GATE Revision Series 2023

Aditya Vangala (M.Tech (Res) IISc Bangalore)
Faculty at Ace Engineering Academy
Subjects: Engineering Maths for all branches and Signals and Systems for ECE, EE and IN.
Email Id: adityasrnvs@gmail.com
Telegram Id: adityasrnvs

Aditya Vangala

ACE

# Linear Algebra

C S E

L U

1. Basics
2. Determinants
3. Rank
4. System of linear eq$^{ns}$
5. eigen values and eigen values

ECE

vector space.

Aditya Vangala

ACE

$A_{m \times n}$ , $B_{n \times p}$

Total multiplications $= m \cdot n \cdot p$

Total Additions $= m(n-1)p$

Aditya Vangala

ACE

Idempotent matrix : $A^2 = A$

Involutary matrix $A^2 = I$

Nilpotent matrix $N^P = O$

Orthogonal matrix $A \cdot A^T = A^T \cdot A = I$ or $A^T = A^{-1}$

Sym matrix $A^T = A$ or $a_{ij} = a_{ji}$

Skew Sym $A^T = -A$

Hermitian $\bar{A}^T = A$

Skew Herm $\bar{A}^T = -A$

Aditya Vangala

ACE

Identity matrix :- $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

upper triangular $\begin{pmatrix} u_{11} & u_{12} & u_{12} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

Lower $\begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix}$

Diagonal

$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$

Aditya Vangala

ACE

## Determinants :-

$$\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = a(ez - fy) - b(dz - fx) + c(dy - ex)$$

Aditya Vangala

ACE

# Elementary Row operations :-

Type 1 : Interchange Row $i$ and $j$

$$|B| = -|A|$$

Type 2 : $R_i \to \alpha R_i$ , $\alpha \neq 0$

$$|B| = \alpha|A|$$

$C = \alpha A$

$|C| = \alpha^n |A|$

Type 3 : $R_i \to R_i + \alpha R_j$

$$|B| = |A|$$

$R_1 \to 2R_1 + R_2$ ✗

$R_1 \to R_1 + 2R_2$ ✓

Aditya Vangala

ACE

Since 1995

operations that do not change Det

$R_1 \to R_1 + 2R_2$

$R_1 \to R_1 + R_2 + R_3 + \cdots + R_n$

Operations that change the Det value

$R_1 \to 2R_1 + R_2$

Aditya Vangala

ACE

1. $|A \cdot B| = |A| \cdot |B|$

2. $|A^2| = |A|^2$

3. $|A^n| = |A|^n$

4. $|A+B| \neq |A| + |B|$

5. $\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 0 \quad \begin{vmatrix} a & b & c \\ d & e & f \\ \alpha a & \alpha b & \alpha c \end{vmatrix} = 0 \quad \begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{vmatrix} = 0$

Aditya Vangala

ACE

A is skew sym matrix of odd order $|A| = 0$

$$\begin{vmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{vmatrix} = u_{11} \cdot u_{22} \cdot u_{33}$$

Aditya Vangala

ACE

Rank :-

1. Convert the given matrix to row echelon form.

2. Rank is n.o of non-zero rows.

Elementary row operations do not change rank of the matrix.

Aditya Vangala

ACE

Linearly independent and dependent vectors :-

$|v_1 \; v_2 \; v_3 \cdots v_n| \neq 0$ L.I

$|v_1 \; v_2 \; v_3 \cdots v_n| = 0$ L.D

System of Linear eqns :-



Homogeneous sys

$Ax = 0$

Trivial solution

$R(A) = n$

Non-trivial Solution

$R(A) < n$

$n$ is number of variables.

Aditya Vangala

ACE

Non-Homogeneous sys

$Ax = b$

No solution

$R(A|b) \neq R(A)$

unique solution

$R(A|b) = R(A) = n$

Infinitely many solution

$R(A|b) = R(A) < n$

# Eigen Values and Eigen vectors :-

Aditya Vangala

ACE

$|A - \lambda I| = 0$

$(A - \lambda I)X = 0$

$$\boxed{A \cdot X = \lambda \cdot X}$$

$\lambda \longrightarrow A$

$\lambda^2 \longrightarrow A^2$

$\lambda^m \longrightarrow A^m$

$f(\lambda) \longrightarrow f(A)$

$a_2\lambda^2 + a_1\lambda + a_0 \longrightarrow a_2A^2 + a_1A + a_0I$

$\frac{1}{\lambda} \longrightarrow A^{-1}$

$\frac{|A|}{\lambda} \longrightarrow \text{Adj } A$

Aditya Vangala

ACE

Trace(A) = Sum of eigen values

|A| = Product of eigen values.

Real Sym matrix ⇒ All eigen values are real

Skew Sym ⇒ Eigen values are either 0 or Purely Imaginary

Diagonal, triangular matrix ; eigen values are principal diagonal itself.

Aditya Vangala

ACE Since 1995

Sym matrix :-

$\lambda_1 , \lambda_2$

$e_1 , e_2$

$e_1^T \cdot e_2 = 0$

Cayley Hamilton theorem :-

$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

$|A - \lambda I| = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$A^2 - 5A + 4I = O$

$A^2$

1. $5A - 4I$
2. $5A + 4I$

⋮

Aditya Vangala



$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\lambda = 2, 3, 7$

$A.m(\lambda = 2) = 1$

$A.m(\lambda = 3) = 1$

$A.m(\lambda = 7) = 1$

$\lambda = 1, 1, 1$

$A.m(\lambda = 1) = 3$

Aditya Vangala

ACE

$G.M = n - R(A - \lambda I)$

$=$ Number of L.I eigen vectors of A corresponding eigen value $\lambda$

If $A = \begin{bmatrix} 10 & 2k+5 \\ 3k-3 & k+5 \end{bmatrix}$ is a symmetric matrix, the value of k is .

Aditya Vangala

ACE

(GATE-2022)

(A) 8

(B) 5

(C) -0.4

(D) $\frac{1+\sqrt{1561}}{12}$

$A^T = A$

$A^T = \begin{pmatrix} 10 & 3k-3 \\ 2k+5 & k+5 \end{pmatrix}$

$2k+5 = 3k-3$

$5+3 = 3k-2k$

$k = 8$

The two vectors [ 1 1 1 ] and [ 1 $a$ $a^2$ ] where $a = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$ and $j = \sqrt{-1}$ are

$V_1$ $V_2$ $\omega$

Aditya Vangala

ACE

(GATE-2011)

(a) Orthonormal

(b) Orthogonal

(c) Parallel

(d) Collinear

$1, \omega, \omega^2$ are cube roots of unity.

$1 + \omega + \omega^2 = 0$

$V_1 \cdot V_2 = 1.1 + 1.a + 1.a^2$

$= 1 + a + a^2$

$= 0$

Dot product $= 0$

$|V_1| = \sqrt{1^2 + 1^2 + 1^2}$

$= \sqrt{3}$

Euclidean norm (length) of the vector $[4 \quad -2 \quad -6]^T$ is

Aditya Vangala

ACE

GATE-2019

(a) $\sqrt{56}$

(b) $\sqrt{24}$

(c) $\sqrt{48}$

(d) $\sqrt{12}$

$$\|V\| = \sqrt{4^2 + (-2)^2 + (-6)^2}$$

$$= \sqrt{16 + 4 + 36}$$

$$= \sqrt{56}$$

Aditya Vangala

ACE

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_1 \to R_1 + R_2 + R_3 + R_4$

$$\begin{vmatrix} 5 & 5 & 5 & 5 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$$5\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$5\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$= 5(1 \times 1 \times 1 \times 1)$

$= \underline{\underline{5}}$

Consider the matrix

$$J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$I_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Aditya Vangala

ACE

which is obtained by reversing the order of columns of the identity matrix $I_6$. Let $P = I_6 + \alpha J_6$ where $\alpha$ is a non-negative real number. The value of $\alpha$ for which $|P| = 0$ is (GATE-14)

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For $\alpha = 1$

$|P| = 0$

Aditya Vangala

ACE

The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has $|A| = 100$ and $trace(A) = 14$. The value of $|a - b|$ is

GATE-2016

$$|A| = 0(\;) + 0(\;) + 0(\;) + b(-1)^{4+4} \begin{vmatrix} a & 0 & 3 \\ 2 & 5 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 100$$

$$b\left(a(10) - 0(\;) + 3(0)\right) = 100$$

$$10ab = 100$$

$$ab = 10$$

$$\text{Trace}(A) = 14$$

$$a + 5 + 2 + b = 14$$

$$a + b = 7$$

$$ab = 10$$

$$(a-b)^2 = (a+b)^2 - 4ab$$

$$= 7^2 - 4(10)$$

$$= 49 - 40$$

$$(a-b)^2 = 9$$

$$a - b = \pm 3$$

$$|a-b| = 3$$

The inverse of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ is

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

Aditya Vangala

ACE

GATE-2019

(a) $\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & \frac{-4}{5} & \frac{-9}{5} \\ -3 & \frac{4}{5} & \frac{14}{5} \\ 1 & \frac{-1}{5} & \frac{-6}{5} \end{bmatrix}$

(c) $\begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & \frac{-4}{5} & \frac{-14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}$ (d) $\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$

$A \cdot A^{-1} = I$ $A^{-1} \cdot A = I$

Aditya Vangala

ACE

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix} =$$

$-20 + 45 - 20 = 5$ ✗

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & \frac{-4}{5} & \frac{-14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} =$$

$-4 + 9 - 4 = 1$ ✓

The rank of the matrix is $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Aditya Vangala

ACE

2 min

GATE-17

$R_4 \to R_4 + R_1 + R_2 + R_3 + R_5$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$R_2 \longleftrightarrow R_3$, $R_4 \longleftrightarrow R_5$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(A) = 4$

Aditya Vangala

ACE

$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$

If the rank of the matrix $A$ is $N$ then the rank of matrix $B$ is

GATE-2014

(a) $\frac{N}{2}$

(b) $N-1$

(c) $N$

(d) $2N$

$B = A \cdot A^T$

$B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{pmatrix}$

$R(A \cdot A^T) = R(A) = N$

$X = [x_1 \ x_2 \ \cdots \ x_n]^T$ is an $n$-tuple non-zero vector.
The $n \times n$ matrix $V = XX^T$

Aditya Vangala

ACE

GATE-2007

(a) has rank zero

(b) has rank 1

(c) is orthogonal

(d) has rank n

If the vectors (1,-1,2), (7,3,x) and (2,3,1) in $R^3$ are linearly dependent then $x = ?$ GATE-21

Aditya Vangala

ACE

$$\begin{vmatrix} 1 & -1 & 2 \\ 7 & 3 & x \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$1(3-3x)+1(7-2x)+2(21-6)=0$$

$$3-3x+7-2x+30=0$$

$$40=5x$$

$$x=8$$

Aditya Vangala

ACE

Given $M = \begin{bmatrix} 2 & 3 & 7 \\ 6 & 4 & 7 \\ 4 & 6 & 14 \end{bmatrix}$, which of the following statement(s) is/are correct?

