Diversity Induced Resonance in Discrete Chaotic Maps

By –

Tejas Adsul

Savyaraj Deshmukh

Original Paper by –
Mr. Nikunj Maheshwari and
Prof. Punit Parmananda

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Some Terminology

- **Resonance** is a phenomenon where a system is driven to oscillate with a higher amplitude at specific frequencies due to another vibrating system or an external force.

- **Stochastic Resonance** is a phenomenon where adding noise helps improve the original signal. The frequencies in the noise corresponding to the original signal's frequencies will resonate with each other, thus improving the strength of the original signal. The noise can later be removed to obtain the original signal.

Some Terminology

- **Coherence Resonance** is the phenomenon where the oscillations in a system are driven at almost the same frequency by adding an optimum level of noise.

- **Diversity** refers to addition of noise in the parameters of the coupled oscillators. Tessone et. al have shown that coupled oscillators show collective resonant behaviour when diversity is introduced. This is called **Diversity Induced Resonance**.

Some Terminology

- A **Logistic Map** is a recurrence relation of degree 2, mathematically written as
$$\boldsymbol{x_{n+1} = r x_n (1 - x_n)}$$
- It is a discrete-time analog of the logistic equation used for population growth.
- In this paper, resonance effects of the discrete coupled oscillators, when diversity is introduced, are studied.

- **Coherence Factor** is a quantity used to describe the power spectrum quantitatively. It is given as $\boldsymbol{\beta = \frac{H\omega}{\Delta\omega}}$ where H is the height of the dominant peak of the cumulative spectrum, $\Delta\omega$ is the width of the dominant peak and ω is the peak frequency.

Governing Equation

- A system of N logistic maps, which are coupled globally, are studied.
- The recurrence relation for each of these maps is given by

$$\boldsymbol{x_{i,n+1} = (1-\epsilon)a_i x_{i,n}\left(1 - x_{i,n}\right) + \frac{\epsilon}{N}\sum_{j=1}^{N} a_j x_{j,n}(1 - x_{j,n})}$$

$$\boldsymbol{\forall i = 1, 2, \ldots, N}$$

- Here, ϵ is the coupling strength and a_i are the individual map parameters, sampled from the Uniform Distribution $[\hat{a} - \delta, \hat{a} + \delta]$. δ is the measure of diversity in the system.
- The bifurcation diagram (which we'll analyze in the later slides) for uncoupled system, shows Period 5 dynamics for $\hat{a} \in [3.737, 3.745]$. The dynamics are chaotic beyond this window on both sides.

Analysis by parameterising $\widehat{\boldsymbol{a}}$, $\boldsymbol{\delta}$, $\boldsymbol{\epsilon}$

- A total of 5000 time series are generated.
- For every oscillator, the first 1000 steps are left uncoupled so that each one attains its attractor.
- Then, all the oscillators are coupled and the next 1000 steps are ignored as transients.
- The remaining 3000 steps are taken into consideration as a time series corresponding to the given values of the parameters.
- Every step of the N oscillators is averaged out and this average is used for understanding the dynamics of the system.

Analysis by parameterising $\hat{a}$, δ, ϵ

- After the 5000 time steps of all the N oscillators are calculated and averaged out, its Fourier spectrum and hence the power spectrum is calculated.
- This is repeated 5000 times. The mean of all the 5000 power spectra gives the cumulative power spectrum.
- The Coherence Factor is calculated from the cumulative power spectrum.
- Two different sampling methods are used to understand the dynamics of the system.

Sampling Method I

- Here, the parameter a_i is given random values from the Uniform Distribution $[\hat{a}-\delta, \hat{a}+\delta]$.
- The following plots are then obtained for the specified parameter values:

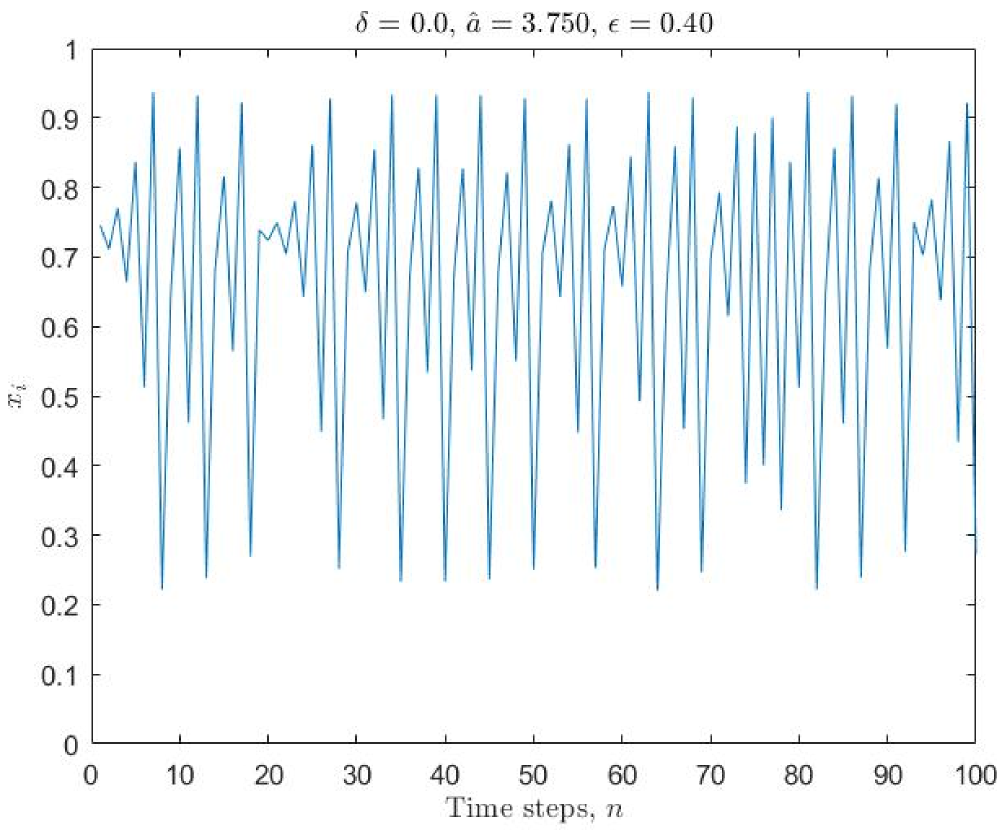


Fig. 1

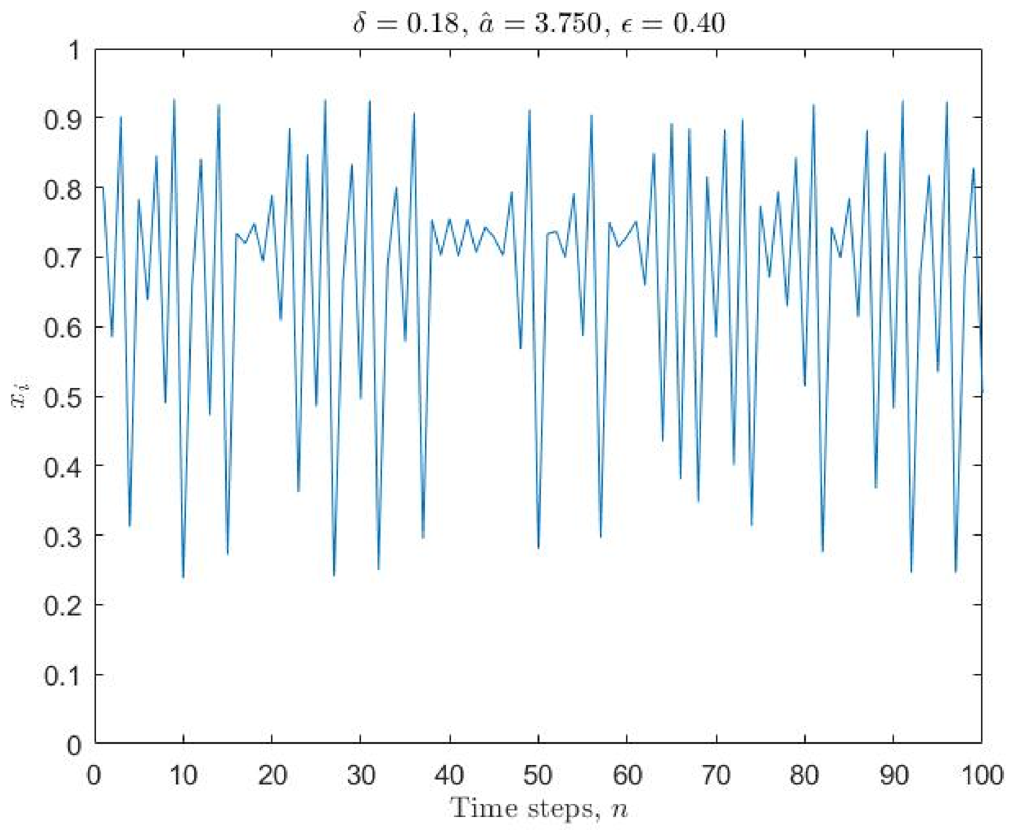


Fig. 2

As can be seen from the **Fig.1** and **Fig. 2**, the dynamics are chaotic for low and high diversities.

