EECE 5552
Assistive Robotics

Assignment – 2

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Problem 1(a):-

$$R_{x\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad R_{z\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{y\psi} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}$$

Rotation matrix $R = R_{y\psi} \cdot R_{x\phi} \cdot R_{z\theta}$

Problem 1(b):-

$$R_{x\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad R_{z\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{x\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

Resultant Rotation matrix,

$$R = R_{z\theta} \cdot R_{x\phi} \cdot R_{x\psi}$$

Problem 2(a):-

Q 2) To prove:- SO(3) with the given conditions follows group with matrix multiplication operation.

To prove the asked, let us take 3 SO(3) matrices as shown below:-

$$R_1^0 = \begin{bmatrix} x_1 x_0 & y_1 x_0 & z_1 x_0 \\ x_1 y_0 & y_1 y_0 & z_1 y_0 \\ x_1 z_0 & y_1 z_0 & z_1 z_0 \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} x_2 x_1 & y_2 x_1 & z_2 x_1 \\ x_2 y_1 & y_2 y_1 & z_2 y_1 \\ x_2 z_1 & y_2 z_1 & z_2 z_1 \end{bmatrix}$$

$$R_2^0 = \begin{bmatrix} x_2 x_0 & y_2 x_0 & z_2 x_0 \\ x_2 y_0 & y_2 y_0 & z_2 y_0 \\ x_2 z_0 & y_2 z_0 & z_2 z_0 \end{bmatrix}$$

Now, let us check if given conditions satisfy the SO(3) matrices declared.

(a) $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$

$R_1^0 * R_2^1 \in SO(3)$ for all $R_1^0, R_2^1 \in SO(3)$

$$\begin{bmatrix} x_1 x_0 & y_1 x_0 & z_1 x_0 \\ x_1 y_0 & y_1 y_0 & z_1 y_0 \\ x_1 z_0 & y_1 z_0 & z_1 z_0 \end{bmatrix} \times \begin{bmatrix} x_2 x_1 & y_2 x_1 & z_2 x_1 \\ x_2 y_1 & y_2 y_1 & z_2 y_1 \\ x_2 z_1 & y_2 z_1 & z_2 y_1 \end{bmatrix}$$

Since these are base vectors,

$x_1 y_1 = 0$ (General properties, orthogonal)

$x_1 . x_1 = 1$

~~Allying~~ Applying these,

$$= \begin{bmatrix} x_2 x_0 & y_2 x_0 & z_2 x_0 \\ x_2 y_0 & y_2 y_0 & z_2 y_0 \\ x_2 z_0 & y_2 z_0 & z_2 z_0 \end{bmatrix}$$

$$= R_2^0$$

which belongs to SO(3)

(b) Consider another SO(3)

$$R_3^2 = \begin{bmatrix} x_3 x_2 & y_3 x_2 & z_3 x_2 \\ x_3 y_2 & y_3 y_2 & z_3 y_2 \\ x_3 z_2 & y_3 z_2 & z_3 z_2 \end{bmatrix}$$

~~To~~ Condition given:- $(x_1 * x_2) \times x_3 = x_1 * (x_2 * x_3)$

$$\Rightarrow (R_1^0 * R_2^1) * R_3^2 = R_1^0 * (R_2^1 * R_3^2)$$

Solving the L.H.S part,

$$(R_1^0 * R_2^1) = \begin{bmatrix} x_2 x_0 & y_2 x_0 & z_2 x_0 \\ x_2 y_0 & y_2 y_0 & z_2 y_0 \\ x_2 z_0 & y_2 z_0 & z_2 z_0 \end{bmatrix}$$

$$(R_1^0 * R_2^1) * R_3^2 = \begin{bmatrix} x_3 x_0 & y_3 x_0 & z_3 x_0 \\ x_3 y_0 & y_3 y_0 & z_3 y_0 \\ x_3 z_0 & y_3 z_0 & z_3 z_0 \end{bmatrix}$$

