LOGIC

Logic is the basis of all mathematical reasoning, and of all automated reasoning.

It has the practical applications

- To the design of computing machines
- To the specification of systems
- To artificial intelligence
- To programming languages and to other areas of computer science.

Propositions:-

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, But not both.

Eg:-

1. Washington D.C. is the capital of the united states of America.

2. Bombay is the capital of India.

3. $1 + 1 = 2$

4. $2 + 2 = 3$

Eg:-

1. What time is it?

2. Read this carefully

3. $x + 1 = 2$

4. $x + y = z$

Propositional Variables : p, q, r, s, ...

Truth Values : True → T

False → F

Propositional Calculus : The area of logic that deals with propositions is called as propositional logic (or) propositional calculus.

Compound Propositions : Many Mathematical statements are constructed by combining one or propositions by using logical operators, known as compound propositions.

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Logical Operators:

$\wedge$ (and)	-	Conjunction
$\vee$ (or)	-	Disjunction
$\rightarrow$ (Implies)	-	Implication
$\leftrightarrow$ (Bi-implies)	-	Bi-conditional

Negation:

p : Mumbai is the capital of Maharashtra

~p : Mumbai is NOT the capital of Maharashtra

Truth Table:

Compound Proposition:

Construct Truth Tables of the following compound proposition:

$\sim(p \wedge q) \vee (\sim p \rightarrow \sim q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$\sim(p \wedge q) \vee (\sim p \rightarrow \sim q)$
T	T	T	F	F	F	T	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	F	~~F~~ T
F	F	F	T	T	T	T	T

Tautology

Construct Truth Tables of the following compound proposition:

$\sim (p \wedge q) \vee (\sim p \rightarrow \sim q)$

Logical Equivalence: Compound propositions that have the same truth values in all possible cases are called logically equivalent.

The compound propositions 'p' and 'q' are called logically equivalent if $p \leftrightarrow q$ is a tautology

$$p \cong q$$

$$P \cong Q$$

GATE: The binary equation □ is defined as follows

p	q	p □ q
T	T	T
T	F	T
F	T	F
F	F	T

$p \vee q$	$\sim p$	$\sim q$	$\sim q \,\square \sim p$	$p \,\square \sim q$
T	F	F	T	T
T	F	T	T	T
T	T	F	F	T
F	T	T	T	F

Then compound proposition $(p \vee q) \Leftrightarrow$

a) $\sim q \,\square \sim p$ b) $p \,\square \sim q$ c) $\sim q \,\square\, p$ d) $p \,\square\, q$

$$p \vee q \cong p \,\square \sim q$$

Tautology, Contradiction, Contingency, Satisfiable and Unsatisfiable:

Tautology $\downarrow$ t

Contradiction $\downarrow$ f

Contingency $\downarrow$ Neither

Satisfiable $\downarrow$ $\geq$ t

Unsatisfiable $\downarrow$ Not Satis (f)

Eg: Which of the following is NOT a Tautology

a) $\sim(p \rightarrow q) \rightarrow p$

b) $\sim(p \rightarrow q) \rightarrow \sim q$

c) $[\sim p \wedge (p \vee q)] \rightarrow q$

d) $(p \rightarrow q) \wedge q \rightarrow p$

(a)

$$\overbrace{\sim(p \rightarrow q)}^{\text{True}} \longrightarrow \overbrace{p}^{\text{False}}$$

$$\sim(F \rightarrow T/F) \longrightarrow F$$

$$\sim(T) \longrightarrow F$$

$$F \longrightarrow F = t = \text{NOT a tautology}$$

Eg: Which of the following is NOT a tautology

a) $(p \wedge q) \rightarrow (p \vee q)$

c) $\sim p \rightarrow (p \rightarrow q)$

b) $(p \wedge q) \rightarrow (p \rightarrow q)$

d) $p \rightarrow (p \wedge q)$ ✓

Conditional Statements:

The statement $p \rightarrow q$ is called a conditional statement, because $p \rightarrow q$ asserts that 'q' is True on the condition that 'p' holds.

The conditional statement $p \rightarrow q$ is the proposition "If p, then q". The conditional statement $p \rightarrow q$ is false when 'p' is true and 'q' is false, otherwise True.

Variety of Terminology to express $p \to q$:

"if p then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q when ever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless ~ p"	"q provided that p"

Converse, Inverse and contra positive:

Conditional : $p \longrightarrow q$

converse : $q \longrightarrow p$

Inverse : $\sim p \longrightarrow \sim q$

contra positive : $\sim q \longrightarrow \sim p$

Eg: In Triangle ABC, If AB = AC then $\angle B = \angle C$

Eg: The home Team wins whenever it is raining"

GATE:

What is the converse of the following assertion?

"I stay only if you go"

a) I stay if you go

b) If I stay then you go

c) If you do not go then I do not stay

d) If I do not stay then you go

Laws (or) Properties (or) Logical Equivalences:

	Rules	Name
1	$p \vee p \cong p$	Idempotent law
	$p \wedge p \cong p$	
2	$p \vee q \cong q \vee p$	Commutative law
	$p \wedge q \equiv q \wedge p$	
3	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	Associative law
	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	
4	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive Law
	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
5	$\sim (p \vee q) \equiv \sim p \wedge \sim q$	De Morgan's Law
	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	

1	$p \vee p \cong p$	Idempotent law
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	$p \wedge q \equiv q \wedge p$	
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	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
5	$\sim (p \vee q) \equiv \sim p \wedge \sim q$	De Morgan's Law
	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	
6	$p \rightarrow q \equiv \sim p \vee q$	Implication law

Laws (or) Properties (or) Logical Equivalences:

	Rules	Name
7	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Bi-implication law
8	$p \vee (p \wedge q) \equiv p$	Absorption law
	$p \wedge (p \vee q) \equiv p$	
9	$p \wedge t \equiv p$	Identify law
	$p \vee f \equiv p$	
10	$\sim(\sim p) \equiv p$	Double Negation (or) Involutary law
11	$p \vee \sim p \equiv t$	Negation law (or) Complements (or) Inverse law
	$p \wedge \sim p \equiv f$	

7	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$	Bi-implication law
8	$p \vee (p \wedge q) \equiv p$	Absorption law
	$p \wedge (p \vee q) \equiv p$	
9	$p \wedge t \equiv p$	Identify law
	$p \vee f \equiv p$	
10	$\sim(\sim p) \equiv p$	Double Negation (or) Involutary law
11	$p \vee \sim p \equiv t$	Negation law (or) Complements (or) Inverse law
	$p \wedge \sim p \equiv f$	
12	$p \vee t \equiv t$	Domination law
	$p \wedge f \equiv f$	

$p \vee t \equiv t$

FIRST - ORDER LOGIC

* First-order logic was quantified variables over non-logical objects and allows the use of sentences that contain variables.
* First-order logic is another way of knowledge representation in artificial intelligence.
* It is an extension propositional logic
* First-order is also known as predicate logic (or) First-order predicate logic.
* First-order logic is a powerful language that develops information about the objects in a more easy way and can also express relationship between those objects.

Argument:

The argument is a set of statements (or) propositions which contains premises and conclusion.

All the statements are called "premises" except the final one. The last statement is called conclusion.

$$\underbrace{[p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots \wedge p_n]}_{\text{true}} \Longrightarrow q$$

$$\begin{array}{c} a \\ b \\ c \\ \hline \therefore z \\ \hline \end{array}$$

$a \wedge b \wedge c = \geq t$

$[(a \wedge b \wedge c) \rightarrow z]$

tautology

Representation:

1) $(p_1 \wedge p_2 \wedge p_3 \wedge p_4 \,.\,.\,.\,.\,.\,.\,.\,.\, \wedge p_n) \Rightarrow q$

2)
$$\begin{array}{l} p1 \\ p2 \\ p3 \\ . \\ . \\ . \\ p_n \\ \hline \therefore\ q \\ \hline \end{array}$$

3) $\{p_1, p_2, p_3, p_4, \ldots\ldots, p_n\} \rightarrow q$

Consistency of Premises:

The premises $p_1, p_2, p_3, \ldots\ldots, p_n$ are said to be consistent iff their conjunction $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots\ldots \wedge p_n)$ is true in atleast one possible situation, otherwise they are in-consistent (NOT consistent).

Q. In which of the following cases the premises are NOT inconsistent

✗ a) $\{P \to Q, P \to R, Q \to \sim R, P\}$

✗ b) $\{R \veebar M, \sim R \veebar S, \sim M, \sim S\}$ $R \vee M$

✓ c) $\{P \to Q, Q \to R, \sim R \vee S, \sim P \to \sim S, \sim S\}$

✓ d) $\{Q \to P, \sim R, P \to S, Q \vee R\}$

$$ⓐ \quad \{ \overbrace{\underbrace{\underset{T \to T}{P \to Q}}_{T}}^{T\checkmark}, \overbrace{\underbrace{\underset{T \to T}{P \to R}}_{T}}^{T}, \overbrace{\underbrace{\underset{T \to F}{Q \to \sim R}}_{F}}^{T\checkmark}, \overbrace{\underbrace{\underset{T}{P}}_{T}}^{T} \}$$

$$T \wedge T \wedge F \wedge T = F$$

NOT consistent

Validity of the Argument:

An argument is said to be valid. If the conclusion can be derived from the premises by applying the rules of inference.

i.e., An argument $(p_1 \wedge p_2 \wedge p_3 \wedge \dots\dots\dots \wedge p_n) \rightarrow q$ is valid if It is a tautology.

Rules of Inference:

Sl. No	Rule	Name
1	$\dfrac{p}{\therefore p \vee q}$	Addition
2	$\dfrac{p \wedge q}{\therefore p}$	Simplification
3	$\dfrac{p \quad q}{\therefore p \wedge q}$	Conjunction

Rules of Inference:

$F \quad T$

$p \vee q = T$

$\sim p = T$

$\therefore q$

p = Father earns

q = Mother earns

Sl. No	Rule	Name
4	$p \vee q$ $\sim p$ $\therefore q$	Disjunctive Syllogism
5	$p \vee q$ $\sim p \vee r$ $\therefore q \vee r$	Resolution
6	$p \rightarrow q$ p $\therefore q$	Modus Ponens

Rules of Inference:

Sl. No	Rule	Name
7	$p \rightarrow q$ $\sim q$ $\therefore \sim p$	Modes Tollens
8	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	Hypothetical Syllogism
9 ✓	$p \vee Q$ $p \rightarrow r$ $Q \rightarrow r$ $\therefore R$	Dilema

$$\begin{array}{l} a \longrightarrow b \\ c \longrightarrow b \\ \underline{a \vee c} \\ \therefore b \end{array}$$

a : Mother – $\overset{b}{\text{you}}$

c : Father – you

$\underline{a \vee c}$

$\therefore b$

Rules of Inference:

Sl. No	Rule	Name
10	$p \to Q$ $R \to S$ $p \vee R$ $\therefore Q \vee S$	Constructive Dilemma
11	$p \to Q$ $R \to S$ $\sim Q \vee \sim S$ $\therefore \sim p \vee \sim R$	Destructive Dilemma

Consider the following logical inferences

I_1 : If it rains then the cricket match will not be played.

p (written under "it rains"); q (written under "will not be")

The cricket match was played.

