) = A \cup B$

$(A \cdot B) + (A \cdot \overline{B}) + (\overline{A} \cdot B) + (\overline{A} \cdot \overline{B})$

$= A \cdot (B + \overline{B}) + \overline{A} \cdot (B + \overline{B})$

$= A + \overline{A}$

$= 1$

$= U \neq A \cup B$ —— False.

Ex: If $A \subset B$ then, which of the following is not true ?

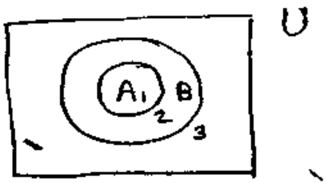
a) $B^c \subset A^c$

b) $A \cup B = B$

c) $A \cap B = A$

~~d) $A \cap B = A$~~

d) $B - A = \phi$



$A \subset B$

Ex: If $A \oplus B = (A - B) \cup (B - A)$ which is False ?

a) $A \oplus A = \phi$

b) $A \oplus \phi = A$

c) $(A \oplus B) \oplus B = A$

$L.H.S. = A \oplus (B \oplus B)$
$= A \oplus \phi$
$= A = R.H.S.$

d) $A \oplus B = A$ iff $B = \phi$

$\Rightarrow A \oplus B = A$
$A \oplus B = A \oplus \phi$

(e) $A \oplus B = (A \cap B^c) \cap (B \cap A^c)$

Ex : Let ϕ be the empty set , then $|P\{P(\phi)\}| = ?$

$$\phi = \{\ \}$$

$$P(\phi) = \{\phi\}$$

$$P\{P(\phi)\} = \{\phi, \{\phi\}\}$$

$$|P\{P(\phi)\}| = 2$$

Ex : Consider $A = \{ s | s \text{ is a set} \}$

$B = \{ s | s \text{ is a set and } s \notin s \}$

which is true ?

a) A is a set and B is not a set

b) B is a set and A —— " ——

c) Both A and B are sets

~~d)~~ Both A and B are not sets .

⇒ * Set of set is not set . (An set cannot be element of itself)

Ex : Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$ are represented by the bit strings 1111100000 and 1010101010 resp. $(U = \{1, 2, 3, \dots 10\})$

$= \{1111111111\}$ (missing bit 0)

~~a)~~ $A \cup B = \underset{1\,2\,3\,4\,5\,6\,7\,8\,9\,10}{1111101010}$

$A \cup B = \{1, 3, 5, 2, 4, 7, 9\}$

~~b)~~ $A \cap B = \underset{1\ \ 3\ \ 5}{1010100000}$

$A \cap B = \{1, 3, 5\}$

~~c)~~ $A - B = \underset{2\ \ 4}{0101000000}$

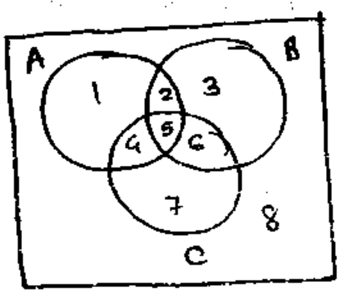
$A - B = \{2, 4\}$

~~d)~~ $B^c = 0000011111$

$B^c = (2, 4, 6, 8, 10)$

$= 0101010101$

Ex: Which is False ?



a) $(A-B)-C = (A-C)-(B-C)$

$\{1,4\} - \{4,5,6,7\}$

$= \{1\}$

$\{1,2\} - \{2,3\}$

$= \{1\}$

True

b) $(A-B)-C = (A-C)-B$

$\{1,4\} - \{4,5,6,7\}$ | $\{1,2\} - \{2,3,5,6\}$

$= \{1\}$ | $= \{1\}$

TRUE

c) $(A \cap B) - (B \cap C) = (A-(A \cap C)) - (A-B)$

$\{2,5\} - \{5,6\}$ | $[\{1,2,4,5\} - \{4,5\}] - \{1,4\}$

$= \{2\}$ | $= \{1,2\} - \{1,4\}$

$= \{2\}$

TRUE

d) $(A-B)-C = A-(B \cap C)$

$\{1,4\} - \{4,5,6,7\}$ | $\{1,2,4,5\} - \{5,6\}$

$= \{1\}$ | $= \{1,2,4\}$

$\neq$

false

Multiset :- A unordered collection of object in which any object can appears more than once is called Multiset.

E.g. $\{a, a, b, b, b, c, c, c, c, d\}$

$= \{2a, 3b, 4c, d\}$

Let, $P = \{m_1 a_1, m_2 a_2 \cdots m_k a_k\}$

where m_i = multiplicity of a_i

$Q = \{n_1 a_1, n_2 a_2 \cdots n_k a_k\}$

$P \cup Q$ = A multiset in which the multiplicity of $a_i = \max\{m_i, n_i\}$

$P \cap Q$ = —— " —— of $a_i = \min\{m_i, n_i\}$

$P - Q$ = A multiset of $a_i = \begin{cases} m_i - n_i & ; \text{ if } m_i > n_i \\ 0 & \text{otherwise} \end{cases}$

$P + Q$ = A multiset of $a_i = m_i + n_i$

Ex: $A = \{3a, 2b, 1c\}$, $B = \{2a, 3b, 4d\}$

$A \cup B = \{3a, 3b, 1c, 4d\}$

$A \cap B = \{2a, 2b\}$

$A - B = \{a, c\}$

$A + B = \{5a, 5b, 1c, 4d\}$

Ex: The set $A = \{1, 2, 3 \cdots n\}$ then how many multisets are possible with the element A.

a) 2^n b) n^n c) $2^{(n^2)}$ d) unlimited

→ Each no. can appears many no. of times

∴ Multisets are unlimited.

Ex: If $A = \{1, 2, \cdots n\}$, then how many multisets of size 4 are possible with elements of A, so that $\underbrace{\text{atleast one}}_{\geq 1}$ element appear exactly two.

→ $\{a, b, c, d\}$ — $\{a, a, b, c\}$

— $\{a, a, b, b\}$

~~Case 1 :- only one elements appears exactly twice~~

In this case the multiset is ~~of the~~ form $\{a, a, b, c\}$

No. of multiset in this case $= {}^nC_3 \cdot {}^3C_1$ or ${}^nC_1 \; {}^{(n-1)}C_2$

Case 2 : Two elements appears exactly twice

No. of Multisets in this case $= {}^nC_2$

Req. No. of multisets $= {}^nC_3 \cdot {}^3C_1 + {}^nC_2$

$$= \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \times 3 + \frac{n(n-1)}{2}$$

$$= \frac{n(n-1)^2}{2}$$

Cartesian Product :-

- If A and B are two sets, then the cartesian product of A and B defined as $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- If no. of elements in A is m and no. of elements in B is n then no. of elements in $A \times B$

  $|A \times B| = mn$

- In General $(A \times B) \neq (B \times A)$
- If $(A \times B) = (B \times A)$ then $A = B$ or $A = \emptyset$ or $B = \emptyset$

Relation :-

- A and B are two sets then Every subset of $(A \times B)$ is called a relation from A to B

$A \times B = \{(a_1, b_1)(a_1, b_2)(a_1, b_3)$
$(a_2 b_1)(a_2 b_2)(a_2 b_3)\}$

$A \rightarrow B$
$a_1 \quad b_1$
$a_2 \quad b_2$
$\quad\;\; b_3$

* If $|A| = m$ and $|B| = n$, then number of relations possible from A to B $= 2^{mn}$

$A \rightarrow A$
$a_1 \quad a_1$
$a_2 \quad a_2$
$a_3 \quad a_3$

* A relation on set A to A is called a relation on A
* If $|A| = n$ then no. relations possible on $A = 2^{n^2}$

Inverse of a relation :-

If R is a relation from A to B, then

$R^{-1} = \{$ set of all order pairs $(b, a) \mid \cancel{(a, b)}\ (a, b) \in R \}$ is

a relation from B to A.

Complement of a relation :-

If R is a relation from A to B, then

$\bar{R} = (A \times B) - R$

Diagonal Relation :-

A relation R on set A is said to be diagonal relation, if $R = \{(x, x) \mid x \in A\}$

$= \Delta A$

$A \longrightarrow A$

$a_1 \longrightarrow a_1$

$a_2 \longrightarrow a_2$

$a_3 \longrightarrow a_3$

$R = \{(a_1, a_1)\ (a_2, a_2)\ (a_3, a_3)\}$

Reflexive Relation :-

A relation R on set A is said to be reflexive if

$x R x \quad \forall x \in A$

i.e. $(x, x) \in R \quad \forall x \in A$

* A diagonal relⁿ on set A is reflexive and every superset of diagonal relⁿ is also reflexive.
* Reflexive relⁿ may contains non-diagonal relⁿ.

Ex : $A = \{1, 2, 3\}$ then $R_1 = \{(1,1)(2,2)(3,3)\}$ — smallest reflex relⁿ (Δ_A)

$R_2 = \{(1,1), (2,2)(3,3)(1,2)(3,1)\}$

$R_3 = A \times A$ — Large reflex relⁿ

Ex: If $A = \{1 \ldots n\}$ then no. of Reflexive relⁿ possible on $A = ?$

→ No. of non diagonal pairs $= n^2 - n = n(n-1)$

No. of reflexive relⁿ possible $= 2^{n(n-1)}$

→ The relation $\leq$ is reflexive on any set of real no.s.

— The relation $\subseteq$ is reflexive on any collection of sets.

— The relation 'divides' or 'is a divisor of' denoted by '$|$' is reflexive on any set of non zero real no.s.

IRREFLEXIVE RELATION :- A reln R on a set A is said to be irreflexive if $x \not R x \quad \forall x \in A$

i.e. $(x, x) \notin R \quad \forall x \in A$

Ex : $A = \{1,2,3\}$ then $R_1 = \{\phi, (1,2), (2,3), (3,2)\}$

* Smallest irreflexive reln on $A = \phi$ relation.
* Largest ——— " ——— $= (A \times A) - \Delta_A$
* if $A = \{1 - - n\}$ then no of irreflexive reln $= 2^{n(n-1)}$
* The relations "is less than" and "is greater than" are irreflexive on set of all real no.s.
* $R = \{(1,1), (2,3)\}$ on $A = \{1,2,3\}$ is ~~nor~~ neither reflexive nor irreflexive.

Symmetric Relation :- A reln R on a set A is said to be symmetric if (xRy) and (yRx) $\forall x, y \in A$. That is if $(x,y) \in R$ then $(y,x) \in R \quad \forall x, y \in A$. ↪ y related to x

Ex : if $A = \{1,2,3\}$ then $R = \{\{(1,1), (3,3), (1,2), (2,1)\}$ is a symmetric reln on A.

$R = \{(1,2), (2,1), (1,3)\}$ is not symmetric because $(1,3) \in$ but $(3,1) \notin R$.

* smallest reln on A = Empty reln (ϕ)
* Largest —— " —— $= A \times A$
* if $A = \{1 - - - n\}$ then no. of symmetric relations are possible on $A = 2^n \cdot 2^{[n(n-1)]/2}$

$= 2^{[n(n+1)]/2}$

* The relation x is "a complement of y" is symmetric in Boolean algebra.
* The relation x is "perpendicular to" is symmetric for any two lines in a plane.
* The relation x is "brother of" is symmetric for any

Antisymmetric Relation :- A relation R on a set A is said to be antisymmetric if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$

* if R is not antisymmetric if there exists $a, b \in A$ such that $(a,b) \in R$ and $(b,a) \in R$ but $a \neq b$.
* In antisymmetric relⁿ only one of the pair of (x,y) or (y,x) can be present or both of the pairs may be ~~present~~ absent.

Eg: if $A = \{1,2,3\}$ then $R = \{(1,1), (2,2), (2,1), (2,3)\}$ is a Antisymmetric relation.

* smallest Antisymmetric relⁿ on A = Empty set ϕ.
* No. of elements in a largest antisymmetric relation on

$$A = \frac{n(n+1)}{2} \qquad (n(n+1)/2)$$

* The relⁿ is $a \leq b$ and $b \leq a$ then $a = b$ is antisymmetric relⁿ on the set of all real no.
* if $A = \{1 \cdots n\}$ the no. of antisymmetric relⁿ possible on A

$$= 2^n \cdot 3^{[n(n-1)]/2}$$

* $R = \{(1,1), (3,3)\}$ — symmetric also ~~also~~ antisymmetric.
* $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (3,1)\}$ — largest antisymm.

$$\{x, y\} \rightarrow \left.\begin{array}{l} (x,y) \\ (y,x) \\ \text{—} \end{array}\right\}$$ 3 possibilities with non diagonal pairs

Ex. If $A = \{1,2,3\}$ then how many relⁿ on A are not antisymmetric.

$$\rightarrow \quad = 2^{(n^2)} - 2^n \cdot 3^{n(n-1)/2} \qquad \text{where } n = 3$$

$$= 2^9 - 2^3 \cdot 3^3$$

$$= 512 - 216$$

$$= 296$$

* The relⁿ $\leq$ is antisymmetric on any set of real no.s.
* The relⁿ $\subseteq$ is antisymmetric on any collection of sets.
* The relⁿ $|$ is antisymmetric on any set of +ve integers.

Asymmetric Relation :- A relation R on a set A, is said to be asymmetric if ($x R y$) then ($y \not R x$) $\forall x, y \in A$

i.e. if $(x, y) \in R$ then $(y, x) \notin R$ $\forall x, y, \in A$

* Every Asymmetric relation is also antisymmetric.
* In asymmetric relation diagonal pairs are not allowed where as in antisymmetric relation diagonal pairs can be present.

Ex: $A = \{1, 2, 3\}$ then $R_1 = \{\ \}$ — smallest

$R_2 = \{(1,2), (2,3)\}$

$R_3 = \{(1,2), (3,2), (1,3)\}$

* If $A = \{1 \dots n\}$ then no. of asymmetric relations possible on $A = 3^{n(n-1)/2}$
* The relation $<$ is asymmetric on any set of real no-s.
* —— " —— $>$ —— " —— " —— " ——.

Transitive Relation :- A relation R on a set A is said to be transitive if ($x R y$ and $y R z$) then ($x R z$) $\forall x, y, z \in A$.

x related to z

* The relation $\leq$ is transitive on any set of real no-s.
* —— " —— $|$ is —— " —— " ——.
* —— " —— $\subseteq$ is —— " —— collection of sets.

Ex: If $A = \{1, 2, 3\}$ then $R_1 = \{\ \}$ — smallest

$R_2 = \{(1,1), (2,2)\}$

$R_3 = \{(1,2), (2,3), (1,3)\}$

$R_4 = A \times A$ — largest.

$R_5 = \{(1,2), (2,1), (1,1), (2,2)\}$

Ex: If $A = \{1, 2\}$ then no. of transitive rel^n possibles on $A = ?$

→ Rel^n which are not transitive are $R_1 = \{(1,2), (2,1)\}$

$R_2 = \{(1,2), (2,1), (1,1)\}$

$R_3 = \{(1,2), (2,1), (2,2)\}$

$\therefore$ Transitive $rel^n = 2^{n^2} - 3$

$= 2^4 - 3 = 16 - 3$

$= 13$

<u>Equivalence Relation</u> :- A rel^n R on a set A is said to be an equivalence rel^n if R is refexive, transitive and symmetric.

Ex : If $A = \{1,2,3\}$ then how many equivalence rel^n are possible on A = ?

$\rightarrow$

$R_1 = \{(1,1)(2,2)(3,3)\}$ — smallest equivalence rel^n on A

$R_2 = \{(1,1)(2,2)(3,3), (1,2)(2,1)\}$

$R_3 = \{(1,1),(2,2),(3,3),(2,3),(3,2),\}$

$R_4 = \{(1,1),(2,2),(3,3), (1,3),(3,1)\}$

$R_5 = A \times A$ — largest equivalence rel^n of A.

$\therefore$ 5 equivalence rel^n are possible.

Ex : If $A = \{1,2,3,4\}$ then no. of equivalence rel^n possible on A = ?

$\rightarrow$ 15.

Ex : A rel^n R on set of all real no. is defined by

$x R y \Leftrightarrow (x-y)$ is integer then R is an equivalence relation.

$\rightarrow$ 1) we have $x - x = 0$, an integer

$\Rightarrow x R x \quad \forall x$

$\therefore$ R is reflexive.

2) Let $x R y$

$\Rightarrow (x-y)$ is an integer

$\Rightarrow (y-x)$ ——"——

$\Rightarrow y R x$

$\Rightarrow$ R is symmetric

3) Let $x R y$ and $y R z$

$\Rightarrow (x-y)$ & $(y-z)$ is an integer

Now, $(x-z) = (x-y) + (y-z)$

$\Rightarrow (x-z)$ is an integer

~~Ex: A reln R on set of all integers is defined by~~

$x R y \Leftrightarrow (x-y)$ is an even no. then R is an equivalence reln.

Ex: A reln R on set of all integers is defined by

$x R y \Leftrightarrow (x-y)$ is divisible by 5 then R is an equivalence reln.

<u>Partial ordering relation</u> :-

A relation R on a set A is said to be partial ordering relation if R is reflexive, antisymmetric, and transitive.

<u>Partially ordered set</u> (Poset) :- A set 'A' with partial ordering relation R defined on A, is called Poset and denoted by [A ; R]

~~Ex: If A~~

* The relation $\le$ is a partial ordering relation, on any set of real no.s A. i.e. $[A ; \le]$ is a poset.
* The relation $|$ is a partial ordering reln on any set of +ve integer no.s. i.e. $[A ; |]$ is a poset.
* The relation $\subseteq$ is a partial ordering reln on any collection of sets 'S' i.e. $[S ; \subseteq]$ is a poset.

<u>Totally ordered set</u> :- A poset [A ; R] is called totally ordered set, if every pair of elements are compared w.r.t. R.

- Also called as linearly ordered set or chain.

i.e. $a R b$ or $b R a$ $\forall$ $a, b \in A$

Ex: If A is any set of real no. then the poset $[A ; \le]$ is a totally ordered set.

Ex: Let $A = \{1, 2, 3 ---- 10\}$ then poset $[A ; |]$ is not a totally ordered set.

Ex: —"— $A = \{1, 4, 8, 16, 32\}$ —— " —— " —— is a totally ordered set.

Ex: If $A = \{1,2\}$ then the poset $[P(A); \subseteq]$ is not a totally ordered set

$\rightarrow$ $P(A) = \{\phi, \underbrace{\{1\}, \{2\}}_{\text{not comparable}}, \{1,2\}\}$

Ex: If $A = \{\phi, \{1\}, \{1,2\}, \{1,2,4\}\}$ the poset $[S; \subseteq]$ is a totally ordered set.

Ex: Let $A = \{1,2,3\}$ R_1 R_2 R_3 R_4 are relations on A.
which of the following is false ?

a) $R_1 = \{(1,1), (3,3)$ is symmetric and antisymmetric.

b) $R_2 = \{(1,2), (2,1), (3,2)$ is neither symmetric nor antisymmetric

c) $R_3 = \{(1,2), (2,3), (3,2)\}$ is not symmetric but antisymmetric.

d) $R_4 = \{(1,2), (2,1), (3,3)\}$ is symmetric but not antisymmetric.

$\therefore$ All stat. are true.

Ex: A relⁿ R on a set $A = \{1,2,3,4\}$ is given by

$R = \{(1,1), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$

is ___

(a) Equivalence Relation

(b) irreflexive, symmetric, transitive

(c) irreflexive, symmetric, antisymmetric

(d) Transitive

not becoz (2,2) diagonal present

Not For (2,1) (1,2) not present

not becoze (2,3) (3,2)

Ex: Relation R on set of all integers is defined by

$a R b \Leftrightarrow b = a^k$ for some +ve integer K. then which is true ?

(a) R is an equivalence relation

(b) R is a partial ordering relⁿ

(c) R is total ordering relⁿ

(d) R is a compatibility relⁿ

$\hookrightarrow$ reflexive & symmetric but not transitive

→ i) we have, $a = a^1$

$\Rightarrow a R a \ \forall a$

$\Rightarrow R$ is reflexive

Reln is not symmetric on set of all integers

Ex: $8 R 2$ but $2 R 8$ not

$\Downarrow$

$2^3 = 8$ but $2 = 8^?$

2) Let aRb and bRa

$\Rightarrow b = a^{k_1}$ and $a = b^{k_2}$ —— ①

$\Rightarrow \quad a = (a^{k_1})^{k_2}$

$a = a^{k_1 k_2}$

$\Rightarrow k_1 k_2 = 1$

$\Rightarrow k_1 = 1$ & $k_2 = 1$

$\therefore b = a$ and $a = b$ (from ①)

$\therefore R$ is antisymmetric.

3) $b = a^{k_1}$ and $c = b^{k_2}$ —— ①

$\Rightarrow c = a^{k_1 k_2}$

$\Rightarrow a R c$

$\therefore R$ is transitive.

The given reln is not a total order.

Ex: $2 \not R 3$ & $3 \not R 2$

Ex: If $A = \{a, b, c\}$ then $R_1 = \{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

is compatibility relation.

Ex: If A = {cat, bat, book, dog} then reln R on A is defined by $word_1 R word_2 \Leftrightarrow$ The two words have a common letter

then R is compatibility reln on A.

→ cat R bat and bat R book

but cat $\not R$ book (no common letters)

Ex : which of the following statements is not true ?

a) If a reln R is symmetric and transitive then it is Reflexive — FALSE

→ If $A = \{1,2,3\}$ then

$R = \{(1,1), (2,2), (1,2), (2,1)\}$

symmetric, transitive but not reflexive.

b) if R is reln on A is irreflexive and transitive then R is antisymmetric. — TRUE

$A = \{1,2,3\}$

$R = \{\ \}$

→ Let R is irreflexive and transitive but not antisym.

Let $(x,y) \in R$ and $(y,x) \in R$

(if R is not antisym. then we get two pairs (x,y) and $(y,x) \in R$).

$\Rightarrow$ $(x,x) \in R$ — by transitivity

$\Rightarrow$ R is not irreflexive.

which is contradiction.

∴ given stat. is true.

c) If R is antisymmetric relation on A, then $(R \cap S)$ is also antisymmetric for any reln S on A. — TRUE

→ $R = \{(1,1), (2,3), (3,1)\}$

Every subset of antisym. is antisym.

$(R \cap S) \subseteq R$

∴ $R \cap S$ is also antisym.

d) If R is reflexive then $\overline{R}$ is irreflexive. — TRUE

→ $\overline{R} = (A \times A) - R$

Ex: which of the following stat. is not true?

a) If R and S are reflexive relⁿ on A, then $(R \cap S)$ and $(R \cup S)$ are reflexive → <u>TRUE</u>

→ for $(R \cap S)$ & $(R \cup S)$ — diagonal pairs are present.

b) If R and S are ~~anti~~ symmetric then $(R \cup S)$ & $(R \cap S)$ are also symmetric → <u>TRUE</u>.

→ $R = \{(a,b), (b,a)\}$

$S = \{(a,b), (b,a), (a,c), (c,a)\}$

$R \cap S = R$

$R \cup S = \{\text{all}\}$

∴ symmetric.

c) If R and S are transitive then $(R \cup S)$ & $(R \cap S)$ are transitive.

↳ <u>FALSE</u>

→ $R = \{(a,b)\}$ → always transitive

$S = \{(b,c)\}$

$R \cup S = \{(a,b), (b,c)\}$
↓
(a,c) absent

∴ $R \cup S$ is not transitive

$R \cap S = \{\}$ → transitive

d) If R and S are antisym. relⁿ on A, then $(R \cup S)$ need not be antisym. but $(R \cap S)$ is always antisym.

→ $R = \{(a,b)\}$

$S = \{(b,a)\}$

$R \cup S = \{(a,b), (b,a)\}$ — Not antisym.

$R \cap S = \{\}$ — antisym.

Transitive closure :- Let R be any rel^n on set A

Transitive closure of R is denoted by R^* is defined by = the smallest transitive rel^n on A which contains R.

Ex: Let $A = \{1,2,3\}$ and $R = \{(1,2),(2,3)\}$

then $R^* = \{(1,2),(2,3),(1,3)\}$

if R is transitive rel^n then $R^* = R$

Reflexive closure :-

Reflexive closure of $R = R^{\#}$

= The smallest reflexive relation on A which contain R

$= R \cup \Delta_A$

Ex: $A = \{1,2,3\}$

$R = \{(1,2),(2,3)\}$

then $R^{\#} = \{(1,2),(2,3),(1,1),(2,2),(3,3)\}$

Symmetric closure :-

Symmetric closure of $R = R^{+}$

= The smallest symmetric rel^n on A which contain R.

$= R \cup R^{-1}$

Ex: $A = \{1,2,3\}$

$R = \{(1,2),(2,3)\}$

then $R^+ = \{(1,2),(2,3),(2,1),(3,2)\}$

Note :-

1) Symmetric closure of reflexive ~~clouse~~ closure of R :-

= Reflexive closure of symmetric closure of R.

Ex: $A = \{1,2,3\}$

$R = \{(1,2),(1,1),(2,2),(3,3)\}$ — Reflexive

then [illegible] $(2,1)\}$ — Symmetric

② Transitive closure of the reflexive closure of R

= Reflexive closure of the transitive closure of R .

Ex: $A = \{1, 2, 3\}$

$R = \{(1,2), (2,3)\}$

$R_1 = \{(1,2), (2,3), (1,3)\}$ — transitive

$R_2 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,3)\ (1,3)\}$

— Reflexive clos of trans. closure .

③ Transitive closure of the symmetric closure of R need not be symmetric closure of the transitive colsure of R .

Let $A = \{1, 2, 3\}$

$R = \{(1,2)\ (2,3)\}$

$L.H.S. = \{(1,2)\ (2,3)\ (2,1)\ (3,2)\ (1,3)\ (3,1)$
$(1,1)\ (2,2)\ (3,3)\}$

$= A \times A$

$R.H.S = \{(1,2)\ (2,3)\ (1,3)\}$ — transitive

$= \{(1,2)\ (2,3)\ (1,3)\ (2,1)\ (3,2)(3,2)$

└ symm. clos. of transitive clo

$\neq A \times A$

LHS $\neq$ RHS

Ex: $R = \{(x,y) \mid y = x+1$ and $(x,y) \in \{0, 1, 2, \ldots \infty\}\}$

then the reflexive transitive closure of R = ?

→ $R = \{(0,1), (1,2), (2,3), (3,4) \ldots\}$

$(0,2), (0,3), (0,4) \ldots$

$(1,3)\ (1,4)\ (1,5) \ldots$

$(2,4)\ (2,5)\ (2,6) \ldots$

$(0,0)\ (1,1)\ (2,2)\ (3,3) \ldots$

$\therefore \{(x,y) \mid y \geq x$ and $(x,y) \in (0, 1, 2, \ldots)\}$

Ex: Let $A = \{1,2,3\}$ and a relation R on A is defined by $R = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$ then transitive closure of R is —

→ The matrix corresponding to given relⁿ is

Warshal's Algorithm

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{array} \xrightarrow{+(3,3)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & \oplus 1 \end{bmatrix}$$

	I	II	III
Column	{1,3}	{2,3}	{1,3}
Row	{1,3}	{2}	{1,2,3}

(1,1) {2,2} {1,1} {1,2} {1,3}
(1,3) {3,2} {3,1} {3,2} {3,3}
(3,1)
(3,3)

$\therefore \quad R^* = \{(1,1),(1,3),(2,2),(3,1),(3,2)(3,3)(1,2)\}$

$= (A \times A) - \{(2,1),(2,3)\}$

Ex: Let $A = \{a,b,c,d\}$ and relⁿ R on the set A is defined by $R = \{(a,d),(b,a),(b,c),(c,a),(c,d),(d,c)\}$ Find transitive closure of R.

→ Matrix =
$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array} \xrightarrow[\substack{+\{d,a\} \\ +\{d,d\}}]{+\{b,d\}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \cancel{0}\,1 & 0 & 1 & \cancel{0}\,1 \end{bmatrix}$$

	I	III	II	IV	
column	{b,c}	{b,d}	{ }	~~{a,c}~~	{a,b,c,d}
Row	{d}	{a,d}	{a,c}	~~{c}~~	{a,c,d}

{b,d} {b,a} {d,a}
{c,d} (b,d) (d,d)

{a,c} {a,a} {a,c} {a,d}
{c,c} {b,a} {b,c} {b,d}
{c,a} {c,d} {d,a}
{d,c} {d,d}

$R^* = \{(a,d),(b,a),(b,c),(c,a),$
$(c,d),(d,c),(b,d),(a,a),$
$(a,c)(d,a)(d,d),(c,c)\}$

Equivalence classes :- Let R be an equivalence Relⁿ for a set A, for any element $x \in A$, equivalence class of A denoted by $[x]$, is defined as

$$[x] = \{ y \mid y \in A \text{ and } xRy \}$$

Note : * We can have $[x] = [y]$ even though $x \neq y$

* Set of all distinct equivalence classes of A define a partition of A.

Partition of a set :- Let A be a set with n-elements $(n \geq 2)$. A subdivision of A, $\{A_1, A_2, \ldots A_k\}$ into nonempty and non overlapping subset is called a partition

if $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = A$

Ex: $A = \{1, 2, 3, 4, 5\}$

$P_1 = \{ (1,2), (3,4), (5) \}$ (No common element is present).

Ex: An equivalence relation on the set $A = \{a, b, c, d, e, f\}$ is defined by $R = \{ (a,a), (b,b), (c,c), (d,d), (e,e), \cancel{(a,d)}, (f,f)$ $(a,f)(f,a)(b,e)$ $\cancel{(d,a)}, \cancel{(b,a)}, \cancel{(a,b)}$ $\cancel{(e,d)}, \cancel{(d,e)}$ $(e,b) \}$

Find partition of A defined by R.

$\rightarrow$

$$\left.\begin{aligned} [a] &= \cancel{\{a,b,d\}}\ \{a, f\} \\ [b] &= \cancel{\{b, a}\ \ \{b, e\} \\ [c] &= \{c\} \\ [d] &= \{d\} \\ [e] &= \cancel{\{e\}}\ \{e, b\} \\ [f] &= \{a, f\} \end{aligned}\right\} \text{distinct}$$

Required partition $= \{ (a,f), (b,e), (c), (d) \}$

Ex: Let $A = \{1, 2, 3, 4, 5\}$ and a partition of A is given as $P = \{ (1,3), (2,5), (4) \}$ Find equivalence relⁿ of A w.r.to. P.

$\rightarrow$ $R = \{ (1,3) \times (1,3), (2,5) \times (2,5), (4) \times (4) \}$

$R = \{ (1,1), (1,3), (3,1), (3,3), (2,2), (2,5), (5,2), (5,5), (4,4) \}$

Ex : Let A = set of real no.s , An equivalence relⁿ R on A is defined by $x R y \iff (x-y)$ is an even no. then $[1] = ?$

→ $[x] = \{ y \mid (x-y) \text{ is an even no.} \}$

$[1] = \{ y \mid (1-y) \text{ is } —\text{''}— \}$

$= \{ \pm 1, \pm 3, \pm 5, ---- \}$

= set of all odd numbers.

$[2] = \{ y \mid (2-y) \text{ is } —\text{''}— \}$

$= \{0, \pm 2, \pm 4, \pm 6 ---- \}$

= set of even no.s .

Ex : Let A = set of real no.s, An equivalence relⁿ R on A is defined by $x R y \iff (x-y)$ is integer then

i) what is the equivalence class of 1 ?

ii) $[1/2] = ?$

→ $[x] = \{ y \mid x-y \text{ is integer} \}$

$[1] = \{ y \mid (1-y) \text{ is integer} \}$

= set of all integers

$[1/2] = \{ y \mid (1/2 - y) \text{ is integer} \}$

$= \{ \pm 1/2, \pm 3/2, \pm 5/2 --- \}$

= odd multiple of 1/2

$= \{ (n + 1/2) \mid n \text{ is any integer} \}$

Ex : Let A = set of all integers

$x R y \iff (x-y)$ is divisible by 3 . How many distinct equivalence classes are possible on A w.r.to R ?

→ $[x] = \{ y \mid (x-y) \text{ is divisible by } 3 \}$

$[0] = \{ --- -9, -6, -3, 0, 3, 6, 9 --- \}$

$[1] = \{ ---- -8, -5, -2, 1, 4, 7, 10 --- \}$

$[2] = \{ --- -7, -4, -1, 2, 5, 8, 11, --- \}$

$A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A\}$

Lattice Properties :- A lattice is denoted by $[L, \cup, \cap]$
$\forall\ a, b, c \in L$

1) $a \cup b = b \cup a$
$a \cap b = b \cap a$

2) $(a \cup b) \cup c = a \cup (b \cup c)$
$(a \cap b) \cap c = a \cap (b \cap c)$

3) $a \cup a = a$
$a \cap a = a$

4) $a \cup (a \cap b) = a$
$a \cap (a \cup b) = a$
$b \cup (b \cap a) = b$
$b \cap (b \cup a) = b$

5) In a lattice L, $a \cup b = b \iff (a \cap b) = a \quad \forall\ a, b \in L$

Sub lattice : Let $[L, \cup, \cap]$ be a lattice, a subset M of L is called a sublattice of L if

i) $[M, \cup, \cap]$ is a lattice

ii) LUB of (GLB) of any two elements a and b of M is same as LUB of a and b in L.

Distributive lattice : A lattice L is said to be distributive if

$$\left.\begin{aligned} a \cup (b \cap c) &= (a \cup b) \cap (a \cup c) \\ a \cap (b \cup c) &= (a \cap b) \cup (a \cap c) \end{aligned}\right\} \forall\ a, b, c \in L$$

Bounded lattice : Let M be lattice w.r.t. Rel^n R
if there exist an element $I \in L$ such that
$a R I \quad \forall\ a \in L$ then I is called upper bound of L

Similarly, if there exist element $O \in L$ such that
$O R a \quad \forall\ a \in L$ then O is called lower bound of L

In a lattice L if upper bound and lower bound exists then it is called Bounded Lattice.

– In a bounded lattice the upper bound (and lower bound) in unique.

In a bounded lattice L, following properties hold good :-

$$\left.\begin{array}{l} a \cup I = I \\ a \cap I = a \end{array}\right\} \forall a \in L$$

$$\left.\begin{array}{l} a \cup 0 = a \\ a \cap 0 = 0 \end{array}\right\} \forall a \in L$$

Complement :- Let L be a bounded lattice for element $a \in L$ if there exists an element $b \in L$

such that $\left.\begin{array}{l} a \cup b = I \\ a \cap b = 0 \end{array}\right\}$ then b is called complement of a

* In a lattice complement of an element may or may not exists and need not be unique if exists

Complemented Lattice :- In a lattice if each element has a complement, then it is called complemented lattice.

* In a complemented lattice each element has atleast one complement.

* In a distributive lattice complement of element if exists, then it is unique.

* In a distributive lattice each element has atmost one complement.

Boolean Algebra :- If a lattice is distributive and complemented then it is called Boolean algebra / Boolean lattice.

* In a Boolean Algebra each element has unique complement.

Hasse Diagram (Poset Diagram) :-

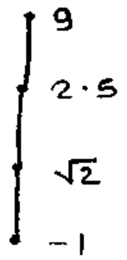
Let [A ; R] be a finite poset on the hasse diagram of A,

if 1) There is a vertex corresponding to each element of A

2) An edge betn two elements a and b is not present iff there is an element x such that $a R x$ and $x R b$

3) An edge betn a and b is present iff $a R b$ and there is ~~an~~ no element x such that $a R x$ and $x R b$.

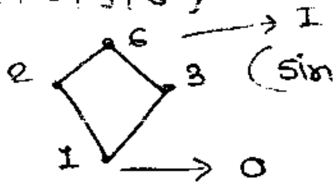
Ex: Draw Hasse dia. for the poset $[A; \leq]$ where $A = \{-1, \sqrt{2}, 2.5, 9\}$



* In a total divided set complement exists only for lower bound and upper bound (Not a complemented lattice).

Ex: Draw Hass dia. for the poset $[D_6 ; |]$ where

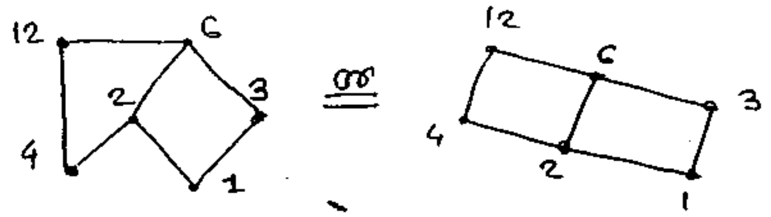
$D_6 = \{1, 2, 3, 6\}$



(Since (2 & 3) present direct edge 1 to 6 not allowed)

Ex: Draw Hass. Dia. for $[D_{12} ; |]$

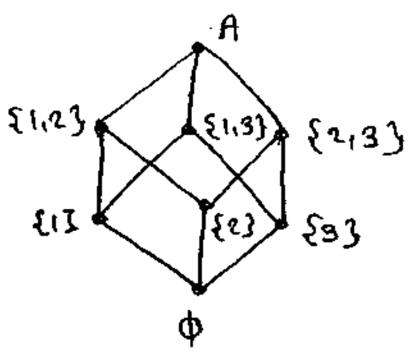
$D_{12} = \{1, 2, 4, 3, 6, 12\}$



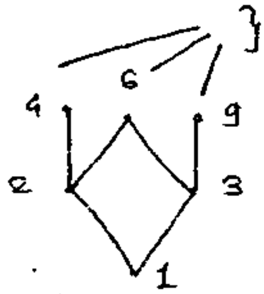
Ex: Draw Hass. Dia. $[P(A) ; \subseteq]$

where $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\} \{1,3\} \{2,3\}, A\}$



Ex: Draw the Hasse Dia. For {1, 2, 3, 4, 6, 9} w.r.t. '|'



} Maximal elements

No upper bound

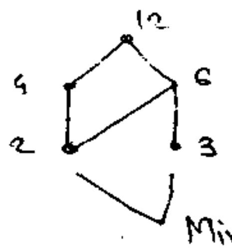
∴ Not lattice.

Lower bound is present.

GLB exists for every pair of elements

∴ It is meet semi-lattice.

Ex: Draw Hasse dia. for {~~1~~ 2, 3, 4, 6, 12} w.r.t. '|'



Not lattice.

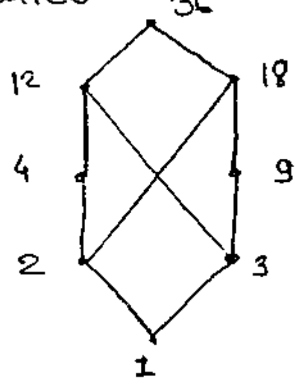
Minimal elements.

But it is join semi lattice

Ex: The poset {1, 2, 3, 4, 9, 12, 18, 36} w.r.t. '|' is ___

a) a join semi lattice b) a meet semi lattice

~~c)~~ a lattice d) none.



Not lattice

for 2 & 3 no LUB.

whenever there are cross edges it is not lattice.