GATE-2022(MSQ)

(A) The rank of $M$ is 2 True

(B) The rank of $M$ is 3 False

(C) The rows of $M$ are linearly independent False

(D) The determinant of $M$ is 0 True

$R_3 = 2R_1$

$|M| = 0$

$R(M) < 3$

$\begin{vmatrix} 2 & 3 \\ 6 & 4 \end{vmatrix} \neq 0$

$R(M) = 2$

(A, D)

Consider the following system of equation

$$3x + 2y = 1$$
$$4x + 7z = 1$$
$$x + y + z = 3$$
$$x - 2y + 7z = 0$$

Which of the following is true? GATE-14

(a) The system has no solution

(b) The system has infinitely many solutions

(c) The system has unique solution

(d) The rank of the augmented matrix of the system is 2

Aditya Vangala

ACE

x, y and z are the 3 variables

$n = 3$

$$(A|b) = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_3 \to R_3 - 4R_2$

$R_4 \to R_4 - 3R_2$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 15 & 21 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(A|b) = 3$

$R(A) = 3$

$n = 3$

$R(A|b) = R(A) = n$

unique solution

The system of linear equations in real $(x, y)$ given by

$$[x \;\; y]\begin{bmatrix} 2 & 5-2\alpha \\ \alpha & 1 \end{bmatrix} = [0 \;\; 0]$$

involves a real parameter $\alpha$ and has infinitely many non-trivial solutions for special value(s) of $\alpha$. Which one or more among the following options is/are non-trivial solution(s) of $(x, y)$ for such special value(s) of $\alpha$? GATE-2022(MSQ)

(a) $x = 2, y = -2$ $\Rightarrow x = -y$

(b) $x = -1, y = 4$ $\Rightarrow x = -\frac{1}{4}y$

(c) $x = 1, y = 1$

(d) $x = 4, y = -2$

Aditya Vangala

Since 1995

ACE

$$[x \;\; y]\begin{bmatrix} 2 & 5-2\alpha \\ \alpha & 1 \end{bmatrix} = (0 \;\; 0)$$

$$2x + \alpha y = 0$$

$$(5-2\alpha)x + y = 0$$

$$\begin{pmatrix} 2 & \alpha \\ 5-2\alpha & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$AX = 0$$

Aditya Vangala

ACE

Since 1995

$$\begin{pmatrix} 2 & \alpha \\ 5-2\alpha & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$AX = 0$$

$$r(A) < n$$

$$r(A) < 2$$

$$|A| = 0$$

$$2 - \alpha(5-2\alpha) = 0$$

$$2 - 5\alpha + 2\alpha^2 = 0$$

Roots $\alpha = \frac{1}{2}, 2$

$$AX = 0$$

1 $\alpha = 1/2$

$$\begin{pmatrix} 2 & 1/2 \\ 4 & 1 \end{pmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\begin{pmatrix} 2 & 1/2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x + \frac{1}{2}y = 0$$

$$2x = -\frac{1}{2}y$$

$$x = -\frac{1}{4}y$$

Aditya Vangala

ACE

Since 1995

② $\lambda = 2$

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1$ ✓

$$\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2x + 2y = 0$

$x = -y$

Consider solving the following system of simultaneous equations using LU decomposition.

$$x_1 + x_2 - 2x_3 = 4$$

$$x_1 + 3x_2 - x_3 = 7$$

$$2x_1 + x_2 - 5x_3 = 7$$

where L and U are denoted as

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Which one of the following is the correct combination of values for $L_{32}$, $U_{33}$, and $x_1$ ?

(a) $L_{32} = 2, U_{33} = \frac{-1}{2}, x_1 = -1$

(b) $L_{32} = 2, U_{33} = 2, x_1 = -1$

(c) $L_{32} = \frac{-1}{2}, U_{33} = 2, x_1 = 0$

(d) $L_{32} = \frac{-1}{2}, U_{33} = \frac{-1}{2}, x_1 = 0$

GATE-2022

Aditya Vangala

ACE

Which of the following is/are the eigenvector(s) for the matrix given below?

Aditya Vangala

ACE

$$\begin{bmatrix} -9 & -6 & -2 & -4 \\ -8 & -6 & -3 & -1 \\ 20 & 15 & 8 & 5 \\ 32 & 21 & 7 & 12 \end{bmatrix}$$

GATE-2022(MSQ)

(a) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$

$AX = \lambda X$

$$\begin{bmatrix} -9 & -6 & -2 & -4 \\ -8 & -6 & -3 & -1 \\ 20 & 15 & 8 & 5 \\ 32 & 21 & 7 & 12 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 9 - 6 - 4 \\ 8 - 6 - 1 \\ -20 + 15 + 5 \\ -32 + 21 + 12 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$AX = 1 \cdot X$

# Probability

$$P(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } S}$$

① $0 \leq P(A) \leq 1$

② $P(S) = 1$

③ $P(A \cup B) = P(A) + P(B) \quad (A \cap B = \phi)$

Aditya Vangala

ACE

$P(A^c) = 1 - P(A)$ ( At least )

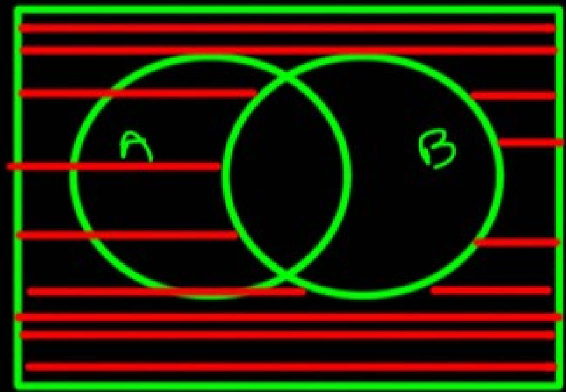
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$
$- P(A \cap B) - P(B \cap C) - P(C \cap A)$
$+ P(A \cap B \cap C)$

Aditya Vangala

ACE

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$




$\leftarrow B^c$

Aditya Vangala

ACE

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

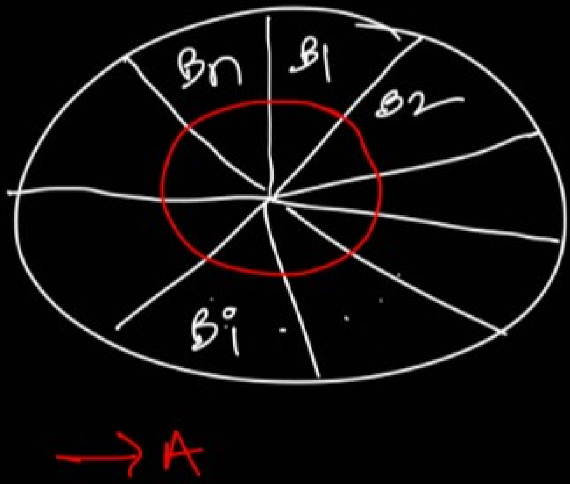
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

If A and B are mutually exclusive

$P(A \cap B) = 0 \qquad P(A|B) = 0 \qquad P(B|A) = 0$

If A and B are independent

$P(A|B) = P(A) \qquad P(B|A) = P(B)$




① $B_1, B_2 \ldots B_n$ are mutually exclusive

② $B_1 \cup B_2 \ldots \cup B_n = S$

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_n)P(B_n)$$

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_n)P(B_n)}$$

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_n)P(B_n)}$$

A box contains the following three coins.

I. A fair coin with head on one face and tail on the other face

II. A coin with heads on both faces.

III. A coin with tails on both faces

A coin picked randomly from the box and tossed. Out of the two remaining coins in the box, one coin is then picked randomly and tossed. If the first toss results in a head, the probability of getting a head in the second toss is GATE-2021

A cab was involved in a hit and run accident at night. You are given the following data about the cabs in the city and the accident.

(i) 85% of cabs in the city are green and the remaining cabs are blue.

(ii) a witness identified the cab involved in the accident as blue.

(iii) it is known that a witness can correctly identify the cab colour only 80% of the time.

Which of the following options is closest to the probability that the accident was caused by a blue cab? (GATE 2018)

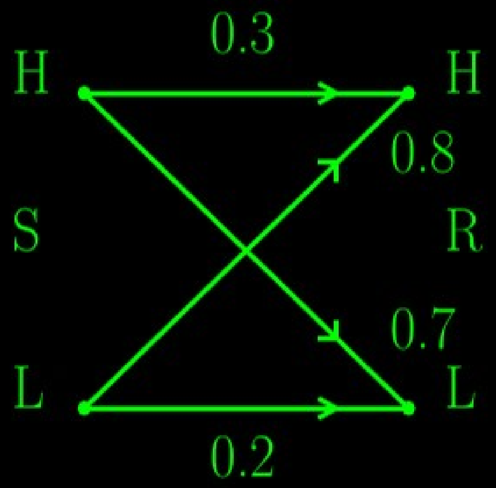
(a) 12% (b) 15% (c) 41% (d) 80%

Aditya Vangala

ACE

A Sender (S) transmits a signal, which can be one of the two kinds: H and L with probabilities 0.1 and 0.9 respectively, to a receiver (R). In the graph below, the weight edge $(u, v)$ is the probability of receiving $v$ when $u$ is transmitted, where $u, v \in \{H, L\}$. For example, the probability that the received signal is L given the transmitted signal was H is 0.7. If the received signal is H, the probability that the transmitted signal was H is

(GATE-2021)




$P(H) = 0.1 \quad P(L) = 0.9$

$P(Rx = L \mid Tx = H) = 0.7$

$P(Rx = H \mid Tx = L) = 0.8$

$P(Rx = H \mid Tx = H) = 0.3$

$P(Rx = L \mid Tx = L) = 0.2$

$$P(Tx = H \mid Rx = H) = \frac{(0.1)(0.3)}{(0.1)(0.3) + 0.9(0.8)}$$

$=$

An urn contains 5 red and 7 green balls. A ball is drawn at random and its colour is noted. The ball is placed back into the urn along with another ball of the same colour. The probability of getting a red ball in the next draw is GATE-16

Aditya Vangala

ACE

(a) $\frac{5}{12}$

(b) $\frac{5}{7}$

(c) $\frac{6}{12}$

(d) $\frac{8}{12}$

$5R + 7G$

$$P(R) = \frac{5}{5+7} = \frac{5}{12}$$

$$P(G) = \frac{7}{5+7} = \frac{7}{12}$$

A bag has r red balls and b black balls. All balls are identical except for their colours. In a trial, a ball is randomly drawn from the bag, its colour is noted and the ball is placed back into the bag along with another ball of the same colour. Note that the number of balls in the bag will increase by one, after the trial. A sequence of four such trials is conducted. Which one of the following choices gives the probability of drawing a red ball in the fourth trial? GATE-2021

(a) $\left(\frac{r}{r+b}\right)\left(\frac{r+1}{r+b+1}\right)\left(\frac{r+2}{r+b+2}\right)\left(\frac{r+3}{r+b+3}\right)$

(b) $\frac{r}{r+b}$

(c) $\frac{r}{r+b+3}$

(d) $\frac{r+3}{r+b+3}$

Aditya Vangala

ACE

$$P(R) = \frac{r}{r+b}$$

Two players A and B alternately keep rolling a fair die. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is GATE-15

Aditya Vangala

ACE

(a) $\frac{5}{11}$

(b) $\frac{1}{2}$

(c) $\frac{7}{13}$

(d) $\frac{6}{11}$

$P(6) = 1/6 \qquad P(6^c) = 5/6$

Break upto 11.20 am

$6$   $1/6$

$A$

$6^c \; 6^c \; 6 \qquad (5/6)(5/6)(1/6)$

$A \quad B \quad A$

$6^c \; 6^c \; 6^c \; 6^c \; 6 \qquad (5/6)^4 \cdot (1/6)$

$A \quad B \quad A \quad B \quad A$

Aditya Vangala

A fair coin is tossed till a head appears for the first time. The probability that the number of required tosses is odd, is GATE-12