Inference : There was no rain

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \\ \hline \end{array}$$

valid by Tollen's

I_2 : If it rains then the cricket match will not be played.

It did not rain.

Inference : The cricket match was played.

$$\begin{array}{l} p \rightarrow q \\ \sim p \\ \hline \therefore \sim q \\ \hline \end{array}$$

X

I_2 : If it rains then the cricket match will not be played.

It did not rain.

Inference : The cricket match was played.

Which of the following is TRUE?

a) Both I_1 and I_2 are correct inferences

b) I_1 is valid, But NOT I_2

c) I_2 is valid, But NOT I_1

d) Neither I_1 nor I_2 is valid

Q. Check the validity of the following argument

$$\{p \rightarrow (r \rightarrow s), \sim r \rightarrow \sim p, p\} \Rightarrow s$$

Q. Check the validity of the following argument

$$\{\sim t \rightarrow \sim r,\ t \rightarrow w,\ r \vee s,\ \sim s\} \Rightarrow w$$

Q. Check the validity of the following arguments

$I_1 : \{R \rightarrow S, P \rightarrow Q, R \vee P\} \Rightarrow (S \vee Q)$

$$\overbrace{\underbrace{\sim R \rightarrow (S \rightarrow \sim T)}_{F} \wedge \underbrace{\sim R \vee w}_{T} \wedge \underbrace{\sim P \rightarrow S}_{T} \wedge \underbrace{\sim w}_{T}}^{F}$$

$I_2 : \{\sim R \rightarrow (S \rightarrow \sim T), \sim R \vee w, \sim P \rightarrow S, \sim w\} \Rightarrow (T \rightarrow P)$

$T \rightarrow (T \rightarrow F)$ $\quad$ $T \vee F$ $\quad$ $T \rightarrow T$ $\quad$ T $\quad$ $\underbrace{\underbrace{T \rightarrow F}_{F}}_{T}$

$\sim R \vee w$

$\sim w$

$\therefore \sim R$

$\sim R \rightarrow (S \rightarrow \sim T)$

$\therefore S \rightarrow \sim T$

$\sim P \rightarrow S$

$\sim P \rightarrow \sim T \equiv T \rightarrow P$

Quantifiers:

Quantifiers are the words the refers to Quantities such as some or all,

and indicates how frequently a certain statement is true.

Types:

$\forall x\, P(x) = P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \cdots = \text{True}$

1.“Universal” (∀) quantifier

$\exists x\, P(x) = P(x_1) \vee P(x_2) \vee P(x_3) \vee \cdots = \text{True}$

2. “There existential” (∃) quantifier

Universal Quantifier:

Let P(x) be statement defined on universe of discourse A, then the universal quantification P(x) is the statement P(x) is true for all x belongs to A

P(x) is true

Negation of Quantifiers:

Example:

Q. Consider the predicate p(x) indicates "x is happy" where universe of discourse is set of all students in a class

a) $\exists x\ p(x)$

b) $\forall \sim p(x)$

c) $\exists x \sim p(x)$

d) $\sim[\forall x \sim p(x)]$

Q. Consider the following predicates

$G(x)$: x is gold ornament

$S(x)$: x is silver ornament

$P(x)$: x is precious

Q. Express the following into appropriate predicate logic "Gold and Silver ornaments are precious".

a) $\forall x[P(x) \rightarrow (G(x) \wedge S(x)]$

b) $\forall x[(G(x) \wedge S(x) \rightarrow P(x)]$

c) $\forall x[(G(x) \vee S(x) \rightarrow P(x)]$

d) $\exists x[(G(x) \vee S(x) \rightarrow P(x)]$

Q. What is the logical translation of the following statement?

"None of my friends are perfect"

a) $\exists x[F(x) \wedge \sim P(x)]$

b) $\exists x[\sim F(x) \wedge \sim P(x)]$

c) $\exists x[\sim F(x) \wedge P(x)]$

d) $\sim \exists x[F(x) \wedge P(x)]$

All my friends are NOT perfect

$\forall x\,[F(x) \longrightarrow \sim P(x)]$ $a \rightarrow b \equiv \sim a \vee b$

$\equiv \forall x\,[\sim F(x) \vee \sim P(x)]$

$\equiv \sim [\exists x\,(F(x) \wedge P(x))]$

Q. Which of the following is NOT logically equivalent to

$\sim \exists x[\forall y(\alpha) \wedge \forall z(\beta)]$?

$\sim \exists x\,[\forall y\,(\alpha) \wedge \forall z\,(\beta)]$

a) $\forall x[\exists z\,(\sim \beta) \rightarrow \forall y\,(\alpha)]$

$\equiv \forall x\,[\exists y\,(\sim \alpha) \vee \exists z\,(\sim \beta)]$

b) $\forall x[\forall z\,(\beta) \rightarrow \exists y\,(\sim \alpha)]$

$\equiv \forall x\,[\forall y\,\alpha \longrightarrow \exists z\,(\sim \beta)]$

c) $\forall x[\forall y\,(\alpha) \rightarrow \exists z\,(\sim \beta)]$

$\equiv \forall x\ \forall z\,(\beta) \rightarrow \exists y\,(\sim \alpha)$

d) $\forall x[\exists y(\sim \alpha) \rightarrow \exists z\,(\sim \beta)]$

$\sim a \vee b \equiv a \rightarrow b$

$x \rightarrow y \equiv \sim y \rightarrow x$

Relations

Cartesian Product: Cartesian product of two sets A and B, written as ‘$A \times B$’ and is defined as

$$A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$$

$$A \times B \times C = \{(a, b, c) / a \in A, b \in B \text{ and } c \in C\}$$

$R \subseteq A \times B$

Relation: A (binary) relation 'R' from A to B is a subset of $A \times B$. If $A = B$, then we can say 'R' is a relation on A

$$R : A \rightarrow A$$

Note:

* If $|A| = m$ and $|B| = n$ then the number of relations possible from A to B $= 2^{mn}$.

* If $|A| = n$ then the number of relations possible on set A $= 2^{n\times n} = 2^{n^2}$

Domain and Range:

If 'R' is a relation from 'A' to 'B'. Then Domain of R = $\{x/x \in A$ and $(x, y) \in R$ for some $y \in B\}$

Range of

$R = \{y/y \in B$ and $(x, y) \in R$ for some $x \in A\}$

Inverse Relation:

$$R^{-1} = \{(b, a) / (a, b) \in R\}$$

Complementary relation: If R is a relation from A to B then the complement of 'R' is

$$\overline{R} = \{(a, b) /(a, b) \notin R\}$$

Types of Relations

Reflexive: A relation 'R' on set A is said to be reflexive if, $x \, R \, x, \forall x \in A$

i.e., $(x, x) \in R \; \forall x \in A$

Examples:

$A = \{1, 2, 3, 4\}$ ✓

$R : A \to A$

$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ ✓

$R_2 = \{(1, 1), (2, 2), (3, 3)\}$ ✗

$R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3)\}$ ✓

$R_4 = \{ \}$ ✗

Symmetric Relation: A relation R on a set A is said to be symmetric

if x R y then y R x $\forall$ x, y $\in$ A

if $(x, y) \in R$ then $(y, x) \in R$ $\forall x, y \in A$

Examples:

$A = \{a, b, c\}$

$R_1 = \{(a, b), (b, a)\}$ ✓

$R_2 = \{(a, b), (b, a), (a, c)\}$ ✗

$R_3 = \{\ \}$ ✓

$R_4 = A \times A$ ✓

Transitive Relation: A relation 'R' on a set A is said to be transitive if (xRy) and (yRz) then (xRz)

$$\forall\, x, y, z \in A$$

If (x, y) and $(y, z) \in R$. Then $(x, z) \in R$

Examples:

Let $A = \{a, b, c, d\}$

$R_1 = \{(a, b), (b, c), (a, \overset{c}{b})\}$

$R_2 = \{(a, a), (b, b)\}$

$R_3 = \phi$

$R_4 = A \times A$

Equivalence: A relation (R, ≤) is said to be an equivalence relation if it is reflexive, symmetric and Transitive.

$$A = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$R_4 = A \times A$$

$$B_4 = 15$$

Compatable = Reflexive & Symmetric

Irreflexive: A relation R on set A is said to be irreflexive if $(x, x) \notin R, \forall x \in A$

$$A = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 2), (1, 3), (2, 4), (4, 3)\}$$

$$R_2 = \{\ \}$$

$$R_3 = (A \times A) - \Delta_A$$

$$R_4 = A \times A$$

$$R_5 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3)\}$$

$$R_6 = \{(1, 1), (2, 2), (2, 3)\}$$

Not reflexive, Not Irreflexive } non-reflexive.

Diagonal Relation: Let A be any set, The diagonal relation on A consists of all ordered pairs (a, b) such that a = b

i.e., $\Delta_A = \{(a, a) / a \in A\}$

If $A = \{1, 2, 3\}$ then $\Delta_A = \{(1, 1), (2, 2), (3, 3)\}$ ✓

$R : A \to A = \{(1, 1), (2, 2), (3, 3), (1, 2), (3, 2)\}$ ✗

Asymmetric Relation: A relation R on a set 'A' is said to be asymmetric, if

If $(x, y) \in R$ then $(y, x) \notin R \quad \forall\, x, y \in A$

$R_1 = \{(1, 2), (1, 3), (3, 2)\}$ ✓

$R_2 = \{(1, 2), (2, 2), (2, 1)\}$ ✗

$R_3 = \{(2, 2), (3, 3)\}$ ✗

$R_4 = \{\ \}$ ✓

Anti-Symmetric: A relation R on a set 'A' is said to be anti symmetric

$$\boxed{\text{If } (\underset{1\,R\,2}{x\,R\,y}) \text{ and } (\underset{2\,R\,1}{y\,R\,x})} \text{ then } \underset{1 \neq 2.}{x = y} \qquad \forall x, y \in A$$

Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 2), (1, 3)\}$ ✓

$R_2 = \{(1, 2), (2, 1)\}$ × $\quad 1 \neq 2$

$R_3 = \{(2, 2), (3, 3)\}$ ✓

$R_4 = \{\ \}$ ✓

Partial-order.

Reflexive.

anti-sym.

transitive.