However, for an optimum value of diversity, which is 0.08 in this case, the system dynamics are observed to be maximally periodic, as can be seen in **Fig. 3**. A part of the plot is enlarged in **Fig. 4**, to show Period 5 dynamics.

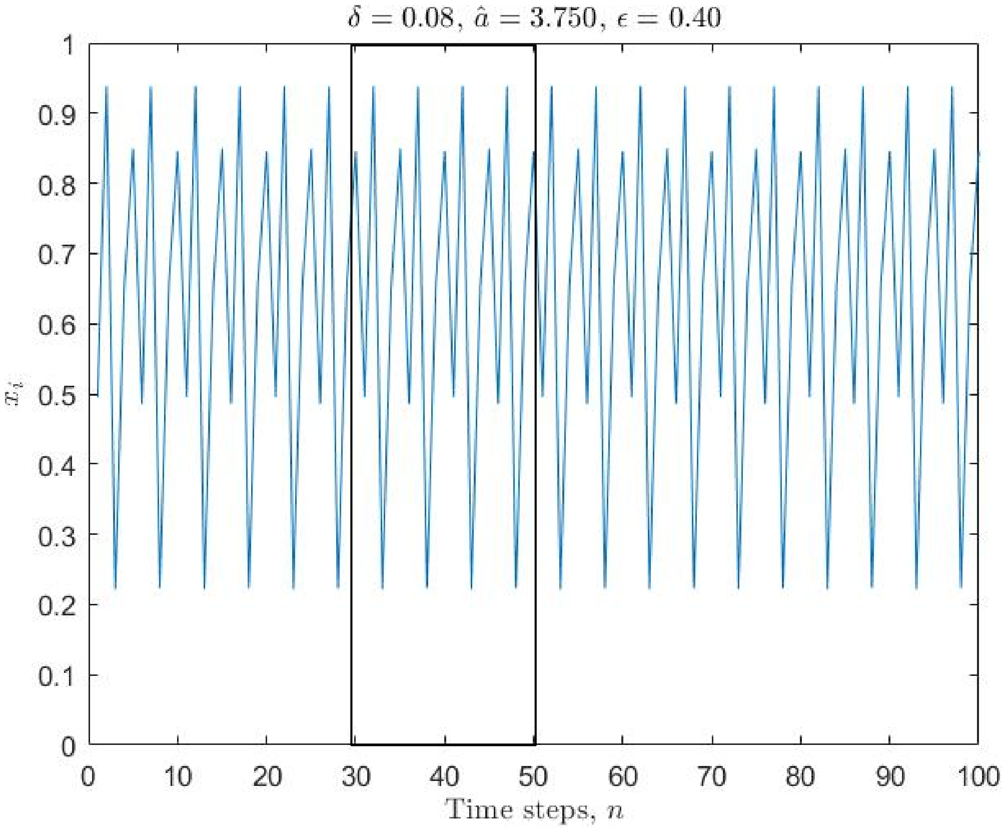


Fig. 3

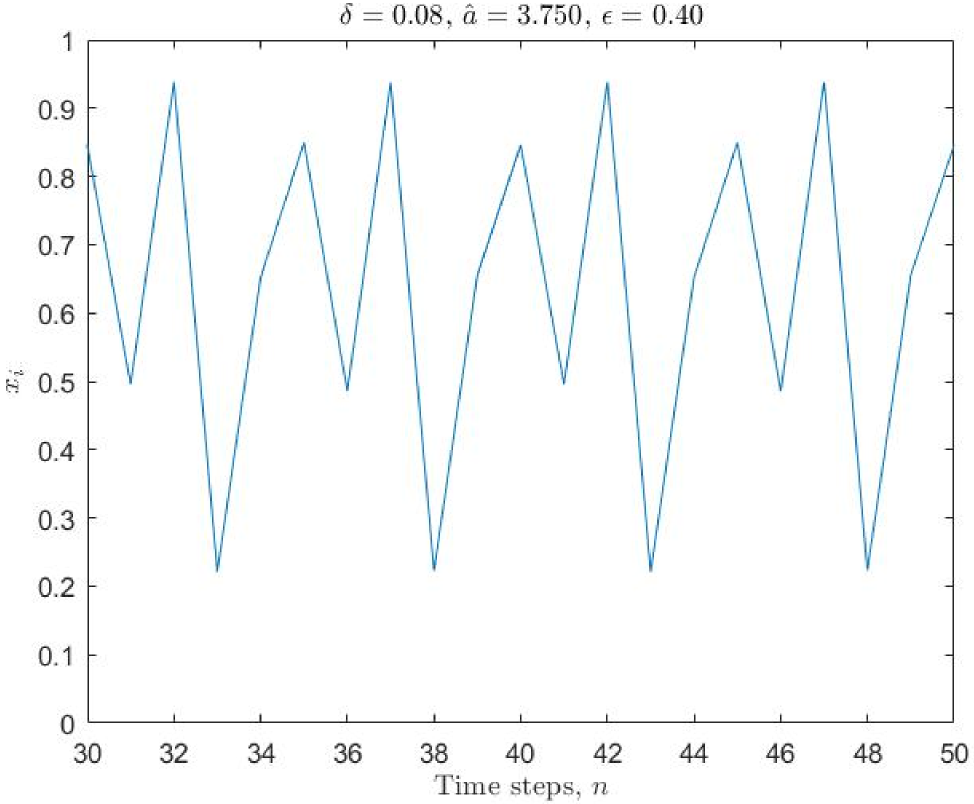


Fig. 4

For a greater understanding of this resonance, we plot the coherence factor calculated from cumulative power spectrum against the diversity.