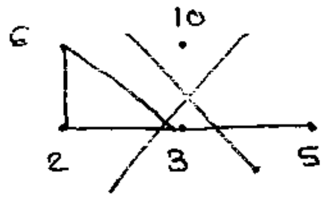
Ex: The poset {2, 3, 5, 6, 10, 15} w.r.t. | is

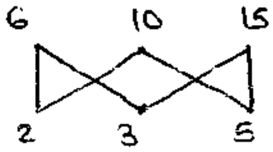
a) a join semi lattice but not a meet semi lattice

b) a meet semi lattice but not a join semi lattice

c) a lattice

✓d) not a semi lattice





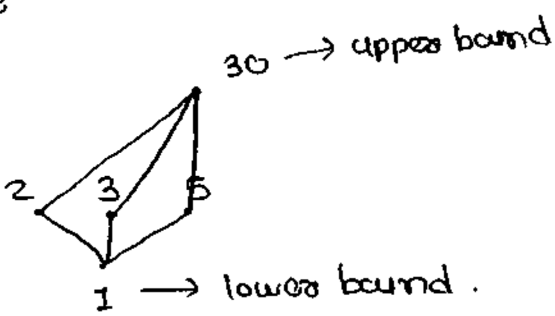
3 - Minimal

3 - Maximal element

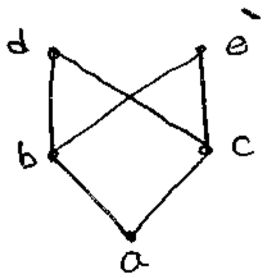
∴ it is not a lattice

Ex: The poset {1, 2, 3, 5, 30} w.r.t. |

→ is a lattice

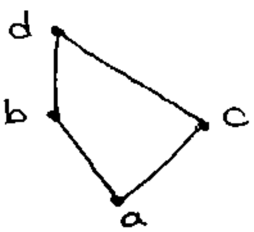


Ex: Consider the poset P = {a, b, c, d, e} shown below which of the following is false ?

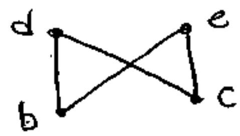


a) P is not a lattice.

b) The subset {a, b, c, d} of P is a lattice.

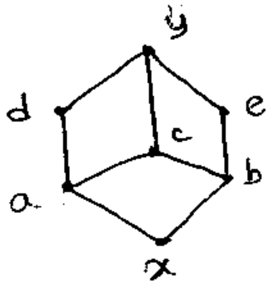


✓c) The subset {b, c, d, e} of P is a lattice.

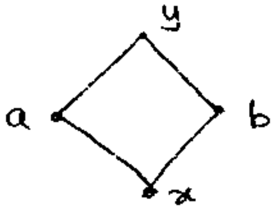


d) The subset {a, b, c, e} of P is lattice.

Ex: Consider lattice $L = \{x, a, b, c, d, e, y\}$ shown below which of the following subsets of L are sub lattice of L.

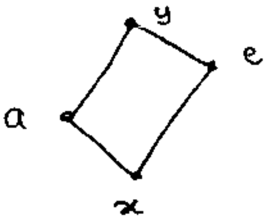


a) $L_1 = \{x, a, b, y\}$



∴ Not sublattice because LUB in this lattice is y but in given diagram it is 'c' of a & b

b) $L_2 = \{x, a, e, y\}$

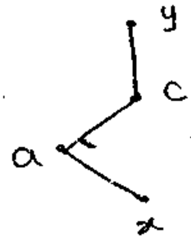


LUB of a & e is y

∴ L_2 is sublattice

for every pair of elements in L_2 LUB & GLB exists and they are same as original lattice.

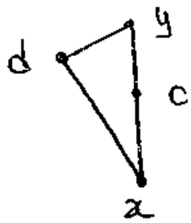
c) $L_3 = \{x, a, c, y\}$



It is a lattice

L_3 is sublattice of L, because for element a & c LUB & GLB are same as L.

d) $L_4 = \{x, c, d, y\}$



Not sub lattice

because for d & c GLB is not same as given lattice.

In L it is 'a' and in this dia. it is 'x'.

Ex: which of the following is not true?

a) The upper bound of the lattice $[D_n ; /]$ is n where n is a +ve integer.

b) The lower bound of the lattice $[D_n ; /]$ is 1 where n is a +ve integer.

c) The lower bound of the lattice $[P(A); \subseteq]$ is ϕ where A is a finite set.

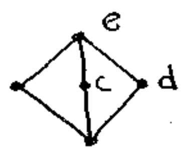
d) The upper bound of the lattice $[P(A); \subseteq]$ is A where A is a finite set.

e) The upper bound of the lattice $[N; \leq]$ is ∞ where $N = \{1, 2, 3, \ldots \infty\}$.

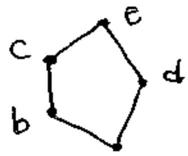
It is false because upper bound not exist and ∞ is not +ve integer

f) The lowerbound of the lattice $[N; \leq]$ is 1 where $N = \{1, 2, 3, \ldots \infty\}$

Ex: Which of the following is not a distributive lattice?



L_1^*



L_2^*

$b \cup (c \cap d) = (b \cup c) \cap (b \cup d)$

$b \cup a \quad | \quad e \cap e$

$b \neq e$

L_1^* Not distributive

∴ ~~Not distributive lattice~~.

For L_2^*,

$b \cup (c \cap d) = (b \cup c) \cap (b \cup d)$

$b \cup a \quad | \quad c \cap e$

$b \neq c$

∴ L_2^* also not distributive.

Theorem: A lattice L is not distributive iff L has a sublattice which is isomorphic to L_1^* or L_2^*.

Ex: which of the following lattice is NOT distributive?

a) $[P(A); \subseteq]$ where A is any finite set.

→ The elements of the power sets are sets. and for any 3 sets the distributive law holds good

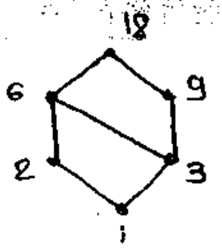
b) Every totally ordered set is a distributive lattice.

→ A totally ordered set can not have sublattice which is

c) $[D_{18} ; |]$

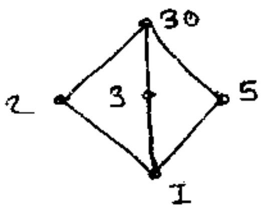
$= \{1, 2, 3, 6, 9, 18\}$

is a distributive lattice.



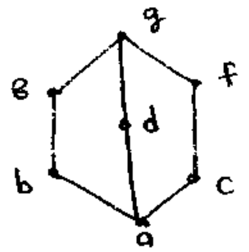
d) $[\{1, 2, 3, 5, 30\} ; |]$

$\therefore$ Not distributive.



$\equiv L_1^*$

Ex. A lattice $L = \{a, b, c, d, e, f\}$ is shown below how many complements does the element 'b' has?



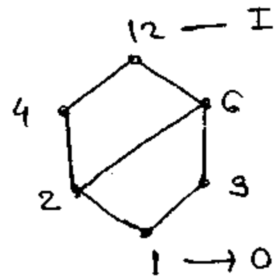
a) 1   b) 2   ~~c)~~ 3   d) 4

The complements of b are d, c, f.

The complements of d are 4 i.e. b, c, e, f.

<u>Note</u>: The given lattice is a complemented lattice but not a distributed lattice.

Ex: For the lattice $[D_{12} ; |]$, which of the following is not true?



a) complement of 3 = 4

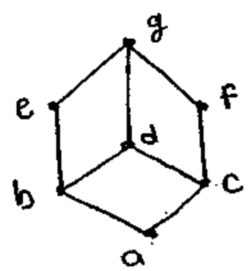
~~b)~~ —— " —— 2 = 6

c) —— " —— 1 = 12

d) —— " —— 6 does not exist.

It is distributive lattice but not complemented lattice.

Ex: For the lattice shown below which of the following is true?



a) distributive but not complemented

b) Boolean algebra

c) complement but not distributive

→ In a lattice, 'd' has no complement

∴ It is not complemented lattice.

'E' has two complements F & C

∴ It is not distributive.

Ex: If A is a finite set, then the poset [P(A) ; ⊆] is a boolean algebra.

→ If $x \in P(A)$, then

Complement of $x = A - x$

· If $A = \{a, b, c\}$ then the poset $\{P(A) ; \subseteq\}$ is a boolean algebra

Complement of $\{a, c\} = A - \{a, c\}$

$= \{b\}$

<u>Square free integer</u> :- A positive integer n is said to be square free, if D_n has no elements that are perfect square (except 1).

Ex: If n is a square free integer then the poset $[D_n ; |]$ is a boolean algebra.

* If $[D_n ; |]$ is a boolean algebra then complement of

$$\bar{x} = \frac{n}{x} \quad \forall x \in D_n$$

Ex: which of the following Posets is not a boolean algebra. ?

a) $[D_{21} ; |]$

$= [\{1, 3, 7, 21\} ; |]$

As 21 is square free integer

∴ D_{21} is boolean algebra.

b) $[D_{90} ; |]$

$90 \Rightarrow 2 \cdot 3 \cdot 3 \cdot 5$

(Not product of distinct prime integer)

∴ Not a square free integer

∴ D_{90} is not boolean algebra.

c) $[D_{110} ; \mid]$

$110 = 2 \cdot 5 \cdot 11$

$$\begin{array}{r|l} 2 & 110 \\ \hline 5 & 55 \\ \hline & 11 \end{array}$$

$\therefore$ square free integer.

$\therefore$ D_{110} is boolean algebra.

d) $[D_{30} ; \mid]$

$30 = 2 \cdot 3 \cdot 5$

D_{30} also boolean algebra.

Ex: In the boolean algebra $[D_{110} ; \mid]$,

complement of $5 = \frac{110}{5}$

$$x = \frac{n}{x} \quad \forall x \in D_n$$

a) 10 b) 11 c) 22 d) 55

Groups :-

Algebraic structure : A non empty set S is called an algebraic structure w.r.t. a binary operation $*$, if $(a * b) \in S \quad \forall a, b \in S$

i.e. $*$ is a closed operation on S

$(S, *)$

$N = \{1, 2, 3, --- \infty\}$

$Z =$ set of all integers

$Q =$ set of rational no.s

$R =$ set of all real no.s

Ex: $(N, +)$ is an algebraic structure.

Ex: $(N, \cdot)$ is an algebraic structure.

Ex: $(N, -)$ not an algebric structure.

Ex: $\{Z, +, \cdot, -\}$ an algebraic structure.

Ex: $\{Q, \div\}$ is not ——— " ———.

Ex: $\{Q^*, \div\}$ is an algebraic structure

$Q^* = Q - \{0\}$ = set of all non zero rational no.

<u>Semi Group</u> :- An algebraic structure $(S, *)$ is called a semigroup iff $a * (b * c) = (a * b) * c$, $\forall$ $a, b, c \in S$

i.e. $*$ is associate on S.

$\Rightarrow$ $(Q^*, \div)$ is not a semi group because $\div$ is not associative

$$a \div (b \div c) \neq (a \div b) \div c$$

$\Rightarrow$ $(Z, -)$ is not a semi group

$$a-(b-c) \neq (a-b)-c$$

$\Rightarrow$ $(Z, +)$ is a semi group

$$a+(b+c) = (a+b)+c$$

$\Rightarrow$ $(N, \cdot)$ is a semigroup.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

<u>Monoid</u> :- A semi group $(S, *)$ is called a monoid if there exist an element $e \in S$, such that

$$a * e = e * a = a$$

i.e. e = Identity element of S w.r.t. $*$

Ex: $\{N, \cdot\}$ is a monoid, with identity element $1 \in N$

Ex : $\{N, +\}$ is not a monoid.

because $0 \notin N$.

Ex : $\{Z, +\}$ is a monoid

because $0 \in Z$

<u>Group</u> :- A monoid $(S, *)$ is called a group if to each element $a \in S$, there is an element $b \in S$, such that

$(a * b) = (b * a) = e$. ~~then~~

b = inverse of a

$= a^{-1}$

Ex : $(Z, +)$ is a group.

Ex : $(Z, \cdot)$ is not a group.

$$a^{-1} = \frac{1}{a}$$

$$a\left(\frac{1}{a}\right) = 1$$

Ex : $(Q, \cdot)$ is not a group

0 has no inverse

Ex : $(Q^*, \cdot)$ is a group

$$Q^* = Q - \{0\}$$

Abelian Group :- (commutative group)

A group (G, *) is said to be abelian, if $a*b = b*a \ \forall a,b \in G$

Ex : $(Z, +)$ is abelian group :- $a+b = b+a \ \forall a,b \in Z$

Ex : $(Q^*, \cdot)$ ——"—— :- $(a\cdot b) = (b \cdot a) \ \forall a,b \in Q^*$

Ex : Set of all non singular matrices of order n is a group w.r.t. matrix multiplication but not a abelian group.

$$A \cdot B \neq B \cdot A$$

Ex : Set of all bijective functions on a set A is a group w.r.t. function composition but not abelian group.

$$(f \circ g) \neq (g \circ f)$$

Ex : Let $A = \{2, 4, 6, 8, \ldots \infty\}$
$B = \{1, 3, 5, 7, \ldots \infty\}$ which of the following is not a semigroup.

a) A w.r.t '+' b) A w.r.t '·'

~~c)~~ B w.r.t. '+' d) B w.r.t. '·'

sum of two odd no. is even.

Ex : Let $A = \{1, 2, 3, \ldots \infty\}$ and * denoted by $a*b = a^b$ then (A, *) is

a) semigroup but not monoid
b) monoid but not a group
c) a group
~~d)~~ not a semigroup.

$$(a*b)*c = (a^b)*c = (a^b)^c = a^{bc}$$

$$a*(b*c) = a*b^c = a^{b^c}$$

$$a^{bc} \neq a^{(b^c)}$$

Ex : Let $A = \{x \mid x$ is a real no. & $0 < x \le 1\}$ then $(A, *)$ is

$\rightarrow$ we have $a \cdot b \in A \;\forall\; a, b \in A$.

'·' is closed operation.

Associativity holds for any 3 real no.s betn 0 to 1

$\therefore$ '·' is associative.

1 is identity element w.r.t. '·' & $1 \in A$

but inverse of elements not exists on group except 1.

$\therefore$ Monoid but not a group

Ex : Let z = set of all integers & a binary operation $*$ is defined as : $a * b$ = maxm of $\{a, b\}$ then $(z, *)$ is —

$\rightarrow$ we have $a * b$ = max of $\{a, b\} \in z$, $\forall a, b \in z$

$\therefore *$ is closed on z.

also $(a*b)*c) = a*(b*c)$

$(3*4)*1 = 3*(4*1)$

$4*1 = 3*4$

$4 = 4$

$\therefore *$ is associative on z

$a*e$ = max. of $\{a, e\} = a \quad \forall a, e \in z$

$\therefore$ The identity element w.r.t. $*$ is the smallest integer which doesn't exists.

$\therefore (z, *)$ is semi group.

Ex : Let Q^+ = set of all the +ve rational no.s & a binary operation $*$ is defined as $(a*b) = \frac{ab}{3}$ then $(Q^+, *)$ is —

$\rightarrow$ $*$ is closed & associative on Q^+

Let e = identity element

$a*e = a \quad \forall a \in Q^+$

$\frac{ae}{3} = a$

$e = 3 \in Q^+$

Let a^{-1} = inverse of a

$a*a^{-1} = e \quad a \in Q^+$

$\frac{a*a^{-1}}{3} = e$

$a^{-1} = a, \_ \in Q^+$

$\therefore (Q^+, *)$ is a group.

Properties :-

In a group $(G, *)$, the following properties hold good

1) The identity element e is unique.
2) The inverse of any element in G is unique.
3) ——— " ——— identity element $e' = e$
4) The cancellation laws hold good

$$(\cancel{a} * b) = (\cancel{a} * c) = b = c$$

$$(a * \cancel{c}) = (b * \cancel{c}) = a = b$$

5) $(a * b)^{-1} = b^{-1} * a^{-1}$

Ex: which of the following is not true?

a) In a group $(G, *)$, if $a * a = a$ then $a = e$

b) In a group $(G, *)$, if $x^{-1} = x \ \forall \ x \in G$ then G is abelian.

consider, $(a * b)^{-1} = (b^{-1} * a^{-1}) \ \forall \ a, b \in G$

$\therefore (a * b) = (b * a) \quad (\because x^{-1} = a)$

$\therefore G$ is abelian.

c) In a group $(G, *)$, if $(a * b)^2 = (a^2 * b^2)$ then G is abelian.

→ Given that, $(a * b)^2 = (a^2 * b^2)$

$\Rightarrow (a * b) * (a * b) = (a * a) * (b * b)$

$a * (b * a) * b = a * (a * b) * b$

$(b * a) = (a * b)$

$\therefore G$ is abelian.

d) In a group $(G, *)$, if $(a * b)^n = (a^n * b^n)$ then G is abelian.

where $n = \{2, 3, 4 \dots\}$ — TRUE

Finite Group :- A group with finite no. of elements.

Order of a group : $[O(G)]$ = No. of elements in G.

Ex : $\{0\}$ is a group w.r.t. '+'

- The only finite group of real no.s w.r.to '+' is $\{0\}$

Ex : $\{1\}$ is a group of order 1 w.r.to multiplication.

Ex : $\{1,-1\}$ —— " —— 2 —— " ——.

$\cdot$	1	-1
1	1	-1
-1	-1	1

$\longrightarrow$ Inverse of $1 = 1$

—— " —— $-1 = -1$

- The only finite group of real no. w.r.t. '*' are $\{1\}$ & $\{1,-1\}$

- In a group of order 2, $a^{-1} = a$, $\forall a \in G$

Ex : The cube root of unity $\{1, \omega, \omega^2\}$ is a group of order 3 w.r.t. '*'

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$x^3 = 1$

$\omega^3 = 1$

$\omega^4 = \omega^3 \cdot \omega$

$= 1 \cdot \omega$

$\omega^4 = \omega$

Inverse of $1 = 1$ $\quad (\because 1 \times 1 = 1)$

Inverse of $\omega = \omega^2$ $\quad (\because \omega \times \omega^2 = 1)$

Inverse of $\omega^2 = \omega$ $\quad (\because \omega^2 \times \omega = 1)$

$\therefore$ It is group of order 3.

Ex : The set $\{1, -1, i, -i\}$ is a group w.r.to. multiplication.

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Inverse of $1 = 1$

—— " —— $-1 = -1$

—— " —— $i = -i$

—— " —— $-i = i$

$\therefore$ It is group of order 4.

Addition Modulo M, $\oplus_M$

If a and b are any +ve integers then $a \oplus_m b$ defined as

$$a \oplus_M b = \begin{cases} (a+b) & \text{if } (a+b) < m \\ r & \text{if } (a+b) \geq m \text{ where } r \text{ is remainder obtained by dividing } (a+b) \text{ with } m. \end{cases}$$

Ex : $\underline{m=6}$

$2 \oplus_6 3 = 5$ $\quad$ $4 \oplus_6 4 = 2$

$3 \oplus_6 4 = 1$

Ex : $\{0, 1, 2, --- m-1\}$ is a group w.r.t. $\oplus_M$

Multipication Modulo M, $\otimes_M$:

If a & b are any two +ve integers then

$$a \otimes_m b = \begin{cases} (ab) & \text{if } (ab) < m \\ r & \text{if } (ab) \geq m \text{ where } r \text{ is remainder obtained by dividing } (ab) \text{ withim } m. \end{cases}$$

Ex: $2 \otimes_7 3 = 6$

Ex: $5 \otimes_7 3 = 1$

Ex: $\{1, 2, 3 \cdots p-1\}$ is a group w.r.to $\otimes_P$ where P is a prime no.

* If n is a +ve integer, then S_n = set of all +ve integers which are less than n and <u>relatively prime</u> to n.

$S_6 = \{1, 5\}$

$S_8 = \{1, 3, 5, 7\}$, $S_7 = \{1, 2, 3, 4, 5, 6\}$

* If n is any +ve integer than S_n is a group w.r.to $\otimes_n$

Ex : The set $\{0, 1, 2, 3, 4, 5\}$ is a group w.r.t. $\oplus_6$ which of the following is NOT TRUE ?

(a) The inverse of $1 = 5$

(b) —— " —— $2 = 4$

(d) —— " —— $4 = 2$

$1 \oplus_6 5 = 0$ but $3 \oplus_6 0 \neq 0$

* In a group of even order i.e. group of even no. of element there exists atleast one element a $(a \neq e)$ such that $a^{-1} = a$

Ex : The set $\{1, 2, 3, 4, 5, 6\}$ is group w.r.t. $\otimes_7$ which is False ?

ⓐ The inverse of $1 = 1$ $\quad 1 \otimes_7 1 = 1 \rightarrow$ Identity element.

ⓑ —— " —— $2 = 4$ $\quad 2 \otimes_7 4 = 1$

ⓒ —— " —— $3 = 5$ $\quad 3 \otimes_7 5 = 1$

ⓓ —— " —— $6 = 2$ $\quad 6 \otimes_7 2 \neq 1$

Ex: The set $\{1, 3, 5, 7\}$ $\otimes_8$ which is False ?

ⓐ The inverse of $1 = 1$ $\quad 1 \otimes_8 1 = 1$

ⓑ —— " —— $3 = 7$ $\quad 7 \otimes_8 3 \neq 1$

ⓒ —— " —— $5 = 5$ $\quad 5 \otimes_8 5 = 1$

ⓓ —— " —— $7 = 7$ $\quad 7 \otimes_8 7 = 1$

Ex : Consider a set $G = \{2, 4, 6, 8\}$ w.r.t. $\otimes_{10}$ which is false ?

ⓐ G is group w.r.t. $\otimes_{10}$

ⓑ The identity element of G is 6

ⓒ The inverse of $2 = 4$

ⓓ —— " —— $4 = 4$

$\otimes_{10}$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

The inverse of $2 = 8$

—— " —— $4 = 4$

—— " —— $6 = 6$

—— " —— $8 = 2$

Ex: which of the following is ~~not true~~? a group ?

a) $\{1, 2, 3, 4, 5\}$ w.r.t. $\otimes_6$

$$2 \otimes_6 3 = 0 \notin A$$

closure property fails

$\therefore$ Not a group.

b) $\{0, 1, 2, 3, 4, 5\}$ w.r.t. $\otimes_6$

for '0' no multiplicative inverse exists

$$0 \otimes_6 x = 0$$

$\therefore$ Not a group.

c) $\{1, 2, 3, 4, 5\}$ w.r.t. $\oplus_6$

$$2 \oplus_6 4 = 0$$

$\therefore$ Not a group

d) $\{1, 2, 3, 4\}$ w.r.t $\otimes_5$

It is a group.

<u>Order of an element of a group</u> :-

Let $(G, *)$ be a group with identity element 'e'

for $a \in G$.

The order of $a = o(a)$ is defined as n

where n is smallest +ve integer such that $a^n = e$

Ex: In the group $\{1, -1\}$ w.r.t. multiplication.

$$o(1) = 1$$
$$o(-1) = 2$$

Ex: In the group $\{1, \omega, \omega^2\}$ w.r.t. multiplication.

$$o(1) = 1$$
$$o(\omega) = 3$$
$$o(\omega^2) = 3$$

Ex: In the group $\{1, \omega, \omega^2\}$ w.r.t. multiplication

* order of a = order of a^{-1} $\forall a \in G$.

Ex: In the group $\{1, -1, i, -i\}$ w.r.t. multiplication

$$o(1) = 1 \qquad o(i) = 4$$
$$o(-1) = 2 \qquad o(-i) = 4$$

* In a group $(G, *)$ order of a = divisor of $o(G)$ $\forall a \in G$.

Ex: In the group $\{0, 1, 2, 3\}$ w.r.t. $\oplus_4$.

order of 0 = 1
— " — 1 = 4
— " — 2 = 2
— " — 3 = 4

$$2^2 = 2 \oplus_4 2 = 0$$
$$1^3 = (1 \oplus_4 1) \oplus_4 1$$
$$= 3$$
$$1^4 = 1^3 \oplus_4 1$$
$$= 3 \oplus_4 1$$
$$1^4 = 0$$

Ex: Group $\{1,3,5,7\}$ w.r.t. $\otimes_8$

$O(1) = 1$ $\quad\quad\quad\quad$ $3^2 = 3 \otimes_8 3 = 1$

$O(3) = 2$

$O(5) = 2$

$O(7) = 2$

<u>Subgroups</u> :- Let $(G, *)$ be a group. A subset H of G is called a subgroup of G, if H is a group w.r.t. $*$

Ex: $G = \{1, -1, i, -i\}$ is a group w.r.t. multipication

$H_1 = \{1, -1\}$ is a subgroup of G.

$H_2 = \{1\}$ is a subgroup.

<u>Note</u> :- For the group $(G, *)$ with identity element e, G and $\{e\}$ are called trivial subgroup of G.

<u>Proper Subgroup</u> :- Any subgroup of $(G, *)$, which is not a trivial subgroup of G, is called as proper subgroup of G.

<u>Theorm 1</u> : Let H be a non empty subset of a group $(G, *)$ H is a subgroup of G iff $(a * b^{-1}) \in H \quad \forall a, b \in H$

<u>Theorm 2</u> : Let H be a non empty finite subset of a group $(G, *)$ then H is a subgroup of G iff $(a * b) \in H \quad \forall a, b \in H$

<u>Theorm 3</u> : (Lagrange's theorm)

If H is a subgroup of a group $(G, *)$, Then $O(H)$ is a divisor of $O(G)$.

Ex: $G = \{1, 2, 3, 4, 5, 6\}$ is a group w.r.t. $\otimes_7$

which of the following are subgroup of G ?

a) $\{1, 6\}$ $\quad\quad$ c) $\{1, 2, 4\}$

b) $\{1, 8, 5\}$ $\quad\quad$ d) $\{1, 3\}$

	1	2	4
1	1	2	4
2	2	4	1
4	4	1	2

Ex : $G = \{0,1,2,3,4,5\}$ is a group w.r.t. $\oplus_6$

which are subgroups ?

✓a) $\{0,3\}$ ✓c) $\{0,2,4\}$

✓b) $\{0,1,5\}$ d) $\{0,1,2,4,5\}$

Ex : which of the following stat. is not true ?

a) Every subgroup of an abelian group is also an abelian group.

✓b) Union of any two subgroups of a group 'G' is also a subgroup of G.

c) Intersection —— ॥ —— .

d) Union of 2 subgroups H_1 & H_2 of a group $(G, *)$ is also a subgroup of G, iff $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Ex : If $(G,*)$ is a group of order P, where P is a prime no. then no. of subgroups in $(G,*)$ is ——

a) $P-1$

b) P

✓c) 2

d) 0

→ Let $(H,*)$ be a subgroup of G.

Let $O(H) = n$

By lagranges theorm n is ~~divisible~~ divisor by p.

$\Rightarrow n = 1$ or $n = P$

$\Rightarrow H = \{e\}$ or $H = G$

<u>Cyclic</u> <u>Group</u> :- Let $(G,*)$ be a group with identity element e. If there exists an element $a \in G$, such that every element of G can written in the form a^n for some integer n. then G is called as cyclic group. and element a is called Generator of G.

Ex : The set $\{1,-1\}$ is a cyclic group with respect to multipication generator -1.

$$\{1, -1\} \quad 1 = (-1)^2, \; -1 = (-1)^1$$

Ex : $\{1, \omega, \omega^2\}$ w.r.t. Multi.

Generator $= \omega, \omega^2$

* In a cyclic group $(G,*)$ with a generator a, a^{-1} is also a generator of G.

Ex: The set $\{1, -1, i, -i\}$ is a cyclic group w.r.t. multiplication.

The generators $= i, -i$

* $\boxed{O(G) = O(a)}$ (* Identity elements can not be generator.)

Ex: The set $\{0, 1, 2, 3\}$ is a cyclic group w.r.t. $\oplus_4$

$$\left.\begin{array}{l} 1^2 = 1 \oplus_4 1 = 2 \\ 1^3 = 1^2 \oplus_4 1 = 2 \oplus_4 1 = 3 \\ 1^4 = 1^3 \oplus_4 1 = 3 \oplus_4 1 = 0 \end{array}\right\} \text{1 is Generator.}$$

Generator is 1 & 3.

3 is inverse of 1.

Ex: The set $\{1, 2, 3, 4\}$ is a cyclic group w.r.t. $\otimes_5$

$$\left.\begin{array}{l} 2^2 = 2 \otimes_5 2 = 4 \\ 2^3 = 2^2 \otimes_5 2 = 4 \otimes_5 2 = 3 \\ 2^4 = 2^3 \otimes_5 2 = 3 \otimes_5 2 = 1 \end{array}\right\} \text{2 is Generator.}$$

As 3 is inverse of 2

$\therefore$ 3 is also generator.

* Let $(G, *)$ be a cyclic group of order n, with generator a.

i) a^m is also a generator of G, if m and n are coprime

i.e. $\gcd(m, n) = 1$

ii) Number of generators in G $= \phi(n) =$ Euler function of n.
$=$ no. of positive integers less than n and relatively prime to n.

Ex: Let $(G, *)$ be a cyclic group of order 8 with generator a then how many generators are there and what are they?

$\rightarrow$ No. of generator $= \phi(8)$ $\quad (\because \{1, 2, 3, 4, 5, 7\})$

$= 4$

The generators are $= a^1, a^3, a^5, a^7$.

Ex: Let $(G, *)$ be a cyclic group of order 70 then no. of Generator in G = ?

$\rightarrow$ $$\phi(70) = 70\left\{\frac{(2-1)(5-1)(7-1)}{2\cdot 5\cdot 7}\right\}$$

$$\begin{array}{r|l} 2 & 70 \\ \hline 5 & 35 \\ \hline & 7 \end{array}$$

* $\phi(n) = n\left[\frac{(P_1-1)(P_2-1)\cdots(P_k-1)}{P_1 \cdot P_2 \cdots P_k}\right]$ where $P_1, P_2, P_3 \cdots P_k$ distinct prime factors of n.

Ex: $\{0,1,2,3,4\}$ is a cyclic group of order 5 then how many generators are there & what are they? (w.r.t. $\oplus_5$)

→ $\phi(n) = n-1$ — if n - prime no.

$\phi(5) = 4$ & The generators are 1, 2, 3, 4

Ex: The set $\{1,2,3,4\}$ is a cyclic group w.r.t. $\otimes_7$ of order 6 then how many generators & what are they?

→ $\therefore \phi(6) = 2$ $\{1, 2, 3, 4, 5, 6\}$

$2^2 = 2 \otimes_7 2 = 4$

$2^3 = 2^2 \otimes_7 2 = 1$

$\Rightarrow O(2) = 3$

$3^2 = 3 \otimes_7 3 = 2$

$3^3 = 3^2 \otimes_7 3 = 2 \otimes_7 3 = 6$

$3^5 = 3^4 \otimes_7 3 = 4 \otimes_7 3 = 5$

$3^6 = 3^5 \otimes_7 3 = 5 \otimes_7 3 = 1$

$\therefore$ 3 is a generator

Also 5 is generator as inverse of 3.

Ex: The group $\{1,3,5,7\}$ w.r.t. $\otimes_8$ is NOT a cyclic group. there is no generator for this group.

$1^2 = 1 \otimes_8 1 = 1$

$3^2 = 3 \otimes_8 3 = 1$

* Following properties hold good in cyclic group
  1. Every group of prime order is cyclic.
  2. Every ~~eh~~ cyclic group is abelian group.
  3. every group of prime order is abelian group.
  4. Every subgroup of a cyclic group is cyclic.

Functions :-

- A relation F from A to B, is called a function, if each element of A is mapped with a unique element in B. & denoted by $F: A \to B$.

    Here $A \to$ Domain of F and $B \to$ co-domain of F.

- Range of $F = \{ y \mid y \in B \text{ and } (x,y) \in F \}$
- Range of $F \subseteq$ co-domain of F.
- If Range = co-domain, then it is an 'onto' function.
- A function $F: A \to A$ is called a function on A.
- If $|A| = m$ and $|B| = n$, then no. of functions possible from A to $B = n^m$.

One to one function (Injection) :-

- A function $f: A \to B$ is said to be one to one, if no two elements in A have same image in B. or if $f(a) = f(b) \Rightarrow a = b$

* If A and B are finite sets then a 1 to 1 function from A to B is possible iff no. of elements in A i.e. $|A| \leq |B|$

* If $|A| = m$ & $|B| = n$ then no. of 1 to 1 functions possible from A to B $(m \leq n)$ ~~then~~ $= P(n,m) = {}^nP_m$

If $|A| = |B| = n$ then $= {}^nP_n = n!$

On-to function

- A function $F: A \to B$ is said to be on-to if each element of B is mapped by atleast one element of A. i.e. Range of F = B = codomain.
- If A and B are finite sets, then an onto function from A to B is possible iff $|B| \leq |A|$
- If $|A| = m$ and $|B| = n$ $(n < m)$ then no. of onto functions possible $= n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - {}^nC_3 (n-3)^m + \cdots + {}^nC_{n-1} (-1)^{n-1} 1^m$

Ex: For $m = 6$ $n = 3$

$= 3^6 - {}^3C_1 \cdot 2^6 + {}^3C_2 \cdot 1^6$

$= 729 - 3(64) + 3$

Note :- If $|A| = |B|$ then every 1 to 1 funⁿ from $A \to B$ is onto and vice versa.

- If $|A| = |B| = n$ then no. of onto functions $= n!$

Bijection :- A function $F : A \to B$ is said to be a bijection, if $F : A \to B$ is 1 to 1 and onto

* If A and B are finite sets then a Biejection from A to B is possible iff $|A| = |B|$
* If $|A| = |B| = n$ then no. of bijection possible from A to B $= n!$

Inverse of a function :- If $F : A \to B$, then the inverse relation from B to A, is called a function (if the inverse relⁿ F^{-1} from B to A is a function), then it is called inverse of F, and denoted by $F^{-1} : B \to A$

* Inverse of a function $F : A \to B$ exists iff $F : A \to B$ is a bijection.

Identity function :- $F : A \to B$ is called an identity function

if $f(x) = x \quad \forall x \in A$

$I : A \to A$ or I_A.

- Every identity function is a bijection.

Function composition :-

$F : A \to A$ and $g : A \to A$, then

i) $(F \circ g) : A \to A$ defined by $(F \circ g)(x) = F\{g(x)\}$

ii) $(g \circ f) : A \to A$ defined by $(g \circ F)(x) = g\{F(x)\}$

- In general $(F \circ g) \neq (g \circ f)$
- If $F : A \to A$ and $I : A \to A$ then

$(F \circ I) = (I \circ F) = F$

- If $F : A \to A$ is a bijection, then $(F \circ F^{-1}) = (F^{-1} \circ F) = I$
- If $F : A \to B$ and $g : B \to C$, then

i) $(g \circ F) : A \to C$

ii) $(F \circ g)$ may not be defined

$(g \circ F) : A \to A$ defined by

only when ~~range of g~~ $\subseteq$ domain of F.

- If $F: A \to B$ then i) $F \circ I_A = f$ and

  ii) $I_B \circ F = F$

- If $F: A \to B$ is a bijection then i) $F \circ F^{-1} = I_B$ and $F^{-1} \circ F = I_A$

  $F^{-1} = B \to A$

Ex: which of the following is a function on set of all real no.s

a) $F_1(x) = 1/x$ $\rightarrow$ for the real no. 0' image does not exists.

b) $F_2(x) = \sqrt{x}$ $\rightarrow$ For -ve real no. in domain image not exists

c) $F_3(x) = \pm\sqrt{x^2+1}$ —— For $1 < \begin{matrix} -\sqrt{2} \\ \sqrt{2} \end{matrix}$ } two images of each element ∴ Not function

d) $F_4(x) = |x|$ $\quad \begin{matrix} -1 \\ 1 \end{matrix} > 1$ (For two element one image)

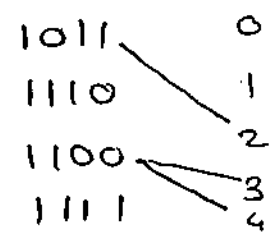
∴ It is a Function.

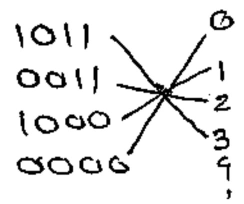
Ex: which of the following relations from set of all bit strings to set of all integers is a function

$F_1(S)$ = The position of a zero bit in the bit string S

$F_2(S)$ = The no. of 1 bits in the bit string S.

$\rightarrow$ $F_1 : S^* \rightarrow Z$ $\qquad$ $F_2 : S^* \rightarrow Z$





f_1 is not a function because a string with 2 or more '0' will have two or more images

For any bit string the no. of 1 bits is non-ve integer

∴ Each bit string has only one image.

∴ F_2 is a function.

Ex: Consider the following function on set of all integers.

$$f(x) = x^3 \text{ and } g(x) = \left\lceil \frac{x}{2} \right\rceil$$

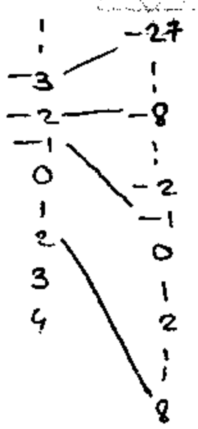
which of the following stat. are true ?

S_1) F is 1 to 1

S_2) F is onto

S_3) g is 1 to 1

→ $F : Z \to Z$



let, $F(a) = F(b)$

$a^3 = b^3$

$a = b$

$\therefore$ F is 1 to 1

F is not onto.

g is not 1 to 1

bcoz the integer 1 & 2 in domain has same image.

g is onto function.

ex. Let $A = R - \{3\}$ and $B = R - \{1\}$ where $F : A \to B$ is defined by $F(x) = \left(\frac{x-2}{x-3}\right)$ which is true ?

a) F is 1-1 but not onto

b) F is onto but not 1-1

c) F is a bijection

d) F is neither 1-1 nor onto

→ $f(a) = F(b)$

$$\frac{a-2}{a-3} = \frac{b-2}{b-3}$$

$(a-2)(b-3) = (b-2)(a-3)$

$a = b$

$\therefore$ one to one function.

Let $f(x) = \frac{x-2}{x-3} = y$

$x - 2 = (x-3)\,y$

$x = \frac{2-3y}{1-y}$ — inverse of $f(x)$

Check for inverse is a function (or) not. if inverse is a function for each value of real no. except 1

$\therefore$ F is onto

$\therefore$ F is bijection.

Ex : For which of the following function inverse is not defined on their range.

A. $f(x) = x^2$ —— Not one to one $\therefore$ No inverse exist

B. $f(x) = x^3$ —— one to one & onto $\therefore$ inverse exists

$$f^{1}(x) = x^{1/3}$$

C. $F_1(x) = \sin x$, $x \in (0, \pi)$ —— Not one to one $\therefore$ inverse not exists.

$$\left.\begin{matrix}\pi/4 \\ 3\pi/4\end{matrix}\right\rangle 1/\sqrt{2} \quad \text{(Two images)}$$

D. $f(x) = 2^x$

$$2^x = y$$

$$x \log e = \log y$$

$$x = \log_2 y$$

$$F^{1}(x) = \log_2 x \quad \text{(Inverse)}$$

* Linear Algebra [3M]

Topics :-
→ Determinants
→ Inverse of a matrix
→ Rank of matrix
→ Homogeneous & non-homogeneous linear eq^n
→ Eigen values & Eigen vectors
→ Cayley - Hamilton theorem.