(a) $\frac{1}{3}$

(b) $\frac{1}{2}$

(c) $\frac{2}{3}$

(d) $\frac{3}{4}$

Aditya Vangala



## Random variables :-

$f(x) = P(X = x)$ PMF

$$F(x) = P(X \leq x) = \sum_{x_i = -\infty}^{x} f(x_i) \quad \text{CDF}$$

$$= \sum_{x_i = -\infty}^{x} P(X = x_i)$$

D.R.V

C.R.V

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$f(x) = \frac{d}{dx} F(x)$$

$$P(X > x) = \int_{x}^{\infty} f(x)\,dx$$

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx = P(x_1 < X < x_2)$$

$$E[X] = \sum_{\forall x} x \, p(X=x)$$

$$E[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx$$

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x) \, dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx$$

$$E[aX] = a\,E[X]$$

$$E(b) = b$$

$$E[aX+b) = a\,E(X) + b$$

$$E[X+Y) = E(X) + E(Y)$$

Aditya Vangala

ACE

$$Var(X) = E\left[(X - E(X))^2\right]$$

$$= E(X^2) - (E(X))^2$$

$$Var(aX) = a^2 \, Var(X)$$

$$Var(b) = 0$$

$$Var(aX + b) = a^2 \, Var(X)$$

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$$

$$Cov(X,Y) = E[XY] - E(X) \cdot E(Y)$$

Aditya Vangala

ACE

If $X$ and $Y$ are independent

$$Cov(X, Y) = 0$$

$$E(XY) = E(X) \cdot E(Y)$$

$$Var(X+Y) = Var(X) + Var(Y)$$

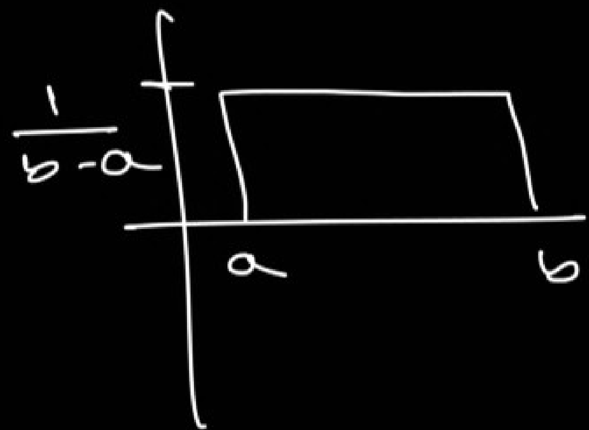
$$Var(X-Y) = Var(X) + Var(Y)$$

Aditya Vangala

ACE

Uniform R.V

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$




$$E(x) = \frac{b+a}{2}$$

$$E(x^2) = \frac{a^2+ab+b^2}{3}$$

$$Var(x) = \frac{(b-a)^2}{12}$$

# Exponential R.V

Aditya Vangala

ACE

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P(X > x) = e^{-\lambda x} \longrightarrow P(X > 2) = e^{-2\lambda}$$

$\rightarrow E(X) = \dfrac{1}{\lambda}$ $\quad E(X^2) = \dfrac{2}{\lambda^2}$

$\rightarrow Var(X) = \dfrac{1}{\lambda^2}$

# Normal or Gaussian R.V

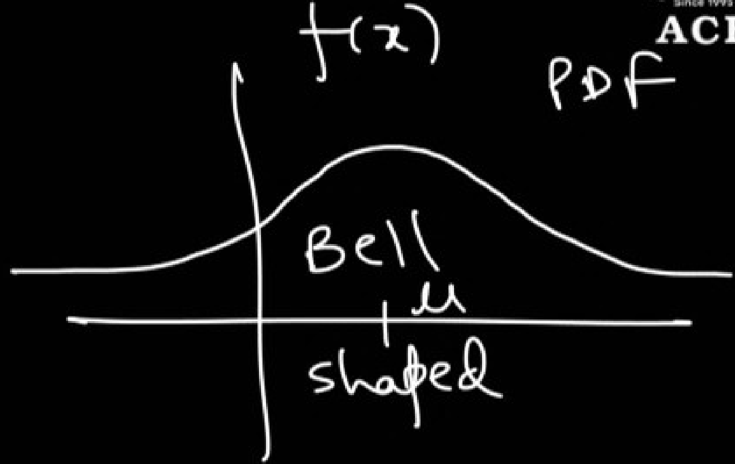
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$E(x) = \mu$

$Var(x) = \sigma^2$

$P(X > \mu) = 1/2$

$P(X < \mu) = 1/2$




PDF

CDF is S-shaped

$Z = \dfrac{X-\mu}{\sigma}$ Standard Normal R.V

$E(Z) = 0$

$Var(Z) = 1$

$P(-1 \leq Z \leq 1) = 0.6827$

$P(-2 \leq Z \leq 2) = 0.9545$

$P(-3 \leq Z \leq 3) = 0.9973$

Aditya Vangala

ACE

Poisson R.V

$$P(X=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}, \quad r = 0, 1, 2, 3 \ldots$$

$$E[X] = \lambda$$

$$Var(X) = \lambda$$

$$E(X^2) = \lambda^2 + \lambda$$

Aditya Vangala

ACE

Since 1995

## Binomial R.V

$$P(X=r) = {}^{n}C_{r}\, p^{r} (1-p)^{n-r}, \quad r = 0, 1, 2, \ldots n$$

$$E[X] = np$$

$$Var(X) = np(1-p)$$

Aditya Vangala

ACE

statistics :-

The probability density function of a random variable X is $P_x(x) = e^{-x}$ for $x \geq 0$ and 0 otherwise. The expected value of the function $g_x(x) = e^{\frac{3x}{4}}$ is

GATE-15

Aditya Vangala

Since 1995

ACE

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$= \int_{0}^{\infty} e^{\frac{3x}{4}} \cdot e^{-x}\, dx$$

$$= \underline{\qquad\qquad}$$

Passengers try repeatedly to get a seat reservation in any train running between two stations until they are successful. If there is 40% chance of getting reservation in any attempt by a passenger, then the average number of attempts that passengers need to make to get a seat reserved is

GATE-2017

Aditya Vangala

ACE

$p = 0.4$

| $X$ | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| $P(X)$ | $p$ | $(1-p)p$ | $(1-p)^2 p$ | $(1-p)^3 \cdot p$ | $(1-p)^4 \cdot p$ | $\cdots$ |

$$E[X] = 1\cdot(p) + 2(1-p)(p) + 3(1-p)^2 p + \cdots$$

$$= \frac{1}{p}$$

$$= \frac{1}{0.4} = \underline{\underline{2.5}}$$

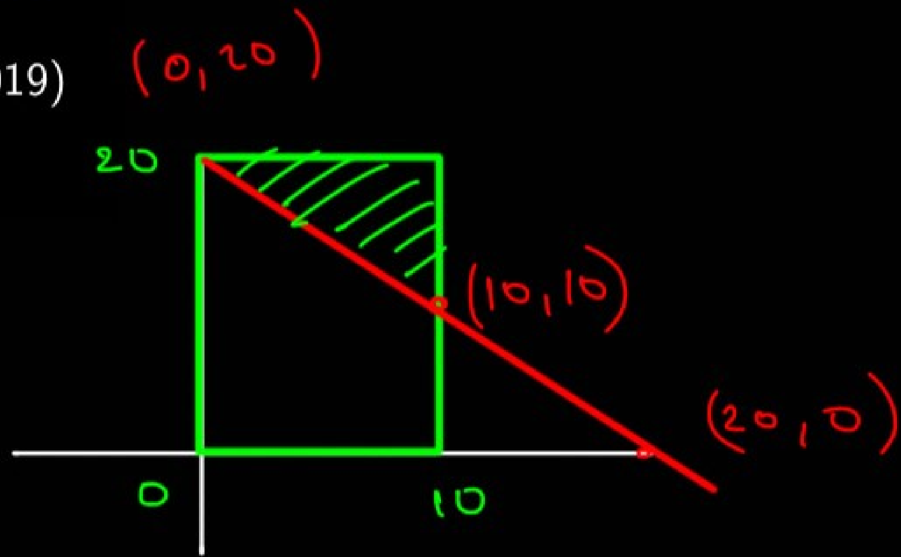
The variable $X$ takes a value between 0 and 10 with uniform probability distribution. The variable $Y$ takes a value between 0 and 20 with uniform probability distribution. The probability of the sum of variables $X + Y$ being greater than 20 is

(GATE 2019)

(a) 0.5 (b) 0 (c) 0.33 (d) 0.25

Aditya Vangala

ACE




$x + y > 20$

$x + y = 20$

$$P(x+y>20) = \frac{\frac{1}{2}(1\cancel{0})(1\cancel{0})}{1\cancel{0} \times 2\cancel{0}} = \frac{1}{4}$$

Consider two exponentially distributed random variables $X$ and $Y$, both having a mean of 0.50. Let $Z = X + Y$ and $r$ be the correlation coefficient between $X$ and $Y$. If the variance of $Z$ equals 0, then the value of $r$ is (round off to 2 decimal places).

(GATE–2020)

Aditya Vangala

ACE

$X \rightarrow E.R.V$

$\frac{1}{\lambda_1} = 0.5 \Rightarrow \lambda_1 = 2$

$Y \rightarrow E.R.V$

$\frac{1}{\lambda_2} = 0.5$

$\lambda_2 = 2$

$$r = \frac{\text{cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$\text{Var}(X) = \frac{1}{\lambda_1^2} = \frac{1}{4}$$

$$\sigma_X = \frac{1}{2}$$

$$\sigma_Y = 1/2$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{cov}(X,Y) = 0$$

$$\frac{1}{4} + \frac{1}{4} + 2\,\text{cov}(X,Y) = 0$$

$$\text{cov}(X,Y) = -1/4$$

$$r = \frac{-1/4}{1/2 \cdot 1/2} = \underline{\underline{-1}}$$

The lengths of a large stock of titanium rods follow a normal distribution with a mean of 440 mm and a standard deviation of 1mm. What is the percentage of rods whose lengths lie between 438 mm and 441 mm

Aditya Vangala

ACE

GATE-19

(a) 81.85%
(b) 68.4%
(c) 99.75%
(d) 86.64%

$$P(438 < L < 441)$$

$$P\left(\frac{438-\mu}{\sigma} < \frac{L-\mu}{\sigma} < \frac{441-\mu}{\sigma}\right)$$

$$P\left(\frac{438-\mu}{\sigma} < z < \frac{441-\mu}{\sigma}\right)$$

Let $X$ be a normal random variable with mean 1 and variance 4. The probability $P(X < 0)$ is

GATE-13

Aditya Vangala

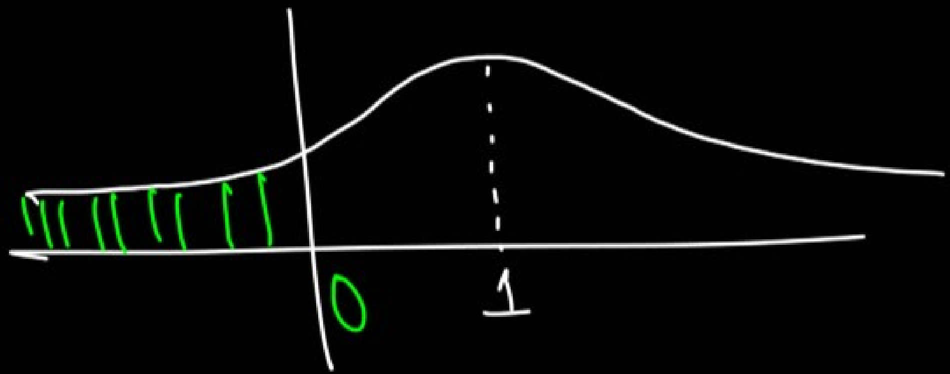
ACE

(a) 0.5

(b) greater than zero and less than 0.5

(c) greater than 0.5 and less than 1.0

(d) 1.0




(b)

An Observer counts 240 vehicles per hour at a specific highway location. Assume that the vehicle arrival at the location is Poisson distributed. The Probability of having one vehicle arriving over a 30 second time interval is