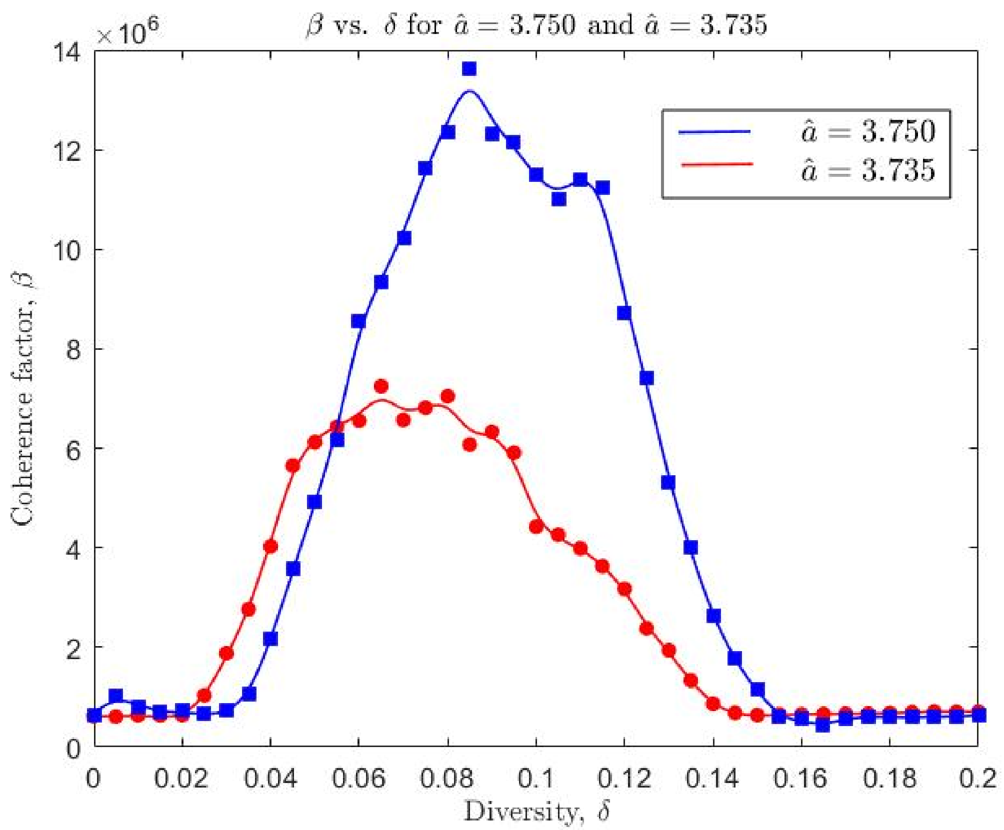


Fig. 5

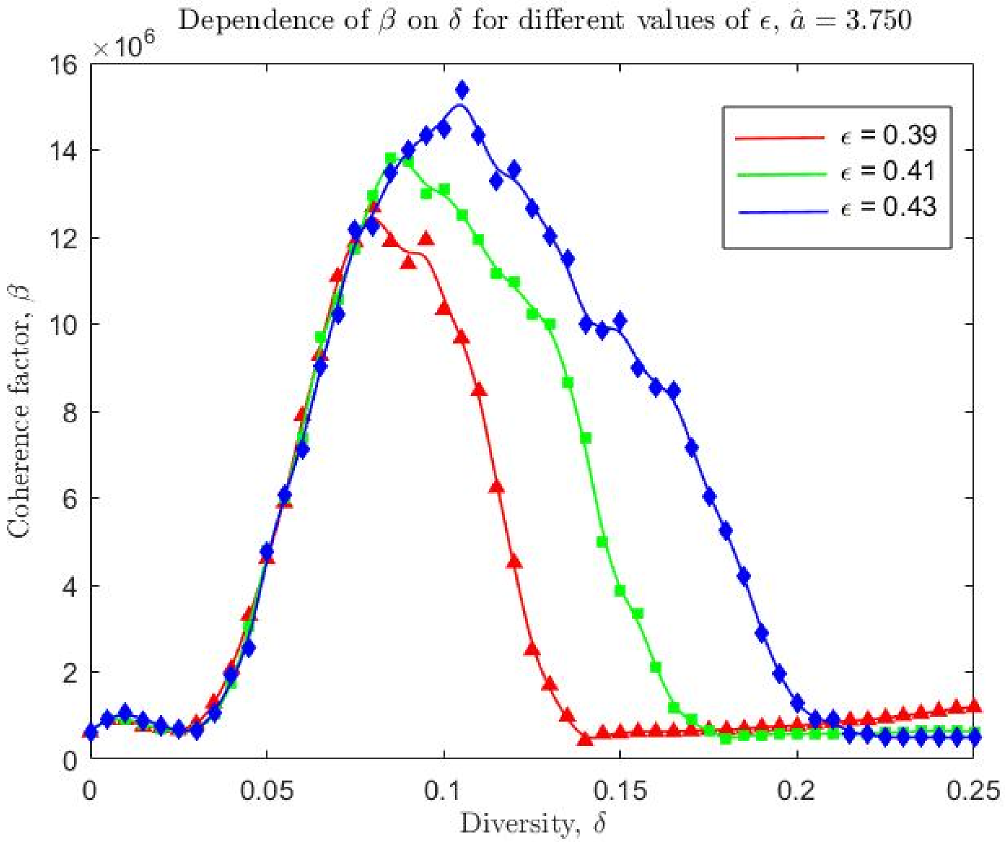


Fig. 6

In **Fig. 5**, we clearly see a peak in the coherence factor for the optimum diversity value. We also get resonance for $\hat{a} = 3.735$, which is on the other side of the Period 5 window.

In **Fig. 6**, we plot the coherence factor for different values of coupling strength. It must be noted that a minimum coupling strength of $\epsilon_{th} \sim 0.38$ is required to synchronize the oscillators. We can see that as the coupling strength is increased, the resonance peak shifts to higher diversity levels and the width of the peak increases.

Sampling Method $\mathbf{II}$

- We saw that in Sampling Method I, the parameter average $\bar{a}$ had a considerable spread about $\hat{a}$. This spread can be calculated by applying the Central Limit Formula to all the samples and saying that $\bar{a}$ follows a Normal Distribution with mean $\mu = \hat{a}$ and standard deviation $\sigma = {}^{\delta}/_{\sqrt{3N}}$. For $\delta = 0.1$ and $N = 100$, we get $\sigma {\sim} 0.006$. This spread is comparable to the Period 5 window, which is 0.012.
- Hence, in this sampling method, $\bar{a}$ was artificially fixed to zero. For odd i, a_i were taken arbitrarily from the Uniform Distribution $[\hat{a} - \delta, \hat{a} + \delta]$. For even i, $a_i = 2\hat{a} - a_{i-1}$. Thus, we always have $\bar{a} = \hat{a}$.