Textbook :-
Matrices by A. R. Vasistha

* Determinants *

* Properties of Determinants :-

Point I : Value of the determinant will not be change if rows and columns are interchange.

i.e., $|A| = |A^T|$

Ex.

$$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10 - 4 = 6$$

$$|A^T| = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6$$

$\therefore |A| = |A^T|$

Point II : Value of the determinant is multiplied by -1 if two rows & two columns are interchange.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Point III : Value of the determinant can be zero in the following cases :-

i) The elements in two rows or two columns are identical.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

same

ii) The elements in two rows or two columns are proportional to each other.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 3 & 6 & 9 \end{vmatrix} = 0$$

proportional

iii) All elements in a row or column are zeros

Ex

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 7 & 3 \end{vmatrix} = 0$$

iv) The elements in a determinant are of consicutive order (continuous order)

→ valid for ~~ont~~ 3×3 & high order matrices

Ex

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

we can start 1^{st} element from any no.

Point IV: The determinant of upper triangular, lower triangular, diagonal, scalar & Identity matrix is the product of its diagonal elements.

1) Upper triangular matrix

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{vmatrix} = 1 \times (-4) \times 7 = -28$$

2) Lower triangular matrix

Ex.

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 7 \end{vmatrix} = 1 \times 3 \times 7 = 21$$

3) Diagonal matrix

Ex.

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 3 \times 2 \times 5 = 30$$

A matrix is diagonal iff at least one diagonal element should be non zero & all other non-diagonal elements should be zero.

4) Scalar matrix

Ex.

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3 \times 3 \times 3 = 27$$

→ Diagonal elements should be same

→ Non - diagonal elements should be zero

5) Identity matrix

Ex.

$$|I| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times 1 \times 1 = 1$$

→ Determinant of Identity matrix is always 1.

Point V : If A is N×N matrix then

$$|KA| = K^n |A|$$

where n = order of matrix A

Ex.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$|3A| = 3^2 \cdot |A| = 9 \times (-2) = -18$$

Point VI : If each element of a row or column contains sum of 2 elements then the determinant can be expressed as sum of two determinants of same order

Ex.

$$|A| = \begin{vmatrix} 1 & 1^2 & 1^3+1 \\ 2 & 2^2 & 2^3+4 \\ 3 & 3^2 & 3^3+5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1^2 & 1^3 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix} + \begin{vmatrix} 1 & 1^2 & 1 \\ 2 & 2^2 & 4 \\ 3 & 3^2 & 5 \end{vmatrix}$$

* Note 1: Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Method I:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Method II:

$$\begin{array}{ccccc} & & \ominus & \ominus & \ominus \\ a_{13} & a_{11} & a_{12} & a_{13} & a_{11} \\ & a_{21} & a_{22} & a_{23} & \\ a_{33} & a_{31} & a_{32} & a_{33} & a_{31} \\ & & \oplus & \oplus & \oplus \end{array}$$

$$|A| = a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{33}a_{21}a_{12} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11}$$

$\longrightarrow$ If determinant contains more no. of zeros use method I.

Ex. 1) Find determinant of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{ccccc} 2 & 1 & 2 & 2 & 1 \\ & 2 & 1 & 2 & \\ 1 & 2 & 2 & 1 & 2 \end{array}$$

$= 8 + 1 + 8 - 4 - 4 - 4$
$= 5$

Ex: 2) $A = \begin{pmatrix} 5 & 0 & 2 \\ \bar{0} & 3^{+} & \bar{0} \\ 2 & 0 & 1 \end{pmatrix}$

$= 3(5 - 4) = 3$

Gate Ex. 3) The following represents eq^n of straight line

$$\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

The line passes through
a) (0,0) ✓ b) (3,4) c) (4,3) d) (4,4)

$x(8) - 2(y) + 4(y - 8)$
$8x - 2y + 4y - 32 = 0$
$8x + 2y = 32$
$4x + y = 16$
$\therefore \; 4(3) + 4 = 16$
$x = 3 \;\&\; y = 4$

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Note 2:– Consider the matrix

$$A = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}_{4\times4} \end{array}$$

Method I :– ((complicated))

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Method II :– (Preferable)

~~Co~~

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}_{4\times4}$$

Procedure:

Step 1 : Among the given element of a matrix, select any non zero element.

step 2 : Make all elements above & below or left & right of the selected element as zero using row & column operations

Therefore, the matrix is

$$A = \begin{pmatrix} a_{11}^{+} & a_{12} & a_{13} & a_{14} \\ 0^{-} & b_{22} & b_{23} & b_{24} \\ 0^{+} & b_{32} & b_{33} & b_{34} \\ 0^{-} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

→ 1^{st} column has more no. of zeros so the determinant along 1^{st} column.

$$|A| = a_{11} \begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix}$$

Ex. 1) Find the determinant of

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 9 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$A = \begin{pmatrix} 1^{+} & -1 & 0 & 1 \\ 0^{-} & 3 & 3 & 3 \\ 0^{+} & 5 & 4 & 7 \\ 0^{-} & 1 & 0 & 2 \end{pmatrix}$$

$$|A| = 1 \begin{vmatrix} 3 & 3 & 3 \\ 5 & 4 & 7 \\ 1^{+} & 0^{-} & 2^{+} \end{vmatrix}$$

$$1\left[21 - 12\right] + 2\left[12 - 15\right]$$

$$9 + 2(-3)$$

$$9 - 6 = 3$$

Problem 1 :- If A has m rows & m+5 columns & B has n rows and 11-n columns. The orders of A and B if AB and BA are defined ?

$\Longrightarrow$

$$(A)_{m \times (m+5)} \quad (B)_{n \times (11-n)}$$

$$(B)_{n \times (11-n)} \quad (A)_{m \times (m+5)}$$

$$\therefore \quad m+5 = n \qquad\qquad m - n = -5$$
$$11 - n = m \qquad\qquad m + n = 11$$
$$2m = 6$$
$$m = 3 \qquad \therefore n = 8$$

Therefore, orders are $A_{(3,8)}$ & $B_{(8,3)}$ resp.

Problem 2 :- If $A = (a_{ij})_{m\times n}$ such that $a_{ij} = i+j, \forall i,j$ then sum of all element of A is ?

$\Longrightarrow$

$$A = \begin{pmatrix} 1+1 & 1+2 & 1+3 & \cdots & 1+n \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ m+1 & m+2 & m+3 & \cdots & m+n \end{pmatrix}$$

according to Point VI

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ 3 & 3 & & 3 \\ \vdots & \vdots & & \vdots \\ m & m & \cdots & m \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & & n \\ \vdots & & & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

$\frac{m(m+1)}{2}$: $\underbrace{\frac{m(m+1)}{2}+\dots+\frac{m(m+1)}{2}}$ | $\underbrace{\frac{n(n+1)}{2}+\frac{n(n+1)}{2}+\dots+\frac{n(n+1)}{2}}$

for n columns $= \frac{n\cdot m(m+1)}{2}$ | for m rows $= \frac{m\cdot n(n+1)}{2}$

Sol:

$$\frac{mn(m+1)}{2}+\frac{mn(n+1)}{2}$$

$$\frac{mn}{2}\left[m+1+n+1\right]$$

$$\frac{mn}{2}\left[m+n+2\right]$$

Problem 8 :- If $A=(a_{ij})_{3\times 3}$, $B=(b_{ij})_{3\times 3}$ such that

$b_{ij} = 2^{i+j}a_{ij}$ $\forall\, i, j$; $|A| = 2$; $|B| = ?$

a) 2^{10} b) 2^{11} c) 2^{12} d) 2^{13} ✓

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2$$

$$|B| = \begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{vmatrix}$$

$$= 2^2\cdot 2^3\cdot 2^4 \begin{vmatrix} 1a_{11} & 2a_{12} & 2^2a_{13} \\ 1a_{21} & 2a_{22} & 2^2a_{23} \\ 1a_{31} & 2a_{32} & 2^2a_{33} \end{vmatrix}$$

$$2^9\cdot 2\cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \cdot 2 = \underline{2^{13}}$$

Problem 4 :— If $A = (a_{ij})_{n\times n}$ such that

i) $a_{ij} = i^2 - j^2$, $\forall i, j$

ii) $a_{ij} = i - j$, $\forall i, j$

Find sum of all elements of A

$\Longrightarrow$

i) $$A = \begin{pmatrix} 0 & -3 & -8 & \cdots & (1^2-n^2) \\ 3 & 0 & -5 & \cdots & (2^2-n^2) \\ 8 & 5 & 0 & \cdots & (3^2-n^2) \\ \vdots & & & & \\ (n^2-1^2) & (n^2-2^2) & (n^2-3^2) & \cdots & 0 \end{pmatrix}$$

A is skew symmetric matrix.

→ In skew symmetric matrix all diagonal elements must be zero & non-diagonal elements should be real no.

Benefit :→ Sum of all elements of skew symmetric matrix is always zero

Ex.

i) $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}_{2\times 2}$ = Sum of all elements of skew sym. matrix = 0

2) $A = \begin{pmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{pmatrix}_{3\times 3}$ = Sum = 0

→ If ith row, jth column elem. of a matrix is in the form $a_{ij} = i^n - j^n$ $(n>0)$ then corresponding matrix is always skew symm.

* Note 3 :-
A matrix A is said to be symmetric if
$A' = A$ or $a_{ij} = a_{ji}$

* Note 4 :-
A matrix A is said to be skew symmetric
if $A' = -A$ or $a_{ij} = -a_{ji}$

Problem 1 :- (UGC 2001) The value of $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix} = ?$

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix} = 0$$

proportional

Problem 2 :- Find the determinant of $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/c & c & ab \end{vmatrix}$

$$= \begin{vmatrix} bc/abc & a & bc \\ ca/abc & b & ca \\ ab/abc & c & ab \end{vmatrix}$$

$$\frac{1}{abc}\begin{vmatrix} bc & a & bc \\ ca & b & ca \\ ab & c & ab \end{vmatrix} = 0$$

same

→ If there are no numbers inside determinant try to make 1 perticular row or column same

Problem 8 :- find the value of $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

$C_1 \rightarrow C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$

$$= x+3a \begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - R_1$; $R_4 \rightarrow R_4 - R_1$

$$= x+3a \begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}$$

according to pt IV

$$= (x+3a)\{1 \times (x-a) \times (x-a) \times (x-a)\}$$

$$= (x+3a)(x-a)^3$$

* **Short cut method :-**

Procedure :-

i) If the diagonal elements are one category of same elements & non diagonal elements are other category of same elements

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2) select the 1^{st} row

3) add all elements of I^{st} row

4) take the product of $(1^{st} - 2^{nd})(1^{st} - 3^{rd})$
$(1^{st} - 4^{th}) \ldots$

5) $(3) \times (4)$

i.e.,

$$|A| = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

Problem: The value of $A = \begin{pmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{pmatrix}$

$$|A| = \begin{vmatrix} x+10 & 2 & 3 & 4 \\ x+10 & 2+x & 3 & 4 \\ x+10 & 2 & 3+x & 4 \\ x+10 & 2 & 3 & 4+x \end{vmatrix} = x+10 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix}$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1 \quad R_4 \to R_4 - R_1$

$$= (x+10) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}$$

$$= (x+10)\, 1 \times x \times x \times x$$

$$= \underline{x^3 (x+10)}$$

Problem 2 : $A = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} 1+a+b+c+d & b & c & d \\ 1+a+b+c+d & 1+b & c & d \\ 1+a+b+c+d & b & 1+c & d \\ 1+a+b+c+d & b & c & 1+d \end{vmatrix}$$

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 1 & 1+b & c & d \\ 1 & b & 1+c & d \\ 1 & b & c & 1+d \end{vmatrix}$$

$R_2 \to R_2 - R_1 \;;\; R_3 \to R_3 - R_1 \;;\; R_4 \to R_4 - R_1$

$$= (1+a+b+c+d) \begin{vmatrix} 1^{+} & b & c & d \\ 0^{-} & 1 & 0 & 0 \\ 0^{+} & 0 & 1 & 0 \\ 0^{-} & 0 & 0 & 1 \end{vmatrix}$$

$$= 1+a+b+c+d \left(1 \, [1] \right)$$

$$= 1+a+b+c+d$$

Problem 3: Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$C_2 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

$$A = \begin{array}{ccc} + & - & + \\ \end{array}$$

$$A = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$= 1\left[(b-a)(c^2-a^2) - (c-a)(b^2-a^2)\right]$

$= (b-a)(c+a)(c-a) - (c-a)(b+a)(b-a)$

$= (b-a)(c-a)\left[c + \cancel{a} - b - \cancel{a}\right]$

$= (b-a)(c-a)(c-b)$

$= \{-(a-b)\}(c-a)\{-(b-c)\}$

$= \underline{(a-b)(b-c)(c-a)}$

* Short cut method [for $(2\times 2)(3\times 3)\dots(n\times n)$]

If matrix is in the form $\begin{vmatrix} 1 & 1 & 1 & 1 & \dots \\ \boxed{a \quad b \quad c \quad d \quad \dots} \\ a^2 & b^2 & c^2 & d^2 & \dots \\ a^3 & b^3 & c^3 & d^3 & \dots \\ \vdots \end{vmatrix}$

then select the 2^{nd} row
& take $(2^{nd} - 1^{st})(3^{rd} - 2^{nd}) \dots (\text{last} - 1^{st})$

i.e., $(a-b)(b-c)(c-d)$

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* Inverse of a matrix *

Let $A = (a_{ij})$ be $n \times n$ matrix

i) **Minor:** Minor of an element a_{ij} is denoted by M_{ij} and is defined as

$$M_{ij} = (n-1)^{th} \text{ order determinant.}$$

ii) **Cofactor:** Cofactor of an element a_{ij} is denoted by A_{ij} and is defined as

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Ex. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 7 & -2 \end{pmatrix}$$

Consider the 2^{nd} row 2^{nd} element

$a_{22} = 3$

→ Minor of a_{22} is

$$M_{22} = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} = -2-8 = -10$$

→ Cofactor of a_{22} is

$$A_{22} = (-1)^{2+2} \times (-10) = -10$$

* Invertible matrix:

A matrix A is said to be invertible if we can find some other matrix B such that $AB = BA = I$ then B is called inverse of matrix A.

$$\cdot\, B \quad I$$

$$\cdots B = A^{-1} I$$

$$I\,B = A^{-1} I$$

$$\boxed{B = A^{-1}}$$

* Singular matrix :—

A matrix is said to be singular if $|A| = 0$

* Non singular matrix :—

A matrix is said to be non singular if $|A| \neq 0$

Inverse of matrix exists if $|A| \neq 0$.

Note 1:
$$A^{-1} = \frac{\text{adj}.A}{|A|}$$

$$\Rightarrow \text{adj}.A = |A| \cdot A^{-1}$$

$$\Rightarrow A \cdot \text{adj}.A = A \cdot |A| \cdot A^{-1}$$

$$\Rightarrow \boxed{A \cdot \text{adj}.A = |A| \cdot I = \text{adj}.A \cdot A}$$

Note 2:
$$A^{-1} = \frac{\text{adj}.A}{|A|}$$

$$\Rightarrow (\text{adj}.A)^{-1} \cdot A^{-1} = (\text{adj}.A)^{-1} \cdot \frac{\text{adj}.A}{|A|}$$

$$\Rightarrow (\text{adj}.A)^{-1} \cdot A^{-1} = \frac{I}{|A|} \qquad \because A \cdot A^{-1} = I$$

$$\Rightarrow (\text{adj}.A)^{-1} \cdot A^{-1} \cdot A = \frac{I \cdot A}{|A|}$$

$$\Rightarrow \boxed{(\text{adj}.A)^{-1} = \frac{A}{|A|}}$$

Note 3: Let A be an $n \times n$ matrix

We know that

$$\text{adj}.A = |A|.A^{-1}$$

$$\Rightarrow |\text{adj}.A| = |\underset{\substack{\downarrow \\ k}}{|A|}.\underset{\substack{\downarrow \\ A}}{A^{-1}}|$$

$$\boxed{|kA| = k^n|A|}$$

$$\Rightarrow |A|^n.|A^{-1}|$$

$$\Rightarrow |A|^n.|A|^{-1}$$

$$\therefore \boxed{|\text{adj}.A| = |A|^{n-1}}$$

Replacing A by adj. A in the above relation

$$|\text{adj}.\text{adj}.A| = |\text{adj}.A|^{n-1}$$

$$= \{|A|^{n-1}\}^{n-1}$$

$$\therefore \boxed{|\text{adj}.\text{adj}.A| = |A|^{(n-1)^2}}$$

Simillarly,

$$|\text{adj}.\text{adj}.\text{adj}.A| = |A|^{(n-1)^3}$$ & so on..

Note 4: We know that

$$A.\text{adj}.A = |A|.I$$

replacing A by adj. A

$$\Rightarrow \text{adj}.A(\text{adj}.\cancel{A}\ \text{adj}.A) = |\text{adj}.A|.I$$

$$\Rightarrow \text{adj}.A(\text{adj}.\text{adj}A) = |A|^{n-1}.I$$

pre multiply both side by A

$$\Rightarrow A.\text{adj}.A(\text{adj}.\text{adj}.A) = A.|A|^{n-1}I$$

$$\boxed{AI = A}$$

$$\Rightarrow |A|.I(\text{adj}.\text{adj}.A) = |A|^{n-1}.A$$

$$\Rightarrow |A| \cdot (\text{adj}\cdot\text{adj}\cdot A) = |A|^{n-1} \cdot A$$

$$\Rightarrow \text{adj adj } A = \frac{|A|^{n-1} \cdot A}{|A|}$$

$$\Rightarrow \boxed{\text{adj adj} \cdot A = |A|^{n-2} A}$$

Note 5 :- A matrix A is said to be orthogonal if

$$\boxed{A \cdot A^T = A^T \cdot A = I}$$

$$A \cdot A^T = I$$

$$A^{-1} \cdot A \; A^T = A^{-1} I$$

$$I \cdot A^T = A^{-1} I$$

$$\boxed{A^T = A^{-1}}$$

Note 6 :- If A is an orthogonal matrix then A^{-1} & A^T are also orthogonal matrices.

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Problem 1 : For the matrix $M = \begin{pmatrix} 3/5 & 4/5 \\ x & 3/5 \end{pmatrix}$ such that $M^T = M^{-1}$. The value of x is ?

$$\frac{3}{5}(x) + \frac{4}{5}\left(\frac{3}{5}\right) = 0$$

$$\boxed{x = -4/5}$$

Note : If A is an orthogonal matrix then its rows & columns are pair wise orthogonal.
But converse of the stmt. may or may not be true.
i.e., If rows & columns of matrix are pair wise orthogonal, then the matrix may or may not be orthogonal.

Problem 2: If $A = (a_{ij})_{5\times5}$ such that

i) $a_{ij} = i-j, \quad \forall i,j$

ii) $a_{ij} = i^2-j^2, \quad \forall i,j$

find A^{-1} in each case.

$\longrightarrow$ i) $a_{ij} = i-j$

$a_{ji} = j-i$

$= -(i-j)$

$= -a_{ij}$

$\therefore \boxed{a_{ij} = -a_{ji}}$ $\quad \therefore A \Rightarrow$ skew symmetric matrix

$\therefore \Rightarrow A^T = -A$

$\Rightarrow |A^T| = |-A|$ $\quad$ as $|KA| = K^n|A|$

$\Rightarrow \quad = (-1)^5 |A|$

$\therefore |A^T| = -1|A|$ $\quad |A| = |A^T| \rightarrow$ property No 1

$|A| = -|A|$

$|A| + |A| = 0$

$2|A| = 0$

$2 \neq 0 \quad \therefore |A| = 0$

$\therefore A^{-1}$ does not exist.

Note: ① Inverse of any odd order skew symm matrix does not exist.

Reason: Since every odd order skew symm matrix is singular i.e., $|A| = 0$

② Inverse of even order skew symm matrix exists.

Reason: Since every even order skew symm. matrix is non singular i.e., $|A| \neq 0$

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ii)

$$A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}_{2\times 2}$$

$$A^T = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$$

$$\boxed{A^T = -A}$$

$$\therefore |A| = \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} = 0 + 9 = (3)^2$$

⇒ The determinant of even order skew symm. matrix is a perfect square. Pro

⇒ If ith row, jth column element of a matrix is in the form $a_{ij} = i^n - j^n$ $(n > 0)$, the corresponding matrix is always skew symmetric.

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Problem 3:- If X and Y are two non zero matrices of the same order such that $XY = (0)_{n\times n}$, then

A) $|X| \neq 0$, $|Y| = 0$

B) $|X| = 0$, $|Y| \neq 0$

c) $|X| \neq 0$; $|Y| \neq 0$

D) $|X| = 0$; $|Y| = 0$

⇒ $XY = 0$

take determinants on b.s

$|XY| = 0$

$|X||Y| = 0$

then $|X| = 0$ or $|Y| = 0$ or both $|X| = 0$ & $|Y| = 0$

ⓒ is omitted

Let $|X| = 0$, $|Y| \neq 0$ $\therefore Y^{-1} \Rightarrow$ exist

$$XY = 0 \Rightarrow XYY^{-1} = 0 \cdot Y^{-1}$$
$$\Rightarrow XI = 0$$
$$\Rightarrow X = 0 \quad \text{(contradiction to hypothesis)}$$
$$\therefore \boxed{|Y| = 0}$$

If we choose $|Y| = 0$, $|X| \neq 0$ then we get

$$\boxed{|X| = 0}$$

$$\therefore \boxed{|X| = |Y| = 0}$$

Problem 4: If A, B, C, D, E, F, G are non-singular matrices of the same order such that $CEDBGAF = I$ then B^{-1} is —

$$\Rightarrow \quad CEDBGAF = I$$
$$C^{-1}CEDBGAF = C^{-1}I$$
$$IEDBGAF = C^{-1}I$$
$$EDBGAF = C^{-1}$$
$$E^{-1}EDBGAF = E^{-1}C^{-1}$$
$$IDBGAF = E^{-1}C^{-1}$$
$$D^{-1}DBGAF = D^{-1}E^{-1}C^{-1}$$
$$BGAF = D^{-1}E^{-1}C^{-1}$$
$$BGAFF^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}$$
$$BGAI = D^{-1}E^{-1}C^{-1}F^{-1}$$
$$BGAA^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}$$
$$BG = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}$$
$$BGG^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}$$
$$\boxed{B = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}}$$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

$$\therefore B^{-1} = (D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1})^{-1}$$
$$\boxed{B^{-1} = GAFCED}$$

* Short cut method :—

$$\underbrace{CE}_{}\ D\ \underbrace{B}_{B^{-1}}\ \underbrace{GAF}_{} = GAFCED$$

$$\underline{C E D B G A F} = F C E D B G$$
$$A^{-1}$$

$$\text{||}^{ly} \quad \underline{C E D B G A F} = C E D B G A$$
$$F^{-1}$$

Prob No. 6. Let k be a +ve real no. & let

$$A = \begin{pmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{pmatrix}_{3\times 3} \qquad B = \begin{pmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{pmatrix}_{3\times 3}$$

find i) det.(adj. B) ii) det. (adj. A)

iii) If det(adj. A) = 10^6, the value of $k = ?$

$\Longrightarrow$

i) $|\operatorname{adj} B| = |B|^{n-1} = 0$

$\because$ determinant of odd order skew symm. matrix is zero.

ii) $|\operatorname{adj}. A| = |A|^{n-1} = |A|^{3-1} = |A|^2$ —— eq^n (1)

$$|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$R_2 \rightarrow R_2 + R_3$

$$= \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 1+2k & -2k-1 \\ -2\sqrt{k} & 2k & -1 \end{vmatrix} \qquad -(1+2k)$$

$C_2 \rightarrow C_2 + C_3$

$$= \begin{vmatrix} 2k-1 & 4\sqrt{k} & 2\sqrt{k} \\ 0^{\ominus} & 0^{\oplus} & -(1+2k)^{\ominus} \\ -2\sqrt{k} & 2k-1 & -1 \end{vmatrix}$$

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$$\Rightarrow -\left\{-(1+2k)\left((2k-1)^2+8k\right)\right\}$$

$$\cancel{(1+2k)\left((2k+1)(2k-1)\right)}$$

$$\Rightarrow (1+2k)\left(4k^2-4k+1+8k\right)$$

$$\Rightarrow (1+2k)\left(4k^2+4k+1\right)$$

$$\Rightarrow (2k+1)(2k+1)^2 = (2k+1)^3$$

put in eq^n (1)

$$\Rightarrow \left|(2k+1)^3\right|^2 = (2k+1)^6$$

iii) $|adj.\,A| = 10^6$

$(2k+1)^6 = 10^6$

$2k+1 = 10 \qquad 2k = 9 \qquad \therefore \boxed{k = 9/2}$

Problem No. 5: If A, B, C, D are non singular matrices of the same order such that $ABCD = I$ then B^{-1} is ?

$$ABCD = I$$

$$\therefore \boxed{B^{-1} = CDA}$$

Prob. No. 7 :—

Find the inverse of foll. matrices —

EE-05 2m 1) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$

EE-95 2m H.W 3) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

EE-98 2m H.W 2) $A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

⇒

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$\Rightarrow |A| = \begin{vmatrix} 1^{(+)} & 0^{(-)} & -1^{(+)} \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 1\left[2+3\right] - 1\left[6-2\right] = 5 - 4 = \underline{1}$$

$$\Rightarrow \text{adj. } A =$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix} \end{array} \begin{array}{l} \longrightarrow \text{III} \\ \longrightarrow \text{I} \\ \longrightarrow \text{II} \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 2 \\ 1 & 3 & 0 & 1 \end{pmatrix} \quad \therefore \text{adj } A = \begin{pmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{pmatrix}$$

Procedure: (3×3)

Select the middle row middle element & move in anticlockwise direction to complete 1 cycle.

The corresponding element will be return in the 1st column seperately

2) Select the 3rd row middle element & move in anti-clockwise direction to complete 1 cycle.
The corresponding element will be return in the 2nd column seperately.

3) Repeat the same process with 1st row also.

4) Copy 1st column as a last column & find det. of smaller matrices.

Procedure : (2 × 2)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \boxed{A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

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Prob. No. 8 :- Given an orthogonal matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ then } (A A^T)^{-1} \text{ is } - ?$$

Orthogonal $\Longrightarrow$ $(A A^T) = I$

$(A A^T)^{-1} = I^{-1}$

but Inverse / adj. of Identity matrix is Identity matrix only.

$\therefore (AA^T)^{-1} = I$

Prob. No. 9 Find the inverse of $A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ Here A is an orthogonal matrix therefore.

$A^{-1} = A^T$

* Rank of Matrix *

* **Sub matrix :** A matrix obtained by deleting some rows or columns or both is called as sub matrix.
Ex. Consider the matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\underbrace{B_1 = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \quad B_3 = \begin{pmatrix} 4 & 5 & 6 \\ 1 & -1 & 0 \end{pmatrix}}_{\text{sub matrices of } A}$$

* **Minor :** The determinant of square sub matrix is called its minor.
Ex.

$$\left.\begin{aligned} |B_1| &= \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = -3 \\ |B_2| &= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12-15 = -3 \end{aligned}\right\} \text{Minors of } A$$

* **Rank :** If the determinant of highest possible square matrix is not equal to zero then the order of the determinant is called Rank of matrix.

Ex. Find the rank of

$$A = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 4 & 2 \\ 1 & -11 & 14 & 5 \end{pmatrix}_{3\times 4}$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix}_{3\times 3} = 1[-14+44] - 3[28-4] - 2[-22+1]$$

$$= 30 - 72 + 42$$

$$= 0$$

$$\begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 2 \\ -11 & 14 & 5 \end{vmatrix}_{3\times 3} = 3[20-28] + 2[-5+22] + 1[-14+44]$$

$$= -24 + 34 + 30 = 40 \neq 0$$

$\therefore$ Rank of A = $\rho(A) = 3$ i.e., order of matrix

* Row Echelon form:–

A matrix A is said to be in Row Echelon form iff

i) zero rows should occupy the last rows, if any.

ii) the no. of zero's before a non zero element of each row is less than no. of such zeros before a non zero element of the next row.

Ex

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 7 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}$$

Condition 1 satisfy

— " — 2 — " —

Note: The rank of Row Echelon form matrix is equal to no. of non zero rows.

$\boxed{\rho(A) = 4}$ ← Linearly Independent Row/Vector

→ These non zero rows are called Linearly Independent rows/vector.

→ To reduce any matrix into row echelon form we should use only row operations.

→ Every upper triangular matrix is in R.E form but every R.E. form will not be an upper triangular matrix.

Ex. 1

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

↓

Upper Δ^{lar} matrix

∴ A is in R.E form

Ex. 2

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↓

Not Upper Δ^{lar} matrix

but A is in R.E Form

Upper Δ^{lar} matrix $\Rightarrow$ R.E form   R.E form $\not\Rightarrow$ Upper Δ^{lar} matrix

* Column Echelon form :-

A matrix A is said to be in column echelon form iff

i) zero ~~rows~~ columns should occupy the last columns, if any.

ii) The no. of zeros above a non zero element of each column is less than the no. of zeros above a non zero element of the next column

Ex. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

Note :-

Rank of matrix in Column Echelon form is equal to no of non zero columns

$$\therefore \boxed{\rho(A) = 4}$$

→ To reduce any matrix into column echelon form, we should use only Column Operations

→ Every lower Δ^{lar} matrix will be in column echelon form.

but every C.E form will not be a lower Δ^{lar} matrix.

Ex. 1

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 9 & 4 & 5 \end{pmatrix}$$

↓

Lower Δ^{lar} matrix

Lower Δ^{lar} M ⇒ C.E form

Ex. 2

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \text{[illegible]} & \text{[illegible]} \\ 2 & \text{[illegible]} & \text{[illegible]} \end{pmatrix}$$

C.E. form ⇏ lower Δ^{lar} matrix

Example: Consider the matrix $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}$

$\Rightarrow$ Row Echelon form

$R_2 \to R_2 - 2R_1 \qquad R_3 \to R_3 - R_1$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix} \qquad R_3 \longrightarrow R_3 - 2R_1$$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 7 & 8 \\ 0 & 0 & 0 \end{pmatrix} \qquad \therefore \boxed{\rho(A) = 2}$$

$\Rightarrow$ Column Echelon form

$C_2 \to C_2 - 3C_1 \qquad C_3 \to C_3 + 2C_1$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -7 & 8 \\ 1 & -14 & 16 \end{pmatrix} \qquad C_3 \longrightarrow 7C_3 + 8C_2$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -7 & 0 \\ 1 & -14 & 0 \end{pmatrix} \qquad \therefore \boxed{\rho(A) = 2}$$

<u>Note:</u> Rank of matrix = no of non zero rows &
no of non zero columns.

Prob. No. 7

2) $A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ $\quad A^{-1} = \dfrac{adj.A}{|A|}$

$|A| = \begin{vmatrix} 5 & 0 & 2 \\ 0^{\ominus} & 3^{\oplus} & 0^{\ominus} \\ 2 & 0 & 1 \end{vmatrix} = 3[5-4] = 3$

$adj.|A| = \begin{pmatrix} 3 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & 0 \\ 3 & 0 & 0 & 3 \end{pmatrix}$

$adj.A = \begin{pmatrix} 3 & 0 & -6 \\ 0 & 1 & 0 \\ -6 & 0 & 15 \end{pmatrix}$ $\quad \therefore A^{-1} = \dfrac{1}{3}\begin{pmatrix} 3 & 0 & -6 \\ 0 & 1 & 0 \\ -6 & 0 & 15 \end{pmatrix}$

3) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad A^{-1} = \dfrac{Adj.A}{|A|}$

$|A| = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0^{\oplus} & 0^{\ominus} & 1^{\oplus} \end{vmatrix} = 1[1+1] = 2$

$adj.A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

$\therefore A^{-1} = \dfrac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

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* Information regarding Rank of matrix :

1) $\rho(O_{n\times n}) = 0$
2) $\rho(I_{n\times n}) = n$
3) $\rho\{\text{adj. } I_{n\times n}\} = n$
4) $\rho(A) = \rho(A^T)$
5) $\rho(A+B) \le \rho(A) + \rho(B)$
6) $\rho(A-B) \geqslant \rho(A) - \rho(B)$
7) $\rho(AB) \geqslant \rho(A) + \rho(B) - n$, if A & B are $n\times n$ matrice
8) $\rho(AB) \le \min\{\rho(A), \rho(B)\}$
9) If A is an $m\times n$ matrix, then $\rho(A) \le \min(m, n)$
10) If $\rho(A_{n\times n}) = n$ then $\rho(\text{adj. } A) = n$
11) If $\rho(A_{n\times n}) = n-1$, then $\rho(\text{adj. } A) = 1$
12) If $\rho(A_{n\times n}) = n-2$, then $\rho(\text{adj. } A) = 0$

2m

Problem 1: If $A = (a_{ij})_{m\times n}$, such that $a_{ij} = i\cdot j$, $\forall\, i, j$ then $\rho(A) = ?$

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & & & & \vdots \\ m & 2m & 3m & \cdots & mn \end{pmatrix}_{m>n}$$

to find $\rho(A)$ convert the matrix into Row Echelon form.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_m \to R_m - mR_1$

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

$\therefore \rho(A) = $ No. of non zero rows $= 1$

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Gate 2010 (in EE)

Problem 2: If $x = (x_1, x_2, \ldots x_n)^T$ is n-tuple non zero vector then

i) $\rho(xx^T)$

ii) $\rho(x^Tx)$

i) $\rho(xx^T)$

$x = (x_1, x_2 \ldots x_n)^T$

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$ $\quad x^T = (x_1\ x_2 \cdots x_n)_{1\times n}$

$\rho(xx^T) \leq \min\{\rho(x), \rho(x^T)\}$

$\rho(xx^T) \leq \min\{1, 1\}$

$\Rightarrow \rho(xx^T) \leq 1$ < 1 ✓, 0 ✗ ($x \to$ Non zero vector)

$\therefore \boxed{\rho(xx^T) = 1}$

Gate

Problem 3: The rank of 5x6 matrix Q is 4 then which of the foll stmt is true.

a) Q will have 4 L.I rows & 4 L.I columns
b) Q will have 4 L.I rows & 5 L.I columns
c) QQ^T is invertible
d) Q^TQ is invertible

$\rho(Q_{5\times 6}) = 4$

$\therefore$ 4 non zero rows or columns

option (a)

option (b) ✗

If det of matrix $\neq 0$ then order of ...

option (c) $(Q)_{5\times 6} \cdot (Q)^T_{6\times 5}$

$(QQ^T)_{5\times 5} \rightarrow$ Invertible

$(QQ^T)^{-1}$ exists

$|(QQ^T)_{5\times 5}| \neq 0 \Rightarrow \rho(QQ^T) = 5$ (contradictⁿ to stmt)

option (d) $(Q^T)_{6\times 5} \; (Q)_{5\times 6}$

$(Q^TQ)_{6\times 6} \rightarrow$ Invertible

$(Q^TQ)^{-1}$ exists

$|(Q^TQ)_{6\times 6}| \neq 0 \Rightarrow \rho(Q^TQ) = 6$ (contradiction to stmt)

* Linearly Dependent & Independent vectors *

$\Longrightarrow$ Two vectors x_1 & x_2 are L.D. if one vector is expressed as multiple of other vector.

x_1 & $x_2 \Rightarrow$ same directional vectors

Example

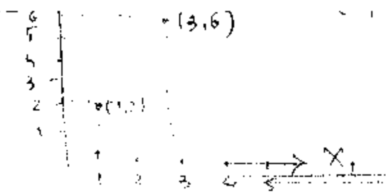
$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$x_2 = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{x_2 = 3x_1}$ or $\boxed{x_1 = \frac{1}{3}x_2}$

$\therefore x_1, x_2 \Rightarrow$ L.D.

collinear.



$\Longrightarrow$ Two vectors in R^2 are L.D. if and only if they are collinear.

$\Longrightarrow$ Three vectors in R^3 are L.D iff they are coplanar.

$\Longrightarrow$ Two vectors x_1 x_2 are L.I. iff it is not possible to express one vector as a multiple of other vector.

Example:

$$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1 \neq k x_2 \quad \text{OR} \quad x_2 \neq k x_1$$

$$\therefore x_1, x_2 \Longrightarrow \text{L.I.}$$

* Linearly Dependent vectors: (for 2 or more than 2 vectors)

A set of r n-vectors $x_1, x_2, \cdots x_r$ are said to be linearly dependent if there exist r scalars $k_1, k_2, \cdots, k_r$ such that

$$k_1 x_1 + k_2 x_2 + \cdots + k_r x_r = 0$$

where $k_1, k_2, \cdots, k_r$ not all zeros.

(atleast one k value is a non zero no.)

* Linearly Independent vectors:

A set of r n-vectors $x_1, x_2, \cdots x_r$ are said to be linearly independent if there exist r scalar $k_1, k_2, \cdots k_r$ such that

$$k_1 x_1 + k_2 x_2 + \cdots + k_r x_r = 0$$

all zeros

* Criteria for L.I & L.D

1) If $\rho(A)$ = no. of given vectors OR $|A| \neq 0$, the given vectors are said to be L.I.

Ex.

Consider the vectors

$x_1 = (1 \;\; 2 \;\; 2)$, $x_2 = (2 \;\; 1 \;\; 2)$, $x_3 = (2 \;\; 2 \;\; 1)$

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(1-4) - 2(2-4) + 2(4-2)$$

$$= -3 + 4 + 4$$

$$= 5$$

$$\therefore |A| = 5 \neq 0 \implies \text{L.I.}$$

2) If $\rho(A) <$ no. of given vectors OR $|A| = 0$, the given vectors are said to be L.D.

$> n$

Ex.

Consider the vectors

$x_1 = (1 \;\; 3 \;\; -2)$, $x_2 = (2 \;\; -1 \;\; 4)$, $x_3 = (1 \;\; -11 \;\; 14)$

$$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = 1(-14+44) - 3(28-4) - 2(-22+1)$$

$$= 30 - 72 + 42$$

$$|A| = 0 \implies \text{L.D.}$$

3) If the given vectors are L.D. then any one of the vector can be expressed as linear combination of other vectors. diff of

4) If the given vectors are L.I then it is not possible to write any one of the vector as linear combination of other vectors.

at least one elem must be nonzero

5) Every non zero vector is L.I. vector.

Ex.

Consider the vector

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$k x = 0$$

$$k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Non zero vector ←

$$\boxed{\therefore k = 0}$$

$\therefore$ x → L.I vector

columns/rows of Identity matrix

6) The set of unit vectors are always L.I

Ex. Consider the set of vectors

$x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$|A| = 1 \neq 0 \implies$ L.I vectors

7) The set of vectors having atleast one zero vector are L.D.

Ex. Consider the set of vectors

$x_1 = (1, 2, 3)$, $x_2 = (0\ 0\ 0)$, $x_3 = (1\ -1\ 4)$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 4 \end{vmatrix} = 0 \quad \therefore x_1 = x_2 = x_3 \Rightarrow \text{L.D.}$$

$|A| = 0 \implies$ L.D

No. of vectors = r
No. of elements in each vector = n

8) A set of r vectors with $r < n$ components (elements) are always L.I, provided the vectors should not be in the same direction.
Ex. Consider the vectors
$(1 \ -1 \ 2 \ 1)$, $(1 \ 2 \ 3 \ 4)$, $(2 \ 3 \ 4 \ 9)$

$r = 3$

$n = 4$

here, $r < n \implies$ L.I

9) A set of r vectors with $r > n$ components then given vectors are L.D.
Ex. Consider the vectors

$x_1 = (1 \ 1 \ 2)$
$x_2 = (1 \ -1 \ 0)$ $\quad r = 4$
$x_3 = (1 \ -1 \ 5)$ $\quad n = 3$
$x_4 = (9 \ -5 \ 4)$ $\quad$ here, $r > n \implies$ L.D.
$\therefore x_1, x_2, x_3, x_4$ are L.D.