GATE-14

Aditya Vangala

$$P(x=r) = \frac{e^{-\lambda} \lambda^{r}}{r!}$$

$$P(x=1) = \frac{e^{-\lambda} \lambda^{1}}{1!}$$

$240v \longleftrightarrow 1$ hour

$240 \longrightarrow 60 \times 60$ sec

$\frac{240}{60 \times 2} \longleftrightarrow 30$ sec

$2 \longleftrightarrow 30$ sec

$\lambda = 2$

A fair coin is tossed 20 times. The probability that head will appear exactly 4 times in the first ten tosses, and tail will appear exactly 4 times in the next ten tosses is

GATE-20

Aditya Vangala

ACE

$$p(x=r) = {}^{n}C_{r}\, p^{r}\,(1-p)^{n-r}$$

$$\left( {}^{10}C_{4} \left(\tfrac{1}{2}\right)^{4} \left(\tfrac{1}{2}\right)^{10-4} \right) \cdot {}^{10}C_{4} \left(\tfrac{1}{2}\right)^{4} \left(\tfrac{1}{2}\right)^{6}$$

An unbiased coin is tossed an infinite number of times. The probability that the fourth head appears at the tenth toss is

Aditya Vangala

ACE

GATE-14

(a) 0.067

(b) 0.073

(c) 0.082

(d) 0.091

$$\left( {}^{9}C_{3} \left(\tfrac{1}{2}\right)^{3} \left(\tfrac{1}{2}\right)^{6} \right) \cdot \tfrac{1}{2}$$

3 heads in

9 ~~to~~ tosses

Let $X_1$, $X_2$ be two independent normal random variables with means $\mu_1$, $\mu_2$ and standard deviations $\sigma_1$, $\sigma_2$, respectively. Consider $Y = X_1 - X_2$, $\mu_1 = \mu_2 = 1$, $\sigma_1 = 1$, $\sigma_2 = 2$. Then

GATE-18

(a) $Y$ is normally distributed with mean 0 and variance 1

(b) $Y$ is normally distributed with mean 0 and variance 5

(c) $Y$ has mean 0 and variance 5, but is not normally distributed

(d) $Y$ has mean 0 and variance 1, but is not normally distributed

Aditya Vangala

Since 1995 ACE

$Y = \alpha_1 X_1 + \alpha_2 X_2$ is also N.R.V

$E(Y) = \alpha_1 \mu_1 + \alpha_2 \mu_2$

$Var(Y) = \alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2$

$E(Y) = 1(1) + (-1)(1) = 0$

$Var(Y) = 1^2 (1) + (-1)^2 (2)^2$

$= 5$

Aditya Vangala

ACE

Consider a binomial random variable $X$. If $X_1, X_2, X_3, \cdots, X_n$ are independent and identically distributed samples from the distribution of $X$ with sum $Y = \sum_{i=1}^{n} X_i$, then the distribution of $Y$ as $n \to \infty$ can be approximated as

GATE-2021

A. Exponential

B. Binomial

C. Bernoulli

D. Normal

$Y = X_1 + X_2 + X_3 + \cdots + X_n$

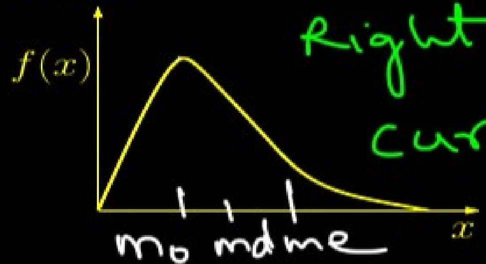
$n \to \infty$ $Y \Rightarrow$ Normal R.V

or

Gaussian R.V

Central limit theorem

A probability distribution with right skew is shown in figure. The correct statement for the probability distribution is GATE 2018




Right skewed curved

(a) Mean is equal to mode

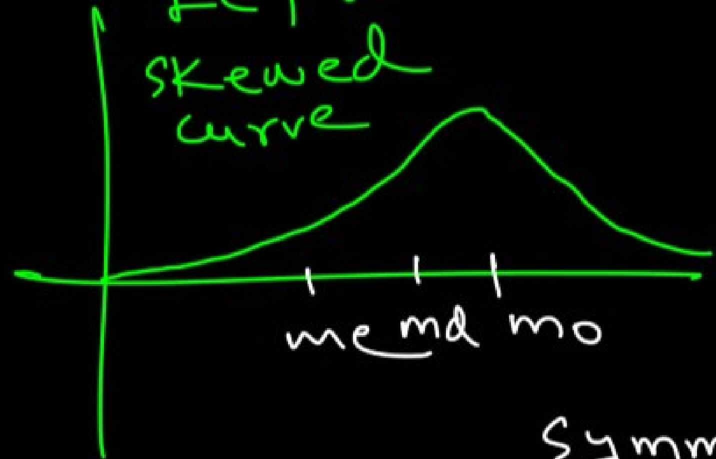
(b) Mean is greater than median but less than mode

(c) Mean is greater than median and mode

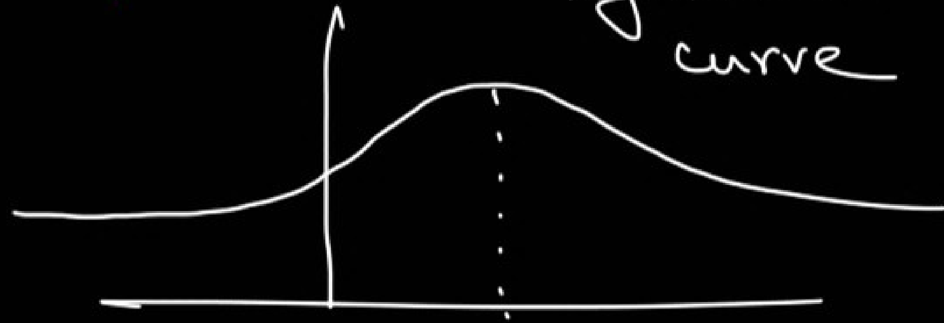
(d) Mode is greater than median

Lack of Sym => Skewness

Left skewed curve




Symmetrical curve



$me = md = mo$

Aditya Vangala

ACE

Marks obtained by 100 students in an examination are given in the table

| S. No | Marked Obtained | Number of students |
|---|---|---|
| 1 | 25 | 20 |
| 2 | 30 | 20 |
| 3 | 35 | 40 |
| 4 | 40 | 20 |

What would be the mean, median, and mode of the marks obtained by the students? GATE-2014

(a) Mean 33, Median 35, Mode 40

(b) Mean 35, Median 32.5, Mode 40

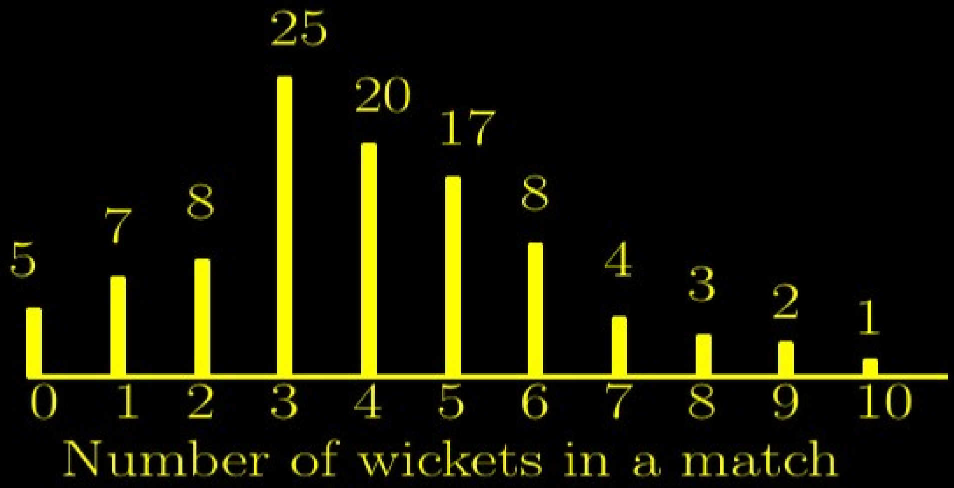
(c) Mean 33, Median 35, Mode 35

(d) Mean 33, Median 32.5, Mode 35

Aditya Vangala

ACE

The bar graph shows the frequency of the number of wickets taken in a match by a bowler in her career. For example, in 17 of her matches, the bowler has taken 5 wickets each. The median number of wickets taken by the bowler in a match is (rounded off to one decimal place).GATE-2022




$\left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\}$ 5 times

$\left.\begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix}\right\}$ 7 times

50 and 51th obs are 4, 4

$$\text{median} = \frac{4+4}{2} = \underline{\underline{4}}$$

Aditya Vangala

ACE

E.E correlation and Regression

E.C.E joint PDF and CDF

Aditya Vangala

ACE

Calculus :-

$$\lim_{x \to a} f(x) = l$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Continuity :-

$$\lim_{x \to a} f(x) = f(a)$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Aditya Vangala

ACE

Indeterminate forms :-

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, \infty^0, 0^0, 1^\infty$

If $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$ and

$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$

If $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} g(x) = \infty$ and

$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$

Aditya Vangala

ACE

Mean value theorems :-

Rolles theorem :- $f'(c) = 0 \quad c \in (a,b)$

Lagranges theorem :- $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Cauchy theorem :- $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

Aditya Vangala

ACE

maximum and minimum

(1) closed Interval

(2) open Interval

Aditya Vangala

ACE

# Integration

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$|x|$, $[x]$, piecewise functions.

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Aditya Vangala

ACE

Since 1995

$$\int_{-a}^{a} f(x)\,dx = \qquad 2\int_{0}^{a} f(x)\,dx \qquad f(-x) = f(x)$$

$$0 \qquad f(-x) = -f(x)$$

Aditya Vangala

ACE

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$\Gamma(1) = 1$

$\Gamma(2) = 1$

$\Gamma(3) = 2!$

$\Gamma(4) = 3!$

$\Gamma(n+1) = n!$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(-1/2) = -2\sqrt{\pi}$

Aditya Vangala

ACE

# Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x\,dx$$

$$= \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

A function $f(x)$ is defined as

$$f(x) = \begin{cases} e^x & x < 1 \\ \ln x + ax^2 + bx & x \geq 1 \end{cases}$$

where $x \in R$.

Which one of the following statements is TRUE?

(GATE-17)

(a) $f(x)$ is NOT differentiable at $x = 1$ for any values of $a$ and $b$.

(b) $f(x)$ is differentiable at $x = 1$ for the unique values of $a$ and $b$.

(c) $f(x)$ is differentiable at $x = 1$ for all values of $a$ and $b$ such that $a + b = e$.

(d) $f(x)$ is differentiable at $x = 1$ for all values of $a$ and $b$.