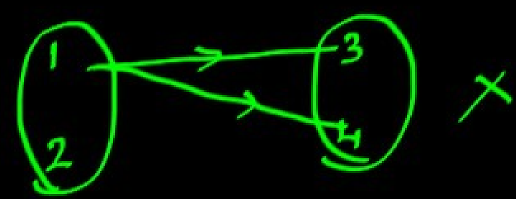
Lattice : $[A : R]$

LUB = Join ($\vee$)

GLB = Meet ($\wedge$)

Functions

Functions: A relation is said to be a function, If every element of its domain has unique order pair



Classification of Functions:

i) One-one (Injective):

A function in which different elements of domain have different images in co-domain, is known as one-one function.

$$^{n}P_{m}$$

$$\begin{array}{ccc} a & \longrightarrow & 1 \\ b & \longrightarrow & 2 \\ c & \longrightarrow & 3 \\ d & \longrightarrow & 4 \\ & & 5 \\ & & 6 \\ m & \leq & n \end{array}$$

$$n \geq m$$

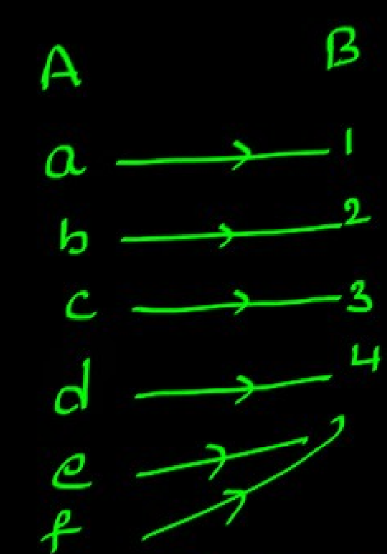
m $\qquad$ n

$$n^{m}$$

ii) Onto (Surjective):-

A function in which its range is same as co-domain is known as onto function.

$$f(A) = B$$



$$m \geq n$$

Range = {1, 2, 3

codomain = {1, 2, 3, 4

$$m \geq n$$

* Number of onto functions from m-element set to n-element set

$$= n^m - {}^nC_1.(n-1)^m + {}^nC_2.(n-2)^m - {}^nC_3.(n-3)^m + \dots\dots\dots + {}^nC_{n-1}(1)^m$$

$n^m - nc_1 (n-1)^m + nc_2 (n-2)^m - \dots + nc_{n-1}$

Bijection: A function which is both one-one and onto is called as a Bijective function.

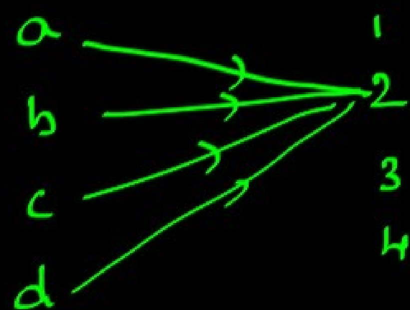
$m = n$

Identity:

A function $I : A \rightarrow A$ is called an identity function

iff $I(x) = x, \forall x \in A$

Constant Function: A function in which range is a singleton set is known as constant function.



$$f(x) = c, \quad \forall x$$

Inverse Function:

Composite Function:

$$f : A \longrightarrow B$$

$$g : B \longrightarrow C$$

$$g \circ f : A \longrightarrow C$$

Q. Suppose X and Y are sets and |X| and |Y| are their respective cardinalities. It is given that there are exactly 97 functions from X to Y. From this one can conclude that

(GATE-96)

a) $|X| = 1, |Y| = 97$

b) $|X| = 97, |Y| = 1$

c) $|X| = 97, |Y| = 97$

d) None of the above

No. of functions $= n^m = |y|^{|x|} = (97)^1$

$|x| = 1, \quad |y| = 97.$

Algebraic Structure

Algebraic Structure: A non-empty set which is equipped with some operations and some properties is known as algebraic structure.

$$(S, *)$$

- Groupoid
- Semi-group
- Monoid
- Group
- Abelian Group

Some Properties:

I. Closure: Let 'S' be the given algebraic structure, '*' is the binary operation and a, b are any two elements in S,

If $a * b \in S$ then we can say (S, *) follows closure property.

$$\forall a, b \in S, \quad a * b \in S$$

$Z = \{\dots -2, -1, 0, 1, 2, \dots\}$

$+$

$2+3 \in Z$

II. Associative:

$$\forall a, b, c \in S, \quad a * (b * c) = (a * b) * c$$

$1+(2+3) = (1+2)+3$

III. Identity:

$$\forall a \in S, \exists e \in S, \ni$$

$$a * e = e * a = a$$

$\exists e \in S, \forall a$

$a*e = e*a = a$

$1+0=1$

$-3+0=-3$

$0+0=0$ e

IV. Inverse:

$\forall a \in S, \quad \exists b \in S, \quad \ni$

$$\boxed{a * b = b * a = e}$$

$5 + \quad = 0$

$6 + \quad = 0$

$-7 + \quad = 0$

V. Commutative:

$\forall a, b \in S$

$$a * b = b * a$$

$2+3 = 3+2$

Classification of Algebraic Structure

Groupoid (1)	Semi-group (2)	Monoid (3)	Group (4)	Sub-Group (4)	Abelian (5)
1) Closure	1) Closure	1) Closure	1) Closure	1) Closure	1) Closure
	2) Associative	2) Associative	2) Associative	2) Associative	2) Associative
		3) Identity	3) Identity	3) Identity	3) Identity
			4) Inverse	4) Inverse	4) Inverse
				$H \subseteq G$	5) Commutative
				$(H,*) \subseteq (G,*)$	

$(Z,+) \subseteq (R,+)$

Sub-Group:

Let (G, *) be a group, H is a subset of 'G' and (H, *) is also group then we can say (H, *) is a subgroup of (G, *)

$$(H, *) \subseteq (G, *)$$

Order of Group (Vs) Order of element:

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* The number of the elements (Cardinality) of a given group is known as order of group, O(G)

* Let (G, *) be a group and an element $a \in G$,

If $\boxed{a^n = e,}$ (where n is a least positive integer). Then 'n' is called order of the element 'a'

$G = \{0, 1, 2, 3, 4, 5\}$ $+_6$

$O(G) = 6$

$O(4) = 3$

Take 4

$4^1 = 4$

$4^2 = 4 * 4 = 4 +_6 4 = 2$

$4^3 = 4^2 * 4 = 2 +_6 4 = 0 = e$

$4^3 = 0 = 4^6 = 4^9 = 4^{12} = ---$

Take 5

$5^1 = 5$

$5^2 = 5 * 5 = 5 +_6 5 = 4$

$5^3 = 5^2 * 5 = 4 +_6 5 = 3$

$5^4 = 5^3 * 5 = 3 +_6 5 = 2$

$5^5 = 5^4 * 5 = 2 +_6 5 = 1$

$5^6 = 5^5 * 5 = 1 +_6 5 = 0$

$o(5) = 6$

$G = \{0, 1, 2, 3, 4, 5\}$

$= \{5^6, 5^5, 5^4, 5^3, 5^2, 5^1\}$

$G = \langle 5 \rangle$

$G = \langle 1 \rangle$

$O(0) \longrightarrow 1$

$O(1) \longrightarrow 6$

$O(2) \longrightarrow 3$

$O(3) \longrightarrow 2$

$O(4) \longrightarrow 3$

$O(5) \longrightarrow 6$

$\boxed{O(e) = 6}$

Divides

$a * b = e$

$5 +_6 1 = 0 = e$

$O(e) = 9 \longrightarrow 1, 3, 9$

$\cancel{2}, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}, \cancel{8}$

Lagrange's

$O(a) \mid O(e)$

$O(H) \mid O(e)$

Generators & Cyclic Group:

Let (G, *) be a group and a $\in$ G, If every element of (G, *) can be expressed as integral power of 'a' then 'a' is called generator of 'G' and group (G, *) is known as cyclic group.

NOTE:

- Every cyclic group is an abelian group. $a*b=b*a$. $\forall a,b$
  gene 4
- If ‘a’ is the generator of group (G, *) then a^{-1} is also generator.
- Every group of order ≤ 6 is an abelian. $\{a,b,c,d\}$
- If (G,*) be a group, $(a*b)^{-1} = b^{-1}*a^{-1}$, $\forall$ a, b $\in$ G.
- Identity element of a group is Unique.
- In a group inverse of a each element is Unique.
- In a group G, if every element has its own inverse then G is abelian. $a^{-1}=a$ $c^{-1}=c$ $b^{-1}=b$
- If group G is commutative iff $(a*b)^2 = a^2 * b^2$, $\forall$ a, b $\in$ G.

$[D_{12} : |]$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

Upper Bound $(2,3) = 6, 12$

Least upper Bound $(2,3)$

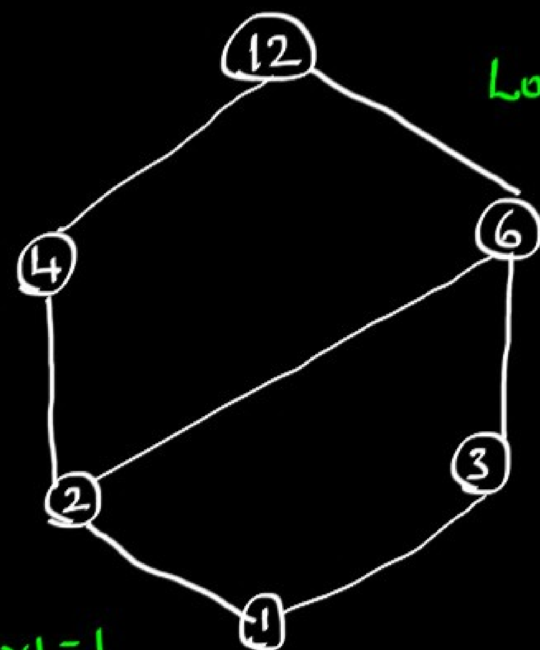
$= 2 \vee 3 = 6$

$LUB(1, x) = x$

$1 \vee 2 = 2$ $1 \vee 4 = 4$

$1 \vee 3 = 3$ $1 \vee 6 = 6$ $1 \vee 1 = 1$

$1 \vee 12 = 12$



Lower Bound $(4,6) = 2, 1$

$GLB(4,6) = 4 \wedge 6 = 2$

* Propositional logic
* First-order logic
* Sets
* Relations
* Partial-ordering
* Functions
* Groups
* Combinatorics
* Graphs

Combinatorics

Combinatorics: Combinatorics is the branch of mathematics, which concern the study of finite (or) countable (or) Discrete Structures.