10) A set of r vectors with $r = n$ components may be L.I or L.D

* Dimension & Basis of the vectors

* Dimension :- It is defined as no. of L.I. vectors.
Dimension = No. of L.I. vectors = no. of non zero rows in Row Echelon form = no. of non zero columns in Column Echelon Form.

* Basis :- It is defined as the set of L.I. vectors.
Basis = set of L.I vectors = set of non zero rows in R.E. form = set of non zero columns in C.E. form.

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Problem 1: Test whether the following vectors are L.D or L.I. Also find their dimension & basis.

$(1 \quad 1 \quad -1 \quad 0)$, $(4 \quad 6 \quad -3 \quad 1)$, $(-6 \quad 2 \quad 2 \quad 2)$, $(9 \quad 9 \quad -6 \quad 3)$

$r = 4$ $n = 4$ here $r = n$ ∴ may be L.D or L.I

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ -6 & 2 & 2 & 2 \\ 9 & 9 & -6 & 3 \end{vmatrix}$$

$R_2 \to R_2 - 4R_1$ $R_3 \to R_3 + 6R_1$ $R_4 \to R_4 - 9R_1$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 3 & 3 \end{vmatrix}$$

zero

row only not previous row

$R_2 \longleftrightarrow R_3$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{vmatrix}$$

$R_4 \to R_4 - 3R_3$

$$A = \left.\begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}\right\} \text{3 L.I vectors (first three rows)}$$

whenever a matrix is reduced to ... E.F form then Rank of matrix = Dimension of ...

Set of L.I vectors are called ...

$\rho(A) = 3 <$ No. of given vectors (4)

$\therefore x_1\ x_2\ x_3\ x_4 \Longrightarrow$ L.D

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} \neq 0$$

Dimension = 3

Basis = $\{(1\ 1\ -1\ 0), (0\ 8\ -4\ 2), (0\ 0\ 1\ 1)\}$

Gate (2m)

Problem: If $q_1, q_2, \ldots, q_m$ are n-dimensional vectors with $m < n$. The vectors are L.D. The matrix Q is $q_1, q_2, \ldots, q_m$ as columns. The rank of Q = ?

(q_1: 1st col, q_2: 2nd col, q_m: mth col)

a) m b) n c) between m & n d) $<m$ ✓

$$q_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$Q = \begin{array}{c} \begin{matrix} q_1 & q_2 & q_3 & \cdots & q_m \end{matrix} \\ \begin{pmatrix} | & | & | & \cdots & | \\ | & | & | & & | \end{pmatrix}_{n\times m} \end{array} \Rightarrow (Q)_{n\times m}$$

$\rho(Q_{n\times m}) \leq \min(n, m)$

but $m < n$ — given

$\therefore \rho(Q_{n\times m}) \leq m$ — m ✗, $<m$ ✓ from criteria (2)

$\therefore \boxed{\rho(Q) < m}$

* Nullity of a matrix :-

* Nullity :- It is denoted by $N(A)$.
& is defined as the difference betⁿ order of matrix and rank of matrix.
i.e.,

$$\boxed{N(A) = n(A) - \rho(A)}$$

Nullity   order   Rank

$\Rightarrow$ Nullity of a non-singular matrix is always zero.

Let A be an $n\times n$ non singular matrix.
Then,

$|A_{n\times n}| \neq 0$

$\therefore \rho(A) = n$

$N(A) = n(A) - \rho(A)$

$= n - n$

$\therefore N(A) = 0$

Problem: The nullity of $A = \begin{pmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{pmatrix} = 1$. The value of $k = ?$

$N(A) = n(A) - \rho(A)$

$1 = 3 - \rho(A)$

$\rho(A) = 2$

for 3×3 matrix
iff $\rho(A) = 3$ then $|A| \neq 0$

$|A| = 0 \Rightarrow$

$$\begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

$k = -1$

$ax + by + cz = 0 \Rightarrow$ Homogeneous

$ax + by + cz = 2 \Rightarrow$ Non - Homogeneous

Problem: The rank of matrix A is 5 & nullity of matrix is 3 then order of matrix = ?

$$N(A) = n(A) - \rho(A)$$

$$3 = n(A) - 5$$

$$n(A) = 8$$

* Non-homogeneous system of Linear Equation *

Consider the following non homogeneous system of m linear eqns in n unknowns.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \text{I}$$

Procedure

1) Write the given system of eqns in the form
$AX = B$

i.e.,

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{sol}^n\text{ matrix}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\text{Column matrix of constants}}$$

k = 8

2) Write the elements of matrix B in the last column of matrix A. The resulting matrix is called Augmented matrix & is denoted by (A|B)

$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

3) Reduce the augmented matrix (A|B) into Row Echelon form & hence find rank of A & rank of (A|B)

4) If $\rho(A) < \rho(A|B)$ OR $\rho(A|B) \neq \rho(A)$, the given system of equations ~~are said to~~ have no solⁿ (inconsistent)

5) If $\rho(A|B) = \rho(A) =$ no. of unknowns, the given system of eqⁿ have unique solution

6) If $\rho(A|B) = \rho(A) <$ no. of unknowns, the given system of eqⁿ have infinite no. of solutions.

7) If the given system of equations have a solⁿ (unique or infinite solⁿ).
The solⁿ can be found by reducing the matrix AB into Row Echelon form & by using back code substitution, the variables $x_1, x_2, \dots x_n$ can be found.

Note: If the total no. of eqⁿˢ < total no. of variables, the given system of eqⁿ have infinite no. of solⁿˢ

These infinite no. of solⁿˢ can be found by assigning $(n-r)$ variables as arbitrary constants.
These $(n-r)$ solⁿˢ are Linearly Independent solⁿˢ.

$$x + y = 3$$

$$r = 1, \quad n = 2$$

$$r < n, \quad n - r = 2 - 1 = 1$$

put $\boxed{y = c} \rightarrow$ L.I sol^n

$$\cancel{x + y = 3}$$

$$x + c = 3$$

$$x = 3 - c$$

$x = 3 - c$ & $y = c$

Note: Consider the system of eq^n.

$$ax + by = e$$

$$cx + dy = f$$

The above system of eq^{ns} have

1) No sol^n if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$

2) Unique sol^n if $\frac{a}{c} \neq \frac{b}{d}$

3) Infinite sol^n if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

ECE 2010 (1M)

Problem 1: The system of eq^{ns}: $4x + 2y = 7$
$2x + y = 6$ have

$$\frac{4}{2} = \frac{2}{1} \neq \frac{7}{6} \qquad \left(\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}\right)$$

$\therefore$ The given system of eq^{ns} have no sol^n

Problem 2: $x + 2y = 5$
$2x + 3y = 9$

$$\frac{1}{2} \neq \frac{2}{3} \neq \frac{5}{9}, \quad \frac{1}{2} \neq \frac{2}{1} \quad \left(\frac{a}{c} \neq \frac{b}{d}\right)$$

$\therefore$ Unique sol^n

roblem B: $x + y = 3$

$3x + 3y = 9$

$\frac{1}{3} = \frac{1}{3} = \frac{3}{9}$ $\therefore$ Infinite sol^n

te (5 = 2m)

roblem 6: How many solutions does the following system of eq^{ns} have

i) $-x + 5y = -1$

$x - y = 2$

$x + 3y = 3$

a) Infinite b) Exactly 2 c) Unique sol^n d) No sol^n

$$(A|B) = \left(\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right)$$

$R_2 \to R_2 + R_1$ $\quad$ $R_3 \to R_3 + R_1$

$$= \begin{pmatrix} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$= \left(\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

A $\quad$ B

$\rho(A|B) = 2$ $\quad$ $\rho(A) = 2$

$\rho(A|B) = \rho(A) =$ No of unknowns $= 2$

$\therefore$ Unique sol^n

Probl

Prob

(2m)

Problem 5: Consider following non homogeneous system of Linear eq^n in 3 variables x_1, x_2, x_3.

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + 5x_3 = 2$$
$$-x_1 + 4x_2 + x_3 = 3$$

The above system of eq^n have —
a) No sol^n b) unique sol^n c) more than 1 but finite no. of sol^{ns} d) Infinite no. of sol^{ns}

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 3 & 2 & 5 & 2 \\ -1 & 4 & 1 & 3 \end{pmatrix}$$

0 sol^n

$R_2 \to 2R_2 - 3R_1$ $\qquad$ $R_3 \to 2R_3 + R_1$

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 7 & 1 & 1 \\ 0 & 7 & 5 & 7 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 7 & 1 & 1 \\ 0 & 0 & 4 & 6 \end{pmatrix}$$

$\rho(A|B) = 3$ $\qquad$ $\rho(A) = 3$

No. of unknowns $= 3$

$\therefore$ Unique sol^n.

2010 (2m)

Problem 6: For the set of eq^{ns} $x_1 + 2x_2 + x_3 + x_4 = 2$
$3x_1 + 6x_2 + 3x_3 + 3x_4 = 6$

which of the foll. stmt is true —
1) There exist only trivial sol^n
2) There are no sol^{ns}
3) Unique non trivial sol^n non zero sol^n
4) Infinite no. of non trivial sol^n

$\Longrightarrow$ No of $eq^{ns}(r) = 2$

No of variables $(n) = 4$

here $r < n$

Infinite sol^n (Non trivial)

Problem 7 : The value of x_3 obtained by solving the foll system of eq^{ns}.

$$x_1 + 2x_2 - 2x_3 = 4$$
$$2x_1 + x_2 + x_3 = -2$$
$$-x_1 + x_2 - x_3 = 2$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 1 & 1 & -2 \\ -1 & 1 & -1 & 2 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$ $\qquad$ $R_3 \to R_3 + R_1$

$$(A|B) = \left(\begin{array}{cccc} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 3 & -3 & 6 \end{array}\right)$$

$R_3 \to R_3 + R_2$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 0 & 2 & -4 \end{array}\right)$$

$$2x_3 = -4$$
$$\boxed{x_3 = -2}$$

(To find x_2

$$-3x_2 + 5x_3 = -10$$
$$-3x_2 - 10 = -10$$
$$-3x_2 = 0$$
$$\boxed{x_2 = 0}$$)

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2011 (am)

Problem 8 : find the values of $\lambda \mu$ for which the following system of eq^{ns}

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$ have

1) No sol^n 3) infinite sol^n

2) Unique sol^n

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right)$$

$R_2 \rightarrow R_2 - R_1$ $R_3 \rightarrow R_3 - R_1$

$$(A|B) = \left(\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda - 1 & \mu - 6 \end{array}\right)$$

$R_3 \longrightarrow R_3 - R_2$

$$(A|B) = \left(\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda - 6 & \mu - 20 \end{array}\right)$$

$$\cancel{\rho(A) = 3}$$
$$\cancel{\rho(A|B) = 3}$$

1) No sol^n : If $\rho(A) < \rho(A|B)$

If $\lambda = 6$, $\mu \neq 20$

$\rho(A) = 2$ $\rho(A|B) = 3$

$\Rightarrow \rho(A) < \rho(A|B) \longrightarrow$ No sol^n

2) Unique sol^n : If $\rho(A) = \rho(A|B) =$ No of unknowns

For $\lambda \neq 6$ & $\mu \longrightarrow$ any value $\mu = 20$ or $\mu \neq 20$

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3) Infinite no. of sol^n :— If $\rho(A) = \rho(A|B) <$ No. of unknowns

No. of unknowns = 3 — must be 2 bcoz rank can't be 0 & 1

$\rho(A)$ & $\rho(A|B)$ should be less than 3

It is possible if

$\lambda = 6$ & $\mu = 20 \implies$ Infinite sol^n set

HW

Problem 9 : Find the values of A & B for which the following system of eq^{ns}

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$ have

1) No sol^n 2) Unique sol^n 3) Infinite sol^{ns}

GATE 2011

Problem 10 : For what values of α & β, the following system of eq^{ns}

$$x + y + z = 5$$
$$x + 3y + 3z = 9$$
$$x + 2y + \alpha z = \beta$$

have infinite no of sol^n ?

a) $\alpha = 2$, $\beta = 7$ c) $\alpha = 3$, $\beta = 4$

b) $\alpha = 7$, $\beta = 2$ d) $\alpha = 4$, $\beta = 3$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right)$$

$R_2 \rightarrow R_2 - R_1$ $R_3 \rightarrow R_3 - R_1$

$$= \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha - 1 & \beta - 5 \end{pmatrix}$$

$R_3 \rightarrow 2R_3 - R_2$

Non-homogeneous system $eq^n \Rightarrow$ draw ...

$$= \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 2\alpha-4 & 2\beta-14 \end{pmatrix}$$

$\rho(A) = \rho(A|B) <$ No. of unknowns

$\rho(A) = \rho(A|B) < 3$

$2\alpha - 4 = 0$ $2\beta - 14 = 0$

$2\alpha = 4$ $2\beta = 14$

$\boxed{\alpha = 2}$ $\boxed{\beta = 7}$

Problem 11 : If A is 3×4 matrix & the non homogeneous system of equations $AX = B$ is inconsistent (No sol) The highest possible rank of A is

$\longrightarrow$

$(A|B)_{3\times5}$

$\rho\{(A|B)\}_{3\times5} \leq \min(3,5)$

$\rho\{(A|B)\} \leq 3$

$= 3$ Highest Possible Rank

But it is given that the given system of eq^{ns} are inconsistent (No sol^n)

For inconsistent $\rightarrow \rho(A) < \rho(A|B)$

$\rho(A) < 3$ $\rho(A) \to 2, 1, 0$

Highest possible rank of A $= 2$

* Homogeneous system of Linear Eq^n *

Consider the following homogeneous system of Linear eq^{ns} in m equations & n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \text{(I)}$$

Procedure to solve problems :-

1) Write the given system of eq^{ns} in the form $AX = 0$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

2) Reduce the matrix A into either row or column echelon form (but always row echelon form is preferable) OR find the determinant of matrix $|A|$.

3) If $\rho(A) =$ No. of unknowns (variables) OR $|A| \neq 0$ ($A \rightarrow$ Non singular matrix), the given system of eq^n have trivial sol^n $(x = y = z = 0)$

4) If $\rho(A) <$ No. of unknowns OR $|A| = 0$, the given system of eq^n have infinite no. of sol^n.

All these infinite no. of sol^n can be found by assigning $(n-r)$ variables as arbitrary constant. These $(n-r)$ sol^{ns} are called L.I. sol^{ns}

$\boxed{AX = B}$

- $\boxed{B = 0}$
  - $\rho(A) = n$ or $|A| \neq 0$ → Trivial sol^n $(x = y = z = 0)$ ⇓ L.D
  - $\rho(A) < n$ or $|A| = 0$ → Non-Trivial → Infinite no. of non zero sol^n ⇓ L.I
- $\boxed{B \neq 0}$
  - $\rho(A) < \rho(A|B)$ → No sol^n
  - $\rho(A|B) = \rho(A) = n$ → Unique sol^n
  - $\rho(A|B) = \rho(A) < n$ → Infinite no. non zero sol^{ns}

Problem 1: For what values of λ the system of eq^{ns}

$$x + y + z = 0$$
$$(\lambda+1)\,y + (\lambda+1)z = 0$$
$$(\lambda^2 - 1)\,z = 0$$

have 2 L.I sol^n ?

$\Rightarrow$ No. of L.I sol^n $= n - r = 2$

$= 3 - r = 2$

$\therefore \boxed{r = 1} \rightarrow$ Rank of matrix

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ \cancel{\lambda+1}\ 0 & \lambda+1 & \lambda+1 \\ 3 & 0 & \lambda^2-1 \end{vmatrix} = 0 \Rightarrow \text{Upper } \Delta^{lar}$$

$$\Rightarrow (\lambda+1)(\lambda^2-1) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-1) = 0$$

$\lambda = -1, 1$ ... not correct
put in matrix in order to get
$\rho(A) = 1$

$\therefore \boxed{\lambda = -1}$

Problem 2 :- The rank of 3×3 matrix A is 1. The homogeneous system of eq^{ns} $AX = 0$ has

a) trivial sol^n b) 1 L.I. sol^n

c) 2 L.I. sol^{ns} d) 3 L.I. sol^{ns}

$(A)_{3\times3} \Rightarrow 3$ eq^{ns} & 3 variables

$n = 3 \quad r = 1 \Longrightarrow r < n$

$\Downarrow$

Infinite sol^n

$\therefore$ L.I. sol^n $= n - r$

$= 3 - 1$

$= 2$

2011 (pm)

Problem 3 :- The system of eq^{ns}: $2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$ have

a) No non trivial sol^n c) 5 Non trivial sol^{ns}

b) Unique non trivial sol^n d) Infinite non trivial sol^{ns}

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 2[1] - 1[0+1] + 1[0-1]$$

$$= 2 - 1 - 1 = 0$$

Infinite sol^n

classmate
Date
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Problem 4: The system of eq^n
$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = \cancel{+[-14+44] - 2[28-4] - 1[-22+1]}$$
$$= \cancel{30 - 72 + 42}$$
$$1[-14+44] - 3[28-4] + 2[-22+1]$$
$$= 0 \quad = 30 - 72 + 42 = 0$$

Problem 5: Find the values of k for which the following system of eq^{ns} have infinite no. of non trivial sol^{ns}.

$\downarrow$
$|A| = 0$

$$(3k-8)x + 3y + 3z = 0$$
$$3x + (3k-8)y + 3z = 0$$
$$3x + 3y + (3k-8)z = 0$$

$\Longrightarrow$ If $|A| = 0$, the given system of eq^{ns} have infinite no. of non zero sol^{ns}.

$$\begin{vmatrix} 3k-8 & 3y & 3z \\ 3 & 3k-8 & 3z \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$$\Rightarrow (3k - 8 + 3y + 3z)\ (3k - 8 - 3y)\ (3k - 8 - 3z) = 0$$
$$(3k - 2)\ (3k - 11)\ (3k - 11) = 0$$

$$3k = 2 \qquad 3k = 11$$
$$\therefore\ k = \frac{2}{3},\ \frac{11}{3}$$

HW

Problem 6 : Find the real value of λ for which the foll system of eq^{ns} have non trivial sol^{ns}

infinite no. of non trivial sol^{n}

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

$\boxed{\lambda = 6}$

VIMP

* Eigen values & Eigen Vectors *

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$\downarrow$

Characteristic matrix

ch. det $\rightarrow$ $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4$

ch. polynomial $= \lambda^2 - 8\lambda + 16 - 4$

ch. eq^{n} $\leftarrow$ $|A - \lambda I| = 0 \implies \lambda^2 - 8\lambda + 12 = 0$

$$\therefore \lambda = 6, 2$$

$\downarrow$

ch. roots or Eigen values

Null vector can not be Eigen vector

* **Eigen values:** let A be an $n \times n$ matrix. λ is a scalar (some constant). The matrix $A - \lambda I$ is called as characteristic matrix.

  $|A - \lambda I|$ is called characteristic determinant or characteristic polynomial.

  The roots of this ch. det. are called char roots OR Eigen values OR Latent roots OR Proper values.

  The set of eigen values of matrix A is called as spectrum of A.

* **Eigen Vector:** If λ is an eigen value of a matrix A then there exist a non zero vector X such that $AX = \lambda X$, then the non zero vector X is called as Eigen vector.

Note:

$$AX = \lambda X$$
$$AX = \lambda X I$$
$$AX - \lambda X I = 0$$
$$AX - \lambda I X = 0$$
$$\boxed{(A - \lambda I)X = 0}$$

Trivial sol^n $\Rightarrow |A - \lambda I| \neq 0$

$\underbrace{\text{Infinite sol}^n}_{\text{Non trivial sol}^n}$ $\Rightarrow |A - \lambda I| = 0$

$\Downarrow$

Eigen Vectors

Note: Consider the matrix

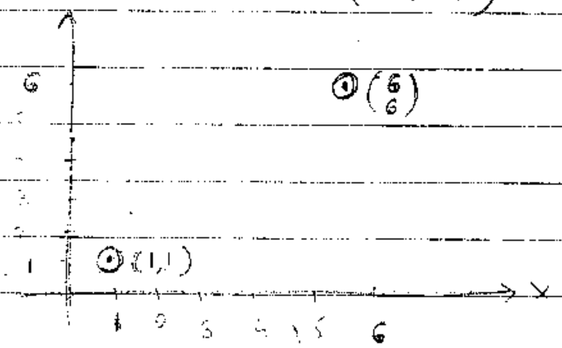
$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}} \underset{\substack{X \\ \text{(Non zero vector)}}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 4+2 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \underset{\lambda}{6} \underset{X}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigen vector corresponding to eigen value $\lambda = 6$ for matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

→ Any pt / vector exist along the same vector will also be an eigen vector corresponding to the same eigen value

Ex. (2,2) (3,3) ... (7,7) (8,8) ...



Eg $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

$$\underset{A}{\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}} \underset{X}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 4+4 \\ 2+8 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix} = 2 \underset{X}{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}$$

Not same

$X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is not eigen vector corresponding to eigen value 2

Problem: Find the eigen values & eigen vector of

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Symmetric matrix

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16$$

$$= -9 - 3\lambda + 3\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda^2 = 25$$

$$\lambda = \pm 5$$

$\therefore$ Eigen values $= 5, -5$

case i) $(A - \lambda I) X = 0$

$$\begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigen vectors

put $\lambda = 5$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 4x_2 = 0 \quad \text{——— (1)}$$

$$4x_1 - 8x_2 = 0 \quad \text{——— (2)}$$

eqn (1) $\Rightarrow$ divide b.s. by (-2)

$$x_1 - 2x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{2}{1}$$

$x_1 = 2x_2$

$$\therefore X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Case ii) when $\lambda = -5$

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 4-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = -5$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$8x_1 + 4x_2 = 0$ ——— (1)

$4x_1 + 2x_2 = 0$ ——— (2)

2 vectors are not der.. only 1 vector is der

$2x_1 + x_2 = 0$

$2x_1 = -x_2$

$\frac{x_1}{x_2} = \frac{-1}{2}$

$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad X_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$X_1^T X_2 = 0$ or $X_1 X_2^T = 0 \Rightarrow X_1$ & X_2 are orthogonal

$X_1^T \cdot X_2 = \begin{pmatrix} 2 & 1 \end{pmatrix}_{1\times 2} \begin{pmatrix} -1 \\ 2 \end{pmatrix}_{2\times 1}$

$(X_1^T \cdot X_2)_{1\times 1} = -2+2 = 0$

$\therefore$ X_1 & $X_2 \Rightarrow$ orthogonal vectors

E.V of Upper Δ, Lower Δ, diag... are
∴ diagonal elem. only

Note 1: The Eigen vectors corresponding to diff. eigen value of a real symmetric matrix are always orthogonal.

Note 2: If the Eigen vectors corresponding to diff. eigen values of any square matrix are always Linearly Independent.

$$X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\therefore X_1 \neq k X_2 \quad \text{OR} \quad X_2 \neq k X_1$$

$$\therefore \left.\begin{matrix} X_1 \\ X_2 \end{matrix}\right\rangle \text{L.I.} \qquad (\text{acc}^n \text{ to criteria (4)})$$

is

In the above ex. the eigen values of matrix are 5, -5 the corresponding eigen vectors are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

* Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

$\lambda = 2, 2 \longrightarrow$ Repeated twice

when $\lambda = 2$ Max. no. of L.I. vects

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = 2$

$$\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0X_1 + 3X_2 = 0$$

$$3X_2 = 0 \qquad \therefore X_2 = 0$$

$x_1 \neq 0$

put $x_1 = c$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \text{ where } c \neq 0.$$

Acc^n to criteria (5)

every non zero vector is L.I vector.

Note:- If some of the eigen value of a matrix are repeated then the eigen vectors corresponding to repeated eigen values may be L.I or L.D.

If an eigen value λ is repeated n times the eigen vectors corresponding to repeated eigen values are always L.I. which are given by

$$\underbrace{p = n - r}_{}\ ; \quad 1 \le p \le m \quad \{\text{max. } m \text{ times}\}$$

$n \rightarrow$ no. of unknowns or variables

$r \rightarrow \rho(A - \lambda I)$

$\boxed{A_{2\times 2}}$

$\lambda_1 \ \& \ \lambda_2$

- $\lambda_1 \neq \lambda_2$
  - x_1, x_2 $\rightarrow$ L.I
- $\lambda_1 = \lambda_2$
  - x_1 $\rightarrow$ L.I
  - $x_1 \ \& \ x_2$ $\rightarrow$ max. 2 times $\rightarrow$ L.I

9.

$$\boxed{A_{3\times 3}}$$

$$\downarrow$$

$$\lambda_1, \lambda_2, \lambda_3$$

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

$\downarrow \quad \downarrow \quad \downarrow$

$x_1 \quad x_2 \quad x_3$

L.I

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

$\downarrow \quad \downarrow \quad \downarrow$

$x_1 \quad x_2 \quad x_3$

L.I $\quad$ L.I or L.D

L.I

$\lambda_1 = \lambda_2 = \lambda_3$

$\downarrow \quad \downarrow \quad \downarrow$

$x_1 \qquad x_1, x_2 \qquad x_1, x_2, x_3$

$\downarrow \quad \downarrow \quad \downarrow$

L.I $\quad$ L.I $\quad$ L.I

$$\begin{pmatrix} c_1 + c_2 \\ c_2 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$x_1 \quad x_2$

L.I

2011 wale Instr. (1m)

Note:-

$$A X = \lambda X$$

Eigen vector of A

Eigen value of A

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot \lambda \cdot X$$

$$I \cdot X = A^{-1} \cdot \lambda \cdot X$$

$$X = \lambda \cdot A^{-1} X$$

$$\frac{X}{\lambda} = A^{-1} X$$

$$A^{-1} X = \frac{1}{\lambda} X$$

Eigen vector of A^{-1}

Eigen value of A^{-1}

If λ is eigen value of A & X is eigen vector of A but $\frac{1}{\lambda}$ is eigen value of A^{-1} and

x is eigen vector of A^{-1}. Therefore, A & A^{-1} have same eigen vectors.

A & A^m have same eigen vectors ($m > 0$) corresponding to eigen values.

$$AX = \lambda X$$
$$AAX = A\lambda X$$
$$A^2X = \lambda\lambda X$$
$$A^2X = \lambda^2 X$$

17/07/2012

* Properties of Eigen Values & Eigen vectors :-

1) Sum of eigen values is equal to trace of matrix
sum of digonal elem

i.e., if $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of a matrix A then

(i) Trace $A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

2) (ii) $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \lambda_n$

Ex. Consider the matrix

$\lambda_1 + \lambda_2 = 7$

$\lambda_1 \cdot \lambda_2 = 6$

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

$$\begin{matrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{matrix}$$

$(5-\lambda)(2-\lambda)-4$

$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6$

$\lambda = 1, 6$

i) $\rightarrow 1 + 6 = 5 + 2$ $\quad\quad$ $\therefore$ Trace of $A = 7$

ii) $\rightarrow 1 \times 6 = 10 - 4 = 6$ $\quad\quad$ $|A| = 6$

3) The eigen values of upper triangular or lower triangular or diagonal or scalar or identity matrix is its diagonal elements.

1) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$; $\lambda = 1, -4, 7$

$\downarrow$

Upper Δ^{lar}

2) $A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 0 & 8 & 9 \end{pmatrix}$; $\lambda = 3, 5, 9$

$\downarrow$

Lower Δ^{lar}

3) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$; $\lambda = 3, 4, 5$

$\downarrow$

Diagonal matrix

4) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$; $\lambda = 3, 3, 3$

$\downarrow$

scalar matrix

5) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $\lambda = 0, 0, 0$

$\downarrow$

scalar / Null matrix

6) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\lambda = 1, 1, 1$

$\downarrow$

Identity matrix

4) The eigen values of A & A^T are same

5) The eigen values of A & $P^{-1}AP$ are same where P is a non singular matrix.

6) The eigen values of real symmetric matrix are real.

7) The eigen values of skew symmetric matrix are purely imaginary OR zeros.

8) The eigen values of orthogonal matrix are of unit modulus i.e., ± 1

9) If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also one of its eigen value

10) If λ is an eigen value of matrix A then

i) $k\lambda$ is an eigen value of kA.

ii) $\frac{1}{\lambda}$ is also an eigen value of A^{-1}.

iii) λ^2 —— " —— A^2.

iv) λ^m —— " —— A^m.

v) $\frac{|A|}{\lambda}$ is an eigen value of adj. A

vi) $\lambda \pm k$ —— " —— $A \pm kI$

vii) $(\lambda \pm k)^2$ —— " —— $(A \pm kI)^2$

viii) $\frac{1}{\lambda \pm k}$ —— " —— $(A \pm kI)^{-1}$

Note: If A is a singular matrix i.e., $|A| = 0$ then one of its eigen value should be zero

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$$
$$\downarrow$$
$$0 \cdot \lambda_2 \cdot \lambda_3 = |A|$$
$$|A| = 0 \qquad \therefore |A| \text{ — singular}$$

2m

Problem 1: $A = \begin{pmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ are $\qquad j = \sqrt{-1}$

A) $3,\ 3+5j,\ 6-j$
B) $-6+5j,\ 3-j,\ 3+j$
x c) $3-j,\ 3+j,\ 5+j$
d) $3,\ -1+3j,\ -1-3j$

by Prop. 1 :

$$\text{Trace } A = \lambda_1 + \lambda_2 + \lambda_3 \qquad [\text{cross check}]$$
$$-1 - 1 + 3 = 3 - 1 + 3j - 1 - 3j$$
$$1 = 1$$

If (1) or more opt^n satisfy [illegible]

Problem 2: The eigen values & eigen vector of a 2×2 matrix are given by

Eigen value   Eigen vector

$\lambda_1 = 8$ $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4$ $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

the matrix is —

a) $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$

to find eigen value given eigen ...
formula $(A - \lambda I) X = 0$

1998 (2M)

Problem 3: The vector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of the matrix $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ The eigen value corresponding to the eigen vector is —

$\Longrightarrow$

$(A - \lambda I) X = 0$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

value consider any 1 row & multiply

$-1 - 4 + \lambda = 0$

$\boxed{\lambda = 5}$

Problem 4: For the matrix $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ the eigen value corresponding to the eigenvector $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2 + 4 - \lambda = 0$

$\lambda = 6$

Problem 5: The min & max eigen values of a matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ are -2 & 6 resp. what would be the 3rd eigen value

$$\text{Trace } A = \lambda_1 + \lambda_2 + \lambda_3$$
$$1+5+1 = -2+6+\lambda_3$$
$$\boxed{\lambda_3 = 3}$$

Problem 6: The matrix $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{pmatrix}$ has an eigen value $= 3$. Sum of other two eigen values is ---

$$1+0+p = 3+\lambda_2+\lambda_3$$
$$p+1-3 = \lambda_2+\lambda_3$$
$$\boxed{\lambda_2+\lambda_3 = p-2}$$

2011

Problem 7: Consider the following matrix

$A = \begin{pmatrix} 2 & 3 \\ x & y \end{pmatrix}$. The eigen values of A are 4 & 8. The values of x, y are

prop 1 $\longrightarrow 2+y = 4+8$
$$y = 12-2 = 10$$

prop 2 $\longrightarrow$ $\cancel{2 \times 10 = 20}$ $\quad 4 \times 8 = 32$

$$\begin{vmatrix} 2 & 3 \\ x & 10 \end{vmatrix} = 32 \qquad x = -4$$

$$20 - 3x = 32$$
$$20 - 32 = 3x \qquad \therefore x = \frac{-10}{3}$$
$$3x = -12 \quad \boxed{x = -4}$$

$$2y - 3x = 4\times 8 \Rightarrow 2\times 10 - 3x = 32$$

$$-3x = 32 - 20$$

$$-3x = 12$$

$$\boxed{x = -4}$$

Pro 8 :- The eigen values of 3×3 matrix are given by 1, −3, 9. Find

i) Trace $(A^2 + A^{-1} - adj.A)$

ii) det $(A^2 + A^{-1} - adj.A)$

$\Rightarrow$ $|A| = 1 \times -3 \times 9 = -27$

$A^2 + A^{-1} - adj.A$

$\rightarrow \lambda^2 + \frac{1}{\lambda} - \frac{|A|}{\lambda}$ by prop.(10)

1

−3 $\quad \lambda^2 + \frac{1}{\lambda} - \frac{|A|}{\lambda}$

9

$$(1)^2 + \frac{1}{1} - \frac{(-27)}{1} = \underline{\underline{29}}$$

$$(-3)^2 + \frac{1}{(-3)} - \frac{(\cancel{-}27)}{(\cancel{-}3)}$$

$$9 - \frac{1}{3} - 9 = \underline{\underline{-1/3}}$$

$$9^2 + \frac{1}{9} - \frac{(-2\cancel{7})^3}{\cancel{9}}$$

$$81 + \frac{1}{9} + 3$$

$$\frac{729 + 1 + 27}{9} = \underline{\underline{\frac{757}{9}}}$$

i) Trace $(A^2 + A^{-1} - adj.A) = 29 - \frac{1}{3} + \frac{757}{9}$

ii) $\det(A^2 + A^{-1} - \text{adj}\, A) = 29 \times \left(\frac{-1}{3}\right) \times \frac{757}{9}$

Prob. 9:- Given $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$, the eigen values of $3A^3 + 5A^2 - 6A + 2I$ are

Upper Δlar

$\therefore \lambda = 1, 3, -2$

$3A^3 + 5A^2 - 6A + 2I \Longrightarrow 3\lambda^3 + 5\lambda^2 - 6\lambda + 2$

put $\lambda = 1$

$3(1)^3 + 5(1)^2 - 6(1) + 2$

$3 + 5 - 6 + 2 = 4$

put $\lambda = 3$

$3 \times 27 + 5 \times 9 - 18 + 2$

$81 + 45 - 16 = 126 - 16 = 110$

put $\lambda = -2$

$-(3 \times 8) \cancel{+} 20 + 12 + 2$

$-24 + 24 = 0$

$3 \times (-8) + 5 \times 4 + 12 + 2$

$-24 + 20 + 12 + 2 = 10$

2009 (em)

Prob. No 10:- The eigen values of 2×2 matrix A are given by -2 & -3 resp. the eigen values of $(x + I)^{-1} \cdot (X + 5I)$

This is not possible

$-\frac{57}{9}$

$\Longrightarrow (\lambda + I)^{-1}(X + 5I) = (X + I)^{-1}\{(X + I) + 4I\}$

$= (X + I)^{-1} \cdot (X + I) + 4I \cdot (X + I)^{-1}$ A + I

$= I + 4(X + I)^{-1}$

$4(X + I)^{-1} + I \Longrightarrow \frac{4}{1 + \lambda} + 1$

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$$\Longrightarrow \frac{4}{1+\lambda}+1$$

put $\lambda = -2$ $\qquad \frac{4}{1-2}+1 = -3$

put $\lambda = -3$ $\qquad \frac{4}{1-3}+1 = -1$

Prob 11 :- The eigen vector of a 3x3 matrix are are orthogonal. What will be the 3rd orthogonal eigen vector.

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$x = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad x_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}. \qquad \text{Let } x_3 = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

here, 3 vectors are orthogonal

$\therefore$ $x^T \cdot x_3 = 0 \qquad x_2^T \cdot x_3 = 0$

$$\begin{pmatrix}1 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$x + z = 0$ —— (i) $\qquad x - z = 0$ —— (ii)

add (i) & (ii)

$2x = 0$

$\therefore x = 0$ $\qquad$ put in (i)

$0 + z = 0$

$z = 0$

$y \neq 0$ $\quad \because$ zero vector can't be eigen vector

$\Rightarrow \therefore y = c$ $\qquad$ c — arbitrary constant & $c \neq 0$

$\therefore$ The 3rd eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

2008 (2m)

Prob. 12 :- The eigen vector of 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ are given by $\begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ b \end{pmatrix}$, what is $(a+b)$?

$$\lambda = 1, 2$$

$AX = \lambda X$ OR $(A - \lambda I)X = 0$

put $\lambda = 1$

$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 1 \\ a \end{pmatrix}} = 1 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

~~1+2a 2a~~ $1 + 2a = 1$

$2a = 0$ $\boxed{a = 0}$

put $\lambda = 2$

$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ b \end{pmatrix} = 2 \begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 2b \end{pmatrix}$$

$1 + 2b = 2$

$2b = 1$

$\boxed{b = 1/2}$

ctor

$c \neq 0$

Method I :-

Case 1 : when $\lambda = 1$

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Prob.

$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2x_2 = 0 \qquad \therefore x_2 = 0$

here $x_1 \neq 0$

$\therefore x_1 = C \qquad C \rightarrow$ non zero arbitrary const

$$X = \begin{pmatrix} C \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ generally C is replaced by 1

Case 2 : when $\lambda = 2$

$$\begin{pmatrix} 1-2 & 2 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x_1 + 2x_2 = 0$

$2x_2 = x_1$

$\frac{2}{1} = \frac{x_1}{x_2}$

$$X_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$x_1 = 2 \qquad x_2 = 1$

$$\frac{x_1}{x_2} = \frac{1}{(1/2)}$$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix} \Rightarrow a = 0$$

$$x_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix} \quad b = 1/2$$

Prob. 13: Find eigen values & eigen vectors of foll.

a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$\Longrightarrow \quad \lambda = 1, 2, 3$

→ put $\lambda = 1$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$x_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

other form of the ... vector

→ put $\lambda = 2$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$x_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

→ put $\lambda = 3$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$x_3 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

for distinct eigenvalues
→ consider ... rows
→ select ... element
→ ...
→ put ...

b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$

$\lambda = 1, -4, 7$

put $\lambda = 1$

$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 2 & 3 & 0 & 2 \\ -5 & 2 & 0 & -5 \end{array}$

$\frac{x_1}{19} = \frac{x_2}{0} = \frac{x_3}{0}$ $\qquad X_1 = \begin{pmatrix} 19 \\ 0 \\ 0 \end{pmatrix}$

put $\lambda = -4$

$\begin{array}{cccc} & & x_2 & x_3 \\ 2 & 3 & 5 & 2 \\ -8 & 2 & 0 & -8 \end{array}$

$\frac{x_1}{28} = \frac{x_2}{-10} = \frac{x_3}{-40}$ $\qquad X_2 = \begin{pmatrix} 28 \\ -10 \\ -40 \end{pmatrix}$

put $\lambda = 7$

$\begin{array}{cccc} & & x_2 & x_3 \\ 2 & 3 & -6 & 2 \\ -11 & 2 & 0 & -11 \end{array}$

$\frac{x_1}{37} = \frac{x_2}{12} = \frac{x_3}{66}$ $\qquad X_3 = \begin{pmatrix} 37 \\ 12 \\ 66 \end{pmatrix}$

Note:— To find the eigen vectors corresponding to nonrepeated eigen value of a matrix, we proceed as follows:

1) Select the 1st two rows only
2) Start from the 1st row middle no. & move in anticlockwise direction to complete 1 cycle. If any element exist in the main diagonal while we are moving in anticlockwise direction, then the eigen value should be subtracted from the corresponding diagonal elements.
These elements will be return seperately in row wise
3) Repeate the same procedure with 2nd row also. & find eigen vectors

* Cayley - Hamiltom theorem *

Statement:
Every square matrix satisfies its own characteristic eqn

Ex. Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

Its char. eqn. is $\lambda^2 - 8\lambda + 12 = 0$

by cayley- Hamiltom theorem, every square matrix satisfies its own char. eqn.

i.e., $A^2 - 8A + 12I = 0$

Note: (2×2)
Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

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$|A - \lambda I| = 0$

$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$(a-\lambda)(d-\lambda) - bc = 0$

$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$

$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

$\Longrightarrow \lambda^2 - \lambda[\text{trace of } A] + |A| = 0$

by cayley-hamilton theorem, it is

$A^2 - A[\text{trace of } A] + |A| \cdot I = 0$

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Note: (3×3)

Consider the 3×3 matrix

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, its characteristic eq^n is—

General procedure:—

$\lambda^3 - \lambda^2(\text{trace of } A) + \lambda\{|\;| + |\;| + |\;|\} - |A| = 0$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $\quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ $\quad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Using cayley Hamilton theorem, we can find

i) Inverse of a matrix
ii) Powers of a matrix.