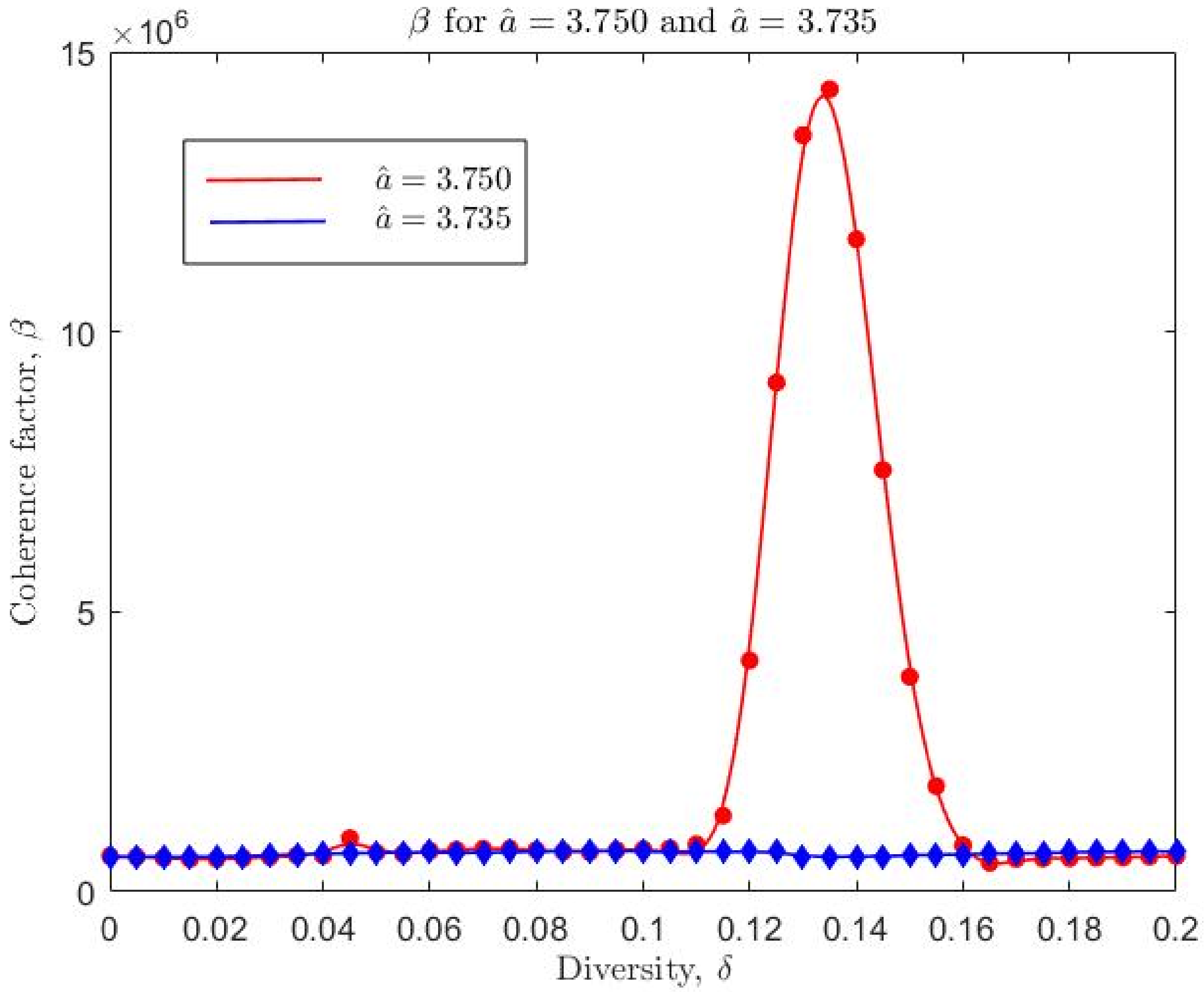


Fig. 7

We see from **Fig. 7**, that in this sampling method II, the resonance for $\hat{a} = 3.750$ appears at a higher diversity value and is much sharper. However, no resonance peak is observed for $\hat{a} = 3.735$, which is contrary to the previous sampling method.