Principle of Mutual Inclusion & Exclusion:

* $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

* $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

* $n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D) - n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D) - n(C \cap D) + n(A \cap B \cap C) + n(A \cap B \cap D) + n(A \cap C \cap D) + n(B \cap C \cap D) - n(A \cap B \cap C \cap D)$

Q. In a class of 200 students, 125 students have taken Programming Language course, 85 students have taken Data Structures course, 65 students have taken Computer Organization course; 50 students have both Programming Language and Data Structures, 35 students have taken both Programming Languages and Computer Organization; 30 students have taken both Data Structures and Computer Organization; 15 students have taken all the three courses.

How many students have not taken any of the three courses?

$n(A \cup B \cup C) = 125 + 85 + 65 - 50 - 35 - 30 + 15$

$= 175$

(GATE-IT-04)

a) 15

b) 20

c) 25

$n(A \cup B \cup C)^c = 200 - 175$

$= 25$

d) 35

Sum Rule:

If E_1, E_2, E_3 are Mutually exclusive events, and E_1 can happen in e_1 ways, E_2 can happen in e_2 ways, …….., E_n can happen in e_n ways, then (E_1 or E_2 or …. Or E_n) can happen in $(e_1 + e_2 + \ldots.. + e_n)$ ways.

$(m + n)$ ways

$3 \times 2 = 6$ ways

Two subtasks = $(m * n)$ ways

Product Rule:

If the independent events E_1, E_2, ……., E_n can happen in e_1, e_2, ……, e_n ways respectively, then the sequence of events E_1 first, followed by E_2, ….., followed by E_n can happen in (e_1, e_2, …., e_n) ways.

Permutation Without Repetition:

A permutation of n-objects taken 'r' at a time is an "ordered selection" (or) "arrangement" of 'r' of the objects.

$$P(n, r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Permutation With Repetition:

The number of r-permutations of n-objects with unlimited repititions is denoted by U(n, r)

$$U(n, r) = n^2 \quad n^r$$

The number of permutations of n objects of which 'n_1' are alike n_2 are alike, …… n_r are alike is

$$\frac{n!}{n_1!n_2!.......n_r!}$$

Combination Without Repetition: A combination of ‘n’ objects taken ‘r’ at a time is an unordered selection of objects.

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Combination With Repetition:

The number of ways of selecting r-objects from available n-objects with unlimited repetition is known as combination with repetition.

$$V(n, r) = C(n + r - 1, r) = {}^{n+r-1}C_r$$

$${}^{n+r-1}C_r.$$

Q. There are 5 bags labeled 1 to 5. All the coins in given bag have the same weight. Some bags have coins of weight 10 gm, others have coins of weight 11 gm. I pick 1, 2, 4, 8, 16 coins respectively from bags 1 to 5. Their total weight comes out to 323 gm. Then the product of the labels of the bags having 11 gm coin is ______ **(GATE-14-Set1)**

$1 + 2 + 4 + 8 + 16 = 31$

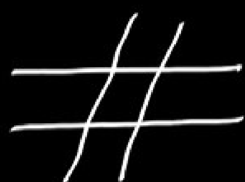
1 ✓ 2 3 ✓ 4 ✓ 5

$31 \times 10\text{ gms} = 310 \xrightarrow{+13} 323$

$31 \times 11\text{ gms} = 341$

$1 \times 3 \times 4 = 12$

Q. A set of 5 parallel lines intersect with another set of 6 parallel lines, Find the possible number of parallelograms in this setup



$$ {}^{5}C_2 * {}^{6}C_2 $$

$$ = 10 * 15 $$

$$ = 150 $$

Division and Distribution:

I. Group sizes are fixed

II. Group sizes equal

III. Group are not fixed

$$\begin{array}{ccc} & 10 & \\ \swarrow & & \searrow \\ 4 & & 6 \end{array}$$

Division

$$\frac{10!}{4!\,6!}$$

Distribution

$$\frac{10!}{4!\,6!} * 2!$$

$$\begin{array}{ccc} & 10 & \\ \swarrow & & \searrow \\ 5 & & 5 \end{array}$$

$$\frac{10!}{5!\,5!\,2!}$$

$$\frac{10!}{5!\,5!\,2!} * 2!$$

10 objects among 3 group

10 chocolates 3 girls $= 3 * 3 * 3 * \ldots = 3^{10}$

Q. How many ways we can distribute $\underbrace{\text{100 distinct}}_{\text{choc}}$ letters among $\underbrace{\text{10 boxes}}_{\text{girls}}$

$$g^c = 10^{100}$$

Q. There are 6 jobs with distinct difficulty levels, and 3 computers with distinct processing speeds. Each job is assigned to a computer such that:

- The fastest computer gets the toughest job and the slowest computer gets the easiest job

- Every computer gets the easiest job

The number of ways in which this can be done is ________

(GATE-21-Set1)

$6-2=$

4 Jobs

$4C_1 * 3C_2$

$\Rightarrow C_1 =$ Fast

$\Rightarrow C_2 =$ slowest

$\rightarrow C_3$

$3^4 - 2^4 = 81 - 16 = 65$

C_3 NOT

3^4

C_1	C_2	C_3
4	0	0
0	0	4 ✓
1	1	2 ✓
2	2	0
3	1	0
2	0	2
⋮		

$4C_1 * 3C_2$

2^4

C_1	C_2	C_3
2	2	0
3	1	0
1	3	0
4	0	0
0	4	0

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Q. n couples are invited to a party with the condition that every husband should be accompanied by his wife. However, a wife need not be accompanied by her husband. The number of different gathering possible at the party is **(GATE-CS-03)**

a) $\binom{2n}{n} * 2^n$

b) 3^n

c) $\frac{(2n)!}{2^n}$

d) $\binom{2n}{n}$

1) H + W

2) W

3) None

$3 * 3 * 3 * \cdots$ (n-times)

$= 3^n$

Note:

- Number of ways of distributing 'n'-identical objects among r-persons each person gets at least one object = ${}^{(n-1)}C_{(r-1)}$ ✓

* Number of ways of distributing n-identical object among r-persons = ${}^{n+r-1}C_{r-1}$ ✓

Q. In how many ways can b blue balls and r red balls be distributed in n distinct boxes? **(GATE-IT-08)**

a) $\frac{(n+b-1)!(n+r-1)!}{(n-1)!b!(n-1)!r!}$ ✓

b) $\frac{(n+(b+r)-1)!}{(n-1)!(n-1)!(b+r)!}$

c) $\frac{n!}{b!r!}$

d) $\frac{(n+(b+r)-1)!}{n!(b+r-1)!}$

n - balls among r - boxes = ${}^{n+r-1}C_{r-1}$

b - blue balls among n - boxes = ${}^{b+n-1}C_{n-1} = \frac{(b+n-1)!}{(n-1)!\,b!}$ ✓

r - red balls among n - boxes = ${}^{r+n-1}C_{n-1} = \frac{(r+n-1)!}{(n-1)!\,r!}$ ✓

Pigeon-hole principle:

If n pigeonholes are occupied by n+1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon.

Generalized pigeonhole principle is: - If n pigeonholes are occupied by kn+1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k+1 or more pigeons.

$(n+1) \longrightarrow n$ holes

Theorem-

I) If "A" is the average number of pigeons per hole, where A is not an integer then

- At least one pigeon hole contains **ceil[A]** (smallest integer greater than or equal to A) pigeons
- Remaining pigeon holes contains at most **floor[A]** (largest integer less than or equal to A) pigeons

Q. A bag contains 4 red balls, 5 green balls, 6 blacks. Find the minimum no.of balls need to from the bag, that guarantees 4 balls are of the same colour?

No of colors = 3

$$\left\lceil \frac{x}{3} \right\rceil = 4$$

$$x = 10$$

Euler Function:

If ‘n’ is a positive integer then $\phi(n)$ = The number of integers x such that $1 \le x \le n$ and ‘n’ and ‘x’ are relatively prime (co prime)

$\phi(n)$ = Number of positive integers, which are less than ‘n’ and co-primes to ‘n’.

$$\phi(n) = n \times \left(1 - \frac{1}{P_1}\right) \times \left(1 - \frac{1}{P_2}\right) \times \left(1 - \frac{1}{P_3}\right) \times \cdots \ldots \ldots ..$$

Number $n = P_1^{\alpha_1} \times P_2^{\alpha_2} \times P_3^{\alpha_3} \times \ldots\ldots\ldots$

P_i = Prime

$\alpha_i \in N$

$12 \longrightarrow 1, 5, 7, 11$

Co-primes $(a, b) = HCF(a,b) = 1$

$$12 = 2^2 \times 3^1$$

$$\phi(12) = 12\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)$$

$$= \overset{4}{\cancel{12}} \times \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}}$$

$$= 4$$

Properties of Euler Function:

I. $\phi(P) = P - 1$, where P is prime

$\phi(13) = 12$

Eg: $\phi(23) = 22$

II. $\phi(m \times n) = \phi(m) \times \phi(n)$, where $\gcd(m, n) = 1$

$\phi(3\times 5) = \phi(3) \times \phi(5)$

$= 2 \times 4$

$= 8$

Eg: $\phi(21) = \phi(3 \times 7)$

$= \phi(3) \times \phi(7) = 2 \times 6 = 12$

III. $\phi(P^n) = P^n - P^{n-1}$, where P is prime

Eg: $\phi(8) = \phi(2^3) = 2^3 - 2^2$

$= 8 - 4 = 4$

$\phi(32) = \phi(2^5) = 2^5 - 2^4$

$= 32 - 16 = 16$

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Q. Number of positive integers which are less than 1368 and co-prime to 1368 is ______

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$\phi(1368)$

$1368 = 2^3 \times 3^2 \times 19^1$

$\phi(1368) = 1368 \times \frac{1}{2} \times \frac{2}{3} \times \frac{18}{19}$

$= 432$

Derangements:

Among the permutations of {1, 2, ….., n} there are some, called **derangements,** in which none of the n integers appears in its natural place (correct place).