Let $f(x)$ be diff at $x=1$

$\Rightarrow f(x)$ is conti at $x=1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$e^1 = \ln 1 + a + b$$

$$\Rightarrow a + b = e \quad \leftarrow 1$$

$$f'(x) = \begin{cases} e^x & x < 1 \\ \frac{1}{x} + 2ax + b & x \geq 1 \end{cases}$$

$$f'(1^-) = f'(1^+)$$

$$e^1 = \frac{1}{1} + 2a + b$$

$$2a + b = e - 1 \quad \leftarrow 2$$

Aditya Vangala

ACE

Since 1995

Aditya Vangala

ACE

$$a + \cancel{b} = e$$

$$\underline{2a + \cancel{b} = e - 1}$$

$$- \quad - \qquad -$$

$$-a = e - (e-1)$$

$$-a = 1$$

$$a = -1$$

$$\Rightarrow a + b = e$$

$$-1 + b = e$$

$$b = e + 1$$

The value of $\lim_{x \to \infty} \left(1 + x^2\right)^{e^{-x}}$ is

(GATE-2015)

Aditya Vangala

ACE

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) $\infty$

$\infty^{0} \rightarrow$ Convert this either to

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

use Logarithm

The limit

$$p = \lim_{x\to\pi}\left(\frac{x^2+\alpha x+2\pi^2}{x-\pi+2\sin x}\right)$$

has a finite value for a real $\alpha$. The value of $\alpha$ and the corresponding limit $p$ are

(GATE-2022)

(a) $\alpha = -3\pi$ and $p = \pi$

(b) $\alpha = -2\pi$ and $p = 2\pi$

(c) $\alpha = \pi$ and $p = \pi$

(d) $\alpha = 2\pi$ and $p = 3\pi$

$$p = \lim_{x\to\pi}\left(\frac{x^2+\alpha x+2\pi^2}{x-\pi+2\sin x}\right) = \frac{\pi^2+\alpha\pi+2\pi^2}{\pi-\pi+2\sin\pi}$$

$$= \frac{3\pi^2+\alpha\pi}{0}$$

Aditya Vangala

ACE

Since 1995

$$3\pi^2+\alpha\pi = 0$$

$$\alpha\cancel{\pi} = -3\pi^{\cancel{2}}$$

$$\alpha = -3\pi$$

Let $f(x) = x^2 - 2x + 2$ be a continuous function defined on $x \in [1, 3]$. The point $x$ at which the tangent of $f(x)$ becomes parallel to the straight line joining $f(1)$ and $f(3)$ is (GATE-2021)

Aditya Vangala

ACE

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Apply LMVT

Suppose that $f : R \to R$ is a continuous function on the interval [-3, 3] and a differentiable function in the interval (-3, 3) such that for every $x$ in the interval $f'(x) \leq 2$. If $f(-3) = 7$ then $f(3)$ is at most (GATE-2021)

Aditya Vangala

ACE

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$a = -3 \quad b = 3$

$$f'(c) = \frac{f(3) - f(-3)}{3 - (-3)}$$

$$f'(c) = \frac{f(3) - 7}{6} \leq 2$$

$$f(3) - 7 \leq 12$$

$$f(3) \leq 19$$

Ans $\underline{\underline{19}}$

The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is (GATE–12)

Aditya Vangala

ACE

(a) 21

(b) 25

(c) 41

(d) 46

Working Rule:

Step 1: Find $p$, $q$, $r$, $s$ and $t$

Step 2: Equate $p$ and $q$ to zero for obtaining stationary points.

Step 3: Find $r$, $s$ and $t$ at each stationary point.

(a) If $rt - s^2 > 0$ and $r > 0$ then $f(x, y)$ has a minimum at that stationary point.

(b) If $rt - s^2 > 0$ and $r < 0$ then $f(x, y)$ has a maximum at that stationary point.

(c) If $rt - s^2 < 0$ then $f(x, y)$ has no extremum at that stationary point and such points are called saddle points.

(d) If $rt - s^2 = 0$ then the case is undecided.

# Double integral

Aditya Vangala

ACE

Since 1995

① $\int_{x=a}^{b} \int_{y=g(x)}^{h(x)} f(x,y)\,dy\,dx \Rightarrow$ vertical strip

② $\int_{y=c}^{d} \int_{x=p(y)}^{q(y)} f(x,y)\,dx\,dy \Rightarrow$ Horizontal strip

Aditya Vangala

ACE

## Applications

$$\iint_R dx\,dy = \text{Area of region } R$$

$$x^2 + y^2 = r^2 \Rightarrow \iint_R dx\,dy = \pi r^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \iint_R dx\,dy = \pi ab$$

The volume beneath the surface $z = f(x, y) > 0$ and above $xy$-plane is

$$V = \iint_R f(x, y)\, dy\, dx$$

Aditya Vangala

ACE

Triple integrals

$$\int_{z=z_1}^{z_2} \int_{y=y_1(z)}^{y_2(z)} \int_{x=x_1(y,z)}^{x_2(y,z)} f(x,y,z)\, dx\, dy\, dz$$

Applications

$$\iiint_T dx\,dy\,dz = \text{volume of the closed surface}$$

sphere $V = \frac{4}{3}\pi r^3$

cylinder $= \pi r^2 h$

Cone $V = \frac{1}{3}\pi r^2 h$

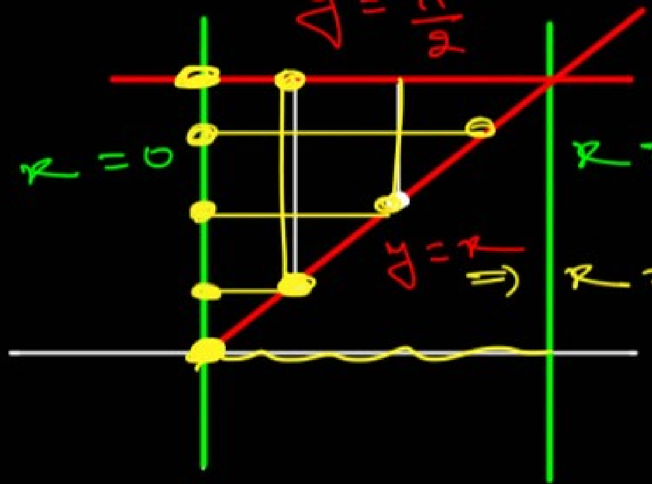
Aditya Vangala

ACE

The value of $\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin y}{y} dy dx$ is

2019

$$\int_{x=0}^{\pi/2} \int_{y=x}^{\pi/2} \frac{\sin y}{y} \, dy \, dx$$




$$\int_{y=0}^{\pi/2} \int_{x=0}^{y} \frac{\sin y}{y} \, dx \, dy$$

$$\int_{y=0}^{\pi/2} \frac{\sin y}{y} \left(x\right)_0^y dy = \int_{y=0}^{\pi/2} \frac{\sin y}{\cancel{y}} \left(\cancel{y-0}\right) dy$$

Aditya Vangala

ACE

$$\int_0^{\pi/2} \sin y \, dy = \left(-\cos y\right)_0^{\pi/2}$$

$$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= -\left(0 - 1\right)$$

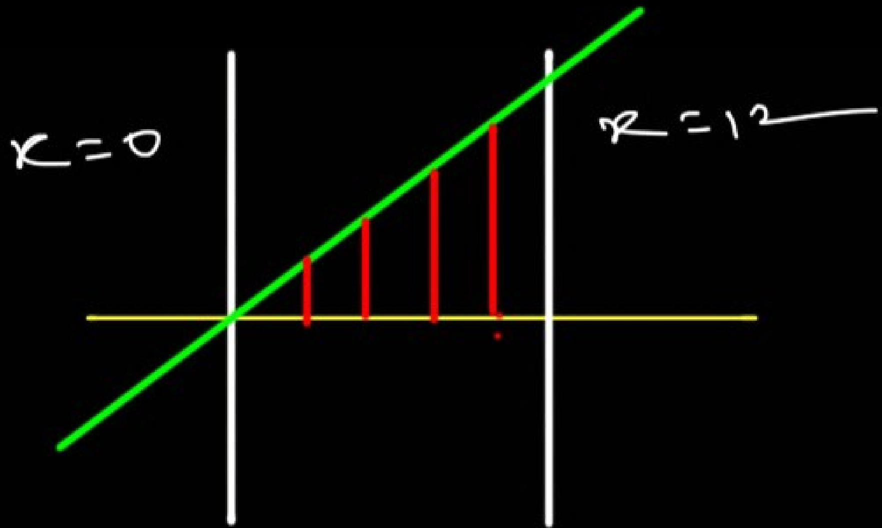
$$= 1$$

Aditya Vangala

ACE

The volume under the surface $z(x, y) = x + y$ and above the triangle in the $xy$ plane defined by $0 \leq y \leq x$ and $0 \leq x \leq 12$ is

GATE-2014




$$V = \iint f(x,y)\,dy\,dx$$

$$= \int_{x=0}^{12} \int_{y=0}^{x} (x+y)\,dy\,dx$$

$$\int_0^{12} \left(xy + \frac{y^2}{2}\right)_0^{x} dx$$

$$\int_0^{12} \left(x \cdot x + \frac{x^2}{2}\right) dx$$

$$\int_0^{12} \frac{3x^2}{2}\,dx = \left(\frac{3x^3}{2\times 3}\right)_0^{12}$$

$$= \frac{12^3}{2} = 864$$

Aditya Vangala
ACE
Since 1995

Vector calculus:—

$\bar{F} = F_1 \hat{i} + f_2 j + f_3 k$

$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$

$\nabla \cdot \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Divergence or Solenoidal $\nabla \cdot \bar{F} = 0$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Curl free
or
Conservative
or
irrotational
$\nabla \times \bar{F} = \bar{0}$

Aditya Vangala

ACE

## Gradient of a Scalar function

$$\nabla f = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) f$$

$$= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \Rightarrow \underline{\underline{\text{vector}}}$$

Gradient fields are always irrotational

$$\nabla \times \nabla f = \bar{0}$$

Aditya Vangala

ACE

The unit normal vector to given Surface $\phi(x, y, z) = 0$ at point P is

$$\bar{n} = \frac{\nabla \phi_P}{|\nabla \phi_P|}$$

The directional derivative of a Scalar function $\phi$ at point 'p' in the direction of vector $\bar{a}$ is

$$\nabla \phi_p \cdot \frac{\bar{a}}{|\bar{a}|}$$

The maximum value of the D.D of a Scalar function $\phi$ at point 'p' is

$$|\nabla \phi_p|$$

Aditya Vangala

The maximum value of D.D of a scalar function $\phi$ occurs in the direction of $\nabla\phi$

Aditya Vangala

ACE

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_C F_1\,dx + F_2\,dy + F_3\,dz$$

$$\bar{F} = F_1\hat{i} + F_2 j + F_3 k$$

$$d\bar{r} = dx\,\hat{i} + dy\,j + dz\,k$$

Aditya Vangala

ACE

Green's theorem :- $\bar{F} = f_1 i + f_2 j$

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_C f_1\,dx + f_2\,dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

Stoke's theorem : $\bar{F} = f_1 i + f_2 j + f_3 k$

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_C f_1\,dx + f_2\,dy + f_3\,dz$$

$$= \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\,ds$$

Aditya Vangala

ACE

## Gauss divergence theorem :-

$$\oiint_S \bar{F} \cdot \bar{n} \, ds = \iiint \nabla \cdot \bar{F} \, dv$$

closed surface integral

Triple integrals or volume integrals.