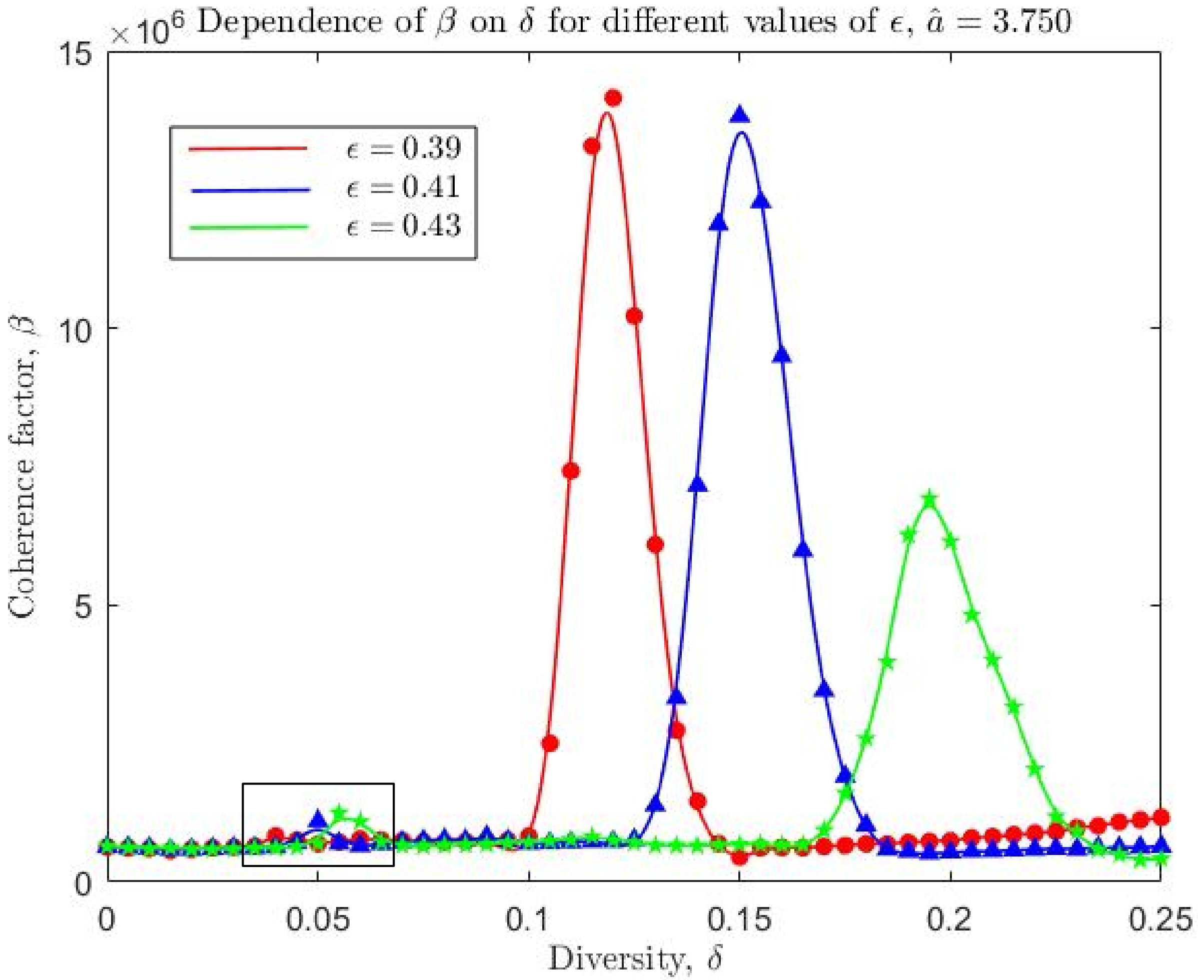


Fig. 8

In **Fig. 8**, coherence factor for three values of ϵ are plotted against diversity. We see that the height of peaks decreases as coupling strength increases. Also, resonance is obtained at a higher diversity value for higher coupling strength.