D_n = The number of derangements of n distinct objects.

$$D_n = n!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots\ldots + \frac{(-1)^n}{n!}\right\} ✓$$

$$\approx n!.e^{-1}$$

3 objects = a b c

1) a b c ✓

2) a c b ✓

3) b a c ✓

4) $\boxed{\text{b c a}}$ *

5) $\boxed{\text{c a b}}$ *

6) c b a ✓

$D_3 = 2$

In particular, $D_5 = 5!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\} = 44$

$D_6 = 265,$

$D_4 = 9,$

$D_3 = 2,$

$D_2 = 1, D_1 = 0$

Note: (1) $D_n = nD_{n-1}+(-1)^n$, $(n \geq 2)$

(2) $D_n = (n-1)\{D_{n-1} + D_{n-2}\}, (n \geq 3)$

$D_n = n\,D_{n-1} + (-1)^n$

$D_n = (n-1)\left(D_{n-1} + D_{n-2}\right)$

$n \geq 3$

$D_7 = 1854$

Q. In how many ways can we distribute 5 distinct balls, $B_1, B_2, \ldots, B_5$ in 5 distinct cells, $C_1, C_2, \ldots, C_5$ such that Ball B_i is not in cell C_i, $\forall$ i = 1, 2, …., 5 and each cell contains exactly one ball? **(GATE-04)**

a) 44 ✓

b) 96

c) 120

d) 3125

$$D_5 = 4(D_4 + D_3)$$
$$= 4(9+2)$$
$$= 44$$

Q. How many ways we can put 5 letters L_1, L_2, L_3, L_4, L_5 in 5 envelops e_1, e_2, e_3, e_4 and e_5 (at one letter per envelope) so that

i) No letter is correctly place is D_5

ii) At least one letter correctly place is $5! - D_5$

iii) Exactly two letters are correctly place is ${}^5C_2 * D_3$

iv) At most one letter is correctly placed is $({}^5C_1 * D_4) + D_5$

v) At least one letter is wrongly placed is $5! - 1$

vi) Exactly one letter is wrongly placed is 0

$\frac{L_5}{e_1}\ \frac{L_2}{e_2}\ \frac{L_3}{e_3}\ \frac{L_4}{e_4}\ \frac{L_1}{e_5}$

$5C_2 * 1$

$= 10$

Recurrence Relations

Let $\{a_0, a_1, a_2, \ldots\ldots, a_n\}$ be a sequence of real numbers, A formula that relates 'a_n' with one (or) more of the previous term is called a recurrence relation

S_n = Sum of first n-terms

$$= 1 + 2 + 3 + \ldots\ldots\ldots + (n-2) + (n-1) + n$$

$$S_n = S_{n-1} + n$$

A.P. : $a, a + d, a + 2d, a + 3d, a + 4d, \text{---------}, a + (n - 1)d$

$$t_n = t_{n-1} + d$$

G.P. : $a, a\,r, ar^2, ar^3, \text{---------}, ar^{n-1}$

$$t_n = t_{n-1} * r$$

Q. Let a_n be the number of n-bit strings that do NOT contain two consecutive 1's. Which one of the following is the recurrence relation for a_n?

(GATE-16-Set1)

a) $a_n = a_{n-1} + 2a_{n-2}$

b) $a_n = a_{n-1} + a_{n-2}$

c) $a_n = 2a_{n-1} + a_{n-2}$

d) $a_n = 2a_{n-1} + 2a_{n-2}$

$a_n = a_{n-1} + a_{n-2}$

i) $\underline{0}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{n-1}$ $\quad a_{n-1}$

ii) $\underline{1}\ \underline{0}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{n-2}$ $\quad a_{n-2}$

Let x_n denote the number of binary strings of length n that contain no consecutive 0s.

(GATE-CS-08)

Q. Which of the following recurrences does x_n satisfy?

a) $x_n = 2x_{n-1}$

b) $x_n = x_{[n/2]} + 1$

c) $x_n = x_{[n/2]} + n$

d) $x_n = x_{n-1} + x_{n-2}$

Q. The value of x_5 is

a) 5

b) 7

c) 8

d) 13

Solution to Recurrence Relation:

1) Substitution Method

2) Master's Method

3) Method of characteristic roots

4) Method of undetermined co-efficient

Master Theorem by Division:

If $T(n) = a\,T\left(\frac{n}{b}\right) + \theta(n^k \,.\, \log_b^p n)$

Where $a > 0$, $b > 1$, $k \geq 0$ and p is real

Case – 1: If $a > b^k$ then $T(n) = \theta\left[n^{\log_b^a}\right]$

Case – 2: If $a = b^k$ and

i) P > -1 then $T(n) = \theta\left[n^{log_b^a} . log_b^{p+1} n\right]$

ii) P = -1 then $T(n) = \theta\left[n^{log_b^a} . log_b^{log_b^n}\right]$

iii) P < -1 then $T(n) = \theta\left[n^{log_b^a}\right]$

Case-3: If $a < b^k$ and

i) $P \geq 0$ then $T(n) = \theta[n^k . \log_b^p n]$

ii) $P < 0$ then $T(n) = \theta(n^k)$

Master Theorem by Division

- **Case 1:** If $a > b^k$
  - $T(n) = \theta\left[n^{\log_b^a}\right]$
- **Case – 2:** If $a = b^k$ and
  - $P > -1$ then $T(n) = \theta\left[n^{\log_b^a} . \log_b^{p+1} n\right]$
  - $P = -1$ then $T(n) = \theta\left[n^{\log_b^a} . \log_b^{\log_b^n}\right]$
  - $P < -1$ then $T(n) = \theta\left[n^{\log_b^a}\right]$
- **Case-3:** If $a < b^k$ and
  - $P \geq 0$ then $T(n) = \theta\left[n^k . \log_b^p n\right]$
  - $P < 0$ then $T(n) = \theta(n^k)$

Q. The recurrence relation $T(1) = 2$

$T(n) = 3T\left(\frac{n}{4}\right) + n$ has the solution $T(n)$ equal to **(GATE-CS-96)**

a) $O(n)$

b) $O(\log n)$

c) $O(n^{3/4})$

d) none of these

Shift Operation E:

The shift operator E is defined as

$$E(a_n) = a_{n+1}$$

$$E^2(a_n) = a_{n+2}$$

$$E^3(a_n) = a_{n+3}$$

$$\dots \dots \qquad \dots$$

$$E^k(a_n) = a_{n+k}$$

II) Methods of characteristic Roots:-

Consider the linear recurrence relation $l_0 a_n + l_1.a_{n-1} + \ldots\ldots + l_k.a_{n-k} = f(n)$ …………….. (1)

Replacing n by n + k, we have

$\Rightarrow l_0.a_{n+k} + l_1.a_{n+k-1} + \ldots\ldots + l_k.a_n = F(n)$

$\Rightarrow l_0.E^k(a_n) + l_1.E^{k-1}(a_n) + \ldots\ldots + l_k.a_n = F(n)$

$\Rightarrow (l_0.E^k + l_1.E^{k-1} + \ldots\ldots + l_k)a_n = F(n)$

$\Rightarrow \phi(E)\, a_n = F(n)$ …………. (2)

Where $\phi(E) = l_0.E^k + l_1.E^{k-1} + \ldots\ldots + l_k$

The characteristics equation is

$$\phi(t) = 0$$

The roots of this equation are called **characteristic roots.**

Let $t = t_1, t_2 \ldots\ldots t_k$ be the characteristic roots

Complimentary Function (C.F):-

This is solution of equation (1)

When $f(n) = 0$,

i.e., the solution of homogenous part of equation (1).

Rules for finding (C.F) are given below.

1. Characteristics roots are real and distinct say $t_1, t_2 \ldots\ldots t_k$

$$C.F = c_1.t_1^{\,n} + c_2.t_2^{\,n} + \ldots\ldots + c_k.t_k^{\,n}$$

2. Roots are real and two roots are equal say $t_1, t_1, t_3, t_4 \ldots., t_k$

$$C.F = (c_1 + c_2.n)t_1^{\,n} + c_3.t_3^{\,n} + \ldots\ldots + c_k.t_k^{\,n}$$

3. Roots are real and 3 roots are equal say $t_1, t_1, t_1, t_4 \ldots\ldots t_k$

$$C.F = (c_1 + c_2.n + c_3.n^2).t_1^{\,n} + c4.t_4^{\,n} + \ldots\ldots + c_k.t_k^{\,n}$$

4. Suppose if all the roots are equal say $t_1, t_1, t_1, \ldots\ldots t_1$

$$C.F = (c_1 + c_2.n + c_3.n^2 + \ldots\ldots + c_k.n^{k-1})t_1^{\,n}$$

5. A pair of roots are complex say $(\alpha \pm i\beta)$

$$C.F = r^n(c_1.\cos(n\theta) + c_2.\sin(n\theta)\}$$

Where

$$r = \sqrt{\alpha^2 + \beta^2} \text{ and}$$

$$\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Particular Solution (P.S):-

From equation (2),

$$P.S = \frac{1}{\phi(E)}\{F(n)\}$$

$F(n) = b^n$ (or) $b^n . n^k$

The solution of equation (1) is

$$a_n = \underbrace{C.F} + \underbrace{P.S}$$

Rule to find P.S:-

When $F(n) = b^n$ we have

$$P.S = \frac{b^n}{\phi(E)} = \frac{b^n}{\phi(b)} \text{ provided } \phi(b) \neq 0$$

Case of failure:-

When $\phi(b) = 0$, use the formula

$$\frac{b^n}{(E-b)^k} = C(n.k).b^{n-k} \ (k = 1, 2, 3, \ldots.)$$

$${}^{n}C_{k} \cdot b^{n-k}$$

Method of undetermined Co-efficient:

$$\phi(E)\, a_n = F(n)$$

$$F(n) = b^n . n^k \checkmark$$

i) If 'b' is not characteristic root then $b \neq t$

$$P.S = \underline{b^n}(\underline{A_0 n^k + A_1 n^{k-1}} + \ldots.. + A_k)$$

ii) If 'b' is the characteristic root with multiplicity 'm' then $b = t$

$$P.S = \underline{b^n}(\underline{A_0 n^k} + A_1 n^{k-1} + \ldots.. + A_k)\underline{n^m}$$

Q. Solve the recurrence equations **(GATE-CS-87)**

T(n) = T(n-1) + n

T(1) = 1

$a_n = a_{n-1} + n$

put $n = n+1$

$a_{n+1} - a_n = n+1$

$E^1(a_n) - E^0(a_n) = n+1$

$(E-1)a_n = n+1$

$\phi(E) = E-1$

$$\phi(t) = t - 1 = 0$$

$$t = 1$$

Method of undetermined Co-efficient:

$$\phi(E)\, a_n = F(n)$$

$$F(n) = b^n . n^k$$

i) If ‘b’ is not characteristic root then

$$P.S = b^n(A_0 n^k + A_1 n^{k-1} + \ldots.. + A_k)$$

ii) If ‘b’ is the characteristic root with multiplicity ‘m’ then

$$P.S = b^n(A_0 n^k + A_1 n^{k-1} + \ldots.. + A_k)n^m$$

ACE

Generating Functions

Transforming problems about sequences into problems about functions is known as generating functions.