* Positive powers of matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

char. eq^n $\longrightarrow$ $\lambda^2 - 8\lambda + 12 = 0$

Cayley Hamilton theorem $\longrightarrow$ $A^2 - 8A + 12I = 0 \longrightarrow (1)$

$$A^2 = 8A - 12I \longrightarrow (2)$$

$$A^3 = 8A^2 - 12A \longrightarrow (3)$$

$$A^4 = 8A^3 - 12A^2 \longrightarrow (4)$$

$$A^5 = 8A^4 - 12A^3 \longrightarrow (5)$$

$$\vdots$$

is —

* Negative powers of matrix

$$A^2 - 8A + 12I = 0$$

$$A^2 \cdot A^{-1} - 8 \cdot A \cdot A^{-1} + 12 \cdot I \cdot A^{-1} = 0$$

$$A - 8I + 12A^{-1} = 0$$

$$12A^{-1} = 8I - A$$

$$A^{-1} = \frac{1}{12}\left[-A + 8I\right]$$

$$A^{-1} \cdot A^{-1} = \frac{1}{12}\left[-A^{-1} \cdot A + 8IA^{-1}\right]$$

$$A^{-2} = \frac{1}{12}\left[-I + 8A^{-1}\right]$$

$$A^{-1} \cdot A^{-2} = \frac{1}{12}\left[- A^{-1} \cdot I + 8 A^{-1} \cdot A^{-1} \right]$$

$$A^{-3} \quad \frac{1}{12}\left[- A^{-1} + 8 A^{-2} \right]$$

Problem 1 :— Find A^8 using Caylay hamilton theorem for

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$|A| = -1 - 4 = -5$

trace of $A = 1 + (-1) = 0$

$A^2 - A\,(\text{trace of } A) + |A| \cdot I = 0$

$A^2 - A(0) + (-5) \cdot I = 0$

$\cancel{A^2 + A - 5I}$

$A^2 - 5I = 0$

$A^2 = 5I$

$(A^2)^4 = (5I)^4 = 5^4 \cdot I^4$

$A^8 = 625\, I$

$= 625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^8 = \begin{pmatrix} 625 & 0 \\ 0 & 625 \end{pmatrix}$$

(4M)

Problem 2:

Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic eq^n.

Consider the matrix

once we get char eq^n after replacing b λ by A 1st time we get char eq^n we should not multiply it by A^{-1}.

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$$

1) A satisfies the relation

(a) $A + 3I + 2A^{-1} = 0$ (b) $A^2 + 2A + 2I = 0$

(c) $(A+I)(A+2I) = 0$ (d) $\exp(A)$

$$|A| = 0 + 2 = 2$$

$$\text{trace of } A = -3 + 0 = -3$$

$$\Longrightarrow A^2 - A(\text{trace of } A) + |A| \cdot I = 0$$

$$A^2 + 3A + 2I = 0$$

$$\cancel{A^2 + 2A}$$

$$A^2 + 2AI + A + 2I = 0$$

$$A(A+2I) + I(A+2I) = 0$$

$$(A+2I)(A+I) = 0$$

2) A^9 equals 511 > 510

a) $511A + 510I$ c) $154A + 155I$

b) $309A + 104I$ d) $\exp(9A)$

$$\Longrightarrow$$

$$A^2 + 3A + 2I = 0$$

$$A^2 = -3A - 2I \quad \text{——— (1)}$$

$$A^3 = -3A^2 - 2A \quad \text{——— (2)}$$

(3 > 2)

$A^{(even)} \rightarrow$ all $-ve$ elem

$A^{odd} \rightarrow$ all $+ve$ elem

$$= -3(-3A-2I)-2A$$

$$\Rightarrow A^3 = 7A+6I \longrightarrow (2) \quad (7>6)$$

$$\Rightarrow A^4 = 7A^2+6A$$
$$= 7[-3A-2I]+6A$$
$$= -21A-14I+6A$$
$$A^4 = -15A-14I \longrightarrow (3)$$
$$(15>14)$$

$$\Rightarrow A^5 = -15A^2-14A$$
$$= -15[-3A-2I]-14A$$
$$= 45A+30I-14A$$
$$A^5 = 31A+30I \longrightarrow (4)$$
$$(31>30)$$

Imp * $A^{(\text{Higher } +ve \text{ Number})} \Rightarrow A^{5000}, A^{10000}$

* Method :—

char eq^n $\Rightarrow \lambda^2-(-3+0)\lambda+2=0$

$$\lambda^2+3\lambda+2=0$$
$$\lambda=-1,-2$$

$\Rightarrow$ If λ values are not repeated then

$$\lambda^n = a\lambda+b \longrightarrow (1)$$

for $\lambda=-1$ $\quad (-1)^n = -a+b \longrightarrow (2)$

for $\lambda=-2$ $\quad (-2)^n = -2a+b \longrightarrow (3)$

$$(-1)^n-(-2)^n = a$$

$$\therefore \boxed{a = (-1)^n-(-2)^n}$$

$$\therefore \boxed{b = (-1)^n+a}$$

$b = (-1)^n + a$

$= (-1)^n + (-1)^n - (-2)^n$

$= 2(-1)^n - (-2)^n$

put a & b in eq^n (1)

$$\lambda^n = \left[(-1)^n - (-2)^n\right]\lambda + \left[2(-1)^n - (-2)^n\right]$$

by c-h theorem

$$\boxed{A^n = \left[(-1)^n - (-2)^n\right]A + \left[2(-1)^n - (-2)^n\right]I}$$

For ex put $n = 9$

$$A^n = \left[(-1)^9 - (-2)^9\right]A + \left[2(-1)^9 - (-2)^9\right]I$$

$$= (-1 + 512)A + (-2 + 512)I$$

$$A^9 = 511A + 510I$$

Problem 3: The char. eq^n. of 3x3 matrix P is given by

$$\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1$$

where I denoted the Identity matrix. The inverse of the matrix P will be —

a) $P^2 + P + 2I$ c) $-(P^2 + P + I)$

b) $P^2 + P + I$ d) $-(P^2 + 2P + 2I)$

$$\lambda^3 + \lambda^2 + 2\lambda + 1$$

$$\implies P^3 + P^2 + 2P + 1 \cdot I = 0$$

$$\cancel{2P = -I - P^2 - P^3}$$

$$P^{-1} \cdot P^3 + P^{-1} \cdot P^2 + 2P^{-1}P + P^{-1}I = 0$$

$$P^2 + P + 2I + P^{-1} = 0$$

$$P^{-1} = -P^2 - P - 2I = -(P^2 + P + 2I)$$

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GATE

Problem (E. value & E. vectors)

How many L.I. eigen vectors of $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$\lambda = 2, 2$

No. of L.I e.vectors $\Rightarrow 1 \leq p \leq \underset{\downarrow\ 2}{m}$ ← No. of times e.value repeated

$p = n - r$

no of unknowns ← n, r → $\rho(A - \lambda I)$

$(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix}$ $\quad \lambda = 2$

$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\rho(A - \lambda I) = 1$

$p = n - r$

$= 2 - 1 = \boxed{1}$

—x—

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पृष्ठ सं. :
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Numericals Methods (2 marks)

Topic :-

1. Solutions to Algebric and Transcandantal Eqn
   - Bisection method
   - ***** - Newton-Raphson Method
   - Regula-false method
   - Secant method

2. Solutions to Systems of Linear Equation
   - * - Gauss Elimination Method
   - * - LU Decomposition

3. Solution to Integration of function
   - ** - Trapezoidal Rule
   - Simpson $\frac{1}{3}$ Rule
   - Simpson $\frac{3}{8}$th Rule

4. Solution to differential Equations
   - * - Euler's method
     - forward
     - Backward
   - Runge Kutta method

1987 - 2012

36 Question

26 Question (Newton-Raphson method)

4 Question (Simpson Rule)

2 Que LU Decom

2 Ques Gauss

2 Ques. Euler

अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

पृष्ठ सं.:
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(I) Mathematical methods are of two types.

(1) Analytical method
(2) Numerical method

(1) Analytical method:-

Eg. 1 Ex. 1. find roots of $x^2-5x+6=0$ using analytical method

→ Analytical sol^n is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\therefore x = 3, 2$

Eg. (2) find $\int_1^2 x\,dx$ using analytical method

→ $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2$

$= \left[\frac{4}{2} - \frac{1}{2}\right]$

$= \frac{3}{2}$

Eg. (3) Solve $\frac{dy}{dx} = x$, Using analytical method.

→ $\frac{dy}{dx} = x \quad \therefore dy = x\,dx$

Integrate $\int \cancel{\frac{dy}{dx}\,dx} \quad \int dy = \int x\,dx$

$y = \frac{x^2}{2} + C$

Note:- - Drawback of analytical method is, it is not applicable for higher degree equations & also not applicable for non-linear Eq's.

TO overcome this, we use NUMERICAL METHODS

- NUMERICAL METHODS provide Approximation value

[I] Solution to Algebric and Transcandantal Equation

* Intermediate Mean Value theorem :-

f(a) is a continuous function defined on [a, b] f(a) & f(b) having opposite signs. In such case there exist at least one Root of f(x) in [a, b]

Let,

$f(x) = x^3 - 4x - 9$, in $[2, 3]$

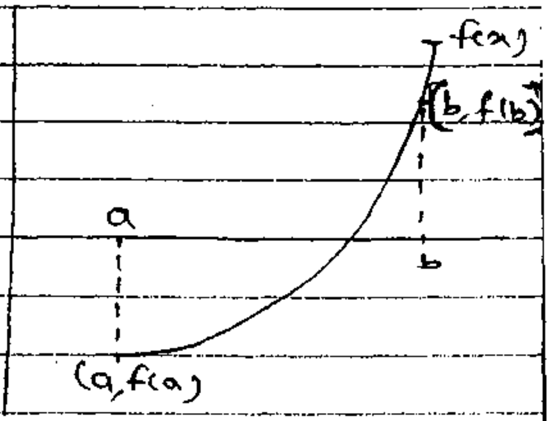
$\therefore f(2) = 2^3 - 4(2) - 9$

$= -9$

$\therefore -9 < 0$

& $f(3) = 3^3 - 4(3) - 9$

$= 6 > 0$



Since, $f(2) < 0$ & $f(3) > 0$

So, there exist atleast one root in $[2, 3]$

∃ - there exists

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* Bisection Method :-

Step I :- Let, $f(x)$ is a continuous on $[a, b]$

Step 2 :- $f(a)$ & $f(b)$ having opposite signs

Say $f(a) < 0$ & $f(b) > 0$

Using intermediate mean value theorem

there exist (∃) atleast one root in $[a, b]$

Step 3 :- Let,

Approximation root is x_1 and

$$x_1 = \frac{a+b}{2}$$

a x_1 b

Case I :-

If $f(x_1) = 0 \Longrightarrow$ 'x_1' is root then

Stop process.

a x_1 x_2 b

Case II :-

If $f(x_1) < 0$ and $f(b) > 0$

then $x_2 = \frac{x_1 + b}{2}$

Continue this process until desired root is found

Case III :-

If $f(x_1) > 0$ and $f(a) < 0$

So ∃ atleast one root between $[a, x_1]$

Say, $x_3 = \frac{a + x_1}{2}$

a x_3 x_1 b

Continue this process until desired root is found

Q find x_2 and x_3 using Bisection method where

$f(x) = x^3 - 4x - 9$, $[2, 3]$

→ Put intervals in $f(x)$

$f(2) = 2^3 - 4 \times 2 - 9$

$= -9$

$\therefore -9 < 0$ $\quad\quad \therefore f(2) < 0$

$f(3) = 3^3 - 4 \times 3 - 9$

$= 6$

$\therefore 6 > 0$ $\quad\quad \therefore f(3) > 0.$

$\exists$ atleast one root between $[2, 3]$

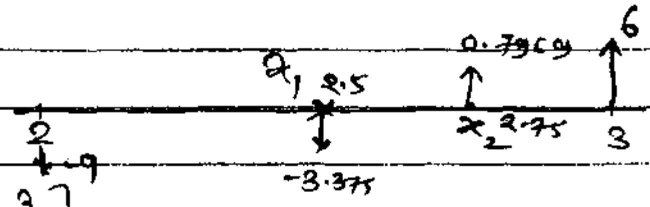
$x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = (2.5)^3 - 4(2.5) - 9$

$= -3.375$

$\therefore -3.375 < 0$ $\quad\quad \therefore f(2.5) < 0.$

Since $f(2.5) < 0$ and $f(3) > 0$



$\exists$ root b/w $[2.5, 3]$

$\therefore$ Let $x_2 = \frac{2.5+3}{2} = \frac{5.5}{2} = 2.75$

$f(x_2) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969$

$\therefore 0.7969 > 0$

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Since $f(2.75) > 0$ & $f(2.5) < 0$

So $\exists$ atleast one root in $[2.5, 2.75]$

Say, $x_3 = \frac{2.5 + 2.75}{2}$

$$\boxed{x_3 = 2.62}$$

* Newton Raphson Method :-

Step 1 :- Let, $f(x)$ is continuous function $[a, b]$

Step 2 :- Newton Raphson iteration formula for finding root of $f(x) = 0$ is,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Q. find Newton raphson iteration formula for square root of +ve real number 'c'.

GATE 2005

$\rightarrow$ Let, $x = \sqrt{c}$

Squaring b.s.

$x^2 = c$

$\therefore x^2 - c = 0$

$\therefore f(x) = x^2 - c \qquad \therefore f(x_n) = x_n^2 - c$

$f'(x) = 2x \qquad f'(x_n) = 2x_n$

$\therefore$ By Newton Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n^2 - c}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + c}{2x_n}}$$

Q. find N-R iteration formula for $f(x) = x^2 - 117 = 0$

GATE 2009 →

$f(x) = x^2 - 117$; $f(x_n) = (x_n)^2 - 117$

$f'(x) = 2x$; $f'(x_n) = 2x_n$

$$x_{n+1} = x_n - \frac{x_n^2 - 117}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + 117}{2x_n}}$$

Q. If $f(x) = x^2 - 13$ and $x_0 = 3.5$, then Value of x_1 using N-R. iteration formula

GATE 2010 →

$f(x) = x^2 - 13$ $\quad f(x_n) = x_n^2 - 13$

$f'(x) = 2x$ $\quad f'(x_n) = 2x_n$

$$\therefore x_{0+1} = x_0 - \frac{x_0^2 - 13}{2x_0}$$

$$x_1 = \frac{x_0^2 + 13}{2x_0}$$

$$x_1 = \frac{(3.5)^2 + 13}{2 \times 3.5} = 3.6071$$

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Q. Newton Raphson iteration formula for finding $\sqrt[3]{c}$

GATE 1996

→ Let $x = \sqrt[3]{c}$

Cubing

$x^3 = c$

$x^3 - c = 0$

$f(x) = x^3 - c$

$f(x_n) = x_n^3 - c$

$f'(x) = 3x^2 - 0$

$f'(x_n) = 3x_n^2$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - c}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + c}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + c}{3x_n^2}}$$

Q. The N-R method is used to find the root of $x^2 - 2 = 0$ & starting value is $x_0 = -1$; The iteration formula will be

GATE 1997

Fastness or Convergence is Rate of Convergence

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(a) Converges to -1 (b) Converges to $-\sqrt{2}$
(c) Converges to $\sqrt{2}$ (d) Not convergent

$f(x) = x^2 - 2$ $\qquad f'(x_n) = 2x_n$

$f(x_n) = x_n^2 - 2$ $\qquad f'(x_n) = 2x_n$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = -1 - \frac{(-1)^2 - 2}{2(-1)}$$

$$= -1 - \frac{1-2}{-2}$$

$$= -1 - \frac{+1}{2}$$

$$= -1 - 0.5$$

$$x_1 = -1.5$$

$$\therefore x_2 = \frac{x_1^2 + 2}{2x_1} = \frac{(-1.5)^2 + 2}{2(-1.5)} = -1.416$$

$$\therefore x_3 = \frac{x_2^2 + 2}{2x_2} = \frac{(-1.416)^2 + 2}{2(-1.416)} = -1.414$$

lly $x_4 = -1.414$ i.e. $-\sqrt{2}$

Recess 10.30 am

ntinue
1:00pm

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Q find Newton Raphson iteration formula, ~~for~~ reciprocal of a where $a > 0$

GATE 2005

→ Let, $x = \frac{1}{a}$

$$\frac{1}{x} = a$$

$$\therefore \frac{1}{x} - a = 0$$

$$\therefore f(x) = 1/x - a$$

$$f'(x) = \frac{-1}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(1/x_n - a)}{(-1/x_n^2)}$$

$$= x_n + x_n^2 (1/x_n - a)$$

$$= x_n + x_n - a x_n^2$$

$$\boxed{x_{n+1} = 2x_n - a x_n^2}$$

Q Given $a > 0$, we wish to compute N-R iteration formula for reciprocal of a for $a = 7$ and $x_0 = 0.2$ then first two iteration will be

GATE 2005

(a) 0.11, 0.1299 (b) 0.12, 0.1392

$0.4 - 7 \times 0.04$ अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

$= 0.4 - 28 = \frac{.24}{0.12}$

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Given $x = \frac{1}{a}$

$\therefore \frac{1}{x} - a = f(x)$

$\therefore x_{n+1} = 2x_n - ax_n^2$

$x_{0+1} = 2x_0 - ax_0^2$

$= 2 \times 0.2 - 7(0.2)^2$

$x_1 = 0.12$

$x_{1+1} = x_2 = 2x_1 - ax_1^2$

$= 2(0.12) - 7(0.12)^2$

$= 0.24 - 7 \times 0.0144$

$x_2 = 0.1392$

H.W

Q 395 $f(x) = x - \cos x$, then $x_{n+1} =$ ______ ?

$\Rightarrow$ Ans. $\Rightarrow x_n - \frac{(x_n - \cos x_n)}{1 + \sin x_n}$

HW Q 005 $f(x) = x.e^{x} - 2$, $x_0 = 0.8679$ then $x_1 =$?

$\Rightarrow$ Ans. $\Rightarrow x_1 = 0.853$

H.W Q 007 $f(x) = x^3 - x^2 + 4x - 4 = 0$, $x_0 = 2$, then $x_1 =$?

$\Rightarrow$ Ans $\Rightarrow x_1 = 4/3$

HW Q 008 $f(x) = e^{x} - 1$, $x_0 = -1$, then $x_1 =$ ______ ?

$\Rightarrow$ Ans $\Rightarrow x_1 = 0.71828$

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$$2x + y + z = 10$$
$$y + 3z = 6$$
$$-2z = -10$$

∴ By Using Back Substitution,
we get $z = 5,\ y = -9,\ x = 7$

Case II :-

If $\rho(A) = \rho(AB) = r$
but $r < n$

* * (i) No. of linearly independent solutions $= n - r$.

(ii) No. of linearly dependent solutions $= r$

In this case system has infinitely many solution

Ex:- Same example only 3rd row elements are made Zeros.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here,

$$\rho(A) = 2 = \rho(AB)$$

(i) No. of linearly independent soln $= n - r$
$= 3 - 2$
$= 1$.

(ii) No. of linearly dependent $= r$
$= 2$.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix}$$

$2x + y + z = 10$ ——— (i)

$y + 3z = 6$ ——— (ii)

$\therefore y = 6 - 3z$ —— (i) ——— y is dependent

z is independent

& from (i)

$x = 2 + z$ —— (ii) x is dependent

z is independent

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+z \\ 6-3z \\ z \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

for different values of z we get different sol^n. so system has infinite sol^n.

Bisection $f(x)$ $[a,b]$ $\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-1/2}$

N·R $f(x)$ x_0.

Q $f(x) = x - e^{-x}$, then $x_{n+1} =$?

2008 ⟹ Ans. ⟹ $x_{n+1} = \dfrac{e^{-x_n}(1+x_n)}{1+e^{-x_n}}$

2001 Q $f(x) = x + \sqrt{x-3}$ and $x_0 = 2$, then $x_1 =$ ____ ?

⟹ Ans ⟹ $x_1 = 1.8124$

Q The newton raphson method used to find the root of

1999 the equation and $f'(x)$ is derivative of f then the method is converges

(a) Always (b) Only if 'f' is polynomial

(c) Only if $f(x_0) < 0$

(d) None of the above ⟹ Ans. = (d).

⟶

- Newton Raphson method is useful for finding roots of eqn whether curve is less to x axis. i.e the curves which are generating high slopes we can get better results using N-R method

- If slope is less then N-R method is not providing accurate results

- The N-R method converging to the root if it satisfy the following eqn

$$|f(x) \cdot f''(x)| \leq |f'(x)|^2$$

* Regula false Method :-

Step 1 :- Let, $f(x)$ is Continuous function in $[a, b]$

Step 2 :- Let us Assume that x_0 and x_1 are initial approximation values for the required root such that $f(x_0)$ and $f(x_1)$ having opposite signs
Say $f(x_0) < 0$, $f(x_1) > 0$

Step 3 :- Regula false Iteration formula for finding root of $f(x)$ in $[x_0, x_1]$ is

if $$\boxed{x_n = \frac{f_n \cdot x_{n-1} - f_{n-1} \cdot x_n}{f_n - f_{n-1}}}$$

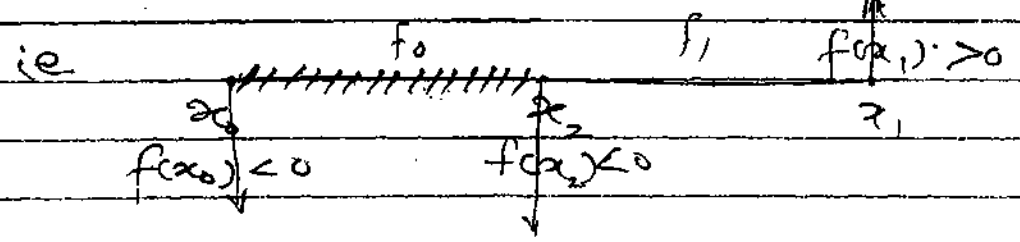
In particular $$\boxed{x_2 = \frac{f_1 \cdot x_0 - f_0 x_1}{f_1 - f_0}} \leftarrow n=1$$

—— (i)

Case I :-
If $f(x_2) = 0 \implies x_2$ is root, then Stop process

Case II :-
If $f(x_2) < 0$ and $f(x_1) > 0$

ie



To ~~find~~ Compute 'x_3', replace 'x_0' by x_2

$$\therefore x_3 = \frac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$$

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Case III :-

If $f(x_2) > 0$ and $f(x_0) < 0$

So to Compute x_3, we have to replace x_1 by x_2, f_1 by f_2 in Eq^n (i)

$$\boxed{x_3 = \frac{f_2 \cdot x_0 - f_0 x_2}{f_2 - f_0}}$$

Continue the process until desired accuracy is found

Q $f(x) = x^3 + x - 1$ and $[0.5 \quad 1]$ then find x_2, x_3 using Regula false method

$[0.5, 1] = [x_0, x_1]$

$f(x) = x^3 + x - 1$

$f(0.5) = (0.5)^3 + 0.5 - 1 = -0.375$ i.e < 0

$f(1) = 1^3 + 1 - 1 = 1$ i.e. > 0

let, $x_2 = \dfrac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$

~~$-0.375 \times$~~

$$= \frac{1 \times 0.5 - (-0.375) \times 1}{1 - (-0.375)}$$

$x_2 = 0.64$

Now, $f_2 = f(x_2) = f(0.64) = (0.64)^3 + 0.64 - 1$

$= -0.0979$

अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

Regula falsi $f(x)$ Assume $[x_0, x_1]$ $\to f(x_0) < 0$, $\to f(x_1) > 0$

Secant $f(x)$ $[x_0, x_1]$, $f(x_0) \cdot f(x_1) < 0$

$$\therefore x_3 = \frac{f_1 \cdot x_2 - f_2 \cdot x_1}{f_1 - f_2}$$

formula $x_3 = \dfrac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$

$$= \frac{-0.0979 \times 0.5 - (-0.375) \times (0.64)}{-0.0979 - (-0.375)}$$

wrong.

$$= \frac{-0.0979 \times 0.5 + 0.375 \times 0.64}{-0.979 + 0.375}$$

$$\boxed{Ans. = 0.672}$$

* Secant Method :-

The difference between Regula false & Secant method is, in Secant method the initial Guess values x_0, x_1 need not satisfy the condition.

Let, $[f(x_0) \times f(x_1)] < 0$

i.e Secant method does not provide 100% Gaurette that the root is existing in the initial guess interval (x_0, x_1)

Iteration formula for finding root of Given Eqn Using Secant method is

$$\boxed{x_{n+1} = \frac{f_n \cdot x_{n-1} - f_{n-1} \cdot x_n}{f_n - f_{n-1}}}$$

$x_0 \curvearrowright x_1 \curvearrowright x_2 \curvearrowright x_3 \curvearrowright x_4$

$$\begin{array}{ccc} f_0 & x_2 & f_1 \\ x_0 & & x_1 \end{array}$$

In particular,

$$\boxed{x_2 = \frac{f_1 \cdot x_0 - f_0 \cdot x_1}{f_1 - f_0}} \quad \text{------ (i)}$$

To compute x_3, x_0 is replaced by x_1
x_1 replaced by x_2 in eq^n (i)

Continue process until desired accuracy of root is found.

Q Using Secant Method, find 1st & 2nd approximation of the real root for the equation $x^3 - 2x - 5 = 0$, with [2, 3]

→

$f(x) = x^3 - 2x - 5$, ~~f(2) =~~

$f(2) = 2^3 - 2(2) - 5 = -1 < 0$ ------ f_0

$f(3) = 3^3 - 2(3) - 5 = 16 > 0$ ------ f_1

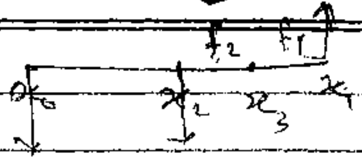
$$x_2 = \frac{f_1 \cdot x_0 - f_0 x_1}{f_1 - f_0} = \frac{16 \times 2 - (-1) \times 3}{16 (-) (-1)}$$

$$= \frac{32 + 3}{17}$$

$$= \frac{35}{17} = 2.058$$

$\therefore f(x_2) = (2.058)^3 - 2(2.058) - 5$

$= -0.3996 \approx -0.3997$



$$x_3 = \frac{f_2 \cdot x_1 - f_1 \cdot x_2}{f_2 - f_1}$$

$$= \frac{-0.3907 \times 3 - 16 \times 2.058}{-0.3907 - 16}$$

$$x_3 = 2.0812$$

*	Method	Order of Convergence
①	Bisection	linear Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^1$ means of order 1
②	Regula false	linear Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^1$ -//- -//- 1
③	Secant Method	Quadratic Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^{1.62}$ -//- -//- 1.62
④	Newton Raphson	Quadratic Convergen. $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^2$ -//- -//- 2

$\alpha = 2$

if $x_n = 2.02$,

$x_{n+1} = 2.004$

Error = Exact − Approxi

$\varepsilon_n = 2 - 2.02$

$= -0.02$

$\varepsilon_{n+1} = -0.004$

$$\boxed{\varepsilon_{n+1} = \varepsilon_n^2}$$

Bisec 1
Reg 1.62
Sec
Newto 2

magnitude

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and

$$x_n = 2.03$$
$$x_{n+1} = 2.06$$

$$\varepsilon_{n+1} = 0.06$$
$$\varepsilon_n = 0.03$$

$$\boxed{\varepsilon_{n+1} = 2\varepsilon_n^1}$$

Note:- Let, us consider n^{th} degree polynomial $f(x) = 0$

(a) The number of +ve real roots for $f(x) \leq$ The number of Sign changes in $f(x) = 0$

(b) The number of -ve real roots for $f(x) \leq$ The number of Sign changes in $f(-x) = 0$

(c) The number of imaginary roots =
$n -$ (No. of +ve roots + No. of -ve roots)

Q Polynomial $P(x) = x^5 + x + 2$ has

GATE 2005

(a) All real roots (c) Three real roots & 2 complex
✓(b) 1 real & 4 complex roots (d) All complex roots

→ from Note (a)

no. of +ve real roots $\leq$ '0' No. of Sign changes in $P(x)$

$\therefore$ No. of +ve real roots $= 0$

from Note (b) No. of -ve real roots $\leq$ No. of Sign changes in $P(-x)$

$$P(-x) = -x^5 - x + 2$$

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$P(x) = 1$ Change

$\therefore$ '1' (-ve) real roots.

Imaginary roots $= n - [(+ve) + (-ve)$ roots$]$

$= 5 - [0 + 1]$

$= 4$

$\therefore$ Ans. 1 Real root & 4 complex root

Q. If $\alpha_1, \alpha_2, \alpha_3 \cdots\cdots \alpha_n$ are n roots of Eq^n.
$a_0 x^n + a_1 x^{n-1} + \cdots\cdots + a_{n-1} x + a_n = 0$
ⓐ $\sum_{i=1}^{n} \alpha_i =$ ____ ⓑ $\sum \alpha_1 \alpha_2 =$ ____

ⓒ $\sum \alpha_1 \alpha_2 \alpha_3 =$ ____ ⓓ $\alpha_1 \cdot \alpha_2 \cdots\cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$

$ax^2 + bx + c = 0$

$\therefore \alpha_1 + \alpha_2 = -\frac{b}{a}$ $\qquad \alpha_1 \cdot \alpha_2 = \frac{c}{a} = \frac{\text{Const.}}{\text{Coeff. of } x^2}$

$\therefore \alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{\text{Const}}{\text{Coeff. of } x^3}$

$\vdots$

$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdots \alpha_n = \frac{\text{Const}}{\text{Coeff. } x^n}$

$\therefore \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdots \alpha_n = \frac{a_n}{a_0}$

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$\therefore ax^3+bx^2+cx+d=0.$

$\therefore \alpha_1+\alpha_2+\alpha_3 = \frac{\cancel{\quad}}{\cancel{\quad}}$

$= \frac{-b}{a} = \frac{-a_1}{a_0}$

$\sum \alpha_1\cdot\alpha_2 = \frac{a_2}{a_0}$

$\therefore \sum \alpha_1\cdot\alpha_2\cdot\alpha_3 = \frac{a_3}{a_0}$

$$\left.\begin{aligned} \therefore \sum_{i=1}^{n}\alpha_i &= \frac{-a_1}{a_0} \\ \sum \alpha_1\cdot\alpha_2 &= \frac{a_2}{a_0} \\ \sum \alpha_1\cdot\alpha_2\cdot\alpha_3 &= -\frac{a_3}{a_0} \\ \alpha_1\cdot\alpha_2\cdots\alpha_n &= (-1)^n\cdot\frac{a_n}{a_0} \end{aligned}\right\} \text{alternate +ve, -ve signs.}$$

Q. It is known that the roots of the non-linear eqn $x^3-6x^2+11x-6=0$ are 1 & 3 then 3rd root will be

GATE 2008

$\longrightarrow$ $\alpha_1+\alpha_2+\alpha_3 = \frac{-b}{a} = \frac{-(-6)}{1} = 6$

$\alpha_1\cdot\alpha_2\cdot\alpha_3 = \frac{c}{a} = \frac{-6}{1} = -6$

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Here, $x^3 - 6x^2 + 11x - 6 = 0$

$\therefore a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$

$$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{c}{a} = \frac{(-1)^3(-6)}{1}$$

$$1 \times 3 \times \alpha_3 = (-1)(-6)$$

$$\alpha_3 = \frac{6}{3}$$

$$\boxed{\alpha_3 = 2}$$

24/08/12

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II Solutions to System of linear Equation

(A) Gauss Elimination :-

⟹ "MATRIX METHOD"

Q. Solve
$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

⟶

Step I :- Construct Augumented matrix

i.e $$[A : B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

A = Coefficient matrix
B = Constant matrix.

Step II :- Convert augumented matrix into an upper triangular matrix using elementary row operations

Here, in above prob we have to do Row operation as, $R_2 - \frac{3R_1}{2}$ and $R_3 - \frac{R_1}{2}$

$$[A : B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{array}\right]$$

$$R_3 - \frac{(7/2)R_2}{1/2} = R_3 - 7R_2$$

$$\therefore \approx \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

Imp
****
pected
Ques.

$\rho(A)$ = No. of Non Zero rows in an uppertriangular matrix of A.

Case I :-

If $\rho(A) = \rho(AB) = r = n$

where, n is no. of unknowns i.e. (x, y, z .... etc)

Then the System is said to CONSISTENT and it has a UNIQUE SOLUTION.

⇒ Continue to prob.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

Here $\rho(A) = \rho(AB) = 3 = n$

So, it has a Unique solⁿ.

$$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ -10 \end{bmatrix}$$

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Case (iii) :-

If $\rho(A) \neq \rho(AB)$ then

System is said to be "INCONSISTENT", then it has "NO SOLUTION"

Ex:- Same example, but in constant matrix in third row a const is present i.e. -5.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$\therefore \rho(A) = 2$

$\rho(AB) = 3$

$\therefore \rho(A) \neq \rho(AB)$ —— "NO SOLUTION"

Gauss Elimination :-

$\Longrightarrow$ "PIVOTAL SOLUTION"

Q. Solve :-

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

Step I :- Construct augmented matrix

i.e. $[A : B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$

from (2, 3, −10, 5, 1, −7) the largest absolute value is −10

absolute = |modulus|

Since, $a_{11} = 2$

Now, scan entire 1st column and select largest absolute value and make it as "Pivot". Exchange pivot element row with 1st row and then eliminate x from row 2 & row 3.

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_2 - \frac{2R_1}{3}$ & $R_3 - \frac{R_1}{3}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 0 & -1/3 & -1 & -2 \\ 0 & 10/3 & 8 & 10 \end{array}\right]$$

Since, $a_{22} = -1/3$

Now, scan entire 2nd column from a_{22} and select largest absolute value

i.e. $\left|\frac{-1}{3}\right| < \left|\frac{10}{3}\right|$ and make it as pivot

Exchange pivot element row with 2nd row and then eliminate y from row 3

a_{11}

a_{22}

a_{33}

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$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 0 & 10/3 & 8 & 10 \\ 0 & -1/3 & -1 & -2 \end{array}\right]$

$$R_3 + \frac{\left(\frac{1}{3}\right)R_2}{\left(\frac{10}{3}\right)} = R_3 + \frac{R_2}{10}$$

$$\approx \left[\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 0 & 10/3 & 8 & 10 \\ 0 & 0 & -1/5 & -1 \end{array}\right]$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 0 & 10/3 & 8 \\ 0 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 10 \\ -1 \end{bmatrix}.$$

By solving, $z = 5$
$y = -9$
$x = 7$

Q. In the solutions of the following set of linear equations by Gauss Elimination using Pivotal soln the pivots for eliminating "x" & "y" respt.

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$5x + y + 2z = 34$
$4y - 3z = 12$
$10x - 2y + z = -4$

(a) 10 & 4
(b) 10 & 2
(c) 5 & 4
(d) 5 & -4

$$\left[\begin{array}{ccc|c} 5 & 1 & 2 & 34 \\ 0 & 4 & -3 & 12 \\ 10 & -2 & 1 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 5 & 1 & 2 & 34 \end{array}\right]$$

$\cancel{34}$
68
+4

$R_3 - R_1/2$

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 0 & \cancel{2} & 3/2 & 36 \end{array}\right]$$

~~Ans.~~ $a_{11} = 10$
$a_{22} = 4$

$\therefore$ Ans. 10 & 4.

(B) LU Decomposition (Method of factorisation) or Do-little method.

Step I :- Let us consider
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Step II :- Matrix representation of given system of equation is

$$AX = B \quad \text{——— (i)}$$

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Let, $A = LU$ ——— (ii)

——— where L = lower Unit Δ'ular matrix

i.e. $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

U = Upper Δular matrix

$= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$\therefore$ (i) $\Rightarrow$ $LUX = B$ ——— (iii)

Let, $UX = Y$ ——— (iv)

where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore$ $LY = B$.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

By solving, we get, y_1, y_2, y_3 in terms of elements of lower Unit Δ'ular matrix.

$$\therefore \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

By Solving, we get x_1, x_2, x_3 in terms of the elements of Upper Δ'ular and lower Unit Δular matrix.

The order computing elements of L & U is
$U_{11}, U_{12}, U_{13}, l_{21}, U_{22}, U_{23}, l_{31}, l_{32}, U_{33}$.

Note:- CROUT'S method is similar to Do-little method except that in Crout's method 'A' is decomposed with lower triangular matrix & Unit upper triangular matrix.

i.e. $$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

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GATE 2011 Q In matrix A is $\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into product of Lower and upper triangular matrices Using Crout's method The properly decomposed L & U matrices respectively

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11} U_{12} \\ l_{21} & l_{21} \cdot U_{12} + l_{22} \end{bmatrix}$$

$\therefore \underline{l_{11} = 2}$

$l_{11} U_{12} = 1$ $\therefore U_{12} = 1/2$

$\underline{l_{21} = 4}$

$l_{21} \cdot U_{12} + l_{22} = -1$ $4 \times \frac{1}{2} + l_{22} = -1$

$\underline{l_{22} = -3}$

$$\therefore \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

III Solution to Integration of function :-

Let us consider the given curve is f(x) and ordinates on x axis is $x = a$ & $x = b$

The area bounded by the given curve and the ordinates is denoted by

$$\int_a^b f(x)\,dx \quad \text{———} \quad (*)$$

Divide $[a, b]$ into "n" equal subintervals where, length of each interval is "h" (Step Size)

$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + h + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 2h + h = x_0 + 3h$

$\vdots$

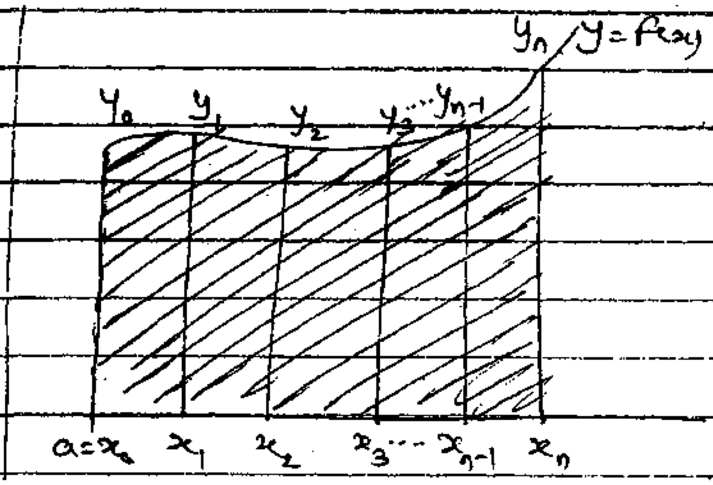
$x_n = x_0 + nh$

i.e. $b = x_0 + nh$

$\therefore x_n = x_0 + nh$

$b = a + nh$

$$\therefore \boxed{n = \frac{b-a}{h}}$$



In Q., Only Simpson rule is mentioned then take Simpson $\frac{1}{3}^{rd}$ rule

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Equation (*) Can be evaluated by using

1) Trapezoidal Rule

$$= \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots\cdots y_{n-1})\right]$$

2) Simpson $\frac{1}{3}$ Rule

$$= \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\cdots\cdots y_{n-2}) + 4(y_1+y_3+y_5+\cdots\cdots y_{n-1})\right]$$

3) Simpson $\frac{3}{8}$th Rule

$$= \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+\cdots\cdots y_{n-3}) + 3(y_1+y_2+y_4+y_5+y_7+y_8+y_{10}+y_{11}\cdots y_{n-1})\right]$$

Recess. 10:40 am

Continue 11:15 am.