Solving R.H.S

$$(R_2^1 * R_3^2) = \begin{bmatrix} x_3 x_1 & y_3 x_1 & z_3 x_1 \\ x_3 y_1 & y_3 y_1 & z_3 y_1 \\ x_3 z_1 & y_3 z_1 & z_3 z_1 \end{bmatrix}$$

$$(R_1^0 * (R_2^1 * R_3^2) = \begin{bmatrix} x_3 x_0 & y_3 x_0 & z_3 x_0 \\ x_3 y_0 & y_3 y_0 & z_3 y_0 \\ x_3 z_0 & z_3 z_0 & z_3 z_0 \end{bmatrix}$$

$$\therefore (R_1^0 R_2^1) * R_3^2 = R_1^0 * (R_2^1 * R_3^2)$$

proved.

ⓒ $I \in X$, $I * x = x * I = x$ for all $x \in X$.

assuming $x = R_1^0$

I is identity matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$I * R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 x_0 & y_1 x_0 & z_1 x_0 \\ x_1 y_0 & y_1 y_0 & z_1 y_0 \\ x_1 z_0 & y_1 z_0 & z_1 z_0 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 x_0 & y_1 x_0 & z_1 x_0 \\ x_1 y_0 & y_1 y_0 & z_1 y_0 \\ x_1 z_0 & y_1 z_0 & z_1 z_0 \end{bmatrix}$$

which is $\cancel{xx} = R_1^0$ //

$$R_i * I = \begin{bmatrix} x_1 x_0 & y_1 x_0 & z_1 x_0 \\ x_1 y_0 & y_1 y_0 & z_1 y_0 \\ x_1 z_0 & y_1 z_0 & z_i z_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 x_0 & y_1 x_0 & z_1 x_0 \\ x_1 y_0 & y_1 y_0 & z_1 y_0 \\ x_1 z_0 & y_1 z_0 & z_1 z_0 \end{bmatrix}$$

which is R_1^0

$\therefore IR_1^0 = R_1^0 I = R_1^0$

(d) for every $x \in X$, there exists $y \in X$,
$\Rightarrow x * y = y * x = I$.

We know from the properties

$$R_0^1 = (R_1^0)^{-1} \in SO(3)$$

So, Correspondingly,
assume $x = R_1^0 \quad y = R_0^1$

$$R_1^0 R_0^1 = R_0^1 R_1^0 = I$$

From cosine Rule,

$$\begin{bmatrix} x_1 x_0 & y_1 x_0 & z_1 x_0 \\ x_1 y_0 & y_1 y_0 & z_1 y_0 \\ x_1 z_0 & y_1 z_0 & z_1 z_0 \end{bmatrix} \begin{bmatrix} x_1 x_0 & x_1 y_0 & x_1 z_0 \\ y_1 x_0 & y_1 y_0 & y_1 z_0 \\ z_1 x_0 & z_1 y_0 & z_1 z_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

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Similarly,

$$R_0^1 R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{//}$$

$$\therefore R_0^1 R_0^1 = R_0^1 R_1^0 = I_{//}$$

$\therefore$ Since All 4 conditions specified for X to be defined as a group are satisfied, The SO(3) with matrix multiplication operator is a group.

Problem 2 (b) :-

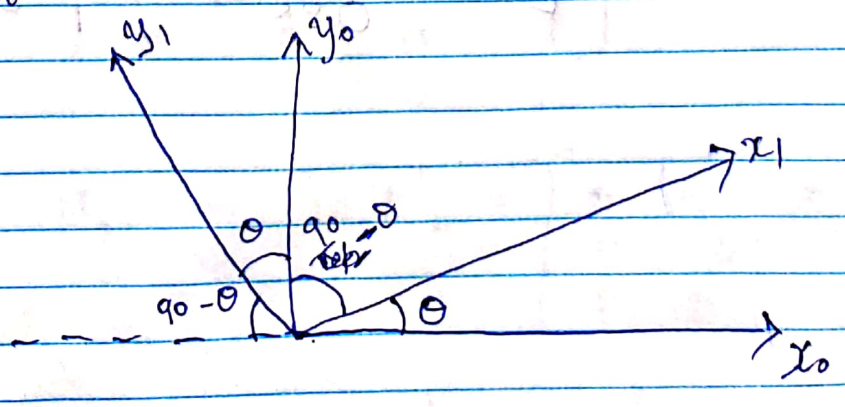
$$A \in SO(2)$$

$$A^T A = I \in R^{2\times 2} \quad -①$$
$$\det\{A\} = 1 \quad -②$$

An SO(n) matrix that satisfies conditions ① & ② is a rotation matrix.