$< a_0\ a_1\ a_2\ a_3\ \ldots\ldots\ a_i\ \ldots\ldots > \leftrightarrow a_0x^0 + a_1x^1 + a_2x^2 + \ldots\ldots\ldots\ldots + a_ix^i + \ldots\ldots\ldots + a_nx^n$

Rule is a_i is acting as co-efficient of x^i

(where indices are 0, 1, 2, 3, ……)

$< 1\ 2\ 3\ 4 \ldots > \longmapsto 1 + 2x + 3x^2 + 4x^3 + \ldots$

ACE

Q. The generating function for choosing n-elements from

$\{a_1\} = (1+x)$

$\{a_2\} = (1+x)$

$\{a_1, a_2\} = (1+x)^2$

$\{a_1, a_2, a_3\} = (1+x)^3$

Q. Generating function for choosing n-elements from $\{a_1, a_2, a_3, a_4\}$ $= (1+x)^4$

Generating function for choosing n-elements of $\{a_1, a_2, a_3, \ldots\ldots, a_k\}$

$$= (1 + x)\,(1 + x)\,(1 + x) \ldots\ldots (1 + x)\ [k - \text{times}]$$

$$= (1 + x)^k$$

* The number of ways of choosing 3-objects of $\{a_1, a_2, \ldots\ldots, a_k\}$

$$= \text{Co-efficient of } x^3 \text{ in } (1 + x)^k \qquad = {}^kC_3$$

* The number of ways of choosing n-objects of $\{a_1, a_2, \ldots\ldots, a_k\}$ $\qquad n \leq k$

$$= \text{Co-efficient of } x^n \text{ in } (1 + x)^k$$

$$= {}^kC_n$$

* $< 1\ 1\ 1\ 1 \ldots\ldots\ldots > \leftrightarrow 1 + x + x^2 + x^3 + \ldots\ldots$

$$\leftrightarrow \frac{1}{1-x} = (1-x)^{-1}$$

* The generating function for choosing n-objects of $\{a_1\}$ with repetitions

$$= 1.x^0 + 1.x^1 + 1.x^2 + 1.x^3 + \ldots\ldots\ldots\ldots$$

$$= 1 + x + x^2 + x^3 + \ldots\ldots\ldots$$

$$= (1-x)^{-1}$$

* The generating function for choosing n-objects of $\{a_2\}$ with repetition $= (1 - x)^{-1}$

ACE

* The generating function for choosing n-objects of

$\{a_1, a_2, a_3, \ldots\ldots, a_k\}$ with repetitions $= (1-x)^{-1} (1-x)^{-1} \ldots\ldots (1-x)^{-1}$ [k times]

$$= (1-x)^{-k}$$

* The number of ways of choosing n-objects of $\{a_1, a_2, a_3, \ldots\ldots, a_k\}$

With repetitions = Co-efficient of x^n in $(1-x)^{-k}$

$$= \frac{(n+k-1)!}{(k-1)!n!} = {}^{(n+k-1)}C_n \qquad {}^{k+n-1}C_n \qquad {}^{n+r-1}C_r$$

$$(1-x)^{-k} = \sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot x^n$$

ACE

*$(1-x)^{-k} = \Sigma_{n=0}^{\infty}\ {}^{(n+k-1)}C_n \cdot x^n$

* $(1-a_x)^{-k} = \Sigma_{n=0}^{\infty}\ {}^{(n+k-1)}C_n \cdot a^n \cdot x^n$ where 'k' is positive integer

$$(1-ax)^{-k} = \sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot a^n \cdot x^n$$

Graph Theory

A graph G is defined as a pair of sets (V, E) where

V = set of all vertices (nodes) in G and

E = set of all edges in G

$|V(G)|$ = Number of vertices in graph G

= Order of G

$|E(G)|$ = Number of edges in graph G

= Size of the graph

Directed graph (diagraph): The elements of E are ordered pairs of vertices. In this case an edge (u, v) is said to be from u to v.

Non-directed graph:

The elements of E are unordered pairs (sets) of vertices. In this case an edge {u, v} is said to join u and v (or) to be between u and v.

Null graph: A null graph of order n is a graph with n vertices and no edges.

Trivial graph: A null graph with only one vertex is called trivial graph.

Loop: An edge drawn from a vertex to itself is called a loop.

Parallel edges: In a graph, if a pair of vertices are joined by more than one edge then those edges are called parallel edges.

Multi-graph: If one allows more than one edge to join a pair of vertices, the resulting graph is then called a multi graph.

Simple graph: A graph with no loops and no parallel edges is called a simple graph.

Degree: Degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

The degree of the vertex ‘v’ is denoted by deg(v).

Note: In a simple graph with n vertices, degree of any vertex cannot exceed (n – 1)

In degree an out degree: In a directed graph an edge (u, v) is said to be incident from u, and to be incident to 'v'.

Within a particular diagraph, the number of edges incident on a vertex is called the **in-degree** of the vertex and the number of edges emerging from the vertex is called its **out-degree.**

- The in-degree of a vertex 'v' in a graph G is denoted by $deg^{+}(v)$. The out-degree of a vertex v is denoted by $deg^{-}(G)$.
- A loop in a diagraph is counted once, for both indegree and outdegree of the vertex.

Degree Sequence: If $v_1, v_2, \ldots\ldots, v_n$ are the vertices of a graph G, then the sequence $\{d_1, d_2, \ldots\ldots, d_n\}$

Where

d_i = degree of v_i is called the degree sequence of G.Usually we order the degree sequence so that the degree sequence is monotonically decreasing.

Sum of Degree Theorem:

If $V = \{v_1, v_2, \ldots\ldots, v_n\}$ is the vertex set of a non directed graph G then

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

$2e$

Corrolories:

1. For directed graph

$$\sum_{i=1}^{n} \deg^{+}(v_i) = \sum_{i=1}^{n} \deg^{-}(v_i) = |E|$$

2. Every undirected graph has an even number of vertices of odd degree.

3. If G is a k-regular graph, then

$$k.|V| = 2.|E|$$

$k.n = 2e$

4. If $k = \Delta(G)$ is the maximum degree of all vertices in a undirected graph G, then $k.|V| \geq 2.|E|$

$k.n \geq 2e$

5. If $k = \delta(G)$ is the minimum degree of all vertices in a undirected graph G, then $k.|V| \leq 2.|E|$

$k.n \leq 2e$

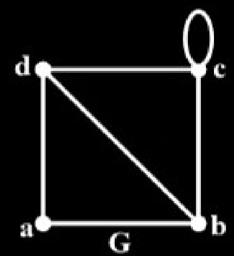
6. $\delta(G).\ |V| \leq 2|E| \leq \Delta(G)\ .\ |V|$

$\leq \quad \leq$

Neighbors: If there is an edge incident from u to v, or incident on u and v, then u and v are said to be **adjacent** (or to be neighbors).

Minimum of all the degrees of vertices in a graph G is denoted by **δ(G)**.

Maximum of all the degrees of vertices in a graph G is denoted by **Δ(G)**. For example,



For the graph shown above $|V| = 4$ and $|E| = 5$

Degree of a = 2

Degree of b = 3

Degree of c = 4

Degree of d = 3

There is a loop at vertex c

$\delta(G) = 2$ and $\Delta(G) = 4$

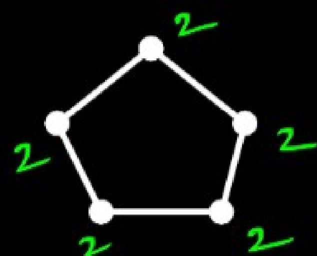
Regular graph: If each vertex of G has degree k, then G is said to be a **regular graph** of degree k (k-regular). For example,

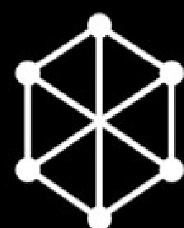
A polygon is a 2-regular graph

And A 3-regular graph is a cubic graph.

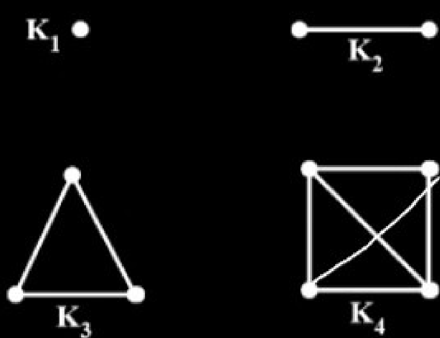
In a graph G, if $\delta(G) = \Delta(G) = k$ then G is a regular graph. For example,

The following graphs are regular





Complete Graph: A simple non directed graph with 'n' mutually adjacent vertices is called a completed graph on n vertices and it is denoted by K_n.



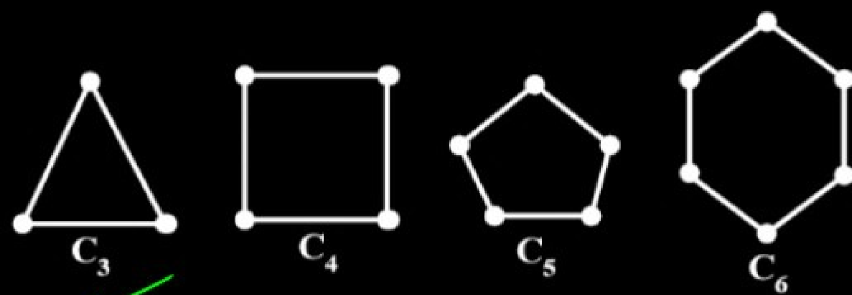
Note: Every complete graph K_n is a regular graph and each of its vertices has degree '$n - 1$'.

Note: Number of edges in a complete graph $K_n = C(n, 2) = n(n - 1)/2$ $\quad {}^nC_2$

Note: A completed graph K_n is a simple graph with n vertices and maximum number of edges.

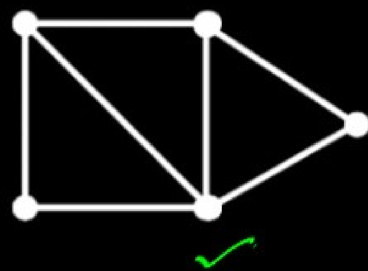
Cycle graph: A cycle graph of order $n(n \geq 3)$ is a simple connected graph whose edges form a cycle of length n.

Its denoted by C_n.

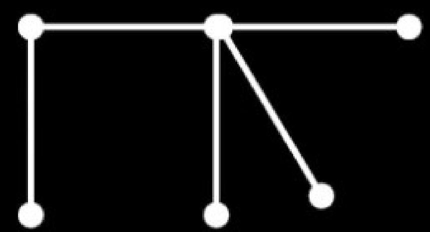


Note: A cycle graph 'C_n' of order n has n vertices and n edges.

Cyclic graph: A graph with atleast one cycle is called cyclic graph.



Acyclic graph: A simple graph having no cycles is called acyclic graph.



Connected graph: An undirected graph G is called connected if there is a path between every pair of distinct vertices in G.