The position vector is given by $\bar{r} = xi + yj + zk$. Let the magnitude of the position vector be $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$. Find the following

$\nabla \bullet (f(r)\bar{r})$



$$\nabla \cdot f(r)\,\bar{r} = 3f(r) + r f'(r)$$

Aditya Vangala

ACE

If $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$ then the value of Div $(e^r \cdot \bar{r})$

(a) $3e^r$

(b) $re^r$

(c) $(3 + r)e^r$

(d) 0

$$\nabla \cdot f(r)\,\bar{r} = 3f(r) + r f'(r)$$

$$= 3e^r + re^r$$

$$= e^r(3 + r)$$

The position vector is given by $\bar{r} = xi + yj + zk$.
Let the magnitude of the position vector be
$r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$. Find the following
$\nabla \times (f(r)\bar{r}) = \bar{0}$

Aditya Vangala

ACE

If $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$ then the value of curl $(r^4 \bar{r})$

Aditya Vangala

Since 1995

ACE

$$\nabla \times (f(r)\bar{r}) = \bar{0}$$

$$\nabla \times (r^4 \bar{r}) = \bar{0}$$

Consider the vector field $\bar{F} = a_x(4y - c_1 z) + a_y(4x + 2z) + a_z(2y + z)$ in a rectangular coordinate system $(x, y, z)$ with unit vectors $a_x, a_y$ and $a_z$. If the field $\bar{F}$ is irrotational (conservative). Then the constant $c_1$ (in integer) is

(GATE-2021)

Aditya Vangala

ACE

2 min

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y - c_1 z & 4x + 2z & 2y + z \end{vmatrix}$$

$$= \hat{i}(\underbrace{2 - 2}_{0}) - j(0 - (-c_1)) + k(\underbrace{4 - 4}_{0}) = 0\hat{i} + 0j + 0k$$

$$\Rightarrow \underline{\underline{c_1 = 0}}$$

The position vector is given by $\bar{r} = xi + yj + zk$.
Let the magnitude of the position vector be
$r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$. Find the following $\nabla f(r)$

Aditya Vangala

ACE

$$\nabla f(r) = f'(r) \cdot \frac{\bar{r}}{r}$$

For a position position vector $\vec{r} = xi + yj + zk$ the norm of the vector can be defined as $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Given a function $\phi = \ln|\vec{r}\,|$, its gradient $\nabla\phi$ is

(GATE 2018)

Aditya Vangala

ACE

$$\nabla f(r) = f'(r)\,\frac{\bar{r}}{r}$$

(a) $\vec{r}$

(b) $\frac{\vec{r}}{|\vec{r}|}$

(c) ~~$\frac{\vec{r}}{\vec{r}\bullet\vec{r}}$~~

(d) $\frac{\vec{r}}{|\vec{r}|^3}$

$$\phi = \ln r$$

$$\nabla \ln r = \frac{1}{r}\cdot\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2} = \frac{\bar{r}}{\bar{r}\cdot\bar{r}}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\bar{r}\cdot\bar{r} = (xi + yj + zk)\cdot(xi + yj + zk)$$

$$= x^2 + y^2 + z^2$$

The directional derivative of $\varphi(x, y, z) = x^2yz + 4xz^2$ at $(1, -2, -1)$ along $2i - j - 2k$ is

(a) $\frac{-37}{3}$

(b) $\frac{37}{3}$

(c) $\frac{3}{37}$

(d) $\frac{-3}{37}$

Aditya Vangala

ACE

$$\nabla\phi_P \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$\bar{a} = 2i - j - 2k$$

$$|\bar{a}| = \sqrt{4+1+4}$$

$$= 3$$

$$\nabla\phi = i(2xyz + 4z^2) + j(x^2z) + k(x^2y + 8xz)$$

$$\nabla\phi_P = i(2(1)(-2)(-1) + 4(-1)^2) + j(1^2(-1)) + k(1(-2) + 8(1)(-1))$$

$$= i(8) - j - 10k$$

$$\text{Ans} = (8i - j - 10k) \cdot \frac{(2i - j - 2k)}{3}$$

$$= \frac{16 + 1 + 20}{3}$$

$$= \frac{37}{3}$$

The directional derivative of the function $f(x, y) = x^2 + y^2$ along a line directed from (0,0) to (1, 1) evaluated at the point $x = 1$, $y = 1$ is

(GATE 2019)

Aditya Vangala



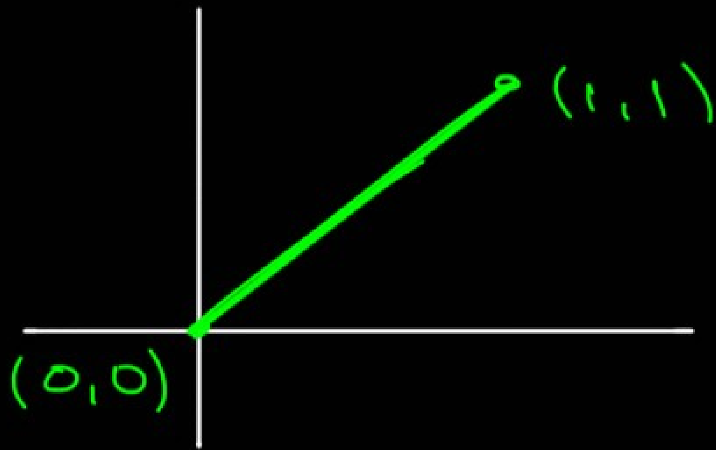
(a) $2\sqrt{2}$

(b) 2

(c) $4\sqrt{2}$

(d) $\sqrt{2}$

$$\nabla f_p \cdot \frac{\bar{a}}{|\bar{a}|}$$




$$\nabla f_p = \hat{i}\,2x + \hat{j}(2y)$$

$$= 2\hat{i} + 2\hat{j}$$

$$\bar{a} = \hat{i}(1-0) + \hat{j}(1-0) = \hat{i} + \hat{j}$$

$$\text{Ans} = \frac{(2\hat{i} + 2\hat{j}) \cdot (\hat{i} + \hat{j})}{\sqrt{1^2 + 1^2}}$$

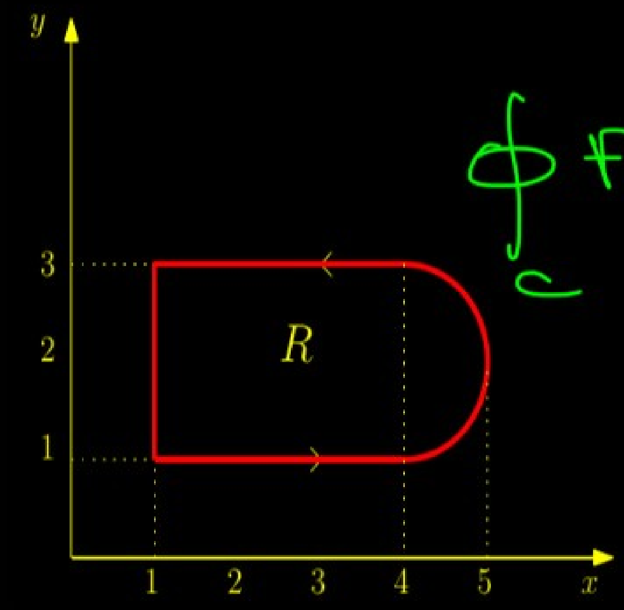
$$= \frac{2+2}{\sqrt{2}} = \underline{\underline{2\sqrt{2}}}$$

Consider the line integral $\int_C (xdy - ydx)$ the integral being taken in a counterclockwise direction over the closed curve $C$ that forms the boundary of the region $R$ shown in the figure below. The region $R$ is the area enclosed by the union of a $2\times3$ rectangle and a semi-circle of radius 1. The line integral evaluates to (GATE 19)

(a) $6+\frac{\pi}{2}$ (b) $8+\pi$ (c) $12+\pi$ (d) $16+2\pi$

Aditya Vangala

ACE




$$\int_C x\,dy - y\,dx$$

$$\oint_C (-y)\,dx + x\,dy$$

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(1-(-1)\right) dx\,dy = 2\iint dx\,dy$$

$$= 2\left(2\times3 + \frac{\pi(1)^2}{2}\right)$$

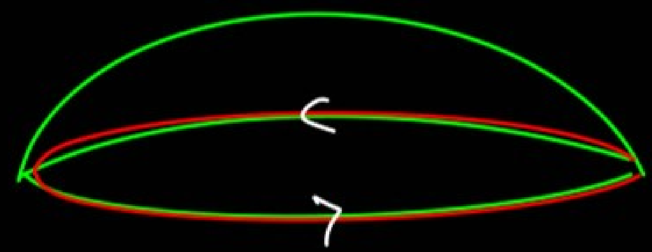
$$= 2\left(6+\pi/2\right) = 12+\pi$$

Given $\vec{F} = z\vec{a_x} + x\vec{a_y} + y\vec{a_z}$. If $S$ represents the portion of the sphere $x^2 + y^2 + z^2 = 1$ for $z \geq 0$, then $\int_S (\nabla \times \vec{F}) \bullet \vec{dS}$ is (GATE 19)

Aditya Vangala

ACE

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$




Along C $\quad z = 0 \quad dz = 0$

$$\oint \bar{F} \cdot d\bar{r} = \oint_C F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \oint_C 0\, dx + x\, dy + y(0) = \oint_C x\, dy$$

Aditya Vangala

ACE

$x^2 + y^2 = 1 \qquad x = 1 \cdot \cos\theta \qquad y = 1 \cdot \sin\theta$

$$\oint_C x\,dy = \int_{\theta=0}^{2\pi} \cos\theta \cos\theta\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}(\theta)_0^{2\pi} + \cancel{\left(\frac{\sin 2\theta}{4}\right)_0^{2\pi}}$$

$0 \qquad 0$

$$= \pi$$

Given a function $\phi = \frac{1}{2}(x^2 + y^2 + z^2)$ in three-dimensional Cartesian space, the value of the surface integral

$$\oiint_S \hat{n} \bullet \nabla \phi ds,$$

where $S$ is the surface of a sphere of unit radius and $\hat{n}$ is the outward unit normal vector on $S$, is

(GATE 2022)

(a) $4\pi$

(b) $3\pi$

(c) $\frac{4\pi}{3}$

(d) 0

Aditya Vangala

$$\phi = \frac{1}{2}(x^2 + y^2 + z^2)$$

$$\nabla \phi = i \frac{2x}{2} + j \frac{2y}{2} + k \frac{2z}{2}$$

$$\nabla \phi = xi + yj + zk$$

$$\oiint \bar{F} \cdot \bar{n}\, ds = \iiint \nabla \cdot \bar{F}\, dV$$

$$= \iiint (1+1+1)\, dV$$

$$= 3 \iiint dV$$

$$= \cancel{3} \times \frac{4}{\cancel{3}} \pi (1)^3$$

$$= 4\pi$$

A vector is defined as $\vec{f} = yi + xj + zk$, where $i$, $j$ and $k$ are unit vectors in Cartesian $(x, y, z)$ coordinate system. The surface integral $\oiint \vec{f} \bullet ds$ over the closed surface $S$ of a cube with vertices having the following coordinates:
$(0,0,0), (1,0,0), (0,1,0), (0,0,1),$
$(1,0,1), (1,1,1), (0,1,1), (1,1,0,)$ is

(GATE 2014)

Aditya Vangala

ACE

$$\oiint \bar{F} \cdot d\bar{s} = \iiint \nabla \cdot \bar{F} \, dv$$

$$= \iiint (0 + 0 + 1) \, dv$$

$$= \iiint dv = \text{volume of cube}$$

$$= 1^3 = 1$$

Aditya Vangala



# Differential eqns

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x$$

$O = 2$

$D = 1$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = e^x \qquad O = 2 \quad D = 1$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/2} + y = 0$$

$$O = 2$$

$$D = 2$$

$$\frac{d^2y}{dx^2} + y = -\left(\frac{dy}{dx}\right)^{1/2}$$

$$\left(\frac{d^2y}{dx^2} + y\right)^2 = \frac{dy}{dx}$$

Aditya Vangala

ACE

# Variable separable method:-

$$\frac{dy}{dx} = (1+y^2)x$$

$$\frac{dy}{1+y^2} = x\,dx$$

$$\int \frac{dy}{1+y^2} = \int x\,dx$$

$$\tan^{-1} y = \frac{x^2}{2} + c$$

$$y = \tan\left(\frac{x^2}{2} + c\right)$$

Reducible to variable separable form:—

Aditya Vangala

ACE

$$\frac{dy}{dx} = \cos(x+y)$$

use substitution $z = x+y$ and reduce to variable form

# Homogeneous function

Aditya Vangala

ACE

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

$y = vx$ or $x = vy$

$\frac{y}{x}$ $\qquad$ $\frac{x}{y}$

Aditya Vangala

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \dots a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = f(x)$$