Gate Q 2010

x	0^{x_0}	0.25^{x_1}	0.5	0.75	1.0
$f(x)$	1	0.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

The value of the integrated betn the limits 0 & 1 using Simpson Rule (if not mentioned then take $\frac{1}{3}^{rd}$ rule)

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Simpson $\frac{1}{3}$ rule $= \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4\cdots)+4(y_1+y_3\cdots)\right]$

$$= \frac{0.25}{3}\left[(y_0+y_4)+2(y_2)+4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[(1+0.5)+2(0.8)+4(0.9412+0.64)\right]$$

$$= 0.7854$$

Note: ① Simpson's Rule is applicable if the number of intervals are "EVEN"

② Simpson's $\frac{3}{8}$th Rule is applicable if the number of intervals are multiples of 3 i.e. ($n = 3, 6, 9, 12,$ etc)

③ Trapezoidal Rule is applicable for any number of intervals

Q. A 2nd degree polynomial f(x) takes a following values

GATE 2006

x	0	1	2
$f(x)$	1	4	15

~~The integration 0 to 2~~ $\int_0^2 f(x)\,dx$ is Evaluated using trapezoidal Rule then the error estimation is

ⓐ $4/3$ ⓑ $-4/3$ ⓒ $2/3$ ⓓ $-2/3$

Error = Exact value − Approximate value

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→ Here,

It is mentioned that 2^{nd} degree polynomial.

$\therefore f(x) = a_0 + a_1 x + a_2 x^2$.

and it takes a followings values — given.

$\therefore f(0) = 1$ — i.e. $x = 0,\ f(x) = 1$.

$\therefore a_0 = 1$

$\therefore f(1) = 4$

$\therefore a_0 + a_1 + a_2 \cancel{x}^2 = 4$

$1 + a_1 + a_2 = 4$

$a_1 + a_2 = 3$. ——— (i)

$f(2) = 15$

$a_0 + a_1 \times 2 + a_2 \times 2^2 = 15$

$2a_1 + 2a_2 = 14$

$a_1 + 2a_2 = 7$ ——— (ii)

$\therefore$ Solve (i) & (ii)

$a_2 = 4$

$a_1 = -1$

$\therefore a_0 = 1,\ a_1 = -1,\ a_2 = 4$

$\therefore f(x) = 1 - x + 4x^2$.

$\therefore$ Exact value $= \int_0^2 f(x)\,dx = \int_0^2 (1 - x + 4x^2)\,dx$

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$$= \left[x - \frac{x^2}{2} + \frac{4x^3}{3} \right]_0^2$$

$$\text{Exact} = \frac{32}{3}$$

Approximate value = Trapezoidal Rule value

$$\therefore \text{T. R. value} = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \cdots) \right]$$

$$= \frac{1}{2}\left[(1+15) + 2(4) \right]$$

$$= 12$$

$\therefore$ Error = Exact − Approximate

$$= \frac{32}{3} - 12$$

$$= \frac{-4}{3}.$$

$\frac{2\pi}{8} \ldots \frac{\pi}{4}$ $\quad 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, 2\pi$

5 significant digit = upto 5th decimal pt.

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ME Q. GATE 2007

$\int_0^{2\pi} \sin x \, dx$ is Evaluated by T.R. Rule with Eight Equal intervals with 5 Significant digits

(a) 0.00000 (b) 1.00000 (c) 0.00500 (d) 0.00025

→ Here, $n=8$, $h = \frac{b-a}{n} = \frac{2\pi - 0}{8} = \frac{\pi}{4}$.

$h = \frac{\pi}{4}$

x	$\sin x$	
$x=0$	$\sin 0 = 0$	y_0
$x+h = 0+\pi/4$	0.70710	y_1
$\pi/2$	1	y_2
$3\pi/4$	0.70710	y_3
π	0	y_4
$5\pi/4$	-0.70710	y_5
$6\pi/4$	-1	y_6
$7\pi/4$	-0.70710	y_7
2π	0	y_8

$$\text{T.R. Rule} = \frac{h}{2}\left[(y_8 + y_0) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{\pi}{4} \cdot \frac{1}{2}\left[(0+0) + 2(0.70710 + 1 + 0.70710 + 0 - 0.70710 - 1 - 0.70710)\right]$$

$= 0.00000$

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Q

ME
GATE
2010

x	0	60	120	180	240	300	360
y	0	1068	-323	0	323	-355	0

Evaluate $\int_0^{2\pi} y\,dx$ Using Simpson's Rule

(a) 542 (b) 995 (c) 1444 (d) 1986

$\rightarrow = \frac{h}{3}\left[(y_0+y_6)+2(y_3+\cancel{y})+4(y_1+y_2+y_4+y_5+\cancel{y})\right]$

$\cancel{h = \frac{b-a}{\pi} = 2\pi - 0}$

$= \cancel{\frac{60}{3}}\left[(0+\right.$

Here $h = 60$.

But the Integration limit is in π term

$\therefore$ ~~take~~ $h = \frac{\pi}{3}$

$\frac{1}{3}$ rd $= \frac{\pi}{3}\left[(y_0+y_6)+2(y_3)+4(y_1+y_2+y_4+y_5)\right]$

$= 995$

x	1	2	3
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$

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MEQ 2011 GATE

The integral $\int_1^3 \frac{1}{x}\,dx$ when Evaluated using $\frac{1}{3}$rd rule on two equal Intervals Each of length 1.

(a) 1.000 (b) 1.111

$\longrightarrow$ $\int_1^3 \frac{1}{x}\,dx$. $h = \frac{b-a}{n}$ —— $n = 2$

$= \frac{3-1}{2}$

$h = 1$

	x	$1/x$
y_0	1	$1/1 = 1$
y_1	2	$1/2$
y_2	3	$1/3$

$\frac{1}{3}$rd $= \frac{h}{3}\left[(y_0 + y_2) + \cancel{\ }\right]$

$\cancel{= \frac{1}{3}\left[0 + 0.5\right]}$

$\cancel{= \frac{0.5}{3}}$

$\cancel{= 0.}$

$\frac{1}{3}$rd $= \frac{1}{3}\left[\left(1 + \frac{1}{3}\right) + 2(0) + 4(y_1)\right]$

$= \frac{1}{3}\left[\frac{4}{3} + 4 \times \frac{1}{2}\right]$

$= \frac{1}{3}\left[\frac{4}{3} + 2\right]$

$= \frac{1}{3}\left[\frac{10}{3}\right]$

$= \frac{10}{9}$

$= 1.111$

Error order T.R rule = h^2
Error order Simpson $\frac{1}{3}$ = h^4
Error order Simpson $\frac{3}{8}$ = h^5.

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Imp Q

Q GATE 2008 The minimum number of Equal lengths of subinterval needed to approximate

$$\int_1^2 x \cdot e^x \, dx$$ to an accuracy of at least $\frac{1}{3} \times 10^{-6}$ Using T.R Rule

(a) $1000 \cdot e$ (b) 1000 (c) $100 \cdot e$ (d) 100

→ $n = ?$
$f(x) = x \cdot e^x$

Note - (a) $\boxed{\text{Error in T.R. Rule} = -\left(\frac{b-a}{12}\right) \cdot h^2 \cdot \max\left[f''(x)\right]}$

(b) $\boxed{\text{Truncation Error in T.R. Rule} = \frac{(b-a)}{12} \cdot h^2 \cdot \max\left[f''(x)\right]}$

— Error order h^2

At least means $\geq$

Here, Error ↑, Accuracy ↓

Given, Accuracy $\geq \frac{1}{3} \times 10^{-6}$

Truncation Error $\leq \frac{1}{3} \times 10^{-6}$

$$\frac{(b-a)}{12} \cdot h^2 \max\left[f''(x)\right] \leq \frac{1}{3} \times 10^{-6}$$

Here,

$$f(x) = xe^x$$
$$f'(x) = xe^x + e^x$$
$$f''(x) = xe^x + e^x + e^x$$
$$\therefore f''(x) = 2e^x + xe^x$$

— Here, in $f''(x)$, e^x is Increasing function as x increases.
& xe^x is also.

$\therefore f''(x)$ is Increasing function in $(1,2)$

$$\therefore f''(2) = 2\times e^2 + 2\times e^2 = 4e^2.$$

Now, $h = \frac{b-a}{n}$.

$$\therefore \frac{(2-1)}{12} \cdot \left(\frac{2-1}{n}\right)^2 \max f''(x) \le \frac{1}{3}\times 10^{-6}.$$

$$\frac{1}{\cancel{12}_{3}} \times \frac{1}{n^2} \times \cancel{4}e^2 \le \frac{1}{\cancel{3}} \times 10^{-6}$$

$$\frac{e^2}{n^2} \le 10^{-6}$$

$$\therefore e^2 \times 10^{6} \le n^2$$

$$n^2 \ge e^2 (10^3)^2$$

$$n \ge e \cdot 10^3$$

$$n \ge 1000 \cdot e.$$

$\therefore$ Ans. $1000 \cdot e$.

Note :-

① Error in $S\text{-}\frac{1}{3}$ rule $= \frac{-(b-a)}{180} \cdot h^4 \cdot \max \left[f^{IV}(x) \right]$

— Error order h^4

② Error in $S\text{-}\frac{3}{8}$ Rule $= \frac{-3}{80} \cdot h^5 \cdot \max \left[f^{IV}(x) \right]$

— Error order h^5.

IV Solutions to differential Equation

Let us consider differential Eq^n.

$$\frac{dy}{dx} = f(x, y) \quad \text{where} \quad y(x_0) = y_0 \quad \text{———} (*)$$

Eq^n (*) can be solved by using :-

1) Eulers method
 - forward Eulers method
 - Backward Eulers method.

2) Runge - kutta method
 - Runge Kutta of 1st order - [Eulers method]
 - Runge Kutta of 2nd order - [modified Eulers method]

Not asked in GATE yet. { - Runge Kutta of 3rd order
 - Runge Kutta of 4th order

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* Eulers Method :- (forward)

Eulers Iterative formula for finding solution curve to the eq^n (*) is

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

In particular for $n=0$

$$\boxed{y_1 = y_0 + h \cdot f(x_n, y_n)}$$

Q. find an approximate value of 'y' corresponding to $x = 0.2$ & $\frac{dy}{dx} = x + y$, $y=1$, when $x=0$. Using Eulers method where $h = 0.1$

make it $f(x,y)$

	x	y	comment
①	$x_0 = 0$	$y_0 = 1$	Initial value.
②	$x_1 = x_0 + h$ $= 0 + 0.1$ $= 0.1$	$y_1 = ?$ $\therefore y_1 = 0.1$	$y_1 = y_0 + h \cdot f(x_0, y_0)$ $= 1 + 0.1\,(x_0 + y_0)$ $= 1 + 0.1\,(0+1)$ $y_1 = 1.1$
③	$x_2 = x_1 + h$ $= 0.1 + 0.1$ $= 0.2$	$y_2 = ?$ $\therefore y_2 = 1.22$	$y_2 = y_1 + h \cdot f(x_1, y_1)$ $= 1.1 + 0.1\,(0.1 + 1.1)$ $= 1.1 + 0.1\,(0.2)$ $= 1.22$

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Q $\frac{dy}{dx} - y = x$. when $y(0) = 0$.
GATE 2010 where $h = 0.1$

Compute $y(0.3)$ using Euler's 1st order method.

⟹ i.e. Euler's forward method.

→ Ans. $\underline{\underline{0.031}}$.

x	y	Comment
$x_0 = 0$	0	$y_1 = y_0 + h\, f(x_0, y_0)$
$x_1 = 0.1$	$y_1 = 0$	$y_1 = 0 + 0.1\,(0+0)$ $y_1 = 0$ $y_2 = y_1 + h\, f(x_1, y_1)$
$x_2 = 0.2$	$y_2 = 0.01$	$y_2 = 0 + 0.1\,(0.1+0)$ $= 0.01$ $y_3 = y_2 + h\, f(x_2, y_2)$
$x_3 = 0.3$	$\boxed{y_3 = 0.031}$	$y_3 = 0.01 + 0.1\,(0.2 + 0.01)$ $= 0.01 + 0.1 \times 0.21$ $= 0.01 + 0.021$ $= 0.031$

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* Eulers Backward Method :-

let, $\frac{dy}{dx} = f(x,y)$ ——— (*)

where, $y(x_0) = y_0$

∴ Eulers Backward iterative formula for solving Eqⁿ (*) i.e $\frac{dy}{dx} = f(x,y)$

$$\boxed{y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})}$$

Here, y_{i+1} is in both L.H.S & R.H.S.
Since, y_{i+1} is defined in function
Therefore, this method is called as Implicit fⁿ.
Euler's method.

Q.1 find an appropriate value for $x = 0.2$ using Implicit Eulers method where, $\frac{dy}{dx} = x+y$, $y(0) = 1$ where step size $h = 0.1$

→ Given, $\frac{dy}{dx} = x+y$

$$\therefore y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$$

$$y_{i+1} = y_i + h \cdot (x_{i+1} + y_{i+1})$$

$$(1-h) \cdot y_{i+1} = y_i + h \cdot x_{i+1}$$

$$y_{i+1} = \frac{y_i + h \cdot x_{i+1}}{(1-h)}$$

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In perticular $x=0$ then

$$\text{i.e.}\ \boxed{y_1 = \frac{y_0 + h\cdot x_1}{(1-h)}}$$

∴ Construct the table for y with respective to values of x

x	y	Comment.
$x_0 = 0$	$y_0 = 1$	Initial Condition
$y_1 = 0.1$	$\boxed{y_1 = 1.222}$	$y_1 = \frac{y_0 + h\cdot x_1}{(1-h)}$ $= \frac{1 + 0.1\times 0.1}{(1-0.1)}$ $y_1 = 1.222$
$x_2 = 0.2$	$\boxed{y_2 = 1.38}$	$y_2 = \frac{y_1 + h\cdot x_2}{(1-h)}$ $= \frac{1.222 + 0.1\times 0.2}{(1-0.1)}$ $= 1.38$

Note:- Eulers Backward method is more stable than forward method.

The exact solution for differential eq^n $\frac{dy}{dx} = x+y$ with $y(0)=1$ is $y = 2e^x - x - 1$

at $x=1$, $y = 3.44$.

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By observing forward & backward Eulers method you can say that Backward method is converging to required value very quickly

GATE Q 2006 The diff. eq^n $\frac{dy}{dx} = 0.25y^2$ is to be solved using backward Euler's method with boundary conditions $y=1$ at $x=0$ and $h=1$ what would be the value of y at $x=1$.

(A) 1.33 (B) 1.67 (C) 2.0 (D) 2.33

→ Here, $\frac{dy}{dx} = 0.25y^2$

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$

$$y_{i+1} - 1\left[0.25\, y_{i+1}^2\right] = y_i$$

$$1 \times 0.25\, y_{i+1}^2 - y_{i+1} + y_i = 0.$$

— Here, value of $h=1$, ∴ No need to construct table

$$0.25\, y_{i+1}^2 - y_{i+1} + y_i = 0.$$

Comparing with $ax^2+bx+c=0$

∴ roots of Eq^n are

$$\cancel{x = -b \pm \sqrt{4}}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2 \cdot a}$$

$$\therefore y_{i+1} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.25)(y_i)}}{2 \times 0.25}$$

$$y_{i+1} = \frac{1 \pm \sqrt{1-y_i}}{0.5}$$

put $i=0$,

$$y_1 = \frac{1 \pm \sqrt{1-y_0}}{0.5} \quad \text{—— } y_0 = 1.$$

i.e. $y(0)=1$ — given

$$\therefore\ y_1 = \frac{1 \pm \sqrt{1-1}}{0.5}$$

$$y_1 = \frac{1}{0.5}$$

$$\boxed{y_1 = 2}$$

Runge Kutta Method :-

GATE 1996 2nd order R-K →

Given diff. eq^n $\frac{dy}{dx} = x - y$ with $y(0)=0$ then value of $y(0.1)$ using 2nd order Runge Kutta method, with step size $h=0.1$

Runge kutta method of 2nd order Iterative formula for finding sol^n curve to eq^n is

$$\boxed{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)}$$

where $\boxed{k_1 = h f(x_0, y_0)}$

$$k_1 = 0.1 \times (x_0 - y_0)$$
$$= 0.1 \times (0-0)$$
$$\boxed{k_1 = 0}$$

$$\boxed{k = h f(x_0+h, y_0+k_1)}$$

$$k_2 = 0.1[(x_0+h) - (y_0+k_1)]$$
$$= 0.1[(0+0.1) - (0+0)]$$
$$\boxed{k_2 = 0.01}$$

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$\therefore y_1 = y(x_0+h) = y(0+0.1) = y(0.1)$

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$$

$$= 0 + \frac{1}{2}(0+0.01)$$

$$= \frac{0.01}{2}$$

$$\boxed{y_1 = 0.005}$$

Q.2. 4th order R-K → Apply Runge-kutta method of 4^{th} order where $\frac{dy}{dx} = x+y$, $y=1$ when $x=0$, $h=0.2$

Compute $y(0.2)$

Runge kutta method of 4^{th} order formula

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)}$$

where

$k_1 = h\cdot f(x_0, y_0)$	$k_2 = h\cdot f(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$	$k_3 = h\,f(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$
$= 0.2(x_0+y_0)$	$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.2}{2}\right]$	$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.24}{2}\right]$
$= 0.2(0+1)$		$= 0.2[0.1+1+0.12]$
$k_1 = 0.2$	$k_2 = 0.24$	$k_3 = 0.244$

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$$\boxed{k_4 = h \cdot f(x_0+h, y_0+k_3)}$$

$$= 0.2\left[x_0+h+y_0+k_3\right]$$

$$= 0.2\left[0+0.2+1+0.2+4\right]$$

$$k_4 = 0.2888$$

$$\therefore y_1 = y_0 + \frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

$$= 1+\frac{1}{6}\left(0.2+2\times 0.24+2\times 0.244+0.288\right)$$

$$\boxed{y_1 = 1.2428}$$

Q.3. 3rd order R-K → Apply Runge kutta method of 3^{rd} order with $\frac{dy}{dx} = x+y$, when $y=1$, $x=0$ & $h=0.2$ then compute $y(0.2)$

Runge kutta method of 3^{rd} order iterative formula for finding solution curve to the Eq^n is.

$$\boxed{y_1 = y_0 + \frac{1}{6}\left(k_1+4k_2+k_3\right)}$$

where

$$\boxed{k_1 = h \cdot f(x_0, y_0)}$$

$$= 0.2(x_0+y_0)$$

$$= 0.2(0+1)$$

$$k_1 = 0.2$$

$$\boxed{k_2 = h \cdot f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)}$$

$$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.2}{2}\right]$$

$$k_2 = 0.24$$

$$\boxed{k_3 = h \cdot f(x_0+h,\ y_0+k')}$$

where, ~~$k'=0$~~

$$\boxed{k' = h \cdot f(x_0+h,\ y_0+k_1)}$$

$$\therefore k' = 0.2\left[0+0.2+1+0.2\right]$$
$$= 0.2(1.4)$$
$$k' = 0.28$$

$$k_3 = 0.2\left[0+0.2+1+0.28\right]$$
$$k_3 = 0.296$$

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 4k_2 + k_3\right]$$

$$\therefore y_0 = 1 + \frac{1}{6}\left[0.2 + 4\times 0.24 + 0.296\right]$$

$$\boxed{y_0 = 1.2426}$$

Q.4. GATE 2007 Euler's method

The diff. eq^n $\frac{dx}{dt} = \frac{1-x}{\tau}$ is evaluated using Eulers method with step size $h = \Delta T$, where $\Delta T > 0$ what is the maximum value of ΔT, To Ensure Stability in So!^{n}

(A) 1 (B) $\frac{\tau}{2}$ (C) τ ✓(D) 2τ

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→ Stable :-

An iterative method is said to be stable if the round off error is remains bounded as $n \to \infty$, where n is no. of Iterations.

The Eulers method formula

$$\boxed{y_{i+1} = y_i + h \cdot f(x_i, y_i)}$$ can be written as

$$\boxed{y_{i+1} = E \cdot y_i + K} \quad \text{———} (*)$$

where,

K = terms which are involved in x or Consts.

Eqn (*) is said to be stable, if $|E| < 1$

i.e $-1 \leq |E| < 1$

$$\frac{dx}{dt} = \frac{1-x}{\tau}$$

Similarly $\frac{dy}{dx} = \frac{1-y}{\tau}$

$$\therefore \frac{dy}{dx} = \frac{1-y}{\tau} = f(x, y)$$

$$\therefore y_{i+1} = y_i + h \cdot f(x_i, y_i)$$

$$= y_i + h \cdot \left(\frac{1-y_i}{\tau}\right)$$

$$y_{i+1} = \underbrace{\left(1 - \frac{h}{\tau}\right)}_{E} \cdot y_i + \underbrace{\frac{h}{\tau}}_{K} \quad \text{———} (I)$$

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Eq^n (I) is stable if $\left|1-\frac{h}{\tau}\right| < 1$

$\therefore \left|1-\frac{h}{\tau}\right| < 1$

$\left|1-\frac{\Delta T}{\tau}\right| < 1$

i.e. $-1 < \left|1-\frac{\Delta T}{\tau}\right| < 1$

Substract 1 throughout to reduce 1 or Cancell 1 from middle term

$\therefore -1-1 < \left|1-1-\frac{\Delta T}{\tau}\right| < 1-1$

$\therefore -2 < \left|-\frac{\Delta T}{\tau}\right| < 0$

$\therefore -2\tau < |-\Delta T| < 0$

$\therefore 2\tau > \Delta T > 0$ —— Removing Signs and changing directions of Equality Signs.

i.e. $0 < \Delta T < 2\tau$

$\therefore \Delta \tau < 2\tau$

Ans. 2τ

Q.1. Simpson Rule — The min. no. of Equal length subintervals needed to approximate $\int_0^2 e^{2x}dx$ to an accuracy of atleast $\frac{8}{45}\times 10^{-8}$ Using Simpson's Rule

(A) 200 e (B) 200 (C) 2000 e (D) 2000

$\rightarrow$ $f(x) = e^{2x}$ ——— $[a, b] = [0, 2]$

Accuracy atleast means

accuracy $\geq \frac{8}{45} \times 10^{-8}$

Here, if Accuracy $\uparrow$ then Error $\downarrow$

$\therefore$ Error $\leq \frac{8}{45} \times 10^{-8}$

In Numerical method error is considered as ~~Simpson's~~ Truncation method

$$\left| \left[\text{Truncation Error} \right]_{\text{Simpson rule}} \right| \leq \frac{8}{45} \times 10^{-8}$$

$$\left| \frac{b-a}{180} \times h^4 \times \text{Max} \left(f^{IV}(x) \right) \right| \leq \frac{8}{45} \times 10^{-8}$$

But $h = \frac{b-a}{n}$

$$\therefore \left| \frac{2-0}{180} \times \left(\frac{2-0}{n} \right)^4 \times \text{Max}. f^{IV}(x) \right| \leq \frac{8}{45} \times 10^{-8}$$

Now, $f(x) = e^{2x}$

$\therefore f'(x) = 2 \cdot e^{2x}$

$f''(x) = 4 \cdot e^{2x}$

$f'''(x) = 8 \cdot e^{2x}$

$f^{IV}(x) = 16 \cdot e^{2x} = 16 \times e^{2 \times 2} = 16 \cdot e^4$ — last limit ~~Atx~~ $x = 2$

$$\therefore \left| \frac{\cancel{2}-0}{\cancel{180}_{4}} \times \left(\frac{2-0}{n} \right)^4 \times \cancel{16} \cdot e^4 \right| \leq \frac{\cancel{8}}{\cancel{45}} \times 10^{-8}$$

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$$\therefore \left|\left(\frac{2}{n}\right)^4 . e^4\right| \leq \frac{8}{1} \times 10^{-8}$$

$$\left|\frac{16}{n^4} . e^4\right| \leq 8 \times 10^{-8}$$

$$\left|\frac{16 . e^4}{1 \times 10^{-8}}\right| \leq n^4$$

$$\left|16 \times e^4 \times 10^{+8}\right| \leq n^4$$

$$\therefore n^4 \geq \left|16 \times e^4 \times 10^{+8}\right|$$

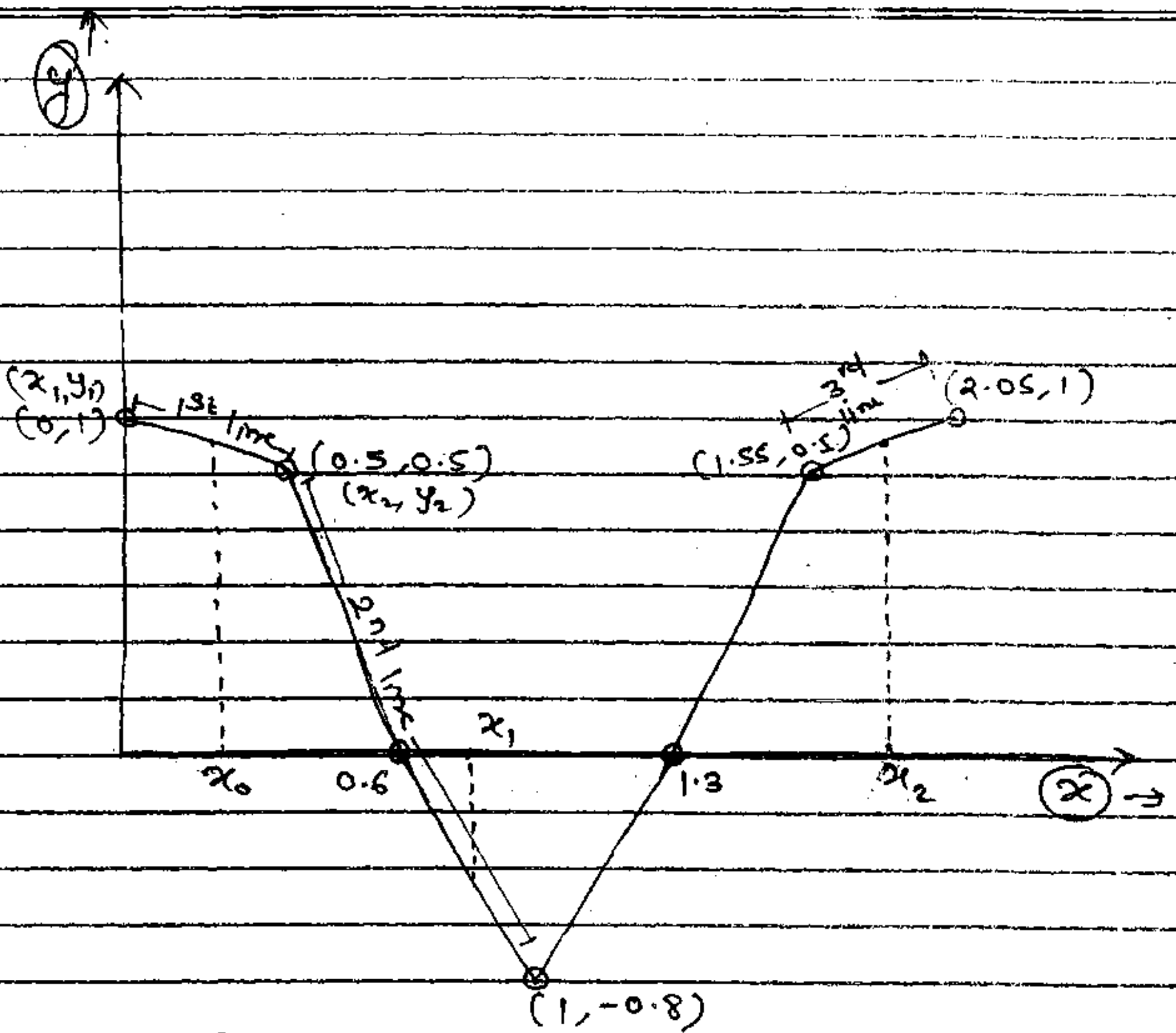
$$\therefore n^4 \geq \left|2^4 \times e^4 \times (10^2)^4\right|$$

$$\therefore n \geq \left|2 \times e^4 \times 10^2\right|$$

$$n \geq 200\, e^4 \text{ ———— Ans.}$$

Q.2. GATE If we use Newton raphson method to find roots $f(x)=0$, Using x_0, x_1 & x_2 respectively As initial guess values & then the roots Obtained would be

(A) 1.3, 0.6, 0.6
(B) 0.6, 0.6, 1.3
(C) 1.3, 1.3, 0.6
(D) 1.3, 0.6, 1.3



→ 1st line for x_0

$m = \frac{y_2 - y_1}{x_2 - x_1}$ (0,1) → (0.5, 0.5)

$m = \frac{0.5 - 1}{0.5 - 0}$ (0,1) — (0.5, 0.5) — original

$m = -1$

Eqⁿ of 1st line ⇒ $(y - y_1) = m(x - x_1)$

$(y - 1) = -1(x - 0)$

$\underline{\underline{x + y = 1}} \Rightarrow y = -x + 1$

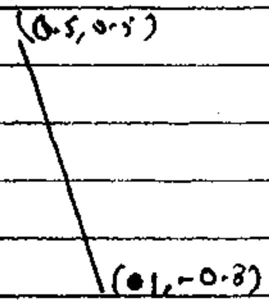
$\Rightarrow y = mx + c$

प्रेम सबसे करो, बुरा किसीका न करो। $\Rightarrow c \to 1$

∴ converges to 1.3

On x axis, $y=0$,
So, $x=1$ which is near to 1.3 than 0.6 on x axis $\therefore x_0$ is converging to 1.3.

— 2^{nd} line for x_1 (0.5, 0.5) (1, −0.8)



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-0.8 - 0.5}{1 - 0.5}$$

$$m = -1.8$$

Eqn of 2nd line $\Rightarrow$ $(y - y_1) = m(x - x_1)$

$$(y - 0.5) = -1.8(x - 0.5)$$

$$y - 0.5 = -1.8x + 0.9$$

$$y + 1.8x = 0.9 + 0.5$$

$$\underline{\underline{y + 1.8x = 1.3}}$$

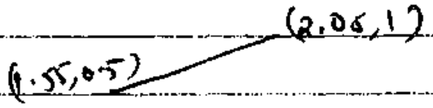
$$\Rightarrow y = -1.8x + 1.3$$

$$\Rightarrow y = mx + c$$

$$\Rightarrow c \to 1.3.$$

$\therefore$ Converges to 1.3.

— 3^{rd} line for x_2 (1.55, 0.5) (2.05, 1)



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0.5}{2.05 - 1.55}$$

$$m = 0$$

Eqn

Eq of 3rd line

$$(y - y_1) = m(x - x_1)$$

$$\cancel{y = 1.55}$$

$$(y - 0.5) = 0(x - 1.55)$$

$$\therefore \quad y - 0.5 = 0$$

$$y = 0.5$$

$$\Rightarrow \therefore y = mx + c$$

$$\therefore c \Rightarrow 0.5$$

Converges to $\underline{0.6}$

PROBABILITY AND STATISTICS

Probability, r.v's
r. processes
Veeravajan
McGraw Hills.

(i) Basics.

(ii) Probability

(iii) Random variable / Expectation.

(iv) Distribution → Discrete
→ Continues.

(V) Mathematical

STATISTICS

→ collection of data.
→ Analysis of data
→ Interpretation of data.

Definition: According to prof. R.A Fisher, Statistics is defined as a collection of data, analysis of data and interpretation of data.

TYPES OF DATA

(i) Grouped and Ungrouped.

(ii) closed and open data.

GROUPED DATA

If data is in the form of class intervals and frequency then the data is known as grouped data, or distributing the frequencies to their corresponding class intervals, then the data is known as Frequency distribution.

UNGROUPED DATA : If the data contains only observations, without any class intervals, then the data is known as ungrouped data or Raw Data.

CLOSED DATA : If the class intervals are in a continuous form without any discontinuity, then the data is known as closed data otherwise open data.

MEAN (AVERAGE)

$$\boxed{\bar{X}_{UGD} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

$$\boxed{\bar{X}_{GD} = \frac{\sum_{i=1}^{n} f_i x_i}{N}}$$

n :→ no of observations.

x : midpoint , $\frac{UL+LL}{2}$

N : sum of frequencies.

f : frequencies.

MEDIAN

- If n is odd, the middle observation itself is the median.
- If n is even, average between the middle observations provided
  - i) Data is rearranged either in increasing or decreasing order.
  - ii) No : of observations above the middle is equal to the number of observations below.

$$M_d = l + \frac{\left(\frac{N}{2} - m\right) \times c}{f}$$

l :- Lower limit for the ideal class

f :- frequency for the ideal class.

m : cumulative frequency for the ideal class.

Q) Find the median for the following frequency data.

C.I	FREQUENCY	CUMULATIVE FREQUENCY
0-5	3	3
5-10	7	10
10-15	11	21 → Idealclass.
15-20	8	29
20-25	2	31
		N = 31

$$\frac{N}{2} = \frac{31}{2} = \underline{\underline{15.5}}$$

$$M_d = 10 + \left(\frac{15.5 - 10}{11}\right) 5$$

$$= 10 + \frac{5.5 \times 5}{11} = \underline{\underline{12.5}}$$

<u>Note:</u> If the first class itself is ideal, the cumulative frequency and frequency are ideal $(m = f) \Rightarrow M_d = l$

MODE

The most frequently repeated observation is known as Mode. 1, 2, 3, 4, 5, 2, 3, 11, 14, 2, 3, 21, 2, 16, 21, 3, 19

M_o = 2,3 ⟶ Bimodal.

For grouped data

$$\boxed{M_o = 3\,Md - 2\,Mean}$$

$$M_o = l + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) c \qquad \Delta_1 = f - f_{-1} \qquad \Delta_2 = f - f_{+1}$$

Q Find the mode for the following frequency distribution.

C.I	Frequency
0-10	11
10-20	14
20-30	17
30-40	8
40-50	5
50-60	3

Ideal class.

Frequency 17 can be treated as Ideal class [Highest frequency]

$\Delta_1 = 17-14 = 3$

$\Delta_2 = 17-8 = 9$

$M_o = 20 + \left(\frac{3}{12}\right) \times 10 = \frac{45}{2} = 22.5$

* If the maximum frequencies are repeated ~~itself,~~ first, last and in between, select 'in between' as the ideal class.
* If the maximum frequencies are repeated in between, select randomly (bimodal).
* If all the frequencies are equal, mode is undefined $\left[\frac{0}{0} \text{ form}\right]$
* If the maximum frequencies are repeated first and last select randomly (bimodal).

MEASURES OF CENTRAL TENDENCIES

Among the 3 measures, mean, mode and median,

Mean is the best measure.

MEASURES OF DISPERSION

→ Range

→ Mean Deviation.

→ Quartile Deviation (Q.D)

→ Standard Deviation. (SD)

→ ~~Coefficient of Variation. (C.V)~~

Measures of dispersion helps us to identify the deviation within the data.

RANGE : [Max — Min.

Greatest value — Least value]

STANDARD DEVIATION

$$\sqrt{\text{Variance}} = (S.D)$$

$$\text{Variance} = (S.D)^2 = \sigma_x^2$$

$$\boxed{\sigma_x^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - (\bar{x})^2}$$

Variance is the Sum of the Squares of deviation from mean. The differences or deviations within the data, is known as Variance.

Note:

i Lesser Variance is more consistent or more uniform.

ii Variance will never be negative.

iii Variance of constant is 0.

iv Sum of the differences from the mean is always Zero.

$$\boxed{\left[\sum_{i=1}^{n}(x_i-\bar{x})\right] = 0}$$

* If the variances are equal for the different groups, greater mean is more consistent.
* Sum of the squares of the deviation from the mean should be minimum.

For grouped data

$$\boxed{\sigma_x^2 = \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i\right)^2}$$

$$\boxed{\sigma_x^2 = \frac{1}{N}\sum f_i (x_i - \bar{x})^2}$$

GROUPED DATA VARIANCE

RELATION B/W QD/MD/SD

$$\boxed{6\,QD = 5\,MD = 4\,SD}$$

$QD = \frac{2}{3}\sigma$, $MD = \frac{4}{5}\sigma$

OEFFICIENT OF VARIATION (C.V)

$$\boxed{C.V = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100}$$

$$\boxed{C.V = \frac{\sigma}{\bar{x}} \times 100}$$

Lesser σ implies lesser C.V $\Rightarrow$ Data is more consistent or uniform.

or identifying the consistency within the data, which can
e measurable — by standard deviation ...

Q, Find mean and variance for the first n natural numbers.

$$\bar{X} = \frac{[1+2+3+\cdots+n]}{n}$$

$$= \frac{\cancel{n}(n+1)}{2\cancel{n}}$$

$$\bar{X} = \frac{n+1}{2}$$

$$\sigma_x^2 = \frac{1}{n}\sum x_i^2 - (\bar{x})^2$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{1}{n}\left[1^2+2^2+3^2+\cdots+n^2\right]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\sigma_x^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] = \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right] = \frac{n+1}{2}\cdot\frac{n-1}{6} = \frac{n^2-1}{12}$$

$$\boxed{\bar{X} = \frac{n+1}{2}}$$

Mean of n natural numbers

$$\boxed{\sigma_x^2 = \frac{n^2-1}{12}}$$

Variance of n natural numbers.

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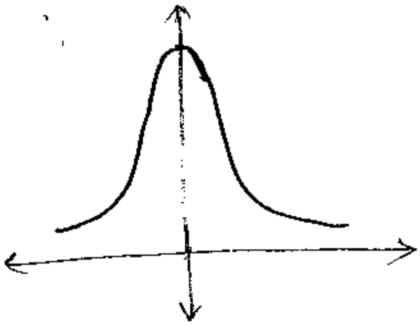
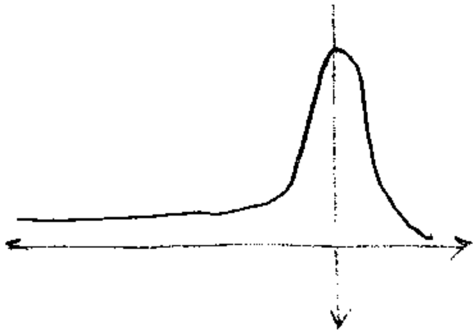
n statistics, geometrical representations or graph representations purely helps us to determine the behaviour of grouped data.)

SKEWNESS : opposite of Symmetry.