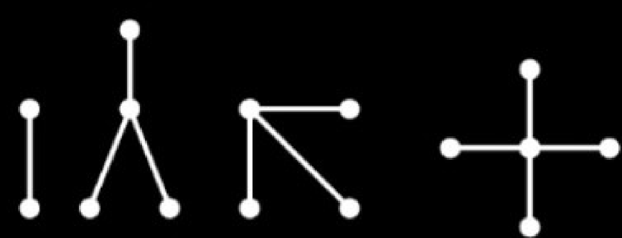
Note: A graph which is not connected has atleast two connected components.

Tree: A connected graph with no cycles is called a tree.

Note: A tree with n vertices has $(n - 1)$ edges.

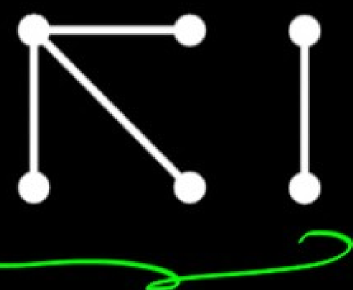
Note: A tree with $n(n > 1)$ vertices has atleast two vertices of degree 1.

Ex:



Note: An acyclic graph which is not connected is called a **forest.**

Ex:



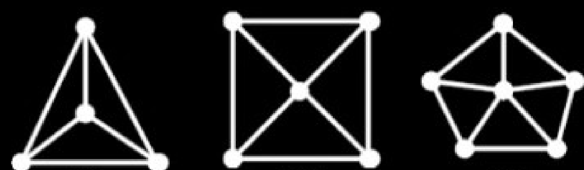
G.

Wheel graph: A wheel graph of order $n(n \geq 4)$ can be obtained from a cycle graph C_{n-1} by adding a new vertex (the hub) which is adjacent to all vertices of C_{n-1}.

It is denoted by W_n.

Note: A wheel graph W_n has 'n' vertices and $2(n-1)$ edges.

Ex:



W_4 W_5 W_6

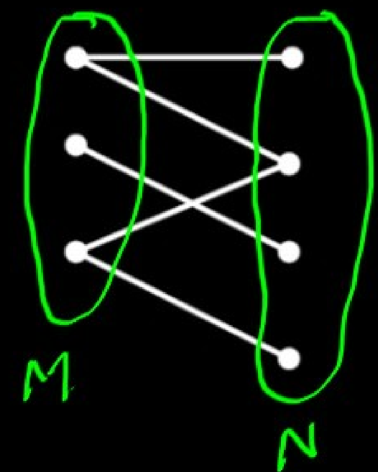
C_3

$W_{10} \longrightarrow C_9$

9

9e
9e
18

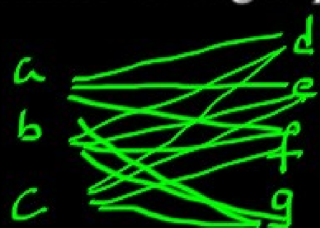
Bipartite graph: A bipartite graph G is a simple non directed graph whose set of vertices can be partitioned in to two sets M and N in such a way that each edge of G joins a vertex in M to a vertex in N.



Properties:

- If $|M| = m$ and $|N| = n$ then the complete bipartite graph is denoted by $K_{m,\,n}$.
- $K_{m,n}$ has ‘$m + n$’ vertices and ‘mn’ edges.
- In general, a complete bipartite graph is not a complete graph.
- $K_{m,n}$ is a complete graph $\Leftrightarrow$ $(m = n = 1)$
- G is a bipartite graph $\Leftrightarrow$ G has no cycles of odd length.
- Maximum number of edges possible in a bipartite graph with n vertices is $\left\lfloor \frac{n^2}{4} \right\rfloor$.

$K_{3,4}$

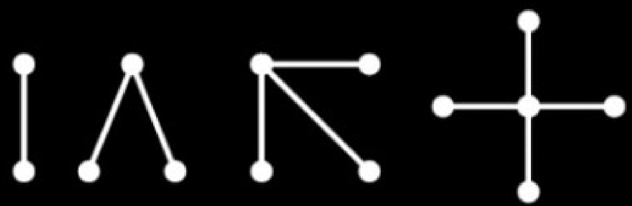


n

$\frac{n}{2}$ $\frac{n}{2}$

$\left\lfloor \frac{n^2}{4} \right\rfloor$

Star graph: A complete bipartite graph of the form $K_{1,\, n-1}$ is called a star graph. For example,



$K_{m,n}$

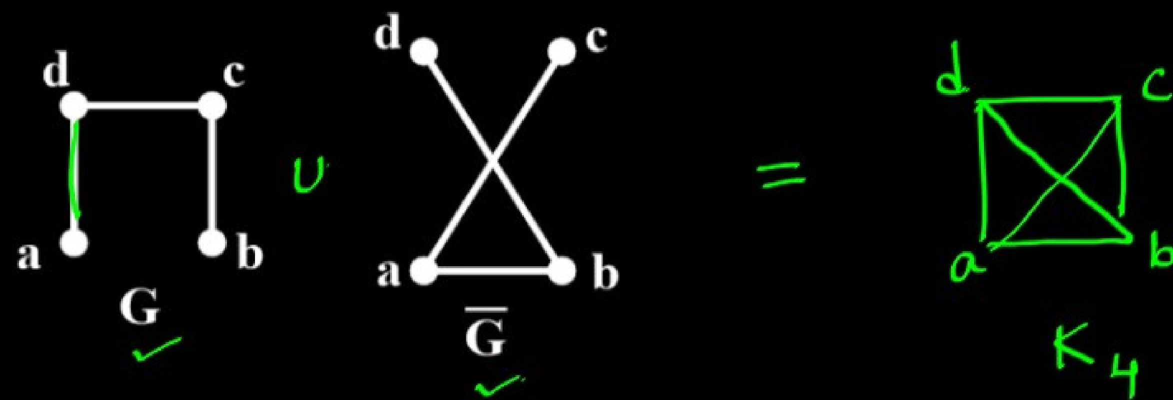
$K_{1,5}$

$K_{3,1}$

$K_{1,1}$

Note: Every star graph is a tree

Complement of a graph: If G is a simple graph then **complement** of G is a simple graph $\overline{G}$ with the same vertices as G and an edge exists in $\overline{G}$ iff it does not exist in G.



Note: $|E(G)| + |E(\overline{G})| = |E(K_n)|$ where $n = |V(G)|$

Havel-Hakimi Result:

Consider the following two sequences and assume that the sequence (i) is in descending order

i) $s, t_1, t_2, \ldots\ldots t_s, d_1, d_2, \ldots\ldots, d_n$

ii) $t_1 - 1, t_2 - 1, \ldots., t_{s-1}, d_1, d_2, \ldots., d_n$

Sequence (i) is graphic $\Leftrightarrow$ sequence (ii) is graphic.

$100 \longrightarrow 99 \longrightarrow 98 \longrightarrow - - - \boxed{5}$

$8 \leftarrow 7 \leftarrow 6 \leftarrow$

Q. The degree sequence of a simple graph is the sequence of the degrees of the nodes in the graph in decreasing order. Which of the following sequences can not be the degree sequence of any graph? **(GATE)**

I. 7, 6, 5, 4, 4, 3, 2, 1 II. 6, 6, 6, 6, 3, 3, 2, 2

III. 7, 6, 6, 4, 4, 3, 2, 2 IV. 8, 7, 7, 6, 4, 2, 1, 1

a) I and II b) III and IV

c) IV only d) II and IV

$\cancel{7}$ 6, 5, 4, 4, 3, 2, 1
$\quad -1 \; -1 \; -1 \; -1 \; -1 \; -1 \; -1$

$\Rightarrow \cancel{5}, \underline{4}, \underline{3}, \underline{3}, \underline{2}, \underline{1}, 0$

$\Rightarrow \cancel{3}, \underline{2}, \underline{2}, \underline{1}, 0, 0$

$\Rightarrow \cancel{1}, 1, 0, 0, 0$

$\Rightarrow 0, 0, 0, 0$

Graphic

Note:

- A path of length ≥ 1 with no repeated edges and whose end points are equal is called a **circuit.**
- A circuit may have repeated vertices other than its end points.
- A **cycle** is a circuit with no other repeated vertices except its end points.
- A **loop** is a cycle of length 1.
- In a simple graph, a cycle that is not a loop must have length at least 3.

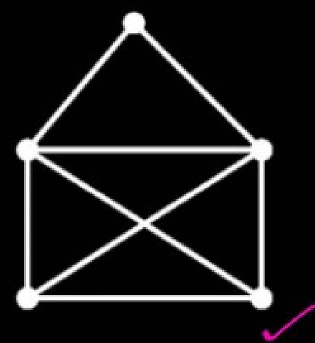
Euler & Hamiltonian:

Euler Graph: If a graph consist Euler path then it is known as Euler graph.

Euler Path: Each edge exactly once, each vertex atleast once.

Euler Circuit: Starting and end vertices same

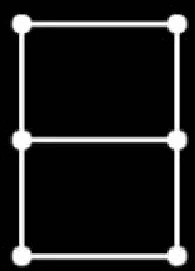
* In a graph if every vertex as even degree then their must exist Euler path

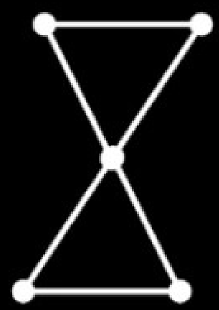


Hamiltonian graph: A graph with Hamiltonian cycle is known as Hamiltonian

Hamiltonian cycle: A cycle which covers all vertices of the graph

Ex:





MINIMAL SPANNING TREE (MST)

$O(e) = O(s)$

100 v　　　100 v & 99 edges

MST
- → kruskal's
- → prim's

Planar Graphs: A simple graph without any edge crossing.

Ex: K_3, K_4, $K_{2,3}$

For Plane Graph:

$$\boxed{r + v = e + 2}$$

$$r = e - v + 2$$

$$e \leq 3v - 6$$

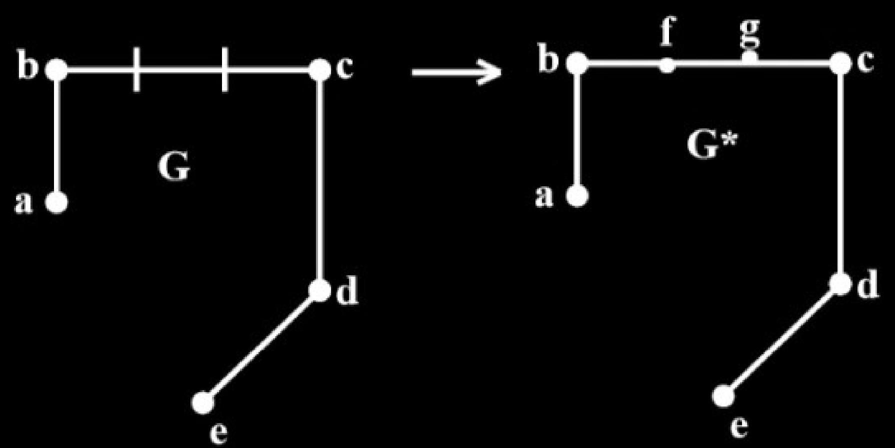
Kuratowski's Theorem:-

It states that "A graph is planar if and only if it does not contain subgraph which is homeomorphic to K_5 (or) $K_{3,3}$

Homeomorphic Graphs:

In a Graph G, if another graph G* taken by dividing edge of G with additional vertices (or) we can say that a Graph G* is obtained by introducing the vertices of degree 2 in any edge of Graph G, then G & G* are said to be Homeomorphic.