↳ Linear D. eqns

(1) $f(x) \neq 0$ Non Homo D. eqn

(2) $f(x) = 0$ Homo D. eqn

Aditya Vangala

ACE

# Exact D.E

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Condition for Exact D.E}$$

**Solution**

$$\underbrace{\int M dx}_{\text{Treat } y \text{ as Constant}} + \underbrace{\int N dy}_{\text{Ignore terms Containing } 'x'} = C$$

## Non exact D.E

$$Mdx + Ndy = 0$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ or constant}$$

$$I.F = e^{\int f(x)dx}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \text{ or constant}$$

$$I.F = e^{\int g(y)dy}$$

First order Linear D.E **

Aditya Vangala

ACE

$$\frac{dy}{dx} + Py = Q$$

$$I.F = e^{\int P\,dx}$$

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\,dx + C$$

$$\frac{dx}{dy} + Px = Q$$

$$I.F = e^{\int P\,dy}$$

$$x \cdot e^{\int P\,dy} = \int Q \cdot e^{\int P\,dy}\,dy + C$$

First order Non linear D. eqns

Aditya Vangala

ACE

$$f'(y)\frac{dy}{dx} + f(y)P = Q$$

$$f(y) = t$$

$$f'(y)\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + Pt = Q$$

Aditya Vangala

ACE

## Higher D.E

$y = y_c + y_p$

$\rightarrow$ $m_1, m_2$ Real & Distinct

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\rightarrow$ $m_1, m_1$ real and equal

$y_c = (c_1 + c_2 x) e^{m_1 x}$

Aditya Vangala

ACE

$\rightarrow \alpha \pm i\beta$ Complex roots

$$y_c = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right)$$

$\rightarrow \alpha \pm \sqrt{\beta}$ Real and irrational

$$y_c = e^{\alpha x}\left(C_1 \cosh\sqrt{\beta}\, x + C_2 \sinh\sqrt{\beta}\, x\right)$$

Aditya Vangala

ACE

## Particular Integral:-

$$g(D)\,y = f(x)$$

(1) $f(x) = e^{\alpha x}$

$$y_p = \frac{e^{\alpha x}}{g(\alpha)}$$

(2) $f(x) = \sin ax \,|\, \cos ax$

$$y_p = \frac{\sin ax}{g(D)} \qquad D^2 = -a^2$$

(3) $f(x) = x^m$

$$y_p = \frac{x^m}{g(D)}$$

$$D^{m+1}(x^m) = 0$$

Aditya Vangala

ACE

(4) $f(x) = e^{\alpha x} V \quad , V = \sin ax \mid \cos ax \mid x^m$

$$g(D)\, y = e^{\alpha x} \cdot V$$

$$y_p = \frac{e^{\alpha x} \cdot V}{g(D)}$$

Replace $D$ with $D+\alpha$

$$y_p = e^{\alpha x} \left( \frac{V}{g(D+\alpha)} \right)$$

Aditya Vangala

ACE

# Cauchy - Euler D.E

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_{n-1} x \frac{dy}{dx} + a_n y = f(x)$$

$x = e^z$ $\quad D = \frac{d}{dx}$ $\quad \theta = \frac{d}{dz}$

$xD = \theta$ , $x^2 D^2 = \theta(\theta-1)$ $\quad x^3 D^3 = \theta(\theta-1)(\theta-2)$

Consider the Differential Equation

$$\frac{dx}{dt} = \sin x$$

with initial condition $x(0) = 0$. The solution to this Ordinary Differential Equation is

GATE-2020

(a) $x(t) = 0$

(b) $x(t) = \sin t$

(c) $x(t) = \cos t$

(d) $x(t) = \sin t - \cos t$

Aditya Vangala

ACE

a) $x(t) = 0$

$x(0) = 0$

$\frac{d}{dt} x(t) = 0$

$\sin 0 = 0$

What are the values of $\alpha$ and $\beta$ that make

Aditya Vangala

ACE

$$dF(x,y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right)dx + \left(xy^{\beta}+1\right)dy$$

an exact differential equation?

ESE-2022

(a) $\alpha = -1,\ \beta = -2$

(b) $\alpha = 1,\ \beta = -2$

(c) $\alpha = -1,\ \beta = 2$

(d) $\alpha = -2,\ \beta = -1$

$$\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$$

$$-\frac{\alpha}{y^2} = y^{B}$$

$$-\alpha y^{-2} = y^{\beta}$$

$$\alpha = -1$$

$$\beta = -2$$

Consider the differential equation given below.

$$\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$$

The integrating factor of the differential equation is

GATE-2021

(a) $\left(1-x^2\right)^{\frac{-3}{4}}$

(b) $\left(1-x^2\right)^{\frac{-1}{4}}$

(c) $\left(1-x^2\right)^{\frac{-1}{2}}$

(d) $\left(1-x^2\right)^{\frac{-3}{2}}$

Aditya Vangala

Since 1995 ACE

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x}{1-x^2}\sqrt{y} = x$$

$$\sqrt{y} = t$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = 2\frac{dt}{dx}$$

$$2\frac{dt}{dx} + \frac{x}{1-x^2}t = x$$

$$\frac{dt}{dx} + \frac{1}{2}\frac{x}{1-x^2}t = \frac{x}{2}$$

$$P = \frac{1}{2}\frac{x}{1-x^2}$$

$$I.F = e^{\int \frac{1}{2}\frac{x}{1-x^2}dx}$$

$$= e^{-\frac{1}{4}\int \frac{-2x}{1-x^2}dx}$$

$$= e^{-\frac{1}{4}\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{-1/4}}$$

$$= (1-x^2)^{-1/4}$$

Consider the initial value problem below. The value of $y$ at $x = \ln 2$ is (rounded off to 3 decimal places)

GATE-2020

Aditya Vangala

ACE

$$\frac{dy}{dx} = 2x - y, \ y(0) = 1$$

$$\frac{dy}{dx} + 1y = 2x$$

$P = 1$, $Q = 2x$

$$I.F = e^{\int 1 dx} = e^{x}$$

$$y \cdot e^{x} = \int 2x e^{x} dx + c$$

$$ye^{x} = e^{x}(2x - 2 + 0) + c$$

$$ye^{x} = e^{x}(2x-2) + c$$

$y(0) = 1$

$$1e^{0} = e^{0}(0-2) + c$$

$$c = 3$$

$$y = (2x-2) + 3e^{-x}$$

$$y = (2\ln 2 - 2) + 3e^{-\ln 2}$$

$$= 2\ln 2 - 2 + 3(1/2)$$

$$= 0.8862$$

Consider the homogeneous ordinary differential equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 0, \; x > 0$$

with $y(x)$ as a general solution. Given that $y(1) = 1$ and $y(2) = 14$ the value of $y(1.5)$ is (rounded off to two decimal places)

GATE-2019

Aditya Vangala

ACE

$x = e^z$

$(\theta(\theta-1) - 3\theta + 3)y = 0$

$(\theta^2 - \theta - 3\theta + 3)y = 0$

$(\theta(\theta-1) - 3(\theta-1))y = 0$

$(\theta-1)(\theta-3) = 0$

Roots 1, 3

$y_c = c_1 e^z + c_2 e^{3z}$

$= c_1 e^z + c_2 (e^z)^3$

$y_c = c_1 x + c_2 x^3$

$1 = c_1 + c_2$

$14 = 2c_1 + 8c_2$

$2 = 2c_1 + 2c_2$

$12 = 6c_2$

$c_2 = 2$

$c_1 = -1$

$y = -x + 2x^3$

$y(1.5) = -1.5 + 2(1.5)^3$

$= 5.25$

Aditya Vangala

ACE

PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \dots = (\ )$$

$B^2 - 4AC < 0$ elliptic

$B^2 - 4AC = 0$ Parabolic

$B^2 - 4AC > 0$ Hyperbolic

Aditya Vangala

ACE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ Laplace eqn

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ Wave eqn

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ Heat eqn or Diffusion eqn

Consider the following partial differential equation (PDE)

$$a\frac{\partial^2 f(x,y)}{\partial x^2} + b\frac{\partial^2 f(x,y)}{\partial y^2} = f(x,y)$$

where $a$ and $b$ are distinct positive real numbers. Select the combination(s) of values of the real parameters $\xi$ and $\eta$ such that $f(x,y) = e^{\xi x + \eta y}$ is a solution of the given PDE.

GATE-2022

(a) $\xi = \frac{1}{\sqrt{2a}}$, $\eta = \frac{1}{\sqrt{2b}}$

(b) $\xi = \frac{1}{\sqrt{a}}$, $\eta = 0$

(c) $\xi = 0$, $\eta = 0$

(d) $\xi = \frac{1}{\sqrt{a}}$, $\eta = \frac{1}{\sqrt{b}}$

Aditya Vangala

ACE

Since 1995

$f = e^{\xi x + \eta y}$

$\frac{\partial f}{\partial x} = \xi e^{\xi x + \eta y}$

$\frac{\partial^2 f}{\partial x^2} = \xi^2 e^{\xi x + \eta y}$

$\frac{\partial^2 f}{\partial x^2} = \xi^2 f$, $\frac{\partial^2 f}{\partial y^2} = \eta^2 f$

$a \cdot \xi^2 \cancel{f} + b\eta^2 \cancel{f} = \cancel{f}$

$a \cdot \xi^2 + b\eta^2 = 1$

Aditya Vangala

ACE

Consider the following partial differential equation for $u(x, y)$ with the constant $c > 1$ :

$$\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0$$

Solution of this equation is

GATE-2017

(a) $u(x, y) = f(x + cy)$

(b) $u(x, y) = f(x - cy)$

(c) $u(x, y) = f(cx + y)$

(d) $u(x, y) = f(cx - y)$

a) $u = f(x + cy)$

$\frac{\partial u}{\partial y} = f'(x+cy)\,c$

$\frac{\partial u}{\partial x} = f'(x+cy)\,1$

b) $u = f(x - cy)$

$\frac{\partial u}{\partial y} = f'(x-cy)(-c)$ $\qquad$ $\frac{\partial u}{\partial x} = f'(x-cy)\,1$

$(-c)f'(x-cy) + c\cdot f'(x-cy) = 0$

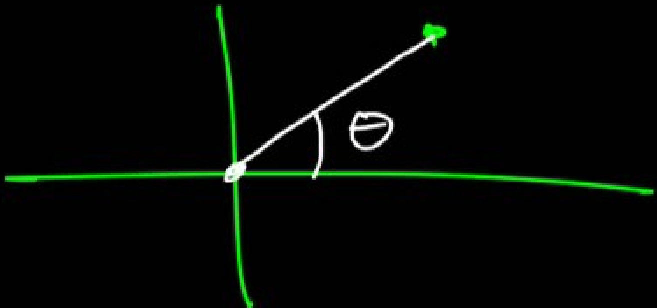
# Complex variables :-

$a+ib$ complex no.

$x+iy$ complex variable

$f(z) = u+iv$ complex function

$z = x + iy$




$r = \sqrt{x^2 + y^2} = |z|$

$\theta$ = Angle made by the line with +ve $x$-axis.