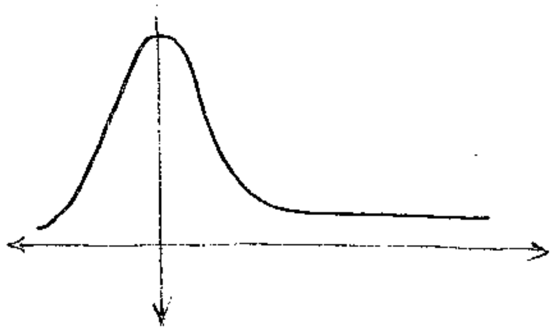
"Lack of symmetry".



Symmetry.



Negatively skewed.



Positively skewed.

→ To check the skewness of a distribution.

→ **PEARSON'S COEFFICIENT OF SKEWNESS**

$$\boxed{S_{KP} = \frac{\text{Mean} - \text{Mode}}{\sigma}}$$

$$\boxed{S_{KP} = \frac{3(\text{Mean} - \text{Median})}{\sigma}}$$

→ practical limit of S_{KP}

$$\boxed{-3 \leq S_{KP} \leq 3}$$

→ For Symmetry $S_{KP} = 0$

Symmetry condition	Negative skewness	positive skewness.
Mode = Median = Mean	Mode > Median > Mean	Mode < Median < Mean.

PROBABILITY

RANDOM EXPERIMENT : Unpredictable outcomes of an experiment is known as a Radom experiment.

eg: Tossing a unbiased coin.

Rolling a Die.

Drawing a card from the pack of 52.

SAMPLE SPACE (S) : The collection of all possible outcomes of an experiment is known as a Sample space. It is denoted by S.

EVENT : The outcomes of an experiment is known as a event. Mathamatically, event is the subset of Sample space.

ition of OBABILITY : The probability of an event is defined as the ratio ~~of~~ b/w the favourable cases to the event and the number of outcomes of an experiment. (The outcomes are mutually exclusive, exhaustive events)

$$\therefore \boxed{P(E) = \frac{m}{n}} \quad \text{where } m \leqslant n$$

AXIOMATIC APROACH / PROBABILITY FUNCTION

le 1 $\quad P(S) = 1$

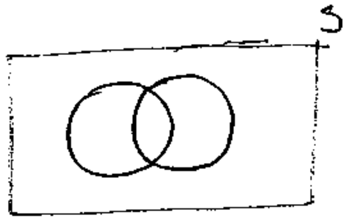
le 2 $\quad 0 < P(E) \leqslant 1$

$P(E) = 0 \quad ; \quad P(\phi) = 0 \longrightarrow$ Impossible Event

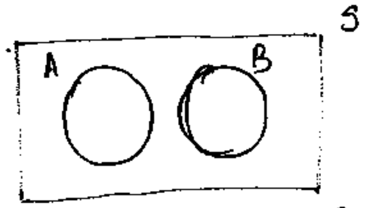
$P(E) = 1$

Rule 3 : $\boxed{P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)}$

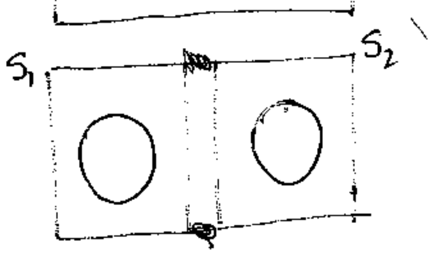
where E_i's are disjoint / Mutually Exclusive.



Dependent.



Mutually Exclusive Events



Independent

Note: Occurance of an event doesnot depends ~~on t~~ upon the occurance of the events in the same sample space, then such events are called Mutually Exclusive event.

→ Let A and B are mutually exclusive events.

$A \cap B = \phi$ & $P(A \cap B) = 0$

→ Occurance of an event doesnot depends ~~on~~ upon the occurance of same event in a different sample space, then those events are called Independent events.

→ Mutually Exclusive events never be independent, Independent events never be ~~not~~ equal to Mutually Exclusive.

RESULTS

1. Compliment theorem.

$$P(A^c) = 1 - P(A)$$

$$P(A) = 1 - P(A^c)$$

2. Addition Theorem.

If A, & B are two events (If nothing specified, take it as ~~ind~~ dependent)

~~P(A∩~~

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

→ If A and B are mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

$$P(A+B) = P(A) + P(B)$$

∴ Here it doesn't mean '+' = '∪'

~~In cuic~~ If '+' Sign is given it is indirectly shown or it is sure that A & B are mutually exclusive. only mutually exclusive events can be added.

$$P(A+B+C) = P(A) + P(B) + P(C)$$

3. Multiplication Theorem for Dependent Events.

If A & B are two events,

$$P(A \cap B) = P(A)\ P(B/A)$$ → Here A must be happened already.

$$= P(B)\ P(A/B)$$ → Here B must be happend already.

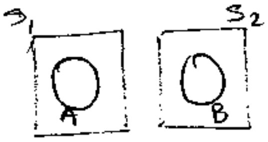
conditional probability

if A, B and C are three events.

$$\boxed{P(A \cap B \cap C) = \underbrace{P(A)\,P(B/A)}_{P(A \cap B)}\;P(C/A \cap B)}$$

3. Multiplication Theorem for Independent events.

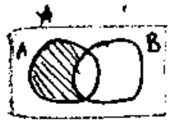
$$P(A \cap B) = P(A) \cdot P(B)$$



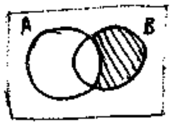
$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

4. If A & B are 2 events.

$$\boxed{P(A \cap B^c) = P(A) - P(A \cap B)}$$ only A

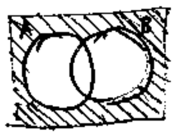


$$\boxed{P(A^c \cap B) = P(B) - P(A \cap B)}$$ only B.

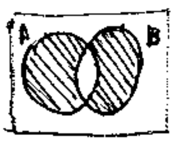


Neither A nor B

$$\boxed{P(A^c \cap B^c) = P(\overline{A \cup B}) = 1 - P(A \cup B)}$$ not in A B



$$\boxed{P(A \Delta B) = P(A \cap B^c) + P(A^c \cap B)}$$ only once



5. $$P(A^c/B) = \frac{P(A^c \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)} = 1 - \frac{P(A \cap B)}{P(B)} = 1 - P(A/B)$$

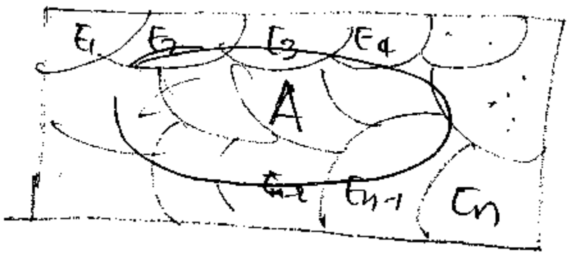
$$P\left(\frac{A}{B^c}\right) = \frac{P(A\cap B^c)}{P(B^c)} = \frac{P(A) - P(A\cap B)}{1 - P(B)} \quad (\because P(B) \neq 1)$$

$$P\left(\frac{A^c}{B^c}\right) = \frac{P(A^c\cap B^c)}{P(B^c)} = \frac{1 - P(A\cup B)}{1 - P(B)} \quad (\because P(B) \neq 1)$$

Note: If A and B are independent events, the probability of $P(A\cap B^c)$, $P(A^c\cap B)$ and $P(A^c\cap B^c)$ are also independent.

6. BAYE'S THEOREM

If $E_1, E_2, E_3, \dots E_n$ are the mutually exclusive events $(P(E_i) \neq 0)$ such that A is an arbitrary events which is a subset of "$\bigcup_{i=1}^{n} E_i$", then $P(A)$ is



$$P(A) = P(E_1\cap A) + P(E_2\cap A) + \dots + P(E_n\cap A)$$

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right) + \dots + P(E_n)P\left(\frac{A}{E_n}\right)$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i)\,P\left(\frac{A}{E_i}\right)}$$ Total probability of unknown event.

Part (ii) of Bayes's theorem : Reverse probability.

$$\boxed{P\left(E_i/A\right) = \frac{P(E_i \cap A)}{P(A)}}$$

$$\boxed{P\left(E_i/A\right) = \frac{P(E_i)\,P\left(A/E_i\right)}{\sum_{i=1}^{n} P(E_i)\,P\left(A/E_i\right)}}$$

Indirectly A is known.

steps in Bayes's theorem.

S1 Identify the known events in the data (Mutually exclusive).

S2 Select the unknown event (It is a part of known events).

S3 Write the probability of unknown in terms of known.

S4 Find the total probability of unknown events.

S5 Compute Reverse probability for known events.

'Atleast'	Minimum	$\geq$
'Atmost'	Maximum	$\leq$
'And'	Product	$\cap$
'OR'	Sum	$\cup$

Application of addition theory

Cases with terminologies

→ 'either - or'

→ 'atleast once'

→ 'OR'

Application of Multiplication theory

Cases with terminologies

→ 'Simultaneously'

→ 'one after other'

→ 'as well as'

→ 'Successively'

→ 'One by one'.

→ 'alternatively'.

→ 'and'.

52 CARD CASE

Total 52 cards

13 Hearts + 13 Diamond + 13 club~~spade~~ + 13 Spade.

Each 13 contains 1, 2, 3, 4, ... 10, J (11), Q (12), K (13)

ie, 10 number cards + 3 Face cards

Total ~~of~~ no:of face cards = 4 x 3 = 12

Here 1 sample space =

Q 3 coins are tossed at a time. Find the probability of getting atmost one head for

$$S = \begin{array}{|c|} HHH \\ HHT \\ HTH \\ HTT \\ THH \\ THT \\ TTH \\ TTT \end{array}$$

$$P(X \leq 1) = P(X < 1) + P(X = 1)$$

$$= \frac{1}{8} + \frac{3}{8}$$

$$= \frac{4}{8} = \frac{1}{2}$$

Q Above data same, Find the probability that atleast one tail.

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

Q Find the probability that atleast one head and one tails

P(atleast 1 head $\boxed{\text{and}}$ atleast 1 tail min).

→ whenever "and" is used as the conjugation b/w any number of events, the all the events must occurs simontaneously in all ~~cases~~ favourable cases.

$$P = \frac{6}{8} = \frac{3}{4}$$

Q, Same data. Find the probability that atleast one head and atmost one ~~⊕~~ tail.

HHH, HHT, HTH, THH $\qquad P = 4/8 = \underline{\underline{1/2}}$

Q, A player tosses 4 coins Find the probability that atleast 2 heads and atleast 2 tails.

$$\frac{4P_4}{2!\,2!} = \qquad \frac{4!}{0!} \qquad \frac{4\times3\times2\times1}{4} = 6$$

HHTT
HTTH
TTHH
THHT
THTH
HTHT

$$= 6 = \boxed{4 \underset{\text{favourable}}{\overset{\text{repeated}}{C_2}}}$$

$$\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$$

H	No of cases		T
favourable case. $4C_0$	= 1	=	$4C_4$
$4C_1$	= 4	=	$4C_3$
$4C_2$	= 6	=	$4C_2$
$4C_3$	= 4	=	$4C_1$
$4C_4$	= 1	=	$4C_0$
	12		

~~Atm~~ Atmost 2 head, Atmost 2 tails = same = $\underline{\underline{\frac{6}{16}}}$

Q, A coin is ~~repea~~ tossed 6 times. Find the probability that the number of heads are more than the number of tails.

$$2^6 = 64$$

$$\underbrace{{}^6C_0 \quad {}^6C_1 \quad {}^6C_2 \quad {}^6C_3 \quad {}^6C_4 \quad {}^6C_5 \quad {}^6C_6}$$

Favourable Cases of Head.

$$P = \frac{{}^6C_6 + {}^6C_5 + {}^6C_4}{64}$$

$$= \frac{1 + \frac{6!}{1 \times 5!} + \frac{6!}{2!\,4!}}{64}$$

$$= \frac{1 + 6 + \frac{\cancel{6}^{3} \times 5}{\cancel{2}}}{64}$$

$$= \frac{1+6+15}{64}$$

$$= \frac{22}{64}$$

Q, A coin is repeated n times. Find the probability that the head appears in the odd terms.

$${}^nC_1 + {}^nC_3 + {}^nC_5 + \dots + {}^nC_{n-1} = 2^{n-1}$$

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots + {}^nC_n = 2^{n-1}$$

$$\text{Req prob} = \frac{2^{n-1}}{2^n} = \underline{\underline{1/2}}$$

1, Two dies are rolled two times. Find the probability that for getting a sum 7

(i) atleast once

(ii) only once

(iii) twice.

(i) $\underbrace{1+6,\ 5+2,\ 4+3}_{3\times 2 = 6}$

$\cancel{6 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6}}$

sum 7 in first : $P(A) = 6/36 = 1/6 \qquad P(A^c) = 5/6$.

sum 7 in second : $P(B) = 6/36 = 1/6 \qquad P(B^c) = 5/6$.

$$P(\text{atleast once}) = P(A \cup B) = 1 - P(A^c \cap B^c) = 1 - P(A^c)P(B^c)$$

$$= 1 - \frac{5}{6} \times \frac{5}{6}$$

$$= 1 - \frac{25}{36}$$

$P(\text{only once})$

$= P(A \cap B^c) + P(B \cap A^c)$

$= P(A)\,P(B^c) + P(B)\,P(A^c)$

$= \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6}$

$= \frac{10}{36}$

$P(\text{twice}) = P(A \cap B) = P(A)\,P(B)$

$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

Q, Two dices are rolled. Find the probability that the first die should contain a prime number or a total of eight.

~~(6,2)(2,6),(5,3)(3,5),(4,4)~~

prime 2, 3, 5,

$2 \leftarrow (2,1)\ (2,2)\ \cdots\ (2,6)$

$3 \leftarrow (3,1)\ (3,2)\ \cdots\ (3,6)$

$5 \leftarrow (5,1)\ \cdots\ (5,6)$

$6 \times 3 = 18 \qquad P = \frac{18}{36}$.

Total 8

(6,2)(2,6)(5,3)(3,5)(4,4)

$P = \frac{5}{36}$

But A & B are dependent.

Hence to find $P(A \cap B)$

ie, Prime in first & sum 8

$(5,3)\ (3,5),\ (2,6) \rightarrow$ 3 cases $\quad P(A \cap B) = \frac{3}{36}$.

we need to find

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{18}{36} + \frac{5}{36} - \frac{3}{36}$$

$$= \frac{20}{36}$$

Two dice are rolled. Find the probability neither sum 9 nor sum 11

$$P(\text{neither sum } 9, \text{ nor } 11)$$

$$= 1 - P(\text{sum } 9, \text{ and sum } 11)$$

sum $9 \rightarrow (5,4)\ (4,5)\ (6,3)\ (3,6) \rightarrow \frac{4}{36}$

sum $11 \rightarrow (5,6)\ (6,5) \rightarrow \frac{2}{36}$

$$P(9^c \cap 11^c) = 1 - P(9 \cup 11)$$

$$= 1 - \left(\frac{4}{36} + \frac{2}{36}\right) = \cancel{\frac{5}{36}}\ \frac{30}{36}$$

Q. A [illegible] of order 2. with the elements 0,(and) or 1. Find the probability that the choosen det is non zero

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad S = 2^4 = 16$$

$\Delta \neq ad - bc$

$\therefore$ Case $\Delta = 1$ [a = d = 1 atleast one of b & c is zero]

$$\Delta = 1 \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

= 3 cases

Case II $\Delta = -1$

$ad = 0 \quad bc = 1$

[b = c = 1, atleast one of a & d is zero]

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \Rightarrow 3 \text{ cases}.$$

$P(\text{non zero } \Delta) = 3/16 + 3/16 = 6/16$

$P(\text{non neg } \Delta) = 1 - 3/16 = 13/16$

~~P(zero Δ) = 1 - 6/16 = 10/16~~

Q, 4 cards are drawn ~~from~~ at random from a pack of 52 cards. Find the probability that

(i) All the 4 cards are drawn from same suit.

(ii) No two cards are drawn from the same suit.

(i) $S = {}^{52}C_4$

$$\frac{\cancel{4(13)}\; 4 \times {}^{13}C_4}{{}^{52}C_4} \quad \text{(at a time)}$$

(ii) $$\frac{{}^{13}C_1 \, {}^{13}C_1 \, {}^{13}C_1 \, {}^{13}C_1}{{}^{52}C_4} = \frac{(13)^2}{{}^{52}C_4} \quad \text{(one by one)}.$$

~~= 13 × 1~~

Q, A card is drawn from a pack of 52 cards. Find the probability that ~~neither a diamond nor a face~~ car

(i) neither a diamond nor a face.

(ii) neither a 10 nor a king

$P(D) = 13/52$

$P(F) = 12/52$

(i) $P(D^c \cap F_c^{\,c}) = 1 - P(D \cup F)$

$= 1 - \left[\frac{13}{52} + \frac{12}{52} - \frac{3}{52}\right] = \frac{30}{\ldots}$ intersection

(ii) $P(10) = \frac{4}{52}$

$P(K) = \frac{4}{52}$

$$P(10^c \cap K^c) = 1 - P(10 \cup K)$$

10 & King are mutually exclusive.

$$= 1 - \frac{4}{52} - \frac{4}{52}$$

$$= 1 - \frac{8}{52}$$

$$= \underline{\underline{\frac{44}{52}}}$$

Q, A and B are the two players rolling a die on the condition that one who gets the two first wining the game. If A starts the game, what are the winning chances of player A, B.

$$\frac{1}{6} + \frac{5}{6} \times \frac{1}{6} \quad \frac{5}{6} \times \frac{1}{6}.$$

$P(2) = 1/6 \qquad P(2^c) = 5/6$

Let getting 2 is p not getting q.

$$P(\text{win } B) = qp + q^3p + q^5p + \dots = qp + q^2qp +$$

$$= p\left[q + q^3 + q^5 + \dots\right] = pq\left[1 + q^2 + q^4 + \dots\right]$$

$$= (qp) \times \frac{1}{(1-q^2)}$$

$$= \frac{1}{6} \times \frac{5}{6} \times \frac{1}{1-\frac{25}{36}}$$

$$= \underline{\underline{5/11}}$$

$$P(\text{win } A) = p + q^2 p + q^4 p + \cdots$$

$$= p\left[1 + q^2 + q^4 + \cdots\right]$$

$$= p \times \frac{1}{1-q^2} = \frac{1}{6} \times \frac{1}{1-\frac{25}{36}}$$

$$= \frac{1}{6} \times \frac{36}{11}$$

$$= \underline{\underline{\frac{6}{11}}}$$

A, B, C are the 3 players in the order. Tossing the same coin on the condition that one who gets the head first winning game. If A starts the game, what are the winning chances of player C in 3rd trial.

$$P(\text{win } C \text{ in } 3^{rd} \text{ trial}) = \cancel{\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)} \times q^3 q^3 q^2 p$$

$P(H) = 1/2$ (p) $\quad P(T) = 1/2$ (q)

$$= \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2}$$

$$= \frac{1}{8} \times \frac{1}{8} \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{512}$$

Q. A number is chosen at random from 100 numbers,

{(00) 01, 02, 03, ... 99}

Let x denote the sum of the digits on the number, and y denotes product of the digits on the number. Find the probability that

$$P\left(x=9 / y=0\right) = \frac{P(x=9 \cap y=0)}{P(y=0)}$$

0 0 – 9

$$P(y=0) = \frac{2/100}{19/100} = \frac{2}{19}$$

9 + 10 — 0.

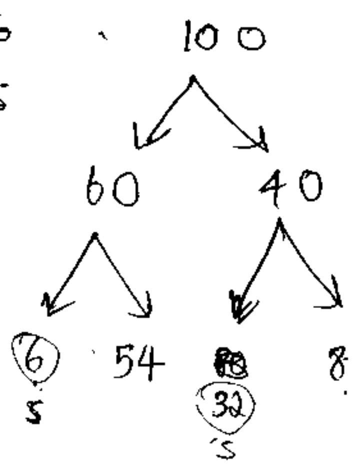
Q. 60% of the employees of the company are college graduates. Of these, 10% are in the sales department. Of the employees who did not graduate from the college are 80% in the sales department. A person is selected at random. Find the probability that

(i) The person is in the sales department.

(ii) Neither in the sales department nor a college ~~departm~~ graduate.

$40 \times \frac{10}{100}$

$40 \times \frac{80}{100}$



(i) $\frac{38}{100}$ ~~P(CG ∩ SD)~~

(ii) $\frac{8}{100}$

(i) $P(SD) = P(CG \cap SD) + P(CG^c \cap SD)$

$$= P(CG)\, P(SD/CG) + P(CG^c) \cdot P(SD/CG^c)$$

$$= 0.6 \times 0.1 + 0.4 \times 0.8$$

$$P(SD) = \underline{\underline{0.38}}.$$

(ii) $P(CG^c \cap SD^c) = 1 - P(CG \cup SD)$

$$= 1 - \left[P(CG) + P(SD) - P(CG \cap SD)\right]$$

$$= 1 - \left[P(CG) + P(SD) - P(CG)\, P\left(SD/_{CG}\right)\right]$$

$$= 1 - \left[0.6 + 0.38 - 0.6 \times 0.1\right]$$

$$= 1 - 0.92$$

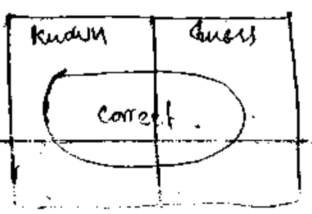
$$= \underline{\underline{0.08}}$$

Q. In answering a multiple choise qn, a student either knows the answer or guess the answer. Let P be the probability that student knowing the answer to the qn and (1-P) be the probability that guessing the ans to a qn. Assume that if the student guess the answer to a question will be correct, with probability 1/5. What is the conditional probability that if the student knew the ans to a qn, given that, he answered correctly?

(qu on inserted above "a multiple")

$P(K) = P$

$P(G) = 1 - P$



E: answering correctly.

$P(K) = P$

$P(G) = 1-P$

$$P(E) = P(E\cap K) + P(E\cap G)$$

$$P(E) = P(E\cap K) + \cancel{P(G)\,P(E/G)}\; P(E\cap G)$$

$$P(E) = P(K)\,P(E/K) + P(G)\,P(E/G)$$

$$= P\,P(E/K) + (1-P)\,1/5$$

$P(E/K) = 1$

Knowing correct then he will correct it for sure.

$$= P\times 1 + (1-P)\,1/5$$

$$= P + 1/5 - P/5$$

$$P(E) = \frac{4P+1}{5}$$

rse probability

$$P(K/E) = \frac{P(K\cap E)}{P(E)}$$

$$= \frac{P\times 1}{\frac{4P+1}{5}} = \frac{5P}{4P+1}$$

In qn. final known even → correct ans.

Unknown event → Knowing ans.

Hence go for Reverse probability.

Q, There are 3 coins. Of these two are unbiased. One is a biased coin with 2 heads. A coin is drawn at random and tossed two times. It appears head on both the ~~sides~~ times, Find the probability that it is from the biased coin.

UB	B	
2	1	= 3

$P(UB) = 2/3 \qquad P(B) = 1/3$.

E: Getting a head two times. is an unknown event.

$$P(E) = P(E \cap UB) + P(E \cap B)$$

$$\cancel{= P(UE)\; P(UB/E) + P(}$$

$$= P(UB)\; P(E/UB) + P(B)\; P(E/B)$$

$P(E/B) = 1$

$$= 2/3 \times \frac{1}{4} + 1/3 \times 1$$

$P(E/UB) = \frac{1}{2} \times \frac{1}{2} = 1/4$

$$= \frac{1}{6} + \frac{1}{3}$$

$$P(E) = \frac{3}{6}$$

Ans: $P(B/E) = \dfrac{P(E \cap B)}{P(E)}$

$$= \frac{P(B)\,P(E/B)}{3/6} = \frac{1/3 \times 1}{3/6} = \frac{2}{3}$$

Ans = $2/3$

5. player A speaking truth 4 out of 7 times, A card is drawn from the pack of 52 cards. He reports that there is a diamond what is the probability that actually there was a diamond.

$\frac{4}{7}$

T: A Player telling truth

~~$P(A) = 4/7$~~

$P(T) = (4/7)$

Lie probability

$P(L) = (3/7)$

D: Reporting a diamond.

$$P(D) = P(T \cap D) + P(L \cap D)$$

$$= P(T)\,P(D/T) + P(L)\,P(D/L)$$

$P(D/T) = 13/52$

$P(D/L) = 39/52$

$$= \frac{4}{7} \times \cancel{13}\,\frac{1}{4} + \frac{3}{7} \times \frac{3}{4}$$

$$= \frac{4+9}{28}$$

$$= \frac{13}{28}$$

$$P(T/D) = \frac{P(T \wedge D)}{P(D)}$$

$$= \frac{\cancel{\frac{4}{7}} \times \cancel{\frac{1}{4}}}{\frac{13}{\cancel{28}}} = \frac{4}{13}$$

Q. A letter is known to have come from either Tatanagar or CALCUTTA ~~Calcutta~~. On the envelope, the just two consecutive letters 'TA' are visible. Find the probability that the letter has come from Tatanagar.

$P(T) = 1/2 \quad : \quad P(C) = 1/2$.

E : Getting a TA consider as single letter.

Given only 1 letter is visible (which is a combination of 2 consecutive) Not more than 1 is visible.

1 2 3 4 5 6 7 8
TA T A N A G A R

CALCUTTA
1 2 3 4 5 6 7

Hence can take either one of 'TA' in TATANAGAR. No need of confusion.

$P(E/T) = 2/8$

$P(E/C) = 1/7$

$$P(E) = P(E \cap T) + P(C \cap E)$$

$$= P(T)\,P(E/T) + P(C)\,P(E/C)$$

$$= \frac{1}{2}\times\frac{2}{8} + \frac{1}{2}\times\frac{1}{7}$$

$$= 11/56$$

Qn $P(T/E) = \dfrac{P(T\cap E)}{P(E)} = \dfrac{\frac{1}{2}\times\frac{2}{8}}{11/56} = \dfrac{7}{11}$

There are 3 bags A, B, C, with Balls Blue, Red, and ~~Green~~ in the form ~~of 1, 2, 3~~ ~~B R G~~

	colour		
Bags	B	R	G
A	1	2	3
B	2	3	1
C	3	1	2

A bag is drawn at random, and two balls are taken from it. They are found to be one blue and one red. Find the probability that the selected balls are from bag C.

$$P(A) = P(B) = P(C) = 1/3.$$

E : Getting one blue and one red.

$$P(E) = P(E \cap A) + P(E \cap B) + P(E \cap C)$$

$$= P(A)\,P(E/A) + P(B)\,P(E/B) + P(C)\,P(E/C)$$

$$= \frac{1}{3}\left[\frac{2}{15} + \frac{6}{15} + \frac{3}{15}\right]$$

$$= \frac{1}{3}\left[\frac{11}{15}\right]$$

$$= \frac{11}{45}$$

$$P(E/A) = \frac{{}^{1}C_1 \times {}^{2}C_1}{{}^{6}C_2} = \frac{2}{15}$$

$$P(E/B) = \frac{{}^{2}C_1 \times {}^{3}C_1}{{}^{6}C_2} = 6/15$$

$$P(E/C) = \frac{{}^{3}C_1 \times {}^{1}C_1}{{}^{6}C_2} = \frac{3}{15}$$

Qn: $P(C/E) = \frac{P(C \cap E)}{P(E)}$

$= \frac{1}{3} \times \frac{3}{15}$ ... $\frac{3}{11}$

Also others can be found out.

$$P(B/E) = \frac{1/3 \times 6/15}{11/45} = \frac{6}{11}$$

$$P(A/E) = \frac{1/3 \times 2/15}{11/45} = \frac{2}{11}$$

LEAP YEAR CONCEPT

LEAP YEAR

366 Days

52 weeks + 2 days

S-M
M-T
T-W
W-T
T-F
F-S
Sat-S

$P(53 \text{ sunday}) = \frac{2}{7}$

NON LEAP YEAR.

365 Days.

52 weeks + 1 day

$P(53 \text{ sundays}) = \frac{1}{7}$

Two dice.

$$P(\text{diff zero}) = P(\text{Doublets}) = \frac{6}{36} = \frac{1}{6}$$

$P(1,1)$

$P(2,2)$

$\vdots$

$P(6,6)$

Three dice

$$P(\text{Triplet}) = \frac{6}{6^3} = \frac{1}{36}$$

$(1,1,1)$

$(2,2,2)$

$\vdots$

$(6,6,6)$

Consider now

$\{1, 2, 3, \ldots, 200\}$

$$P(\text{div } 6 \text{ OR div by } 8)$$

$$P(\text{div } 6) = \frac{33}{200}$$

$$P(\text{div } 8) = \frac{25}{200}$$

$$P(\text{div } 6 \text{ AND div } 8) = \text{LCM}(6,8)$$

$$= P(\text{div by } 24) = \frac{8}{200}$$

Hence $P(\text{div by } 6 \text{ OR div by } 8) = P(6) + P(8) - P(24)$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$

$$= \frac{50}{200} = \frac{1}{4}$$

RANDOM VARIABLE AND EXPECTATION

(r.v)

RANDOM VARIABLE : Connecting the outcomes of an experiment with real values is known as Random Variables.
(It is a rule to assign Real number to the outcome is known as 1D Random Variable)
The corresponding data is known as univariate data.

2-D RANDOM VARIABLE : Connecting 2 outcomes at a time to the one real value provided those two outcomes are drawn from same Sample space.
The corresponding data is known as Bivariate data.

⇒ Similarly the concept of n-D Random variable which corresponds to an n-tuple.

TYPES OF RANDOM VARIABLE

Random Variable

- Discrete Random Variable
  - Probability MASS FUNCTION $P(x)$
    - DISCRETE PROBABILITY DISTRIBUTION
      - Binomial
      - Poisson
- Continous Random Variable
  - Probability DENSITY FUNCTION $f(x)$
    - CONTINUOUS PROBABILITY DISTRIBUTION
      - Normal
      - Uniform.

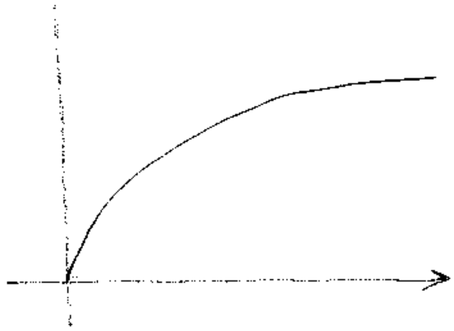
Probability Mass Function $\longrightarrow P(x)$

Probability Density Function $\longrightarrow f(x)$

Distributive Function/Cumulative Function $\longrightarrow F(x)$

$$\frac{dF(x)}{dx} = f(x), \qquad F(x) = \int^{x} f(x)\, dx.$$

Distribution function graph will always be a non decreasing function.



RANDOM PROCESS : Random variable along with time domain.

EXPECTATION

It is actually the mean in the probability ~~function~~ distribution.

$$E(X) = \sum_{x=0}^{n} x \cdot P(x) \quad \text{where } x \text{ is Discrete R.V}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \quad \text{where } x \text{ is Continous r.v}$$

The above relations are derived from Frequency distribution where freq is replaced by probability

$$\bar{X} = \frac{\sum f \cdot x}{\sum f}$$

$$\bar{X} = \frac{\sum P(x) \cdot x}{\sum P(x)}$$

But $\sum P(x) = 1$

$$\therefore \bar{X} = \sum x\, P(x) = E(x)$$

(ii) VARIANCE

From frequency distribution, the variance is given by.

$$\sigma^2 = \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i\right)^2$$

$$\frac{1}{N}\sum f_i x_i = E(x)$$

$$\frac{1}{N}\sum f_i x_i^2 = E(x^2)$$

In general $\boxed{\frac{1}{N}\sum f_i x^r = E(x^r)}$

In probability distribution,

$$\boxed{V(x) = E(x^2) - (E(x))^2}$$

$$\boxed{V(x) = E\left(x - (E(x))^2\right)}$$

$$\boxed{V(x) = \sum x^2 p(x) - \left(\sum x \cdot p(x)\right)^2}$$ where x is a discrete r.V

$$\boxed{V(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int x f(x)\,dx\right)^2}$$ where x is a continuous r.V.

PROPERTIES OF EXPECTATION

(i) If X is a r.v and 'a' is a constant,

then $E(aX) = aE(X)$

If X and Y are r.v's.

(ii) Then $E(X+Y) = E(X) + E(Y)$

$E(X-Y) = E(X) - E(Y)$

(iii) If X & Y are r.v's

$$E(X.Y) = E(X) \cdot E(Y/X)$$

$$= E(Y) \cdot E(X/Y)$$

(iv) If X & Y are Independent random variables,

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

(v) If $Y = aX + b$, where a & b are constants,

Then $E(Y) = aE(X) + b$

(vi) ie, $\boxed{E[\text{constant}] = \text{constant}}$ ie, Mean of constant = That constant itself.

(vii) $\boxed{E\left[E\left[E(X)\right]\right] = \text{constant} = E(X)}$

PROPERTIES OF VARIANCE

i) If X is a r.v and 'a' is a constant,

$$V(aX) = a^2 V(x)$$

$$V(-Y) = (-1)^2 V(Y) = V(Y)$$

If X and Y are independent r.V's.

$$V(X+Y) = V(X) + V(Y)$$

$$V(X-Y) = V(X) + V(-Y)$$

$$\Rightarrow \quad V(X-Y) = V(X) + V(Y)$$

$$\Rightarrow \quad \boxed{V(X \pm Y) = V(X) + V(Y)}$$

If a & b are constants, X & Y are independent r.v's,

$$V(aX - bY) = a^2 V(X) + b^2 V(Y)$$

$$V\left(X/a - Y/b\right) = 1/a^2 \, V(X) + 1/b^2 \, V(Y)$$

If $Y = aX + b$, where a & b are constants,

$$V(Y) = V(aX + b)$$

$$= V(aX) + V(b)$$

$$V(Y) = a^2 V(X) + 0$$

i.e. $V(\text{Constant}) = 0$

If X and Y are two random variables (Dependent r.v's).

$$\boxed{V(X+Y) = V(X) + V(Y) + 2\,CoV(X,Y)}$$

where $CoV(X,Y) \rightarrow$ Covariance of X, Y

where

$$\boxed{CoV(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)}$$

$\boxed{V(X,Y) \text{ is meaningless}}$

$\rightarrow$ $\boxed{CoV(X,X) = V(X)}$

$$\rightarrow \quad CoV(a,b) = E(a\cdot b) - E(a)E(b)$$

$$= ab - a\cdot b$$

$$= \underline{\underline{0}}$$

$\boxed{CoV[a,b] = 0}$ where a and b are constants.

1 If x and y are independent r.v, then covariance of X, Y $= 0$

$CoV(X,Y) = 0$ X, Y are independent

But Converse of the statement is not true.

2. Variance and covariance are independent of change of origin, dependent of change of scale.

$V[aX+b] = a^2 V[x]$

: Mean [Expectation] dependent of origin as well as Dependent of change of scale.

$$E[aX+b] = a^2 V[x] + b$$

SKEWNESS

$$\boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}$$

$\mu_3 \rightarrow 3^{rd}$ central moment.

$\mu_2 \rightarrow$ Variance.

Skewness is defined in terms of 'μ_3'

$$\boxed{\gamma = \sqrt{\beta_1}}$$

γ is also a measure of skewness.

Note :

If $\mu_3 = 0 \implies \beta_1 = 0$ Then the curve is SYMMETRY

If $\mu_3 \rightarrow -ve \implies$ Then the curve is NEGATIVELY SKEWED

If $\mu_3 \rightarrow +ve \implies$ Then the curve is POSITIVELY SKEWED

Q. Find the expectation of the number on a die when it is ~~throug~~ thrown.

$$E(X) = \sum_{0}^{n} x\, p(x)$$

$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= \frac{1}{6}\left[1+2+3+4+5+6\right]$$

$$= \frac{1}{6}\left[\frac{6\times 7}{2}\right]$$

$$= \cancel{2.5}\ \underline{\underline{3.5}}$$

x r.v	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

Q. Find the variance for the single die

$$V(X) = \cancel{E(X)}\ E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{0}^{n} x^2\, p(x)$$

$$= \frac{1}{6}\left[1^2+2^2+3^2+4^2+5^2+6^2\right]$$

$$\frac{13}{7}$$

$$= \frac{\cancel{6}(7)(13)}{6\,\cancel{6}}$$

$$= \underline{\frac{91}{6}}$$

$$\therefore V(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$$

Note :

The mean and variance for the sum of the numbers on the ~~dies~~ dice is

$\boxed{E(X) = \frac{7n}{2}}$ $\boxed{V(X) = \frac{35}{12}n}$ where 'n' is the number dice rolled

Q 3 unbiased dice are thrown Find the mean ~~of~~ and variance for the sum of the numbers on them.

X : sum of number on 3 dies.

$X : 2, 3, 4, \ldots, 18$

$E(X) = \frac{7n}{2} = \frac{7 \times 3}{2} = \frac{21}{2}$

$V(X) = \frac{35}{12}n = \frac{35 \times 3}{12} = \frac{35}{4}$

Two unbiased dies are rolled. Find the expectation for sum 7 on them.

X : sum for the number obtained on 2 dies.

Here X is assuming only 7.

Hence no need of additions.

$E(\text{sum } 7) = 7 \cdot P(7)$

$= 7 \cdot \frac{6}{\ldots}$

6 cases $\begin{cases} 1, 6 \\ 6, 1 \\ 3, 4 \\ 4, 3 \\ \ldots \end{cases}$

Q. A player tosses 3 coins. He wins 500 rupees if 3 heads occur, 300 rs if 2 Heads occurs, 100 rs if only 1 head occur. On the other hand he looses 1500 rs If 3 tails occurs Find value of the game.

X : No :of head possibility

~~$E(X) = 500 \times \frac{1}{8} +$~~

X	3	2	1	0
P(x)	1/8	3/8	3/8	1/8

$\boxed{\text{VALUE OF GAME} = \text{GAIN} - \text{LOSE} = \text{Gain} \times \text{its prob} - \text{Lose} \times \text{its prob.}}$

$$E(X) = 500 \times \frac{1}{8} + 300 \times \frac{3}{8} + 100 \times \frac{3}{8} - 1500 \times \frac{1}{8}$$

$$= \frac{1400 + 300 - 1500}{8}$$

$$= \frac{1700 - 1500}{8}$$

$$= \frac{200}{8} = 25$$

NOTE :

If Game is said to be fair, the expected value of game is said to be 0. (no Loss and no Gain).

Q. A man has given n keys of which one fits the lock. He tries them successively without replacement to open the lock. What is the probability that the lock will be open at the r^{th} trial. Also determine mean and variance.