Ex:



* There are two Kuratowski's Graphs:

1. The complete Graph with 5 vertices

2. A regular connected graph with 6V & 9e ($K_{3,3}$)

Four color Theorem: Every plane graph is four-colorable.

i.e., $\chi(G) \leq 4$

Note: This statement is true for both vertex coloring and map coloring.

Graph Coloring: A coloring of a simple graph is the assignment of colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Chromatic Number: The minimum number of colors needed for vertex colouring of a graph G is called the chromatic number of G, denoted by $\chi(G)$.

Adjacent Regions: In a planar graph two regions are said to be **adjacent** if they have a common edge.

Region colouring (map colouring): An assignment of colors to the regions of a map such that adjacent regions have different colors.

A map 'M' is **n-colorable** if there exists a coloring of M which uses atmost n colors.

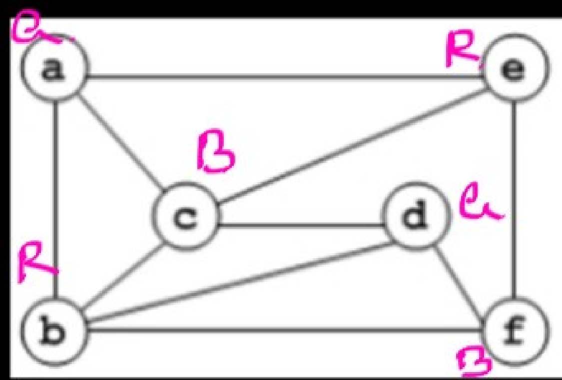
Welch-powell's algorithm:

(for vertex colouring)

Step 1: Arrange the vertices of the graph G in the descending order of their degrees.

Step 2: Assign colours to the vertices in the above order so that no two adjacent vertices have same colour.

Q. The chromatic number of the following graph is ________. (GATE)



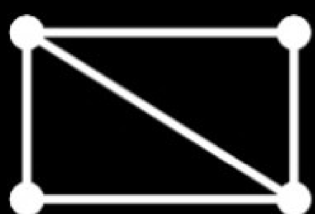
$\chi(G) = 3$

Matchings, Coverings, Independent Sets:

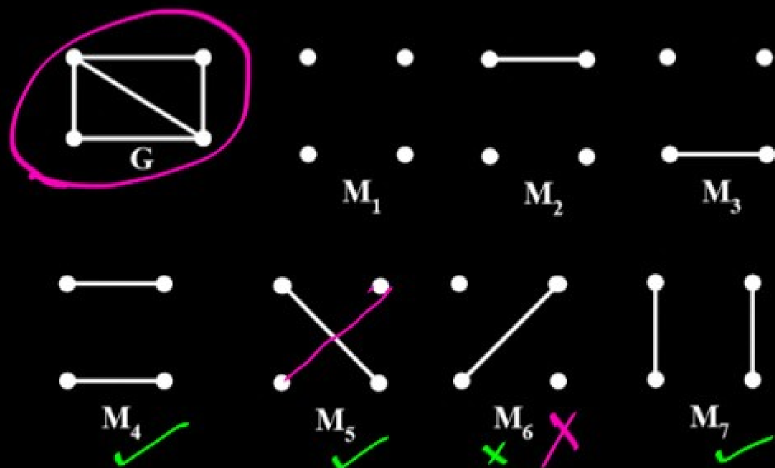
Matching: A matching M, of a graph 'G' is a subgraph such that every vertex of 'G' is incident with atmost one edge in M

$$\deg(v_i) \leq 1 \ \forall \ v_i \in G$$

Ex:



G



* A Matching M is called Maximal if THERE IS NO POSSIBILITY to add any edge to it.

In the example are maximal matchings.

* A maximal matching containing maximum no. of edges is called Largest Maximal Matching.

* The no. of edges in a Largest Maximal Matching is called Matching number

2.

Note:

* Matching number of a Bi-patriate graph $K_{m,n} = \min\{m, n\}$

* Matching number of a complete graph $K_n = \left\lfloor \frac{n}{2} \right\rfloor$

* Matching number of a cycle graph $C_n = \left\lfloor \frac{n}{2} \right\rfloor$

* Matching number of a wheel graph $W_n = \left\lfloor \frac{n}{2} \right\rfloor$

Perfect Matching: A matching is said to be perfect if

$$\boxed{\text{if } \deg(v_i) = 1} \ \forall\, v_i \in G$$

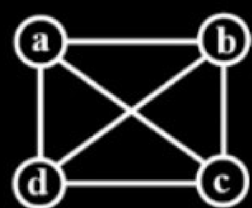
Common graph K_n has perfect matching iff n is even.

$\therefore$ No. of perfect matching in $K_{2n} = \frac{(2n)!}{2^n * n!}$

$$\frac{2n!}{2^n * n!}$$

Ex:

$K_4 =$



* $K_{m,n}$ has perfect matching iff $m = n$

Number of perfect matching $K_{n,n} = n!$

Coverings:

Line Covering: The number of edges, which covers all vertices of a graph.

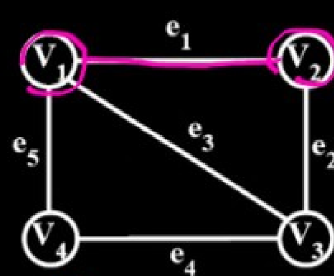
Minimal Line Covering: A line covering is called minimal if there is NO possibility to remove any edge from it.

Smallest minimal Line Covering: A minimal line covering containing minimum no. of edges.

- Number of edges in a smallest minimal line covering is called

  LINE COVERING NUMBER (α_1)

$$\alpha_1 + \beta_1 = \alpha_2 + \beta_2$$



e_1, e_2, e_3, e_4, e_5

$\{e_1, e_3, e_5\}$

$\{e_1, e_4\}$

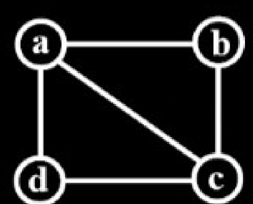
$\{e_2, e_5\}$

* **Vertex Covering:** The set of vertices, which covers all edges

* **Minimal Vertex Covering:** A vertex covering is said to be minimal if the no possibility to remove any vertex from the set.

* **Smallest Minimal Vertex Covering:** A minimal vertex covering containing minimum no. of vertices is called smallest minimal vertex covering.

* The number of vertices in a smallest minimal vertex covering is called VERTEX COVERING NUMBER (α_2)



{a, b, c, d} vertex covering

{a, b, d} minimal vertex covering

{b, d, c} minimal vertex covering

{a, c} smallest minimal vertex covering

$\alpha_2 = 2$

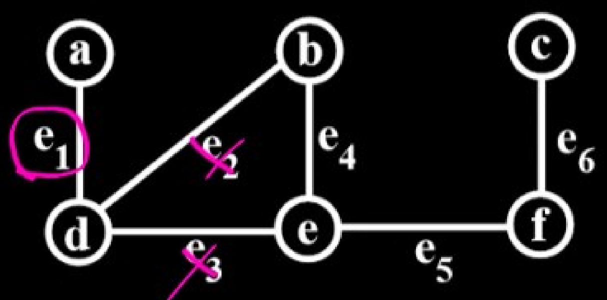
Independent Sets:

* **Line Independent Set:** Set of edges which are NOT adjacent.

* **Maximal Line Independent Set:** A Line Independent set is said to be maximal if it is not possible to add any edge to it.

* **Largest Maximal Line Independent Set:** A maximal Line Independent Set containing maximum no.of edges

* **No. of edges:** In a largest maximal line independent set is called LINE INDEPENDENT NUMBER (β_1).



$e_1 \longrightarrow e_2, e_3$

$e_5 \longrightarrow e_4, e_6$

$\{e_1, e_4, e_6\}$

$\{e_1, e_4\}$ Line Independent Set

$\{e_1, e_5\}$ Maximal Line Independent

$\{e_1, e_6\}$ Line Independent Set

$\{e_2, e_5\}$ Maximal Line Independent Set

$\{e_2, e_6\}$ Maximal Line Independent Set

$\{e_4, e_6\}$ Line Independent Set

$\{e_1, e_4, e_6\}$ Largest Maximal Line Independent

$\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = ord(G)$ $\beta_1 = 3$

I. Math logic

II. Set theory
- S
- R
- P
- f
- G

III. Combinatorics
- Counting
- Pig
- Euler
- Derang
- P & C
- Dividing
- RR GF

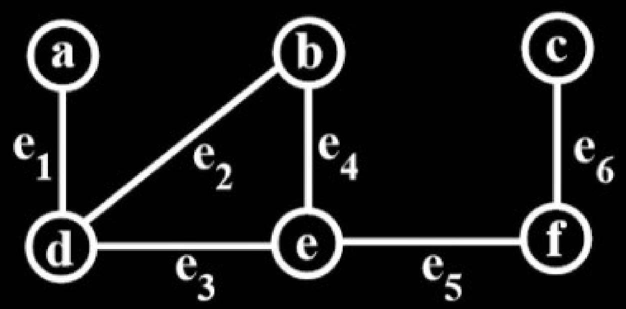
IV. Graphs
- Basic
- Euler
- MST
- Planar
- Graph coloring
- Matching
- Covering
- Independent

Vertex Independent Set: The vertex independent set of a graph 'G' is the vertices of 'G' which are not adjacent with each other.

Maximal Vertex Independent Set: A vertex independent set is said to be maximal if there is no possibility to add any vertex to it.

Largest Maximal Vertex Independent Set: A maximal vertex independent containing maximum number of vertices is called Largest Maximal Vertex Independent Set.

Number of vertices in it is called VERTEX INDEPENDENT NUMBER (β_1).



{a, b} Vertex Independent Set

{a, e} Vertex Independent Set

{a, f} Vertex Independent Set

{a, c} Vertex Independent Set

{d, c} Maximal Vertex Independent Set

{d, f} Maximal Vertex Independent Set

{a, b, c} Largest Maximal Vertex Independent Set

$\beta_2 = 3$