$\therefore$ Hence A can be considered $A = R_1^0$

Assume a rotation of θ degrees by the fixed axes we get



$$R_1^0 = \begin{bmatrix} x_0 x_1 & x_0 y_1 \\ y_0 x_1 & y_0 y_1 \end{bmatrix}$$

we know that

$$x_0 x_1 = \cos\theta \qquad y_1 x_0 = \cos(90+\theta)$$
$$y_0 y_1 = \cos\theta \qquad = -\sin\theta$$
$$x_1 y_0 = \cos(90-\theta)$$
$$= \sin\theta$$

$$\therefore R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Hence proved.

Problem 3 (a) :-

General Homogeneous Transformations

$$H_1^0 = \left[\begin{array}{ccc|c} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ \hline & 0 \in R^{1\times 3} & & 1 \in R \end{array}\right]$$

From the figure given,

$$H_1^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$H_{21} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From, $H_2^1 \Rightarrow R_2^1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ — ①

$d = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ — ②

Applying transpose to (1), we get →

$$(R_2^1)^T = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad — (3)$$

Now, multiply this with d (2).

$$(R_2^1)^T d = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \quad — (4)$$

Now, we can observe that (3) and (4) are the same expressions in H_1^2.

Hence, H_1^2 can be expressed as:

$$H_1^2 = \left[\begin{array}{c|c} (R_2^1)^T_{3\times3} & (R_1^2)^T d_{3\times1} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

Problem 3 (b):-

For the given figure 2;

In general, the DH Params are

$$A_i = \underset{z\theta_i}{Rot} \cdot \underset{z d_i}{Trans} \cdot \underset{x a_i}{Trans} \cdot \underset{x \alpha_i}{Rot}$$

with a_i above Rot, α_i above Trans, d_i above Trans, θ_i above Rot.

$$A_i = \left[\begin{array}{ccc|c} \cos\theta_i & -\sin\theta_i & 0 & \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & \\ \hline & 0 & & 1 \end{array}\right] * \left[\begin{array}{c|c} \text{Homogeneous} & 0 \\ \text{I} & 0 \\ & d_i \\ \hline 0\ 0\ 0 & 1 \end{array}\right] * \quad \text{--- (1)}$$

$$* \left[\begin{array}{c|c} \text{Homogeneous} & a_i \\ \text{I} & 0 \\ & 0 \\ \hline 0\ 0\ 0 & 1 \end{array}\right] * \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & \\ \hline & 0 & & 1 \end{array}\right]$$

Now, we have 6 joints in the given Fig 2. For each joint, let us find the D-H params links.

Joint 1	$\theta = \theta_1$	$d = 0$	$a = 0$	$\alpha = 90^\circ$
Joint 2	$\theta = \theta_2$	$d = 0$	$a = a_2$	$\alpha = 0$
Joint 3	$\theta = \theta_3 + 90$	$d = 0$	$a = a_3$	$\alpha = 0$
Joint 4	$\theta = \theta_4$	$d = 0$	$a = 0$	$\alpha = 90^\circ$
Joint 5	$\theta = \theta_5$	$d = 0$	$a = 0$	$\alpha = -90^\circ$
Joint 6	$\theta = \theta_6$	$d = d_6$	$a = 0$	$\alpha = 0$

Now, substitute the above table in equation (1)

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For link 1,

$$A_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly,

By substituting the values in the table (D-H table) in equation ① we get A_2 A_3 A_4 A_5 A_6 for links 2 to 6.

Final ~~so~~ forward kinematic solution →

$$T_6^0 = A_6^0 = A_1 * A_2 * A_3 * A_4 * A_5 * A_6.$$