$x + iy = re^{i\theta}$

Aditya Vangala

ACE

$$|z_1 \cdot z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$$

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Aditya Vangala

ACE

$f(z) = u + iv$

(1) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(2) $1^{st}$ order Partial Derivatives must be Continuous

$\Bigg\}$ Analytic <u>Function</u>

Aditya Vangala

ACE

$$f(z) = z^2 = x^2 - y^2 + i\,2xy$$

$$f(z) = z^3 = x^3 - 3xy^2 + i\left(3x^2y - y^3\right)$$

Aditya Vangala

ACE

Real part given

Imag part asked

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

$$dv = -\frac{\partial u}{\partial y} \cdot dx + \frac{\partial u}{\partial x} \cdot dy$$

$$v = \underbrace{\int -\frac{\partial u}{\partial y} dx}_{\text{Treat } y \text{ as constant}} + \underbrace{\int \frac{\partial u}{\partial x} \cdot dy}_{\text{Ignore terms containing } x}$$

Aditya Vangala

ACE

$f(z) = ?$

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$

Replace $x$ with $z$

and $y$ with $0$

$f'(z) = g(z)$

$f(z) = \int g(z)\,dz + c$

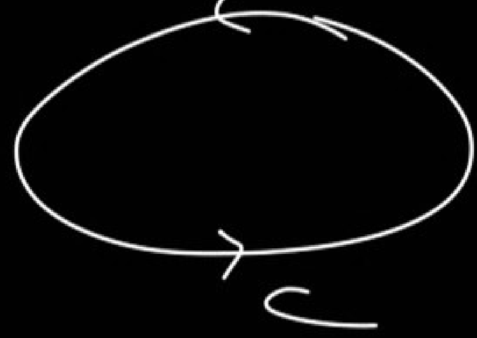
# Complex Integration



## Cauchy Integral Theorem :-

If $f(z)$ is analytic inside and on the simple closed curve $c$ then

$$\oint_c f(z)\,dz = 0$$



## Cauchy's Integral formula :-



$$f(z) = \frac{\phi(z)}{z - z_0}$$

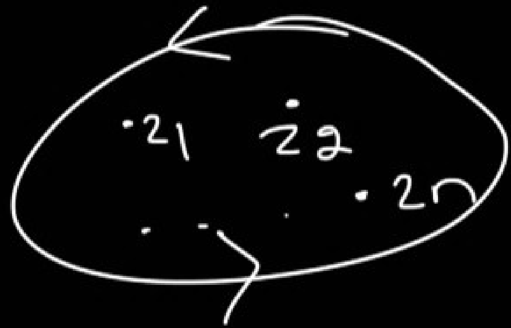

$\phi(z) \to$ Analytic func

$$\int \frac{\phi(z)}{z - z_0}\, dz = 2\pi i\, \phi(z_0)$$

$$\int \frac{\phi(z)}{(z - z_0)^{n+1}}\, dz = 2\pi i\, \frac{\phi^{n}(z_0)}{n!}$$

## Cauchy Residue Theorem:—

$$\oint_C f(z)\,dz = 2\pi i \left\{ \text{sum of residues} \right\}$$



If $z = x + jy$ where $x, y$ are real then the value of $|e^{jz}|$ is

Aditya Vangala

ACE

GATE-2009

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$|e^{j\theta}| = |\cos\theta + j\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

$$|e^{jz}| = |e^{j(x+jy)}| = |e^{jx-y}| = |e^{jx} \cdot e^{-y}|$$

$$= e^{-y}$$

Let $S$ be the set of points in the complex plane corresponding to the unit circle. i.e., $S = \{z : |z| = 1\}$. Consider the function $f(z) = zz^*$ where $z^*$ denotes the complex conjugate of $z$. The $f(z)$ maps $S$ to which one of the following in the complex plane?

GATE-2014

Aditya Vangala

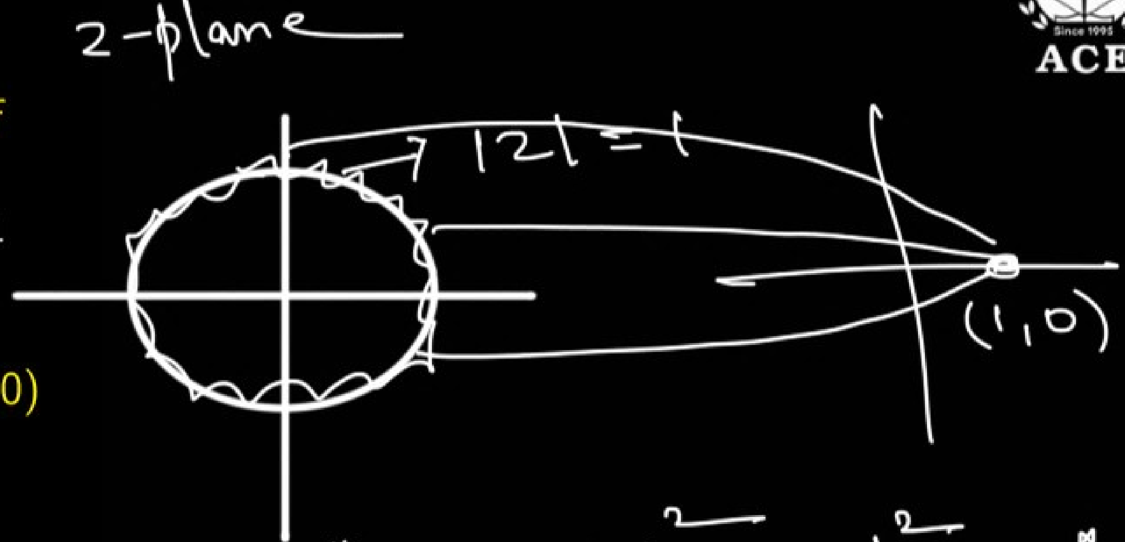
ACE

(a) unit circle

(b) horizontal axis line segment from origin to (1, 0)

(c) the point (1, 0)

(d) the entire horizontal axis




$f(z) = z \cdot z^* = |z|^2 = 1^2 = 1$

$f(z) = 1 + 0i \Rightarrow$ w-plane

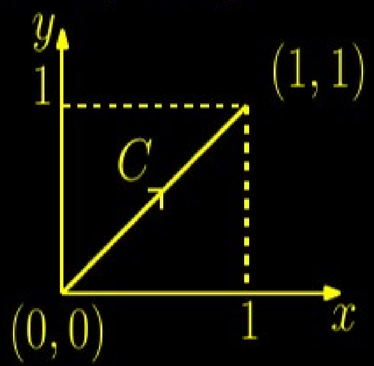
(c)

Consider the line integral

$$I = \int_C (x^2 + iy^2)\,dz$$

where $z = x + iy$. The line $C$ is shown in the figure below. The value of $I$ is (GATE-2017)

(a) $\frac{i}{2}$ (b) $\frac{2i}{3}$ (c) $\frac{3i}{4}$ (d) $\frac{4i}{5}$




$$dz = dx + i\,dy$$

$$I = \int_C (x^2 + iy^2)(dx + i\,dy)$$

Along $C$ $y = x$

$$dy = dx$$

$$I = \int (x^2 + i x^2)(dx + i\,dx)$$

$$= \int_{x=0}^{1} x^2\,dx\,(1+i)^2$$

$$= \left(\frac{x^3}{3}\right)_0^1 (1^2 + i^2 + 2i)$$

$$= (1/3)(2i) = \frac{2i}{3}$$

Aditya Vangala

ACE

Let $C$ represent the unit circle centered at origin in the complex plane, and complex variable, $z = x + iy$. The value of the contour integral $\oint_C \frac{\cosh 3z}{2z} dz$

(where integration is taken counter clockwise ) is

(GATE-2021)

(a) 0

(b) 2

(c) $2\pi i$

(d) $\pi i$

$$\frac{1}{2}\oint \frac{\cosh 3z}{(z-0)} dz$$

$z = 0$

$C$

$$\frac{1}{\cancel{2}} \cancel{2}\pi i \cosh 3(0)$$

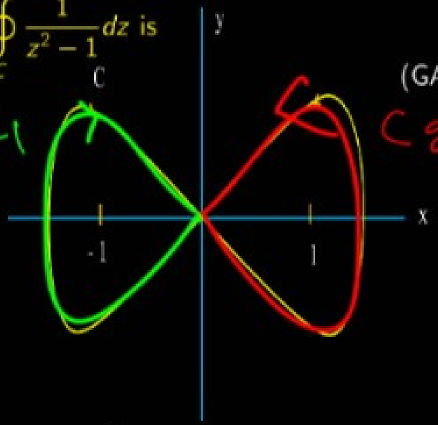
$$\pi i(1) = \underline{\underline{\pi i}}$$

The contour $C$ given below is on the complex plane $z = x + iy$ where $i = \sqrt{-1}$. The value of the integral $\frac{1}{\pi i}\oint_C \frac{1}{z^2-1}dz$ is

(GATE-2018)

Aditya Vangala

ACE




$$\frac{1}{\pi i}\oint_C \frac{1}{(z-1)(z+1)}dz = \frac{1}{\pi i}\oint_{C_1}\frac{1}{(z-1)(z+1)}dz + \frac{1}{\pi i}\oint_{C_2}\frac{1}{(z-1)(z+1)}dz$$

$$= \frac{1}{\pi i}\oint_{C_1}\frac{1/z-1}{z+1}dz + \frac{1}{\pi i}\oint_{C_2}\frac{1/z+1}{z-1}dz$$

$$= \frac{1}{\cancel{\pi i}}\left(-2\cancel{\pi i}\left(\frac{1}{-1-1}\right)\right) + \frac{1}{\cancel{\pi i}}\,2\cancel{\pi i}\left(\frac{1}{1+1}\right)$$

$$= 1 + 1 = 2$$

Aditya Vangala

ACE

Consider the integral

$$\oint_C \frac{\sin x}{x^2(x^2+4)}dx$$

where C is a counter-clockwise oriented circle defined as $|x-i|=2$. The value of the integral is

(GATE-2021)

(a) $\frac{-\pi}{8}\sin(2i)$

(b) $\frac{\pi}{4}\sin(2i)$

(c) $\frac{-\pi}{4}\sin(2i)$

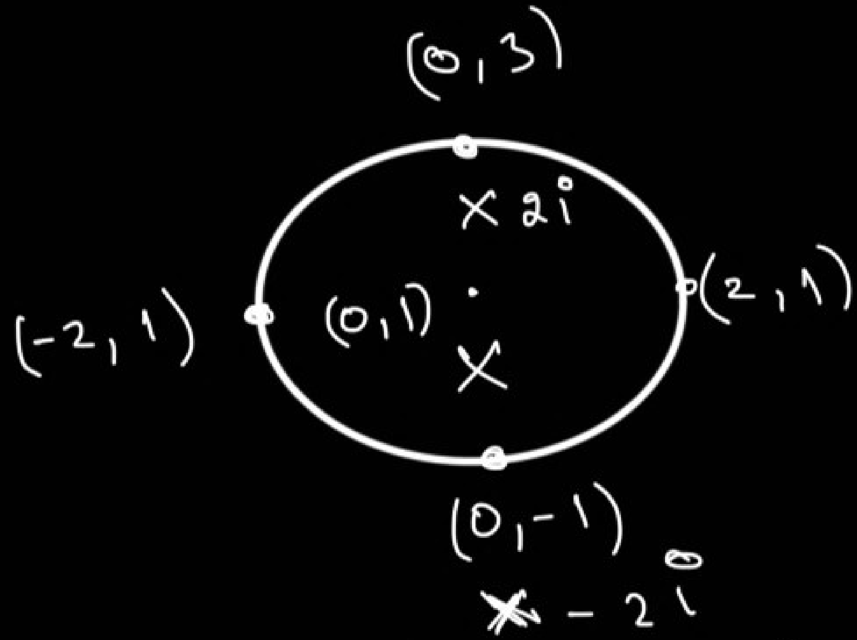
(d) $\frac{\pi}{8}\sin(2i)$

Marks to All

$|x-i|=2$

centre $(0,1)$

Radius $= 2$




$$\oint \frac{\sin x}{x^2(x^2+4)}dx = 2\pi i\left(\text{Residue at } 0 + \text{Residue at } 2i\right)$$