EE1390 - Introduction to AI and ML

Presentation

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Question 45, JEE Advanced 2015

If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then find the value of r^2.

Solution without using matrices

The end points of the latus rectum are (1,2) and (1,-2).
Equation of normal at (1,2) is,

$$x + y - 3 = 0 \tag{1}$$

Equation of normal at (1,-2) is,

$$x - y - 3 = 0 \tag{2}$$

Since theses normals are tangent to the circle, the perpendicular distance from centre of the circle (3,-2) to the tangent= radius. Therefore,

$$r = \frac{|3 - 2 - 3|}{\sqrt{1+1}} \tag{3}$$

Thus, $r^2 = 2$.

Solution using matrices

Equation of a conic section in matrix form is,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0 \tag{4}$$

Equation of the tangent at point p on the curve is,

$$(\mathbf{p}^T\mathbf{V} + \mathbf{u}^T)\mathbf{x} + \mathbf{p}^T\mathbf{u} + F = 0 \tag{5}$$

Equation of parabola $y^2 = 4x$ in matrix form is,

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{6}$$

or,

$$\mathbf{x}^T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{7}$$

Focus of parabola is at (1,0).
$\mathbf{V}= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{u}=\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, F=0.
Equation of circle $(x-3)^2+(y+2)^2 = r^2$ in matrix form is:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2 \begin{pmatrix} 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = r^2 - \begin{pmatrix} 3 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix} 3 & -2 \end{pmatrix}\mathbf{x} = r^2 - 13 \qquad (8)$$

For the circle equation, $\mathbf{V}$=$\mathbf{I}$, $\mathbf{u}$=$\begin{pmatrix}-3\\2\end{pmatrix}$, F=13 - r^2.

Equation of tangent at (1,2) on parabola using formula stated above is,

$$\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}-2\\0\end{pmatrix}=0$$

$$\begin{pmatrix}-2 & 2\end{pmatrix}\mathbf{x}=2 \text{ or, } \mathbf{n}^T\mathbf{x}=1, \textit{where } \mathbf{n}=\begin{pmatrix}-1\\1\end{pmatrix} \tag{9}$$

We need to find vector $\mathbf{m}$ such that $\mathbf{m}^T\mathbf{n}=0$ in order to construct the equation of the normal.

$$\mathbf{m}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\mathbf{n}$$

$$\mathbf{m} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Equation of normal at point $\mathbf{p}$ is $\mathbf{m}^T(\mathbf{x} - \mathbf{p}) = 0$

Equation of normal at (1,2) is,

$$\begin{pmatrix} 2 & 2 \end{pmatrix} (\mathbf{x} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}) = 0$$

$$\begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} = 6$$

Or,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3$$

For the circle, $\mathbf{V}$=$\mathbf{I}$, $\mathbf{u}$=$\begin{pmatrix}-3\\2\end{pmatrix}$, F=13 - r^2.

Writing the equation of tangent for circle using the general format stated above, we get,

$$(\mathbf{p}^T\mathbf{I}+\begin{pmatrix}-3 & 2\end{pmatrix})\mathbf{x} + \mathbf{p}^T\begin{pmatrix}-3\\2\end{pmatrix} + 13 - r^2 = 0$$

Comparing it with the equation of normal,

$$\mathbf{p}+\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\mathbf{p}=\begin{pmatrix}4\\-1\end{pmatrix}$$

Also, $\mathbf{p}^T\begin{pmatrix}-3\\2\end{pmatrix} + 13 - r^2 = -3$

Substituting **p** we get,

$$\begin{pmatrix} 4 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} + 13 \text{ - } r^2 = \text{-3}$$

$$-14 + 13 - r^2 = -3$$

Therefore, $r^2{=}2$

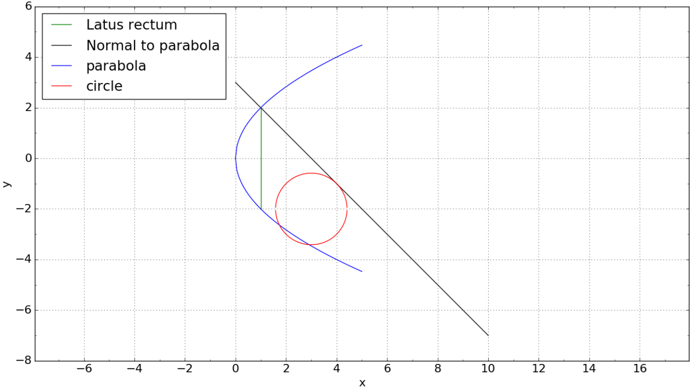

Figure: Graphical representation