Note: with Replacement implies that it is Independent events.

without Replacement implies that it is Dependent Events.

prob of opening the lock in 1^{st} trial $= 1/n$

" " 2^{nd} trial $= 1/(n-1)$

" " 3^{rd} trial $= 1/(n-2)$

$\vdots$

prob of opening lock 1^{st} success in 2^{nd} trial $= \left(1 - 1/n\right)\frac{1}{n-1}$

$$= \frac{\cancel{n-1}}{n} \times \frac{1}{\cancel{n-1}} = \frac{1}{n}$$

prob of opening lock in 1^{st} success in 3^{rd} trial $= \left(1 - 1/n\right)\left(1 - 1/(n-1)\right) \times \frac{1}{n-2}$

$$= \frac{\cancel{n-1}}{n} \times \frac{\cancel{n-2}}{\cancel{n-1}} \times \frac{1}{\cancel{n-2}}$$

$$= \underline{1/n}$$

Hence

prob of opening lock 1^{st} success r^{th} trial $= 1/n$

Variance of n natural numbers $V(x) = \frac{n^2-1}{12}$

consider a value eg:

If keys numbered from 100 — 999

$$\text{Prob}\left(\begin{matrix}450^{th} \text{ trial without replacement} \\ 1^{st} \text{ success in}\end{matrix}\right) = \frac{1}{900}$$ (100-999 means 0→900)

$$\text{Prob}\left(1^{st} \text{ success in } 450^{th} \text{ trial with replacement}\right) = \left(1 - \frac{1}{900}\right)^{449} \frac{1}{900}$$

ie in without replacement, they are dependent, $\cancel{900}\ \frac{\cancel{899}}{900} \times \frac{\cancel{898}}{\cancel{899}} \times \frac{\cancel{897}}{\cancel{898}} = 1/900$

~~they cancel~~ they loss probabilitys of $(r-1)$ trials cancel each other and get only $1/900$.

But in with replacement, each trial is independent event, Hence each ~~loss prob~~ of $(r-1)$ loss probabilities we need to multiply

$$\therefore\ q^{r-1} p^{r}$$

Note: The probability for the 1^{th} success in the r^{th} trial with replacement is

$$\boxed{P\left(1^{st} \text{ success in } r^{th} \text{ trial with replacement}\right) = q^{r-1} p}$$

$q \rightarrow$ Failure prob

$p \rightarrow$ Success prob

Q. if x is said to be a Continuous variable and its probability function ϕ

$$f(x) = Kx^2 \quad 0 < x < 1$$

(i) Find the value of K.

(ii) Find Mean & variance.

(i) ~~$\Sigma f(x)$~~ $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 Kx^2\,dx = 1$$

$$K\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\frac{K}{3} = 1$$

$$\underline{\underline{K = 3}}$$

(ii) Mean = $E(X) = \displaystyle\int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= 3\int_0^1 x^3\,dx$$

$$= 3\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{4}$$

(iii) Variance $V(x) = E(x^2) - (E(x))^2$

$$E(x^2) = \int_0^1 x^2 f(x)\, dx = 3\int_0^1 x^4\, dx = \underline{\underline{\frac{3}{5}}}$$

$$\text{Variance} = \frac{3}{5} - \left(\frac{3}{4}\right)^2$$

$$= \frac{3}{5} - \frac{9}{16}$$

$$= \frac{48-45}{80} = \underline{\underline{\frac{3}{80}}}$$

Q. If x is a continous r.v, and $f(x) = Kx^2 e^{-x}$, $0 < x < \infty$

(i) Find the value K

(ii) Mean & variance.

(i) $$\int_{-\infty}^{\infty} f(x)\, dx = 0 \quad \int_0^{\infty} Kx^2 e^{-x}\, dx = 1$$

Use Gamma Function,

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx, \quad n > 0$$

$$\Gamma n = (n-1)\Gamma n-1$$

$$= (n-1)! \text{ (only when } n \text{ is a integer)}$$

$$\Gamma 1 = 1$$

$\Gamma 0$ = doesnot exist.

So integral becomes.

$$K \Gamma(3) = 1$$

$$K\, 2! = 1$$

$$K = 1/2$$

(ii) Mean

$$E(X) = \int_0^\infty x\, f(x)\, dx$$

$$= \frac{1}{2} \int_0^\infty x\, x^2 e^{-x}\, dx$$

$$= \frac{1}{2} \int_0^\infty x^3 e^{-x}\, dx$$

$$= \frac{1}{2} \Gamma(4)$$

$$= \frac{1}{2} \times 3! = 3$$

Variance,

$$E(X^2) = \int_0^\infty x^2 f(x)\, dx$$

$$= \frac{1}{2} \int_0^\infty x^2 \cdot x^2 e^{-x}\, dx$$

$$= \frac{1}{2} \int_0^\infty x^4 e^{-x}\, dx$$

$$= \frac{1}{2} \Gamma(5)$$

$$= \frac{1}{2} 4! = 12$$

$$\text{Variance} = 12 - 3^2$$
$$= 12 - 9 = \underline{\underline{3}}$$

Q. $f(x) = |x| \quad -1 < x < +1$

$V(x)$ Find variance.

$$V(x) = E(x^2) - (E(x))^2$$

$$E(x) = \int_{-1}^{1} x\, f(x)\, dx = \int_{-1}^{1} x\, |x|\, dx.$$

$$= -\int_{-1}^{0} x^2\, dx + \int_{0}^{1} x^2\, dx$$

$$= \underline{\underline{0.}}$$

odd fn $\times$ even fn
$\therefore = \underline{\underline{\text{odd fn}}}$

$$\underline{\underline{\int_{-a}^{a} \text{odd fn}\, dx = 0}}$$

$$E(x^2) = \int_{-1}^{+1} x^2 f(x)\, dx$$

$$= \int_{-1}^{+1} x^2 |x|\, dx$$

$x^2|x| \rightarrow$ even $\times$ even
$\Downarrow$
even

$$\int_{-a}^{a} \text{even fn} = 2\int_{0}^{a} \text{even fn.}$$

$$= 2 \times \int_{0}^{1} x^3\, dx$$

$$= 2 \times \left[\frac{x^4}{4}\right]_0^1$$

$$= \cancel{2}\ \underline{\underline{2/4}} = 1/2$$

$$\text{Variance} = 2/4 - 0^2 = \cancel{8}\ \underline{\underline{1/2}}$$

Q, If X & Y are the r.v's mean of X is 10, Variance of X = 25. Find positive values of a, b, such that $Y = aX - b$ has Expectation is zero and Variance is 1.

$$E(Y) = E[aX - b] = 0$$

$$aE[X] - E[b] = 0$$

$$a \times 10 - b = 0$$

$$10a - b = 0$$

$$\underline{\underline{b = 10a}}$$

$$V[Y] = V[aX - b] = a^2 V[X] = 1$$

$$= a^2 25 = 1$$

$$\underline{\underline{a = \frac{1}{5}}} \quad \text{Given +ve}$$

$$b = 10a$$

$$= 10 \times \frac{1}{5}$$

$$= \underline{\underline{2}}$$

$$\underline{\underline{a = 1/5, \quad b = 2}}$$

BIVARIATE DATA

Let x, y be two discrete r.v.s,

Their probability together given by Joint probability Mass Function (JPMF)

Let x, y be two continous r.v.s

Their probability together given by Joint Probability Dentity Function (JPDF)

Case (i) Continous R.V's.

→ If x and y are two continous r.v's, and its probability function is known as Joint probability density function is denoted by $f(x,y)$.

→ The marginal density functions are

$$f(x) = \int_y f(x,y)\,dy$$

$$f(y) = \int_x f(x,y)\,dx$$

→ If x and y are 2-D continous Independent r.v's ~~and its probability function is known as Joint~~ Iff,

$$\boxed{f(x,y) = f(x)\,f(y)}$$

ie,

$$\boxed{JPDF = MDf(x) \cdot MDf(y)}$$

Joint Distribution Function ~~D~~ or Cumulative Distribution Function

~~F(x,y) is given by JDF~~

$$\frac{d^2}{dx\,dy} F(x,y) = f(x,y)$$

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{x} f(x,y)\,dx\,dy .$$

→ Conditional probability

$$f(x/y) = \frac{f(x,y)}{f(y)} \quad , \left(f(y) \neq 0\right)$$

$$E(x/y) = \frac{E(x,y)}{E(y)} \quad , \left(E(y) \neq 0\right)$$

Case (ii) <u>Discrete R.V's.</u>

If x and y are two dimensional r.v's and its probability function is known as joint probability function ~~PE~~ is denoted by $P(x,y)$.

The marginal mass functions are

$$P(x) = \sum_{y} P(x,y)$$

$$P(y) = \sum_{x} P(x,y)$$

Q. If x and y are 2-D continuous r.v's and its probability function is given by

$$f(x,y) = x \cdot y$$

$$0 < x < 1$$

$$0 < y < 1$$

(i) $E(X)$; $V(Y)$

(ii) $E(xy)$; $\text{cov}(X,Y)$

(iii) $f(x/y)$; $E(X/y)$

(iv) check X & Y are independent or not

(i) $f(x) = \int_y f(x,y)\,dy = \int_0^1 x \cdot y\,dy = x \cdot y^2/2 \Big|_0^1 = \frac{x}{2}$

$f(y) = \frac{y}{2}$

$E(x) = \int_0^1 x \cdot f(x)\,dx = \int_0^1 x^2/2\,dx = \left[\frac{x^3}{6}\right]_0^1 = 1/6$

$E(y) = \int_0 y\, f(y)\,dy = \int_0^1 \frac{y^2}{2}\,dy = \left[\frac{y^3}{6}\right]_0^1 = 1/6$

$E(y^2) = \int_0^1 y^2 f(y)\,dy = \int_0^1 y^2 \cdot y/2\,dy = 1/8$

$V(Y) = \frac{1}{8} - \left(\frac{1}{6}\right)^2 = \frac{1}{8} - \frac{1}{36} = \frac{36-8}{8 \times 36} = \frac{28}{8 \times 36} = 7/72$

$$E(x \cdot y) = \int_{x=0}^{1} \int_{x=0}^{1} x \cdot y \; f(x,y) \, dx \, dy.$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} x \cdot y \; (x \cdot y) \, dx \, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} x^2 y^2 \, dx \, dy$$

$$= \int_0^1 \frac{x^2}{3} \, dx = \underline{\underline{\frac{1}{9}}}$$

<u>Covariance.</u>

$$Cov(X,Y) = \cancel{E(x\cdot y)} \; E(X \cdot Y) - E(X) \cdot E(Y)$$

$$= 1/9 - 1/6 \times 1/6 = 1/9 - 1/36 = \frac{4-1}{36} = \underline{\underline{\frac{3}{12}}}$$

~~(iv)~~ (iii)

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{x \cdot y}{y/2} = \underline{\underline{2x}}$$

$$E(x/y) = \frac{E(x \cdot y)}{E(y)} = \frac{1/9}{1/6} = \underline{\underline{2/3}}$$

$$f(x,y) = f(x) \, f(y)$$

$x \cdot y \neq \frac{x}{2} \cdot y/2$ $\therefore$ x & y are Dependent r.v's.

Q If x and y are 2-D Discrete r.v's and its joint probability mass function is

X \ Y	−1	0	+1
−1	1/4	0	0
0	0	1/2	0
+1	0	0	1/4

(i) Find $P\left(X+Y=2 / X-Y=0\right)$

$$P\left(X+Y=2 / X-Y=0\right) = \frac{P(X+Y=2 \cap X-Y=0)}{P(X-Y=0)}$$

$$= \frac{P(X=1, Y=1)}{P(X=-1, Y=-1) + P(X=0, Y=0) + P(X=1, Y=1)}$$

$$= \frac{1/4}{1/4 + 1/2 + 1/4}$$

$$P\left(X+Y=2 / X-Y=0\right) = \underline{\underline{1/4}}$$

BINOMIAL DISTRIBUTION

Definition : If x is said to be a binomial random variable. It allows the values from 0 to n with the parameters (n, p) and its probability mass function is

$$B(x, n, p) = P(x) = \begin{cases} {}^nC_x \, p^x \, q^{n-x}, & 0 \leq x \leq n. \\ & p+q = 1 \\ & q = 1-p \\ 0, & \text{otherwise} \end{cases}$$

Conditions

- Observations are independent. (n is small).
- The probability of success is constant (p is large)
- Mean is greater than the variance.

PROPERTIES

$$E(x) = \text{Mean} = np$$

$$V(x) = \mu_2 = npq$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3}$$

$$\beta_1 = \frac{(1-2p)^2}{npq}$$

$$\gamma_1 = \frac{1-2p}{\sqrt{npq}}.$$

∴ In het, $p = 1/2 \longrightarrow$ symmetry.

~~PD~~ $p < 1/2 \longrightarrow$ positively skewed.

$p > 1/2 \longrightarrow$ Negatively skewed.

Moment Generating Function (MGF)

$$\boxed{M_X(t) = E\left[e^{tx}\right] = (q+pe^t)^n}$$

⇒ ~~Momentu~~ Moment Generating Function is used for the taking the sum and differente of 2 r.v's ~~wi~~ along with the Binomial Distribution.

Characteristic Function.

$$\boxed{\phi_X(t) = E\left[e^{itx}\right] = (q+pe^{it})^n}$$

characteristic function is used for finding cumulation and ratio of 2 r.v's with Binomial distribution.

Note:

→ $P = 1/2 \Rightarrow \mu_3 = 0 \Rightarrow \beta_1 = 0$

Then the Curve is Symmetry.

→ If $P < 1/2$, then the curve is Positively skewed.

If $P > 1/2$, then the curve is Negatively skewed.

The moment Generating function is used to find addition and differences b/w the r.v's with their corresponding probability function.

The characteristic function is used to finding the convolution and ratio b/w the r.v's with their probability function.

Sum of Independent Binomial r.v's is also a Binomial random variables.

Q, Find the probability of getting a 9 exactly 2 in 3 times with a pair of dice.

$n = 3$
$x = 2$

$p(\text{sum } 9) = (5,4)(4,5)(6,3)(3,6)$

$= \frac{4}{36} = \frac{1}{9}$

$q = \frac{8}{9}$

required prob $= {}^nC_x \, p^x \, q^{n-x}$

$= {}^3C_2 \left(\frac{1}{9}\right)^2 \left(\frac{8}{9}\right)^1$

$= \frac{8}{243}$

Q, The probability of man hitting the target is $\frac{1}{3}$.

(i) If he fires five times, what is the probability of his hitting the target at least twice.

(ii) How many time must he fire so that the probability of his hitting the target atleast once is more than 90%

(i) $n = 5, \quad p = \frac{1}{3}, \quad q = \frac{2}{3}$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - \left[P(X=1) + P(X=0)\right]$$

In binomial distribution

$$\boxed{\begin{aligned} P(X=0) &= q^n \\ P(X=n) &= p^n \end{aligned}}$$ Always.

$$P(X \geq 2) = 1 - \left[\left(\tfrac{2}{3}\right)^5 + {}^5C_1\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^4\right]$$

$$= 1 - \left[\left(\tfrac{2}{3}\right)^4\left[\tfrac{2}{3} + \tfrac{5}{3}\right]\right]$$

$$= \frac{131}{243}$$

ii) $P(X \geq 1) > 90\%$

$$1 - P(X=0) > 0.9$$

$$P(X=0) < 0.1$$

$$q^n < 0.1$$

$$\left(\frac{2}{3}\right)^n < 0.1$$

Taking log on both sides

Here asked for number of trials

$n = 5.7$ [illegible]

Q. If X and Y are the binomial r.v.

$X \sim B(2, p)$

$Y \sim B(4, p)$

If $P(X \geq 1) = 5/9$

Find $P(Y \geq 1) = ?$

$P(X \geq 1) = 5/9$

$1 - P(X=0) = 5/9$

$1 - (1-p)^n = 5/9$

$(1-p)^n = 4/9$

$q^n = 4/9$

Give $n = 2$

$q^2 = (1-p)^2 = 4/9$

$1-p = 2/3$

$p = 1/3$

For $Y \sim B(4, 1/3)$

$P(Y \geq 1) = 1 - P(Y=0)$

$= 1 - (q)^4$

$= 1 - (2/3)^4 = 1 - \frac{16}{81} = \frac{65}{81}$

Q, 2 dies are rolled 120 times. Find the average no: of times in which the number on the first die exceeds the no on the second die.

$n = 120$

$p = ?$

For finding p,

no: on first die exceeds the second die,

~~(2,1) (3,1) (~~

equal case → $6/36$, remaining $\frac{30}{36}$, Half will be first die > second die.

$\therefore \quad p = 15/36$.

$\therefore$ Average = Mean = $E[X] = np = 120 \times \frac{15}{36} = \underline{\underline{50}}$

Q, If x is a binomial r.v and $E(X) = 4$

$V(X) = 4/3$

Find (i) $P(X \leq 2)$

(ii) comment on β_1

i) $P(X \leq 2)$

$E(x) = 4$

$np = 4$

$npq = \frac{4}{3}$

~~$q = 1/3$ $\quad p = 2/3 \quad \frac{2n}{3} = 4$~~

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \left(\frac{1}{3}\right)^6 + {}^6C_1 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + {}^6C_2 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{3^6}\left[1 + 12 + 60\right] = \underline{\underline{73/729}}$$

(ii) Given Data is -vely skewed since p value is more than $1/2$. $p = 2/3 > 1/2$.

Q, If X is a binomial random variable, then find the value of

$$\sum_{x=0}^{n} \left(\frac{x}{n}\right) {}^nC_x \, p^x q^{n-x}$$

$$= \frac{1}{n} \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \frac{1}{n} \boxed{\sum_{x=0}^{n} x \, P(x)}$$

$$= \frac{1}{n} \times \text{Mean}.$$

$$= \frac{1}{\cancel{n}} \times \cancel{n}P$$

$$= \underline{\underline{P}}$$

POISSON DISTRIBUTION

Probability function is given by

$$\boxed{P(X\,;\,\lambda>0) = P(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!}, & \lambda>0,\ 0\leq x<\infty \\ 0, & \text{otherwise.} \end{cases}}$$

→ It is used when observations are HIGH and success Probability is Low.

→ It is Used to Find RARE OCCURANCES.

→ Poisson's equation is <u>time dependent</u> distribution.

→ ie, it is a Evolutionary Process.

Used to find Defect Probability.

Used to find Arrival Rate.

<u>finition</u>

f X is said to be poisson r.v defined in the interval $0<x<\infty$ with a parameter λ $(\lambda>0)$ and ts probbability mass function is

$$P(X;\lambda>0) = P(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & \lambda>0,\ 0\leq x<\infty \\ 0 & \text{otherwice.} \end{cases}$$

Conditions

- → observations are infinitely large, $(n \longrightarrow \infty)$
- → probability of success is very small $(p \longrightarrow 0)$
- → $np = \lambda \Rightarrow p = \frac{\lambda}{n}$

Then $P(x : n, p) = \dfrac{e^{-np} (np)^x}{x!}$

It is approximation of binomial.

POISSON PROCESS

$$\boxed{P(x ; \lambda , t > 0) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}}$$

PROPERTIES

1. $E(X) = \text{Mean} = \lambda$

$V(X) = \mu_2 = \lambda$

$\mu_3 = \lambda$

$\lambda > 0$

$\therefore \beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{1}{\lambda}$ ie, Poisson's distribution is always positively skewed.

It can never by symmetry

⇒ positively skewed

2. MGF

$$M_x(t) = E\left[e^{t\lambda}\right] = e^{\lambda(e^t - 1)}$$

3. Characteristic function.

$$\phi_x(t) = E\left[e^{itx}\right] = e^{\lambda(e^{t} \pm 1)}$$

te:

In ~~possi~~ poisson's ~~equ~~ distribution.

Mean = Variance = parameter λ.

It is always +vely skewed.

Sum of the independant poisson r.v's is also a oisson's r.v.

Difference b/w the independant poisson's r.v's is not a poisson random variable.

Q) A telephone switch board recieves 20 calls on an avg during an hour. Find the probability for a period of 5 min,

(i) No call is recieved.

(ii) Exactly 3 calls are recieved.

(iii) Atleast 2 calls are recieved.

$\left.\begin{array}{l}\text{reach}\\ \text{arrive}\\ \text{error}\\ \text{defect}\end{array}\right\}$ go for poisson's distributn.

Soln

avg = λ (for 60 min)

For 60 min $\rightarrow \lambda = 20$ calls

For 1 min $\rightarrow \frac{20}{60} = \frac{1}{3} = \lambda$

For 5 min = $\frac{1}{3} \times 5 = 1.65 = \lambda$

(i) $P(X=0) = \frac{e^{-1.65} (1.65)}{0!}$

$= e^{-1.65}$

(ii) $P(X=3) = \frac{e^{-1.65} (1.65)^3}{3!}$

(iii) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)] = 1 - \left[e^{-1.65} + \frac{1.65\, e^{-1.65}}{1!} \right]$

2, If X is a poisson r.v, then Find the Value of

$$\sum_{x=0}^{\infty} \left(\frac{x}{\lambda}\right) \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$\cancel{\frac{1}{\lambda} \times \lambda =}$$

$$\frac{1}{\lambda} \sum_{x=0}^{\infty} x \left(\frac{e^{-\lambda} \lambda^{x}}{x!}\right)$$

$$\frac{1}{\lambda} \times E(X)$$

$$= \frac{1}{\lambda} \times \lambda = 1$$

NORMAL DISTRIBUTION (GAUSSIAN)

$$\boxed{N(x;\mu,\sigma^2) = f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ 0 < \sigma < \infty \end{array} \\ 0, \text{ otherwise.} \end{cases}}$$

Definition: If X is said to be a normal r.v defined in the interval $-\infty < x < \infty$ with mean equal to μ and variance is equal to σ^2, Then the r.v is known as normal r.v. And its density function is

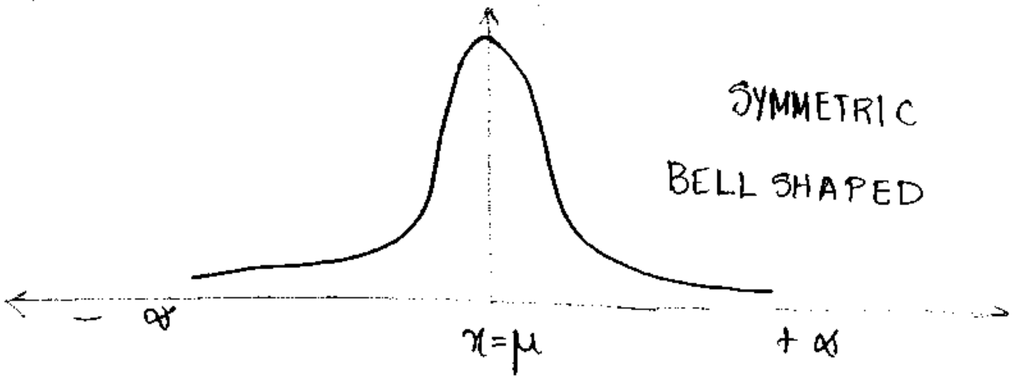
$$N(x;\mu,\sigma^2) = f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ 0 < \sigma < \infty \end{array} \\ 0, \text{ otherwise.} \end{cases}$$

STANDARD NORMAL RANDOM VARIABLE

If X is a normal random variable with mean $= 0$ and variance $= 1$, then the random variable is known as standard normal random variable. It density function is

$$\boxed{N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}$$

Mathematically a standard normal r.v is denoted by Z and is equal to $Z = \frac{X - E(X)}{\sqrt{V(X)}}$ & $\boxed{-3 \leq Z \leq +3}$

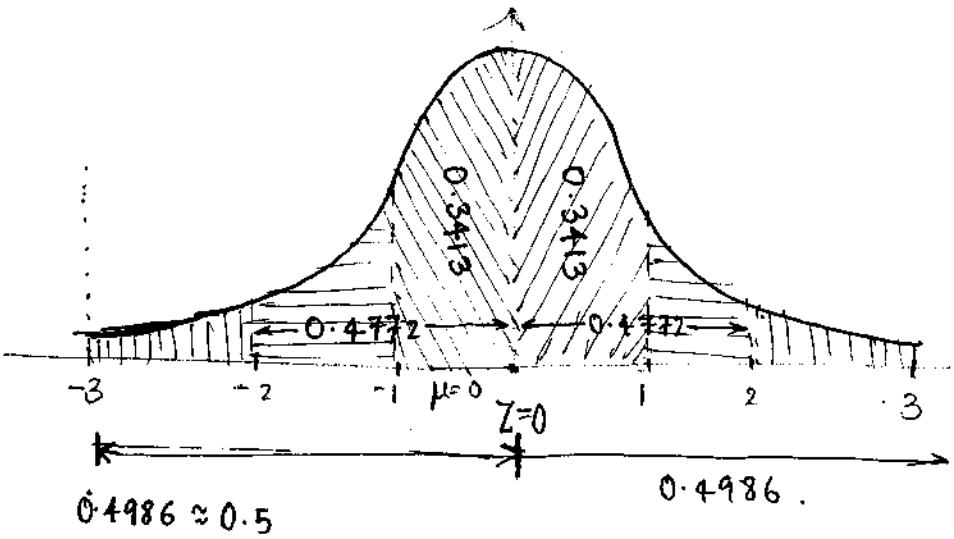


SYMMETRIC

BELL SHAPED

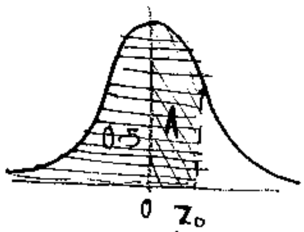
Normal distibution.

Standard Normal Distribution



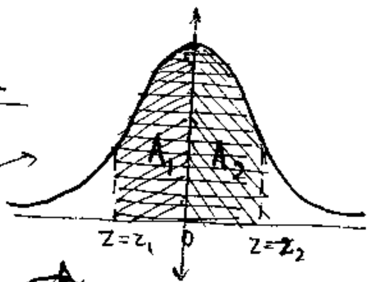
Areas under Normal curve

$P(Z \leq Z_0) = 0.5 + A \quad (Z_0 \text{ +ve})$.

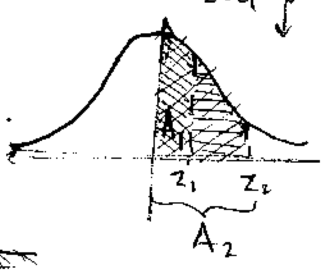


$P(Z \leq Z_0) = 0.5 - A \quad (Z_0 \text{ -ve})$.

$P(Z_1 \leq Z \leq Z_2) = A_1 + A_2 \quad (Z_1 \text{ -ve and } Z_2 \text{ +ve})$



$P(Z_1 \leq Z \leq Z_2) = A_2 - A_2 \quad (Z_1 \text{ and } Z_2 \text{ +ve/-ve})$.



$P(Z \geq Z_0) = 0.5 + A \quad (Z_0 \text{ -ve})$

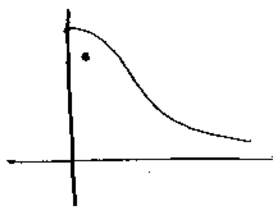
$P(\cancel{Z \geq Z_0}) = \cancel{0.5 - A} \quad (Z_0 \text{ +ve})$

Q, If X is normally distributed with mean $= 20$ and std deviation 3.33. Find the probability b/w $P(21.11 \le x \le 26.66)$. The area under the curve,

$Z_0 = 0$ to $Z = 0.33$ is 0.1293.

$$Z_1 = \frac{X_1 - \mu}{\sigma} \qquad Z_2 = \frac{X_2 - \mu}{\sigma}$$

$$Z_1 = \frac{1}{3} = 0.33 \qquad Z_2 = 2$$



$$P(21.11 \le x \le 26.66) = P(0.33 < Z < 2)$$

$$= 0.4772 - 0.1293$$

$$= \underline{\underline{0.3479}}$$

Q, If x is normally distributed with mean $= 30$ and std deviation is 5. Find $P(|x-30| > 5)$

$P(|X-30| > 5) = \cancel{P(|X|)}$
$1 - P(|x-30| < 5)$

$$\Rightarrow \cancel{(x-30)} \; -5 < x - 30 < 5$$

$$P(25 < x < 35)$$

$$Z_1 = \frac{25 - 30}{5} = \frac{-5}{5} = \underline{\underline{-1}}$$

$$Z_2 = \frac{35 - 30}{5} = \frac{5}{5} = \underline{\underline{1}}$$

$$P(-1 < Z < 1) = A_1 + A_2 = \underline{\underline{0.6826}}$$

2. A die is rolled 180 times. Using the normal distribution, find the probability that the face 4 will turn up atleast 35 times.

$$P(X \geq 35) = ?$$

We can use Binomial Distribution with $n = 180$, $p = 1/6$, $q = 5/6$.

$$E(x) = np = \frac{180}{6} = 30$$

$$V(x) = npq = \frac{180}{6} \times \frac{5}{6} = 25$$

$$Z = \frac{X - \mu}{\sigma} = \frac{35 - 30}{5}$$

$$= \underline{\underline{1}}$$

$$P(Z \geq 1) = 0.5 - 0.3413$$

$$= \underline{\underline{0.1587}}$$

Properties

$E(X) = \text{Mean} = \mu$

$V(X) = \text{Variance} = \mu_2 = \sigma^2$

$\mu_3 = 0 \Rightarrow \beta_1 = 0 \Rightarrow$ Symmetry.

MGF

$M_X(t) = e^{\left(\quad\right)}$

(iv) characteristic

$$\phi_X(t) = e^{\left(it\mu - \frac{t^2\sigma^2}{2}\right)}$$

Properties of std Normal distribution.

→ $X \sim N(0,1)$

→ $E(X) = 0$

→ $V(X) = 1$

→ $M_X(t) = e^{t^2/2}$

→ $\phi_X(t) = e^{-t^2/2}$

→ Sum of the Independent normal r.variable is also a Normal random variable,

→ The difference the independent normal r.v s is also a normal random variable (Linear combination).

UNIFORM DISTRIBUTION [RECTANGULAR]

Definition: If X is a uniform r.v in the interval $a < x < b$ $(a<b)$ and it probabability density function is

$$\boxed{f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ & a<b \\ 0 & \text{otherwise} \end{cases}}$$

MEAN $= \boxed{E(X) = \frac{a+b}{2}}$

VARIANCE $\boxed{V(X) = \frac{(b-a)^2}{12}}$

If X is uniform random variable in the interval $-a < x < a$ and its density function is

$$\boxed{f(x) = \frac{1}{2a}}$$

$$\boxed{\text{Mean} = 0}$$

$$\boxed{\text{Variance} = \frac{a^2}{3}}$$

The shape of uniform curve is rectangular.

CORRELATION / REGRESSION

$\hat{A} = 0.7461$
$\hat{B} = 1.1154$

CORRELATION

Karl pearson's Correlation.

The relation b/w the two dimensional r.v in bivariate data is known as regression (The degree of relation b/w the two variables is known as ~~regression~~ correlation.)

Types of correlation (i) Positive Correlation.

(ii) Negative Correlation.

Positive Correlation: If the changes in the both the variables are in the same direction (increasing or decreasing) Then those variables are known as positively correlated variables.

Negative Correlation: If the changes in the one variable is affecting the changes of other variable in reverse direction, then those variables are known as negatively correlated.

Karl pearson's Correlation equ.

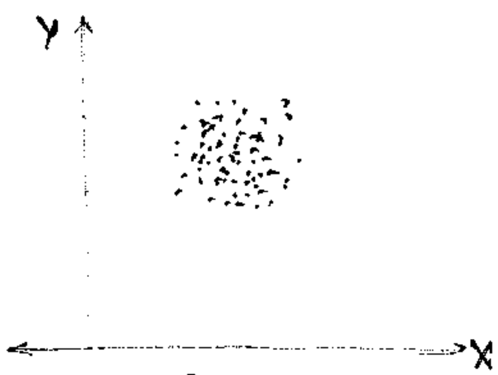
$$r(x,y) = \frac{Cov(x,y)}{\sqrt{x}\sqrt{y}} \quad \text{where } Cov(x,y) = \frac{1}{n}\sum xy - \bar{x}\cdot\bar{y}$$

$$-1 \leq r \leq 1$$

SCATTER DIAGRAM

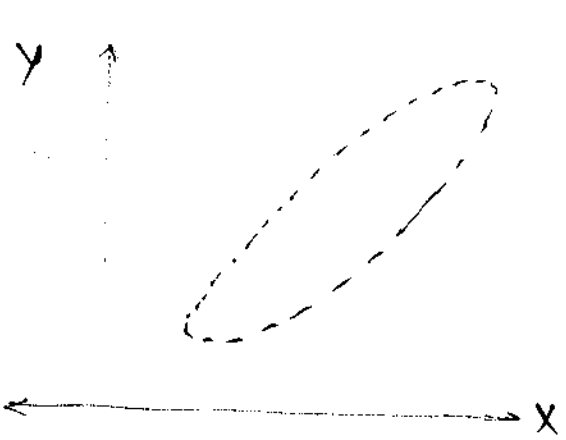
It is a graphical representation of correlation. If the points are very close or thick on the X Y plane then those points are correlated points.

If the points are widely spreaded, then they are said to be uncorrelated.

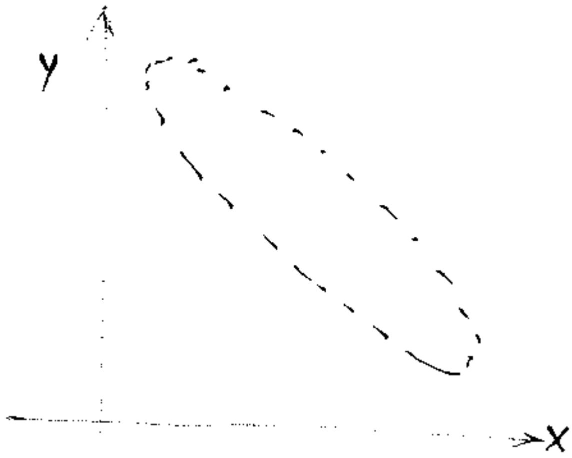


CORRELATED

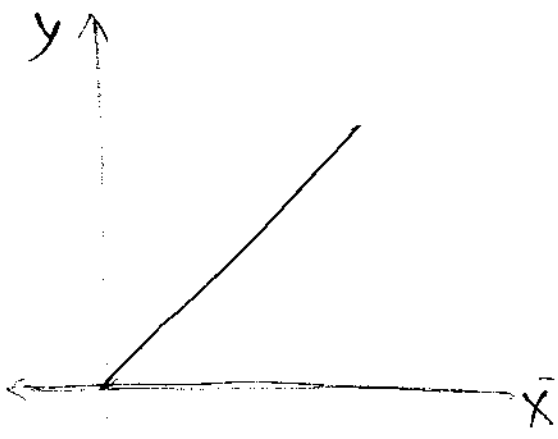
UNCORRELATED



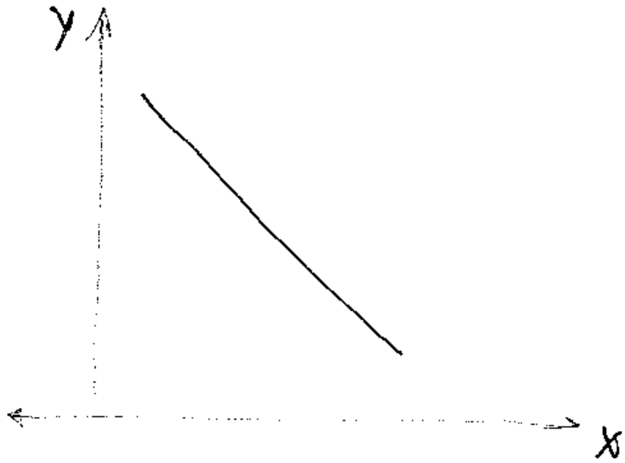
POSITIVELY CORRELATED



NEGATIVELY CORRELATED



PERFECT POSITIVELY



PERFECT NEGATIVELY

Note: → If X & Y are independent r.v's then covariance is zero.

i.e. $\boxed{\text{Cov}(X,Y) = 0 \implies r(X,Y) = 0}$

But converse of the statement is not true.

⇒ Correlation coefficient is independent of origin as well independent of change of scale.

REGRESSION

REGRESSION → Y ON X / X ON Y (Simple regression)

Definition: The linear relationship b/w the 2-D random variable is known as regression.

LINES OF REGRESSION

$$Y - \bar{Y} = r \cdot \frac{\sigma_y}{\sigma_x}(X - \bar{X}) \qquad \text{Y ON X}$$

$$X - \bar{X} = r \cdot \frac{\sigma_x}{\sigma_y}(Y - \bar{Y}) \qquad \text{X ON Y}$$

where
$$\left.\begin{aligned} r \cdot \frac{\sigma_y}{\sigma_x} &= b_{YX} \\ \& \quad r \cdot \frac{\sigma_x}{\sigma_y} &= b_{XY} \end{aligned}\right\} \text{Regression Coefficients.}$$

→ Correlation coefficient is the Geometric mean b/w regression coefficients.

$$\boxed{r = \pm\sqrt{b_{yx}\, b_{xy}}}$$

Note: Both the regression coefficients must have a same sign.

ie, If both +ve $\Longrightarrow$ r is +ve.

If both −ve $\Longrightarrow$ r is −ve

If $b_{yx} > 1 \Rightarrow b_{xy} < 1$ (Vice versa).

If the regression coefficients are equal, their variances also equal.

$\boxed{b_{xy} = b_{yx}} \Rightarrow r \cdot \frac{\sigma_x}{\sigma_y} = r \cdot \frac{\sigma_y}{\sigma_x}$

$$\Rightarrow \boxed{\sigma_y^2 = \sigma_x^2}$$

Regression equations are passes through the point $\bar{X}, \bar{Y}$

Regression Coefficient is independent of change of origin as well as dependent of change of scale.

Angle b/w Regression lines

$$\boxed{\theta = \tan^{-1}\left(\frac{1-r^2}{|r|} \cdot \frac{\sigma_x \sigma_y}{(\sigma_x)^2 + (\sigma_y)^2}\right)}$$

$r = 0 \Rightarrow \theta = \pi/2$

$r = 1 \Rightarrow \theta = 0$ or π

Q, The regression equation are $3x + 2y = 1$.
$2x + 4y = 0$

(i) Find r?

(ii) $\overline{X}, \overline{Y}$

whichever the coefficient in the expression is higher then it is the DEPENDENT VARIABLE.

consider $\uparrow 3X + 2Y = 1$

Dependent variable

$\therefore$ X on Y

X on Y	Y on X
$3X+2Y=1$	$2x+4y=0$
$3X = 1-2Y$	$4Y = -2X$
$X = 1/3 - 2/3\,Y$	$Y = -2/4\,X$
$b_{xy} = -2/3$	$b_{yx} = -1/2$

— Both -ve Hence r also -ve.

$r = \sqrt{\cdot 2/3 \times 1/2}$

$r = -\sqrt{1/3}$

$r = -1/\sqrt{3}$

(ii) Means $\bar{x}$ & $\bar{y}$ satisfies eqn.

$6\bar{x} + 4\bar{y} = 2$

$2\bar{x} + 4\bar{y} = 0$

$4\bar{x} = 2$

$\bar{x} = 1/2$

$\bar{y} = -1/4$

$\therefore$ Means $(\bar{x}, \bar{y}) = (1/2, -1/4)$

; $x - 2y = 2$

$3x - y = 1$

(i) r (ii) $\bar{x}, \bar{y}$

Y on X

$x - 2y = 2$

$x - 2 = 2y$

X on Y

$3x = 1 + y$

$x = 1/3 + y/3$

$$r = \sqrt{\tfrac{1}{2} \times \tfrac{1}{3}}$$

$$r = \frac{1}{\sqrt{6}}$$

(ii) $\bar{x} - 2\bar{y} = 2$

$3\bar{x} - \bar{y} = 1$

$3\bar{x} - 6\bar{y} = 6$

$3\bar{x} - \bar{y} = 1$

$-5\bar{y} = 5$

$\bar{y} = -1$

$3\bar{x} + \cancel{1} = \cancel{1}$

$\bar{x} = 0$

$(\bar{x}, \bar{y}) = (0, -1)$