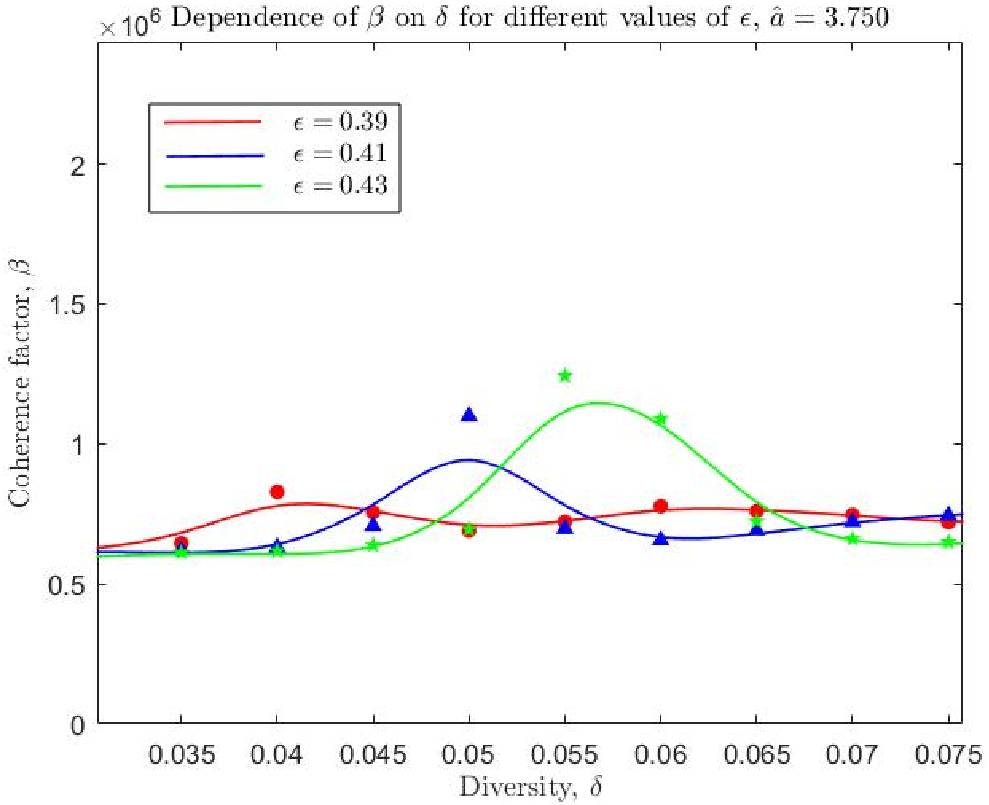


Fig. 9

We also observe multiple resonance peaks of weak intensity at lower diversity values, which can be seen in **Fig. 9**, which is an enlarged version of a part of the plot in **Fig. 8**.

Bifurcation Diagram Analysis

- In order to understand these results, we study the bifurcation diagram of these logistic maps.
- Bifurcation diagram shows the values visited or approached asymptotically by the system as a system parameter is changed.
- The reduced model with the following equations was obtained for the mean field dynamics:

$$\boldsymbol{X_{n+1} = f(X_n, \hat{a}) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} V_n + \frac{\partial^2 f}{\partial x \partial a} W_n}$$

$$\boldsymbol{W_{n+1} = (1-\epsilon)\left(W_n \frac{\partial f}{\partial x} + \frac{\delta^2}{3}\frac{\partial f}{\partial a}\right)}$$

$$\boldsymbol{V_{n+1} = (1-\epsilon)^2\left(V_n \left(\frac{\partial f}{\partial x}\right)^2 + 2W_n \frac{\partial f}{\partial x}\frac{\partial f}{\partial a} + \frac{\delta^2}{3}\left(\frac{\partial f}{\partial a}\right)^2\right)}$$

where $\boldsymbol{f(x, a) = ax(1-x)}$

Bifurcation Diagram Analysis

- In these equations, two additional macroscopic variables have been introduced: W and V, apart from the mean field $\boldsymbol{X} = \langle \boldsymbol{x_i} \rangle$.
- $\boldsymbol{W} = \langle (\boldsymbol{x_i} - \boldsymbol{X})(\boldsymbol{a_i} - \boldsymbol{\hat{a}}) \rangle$, also known as shape parameter, measures the joint dispersion in the map parameters a_i and oscillator coordinates x_i.
- $\boldsymbol{V} = \left\langle (\boldsymbol{x_i} - \boldsymbol{X})^{\boldsymbol{2}} \right\rangle$ measures dispersion only in the oscillator coordinates.
- To obtain these equations, the order parameter expansion approach, suggested by Monte et al. was followed.

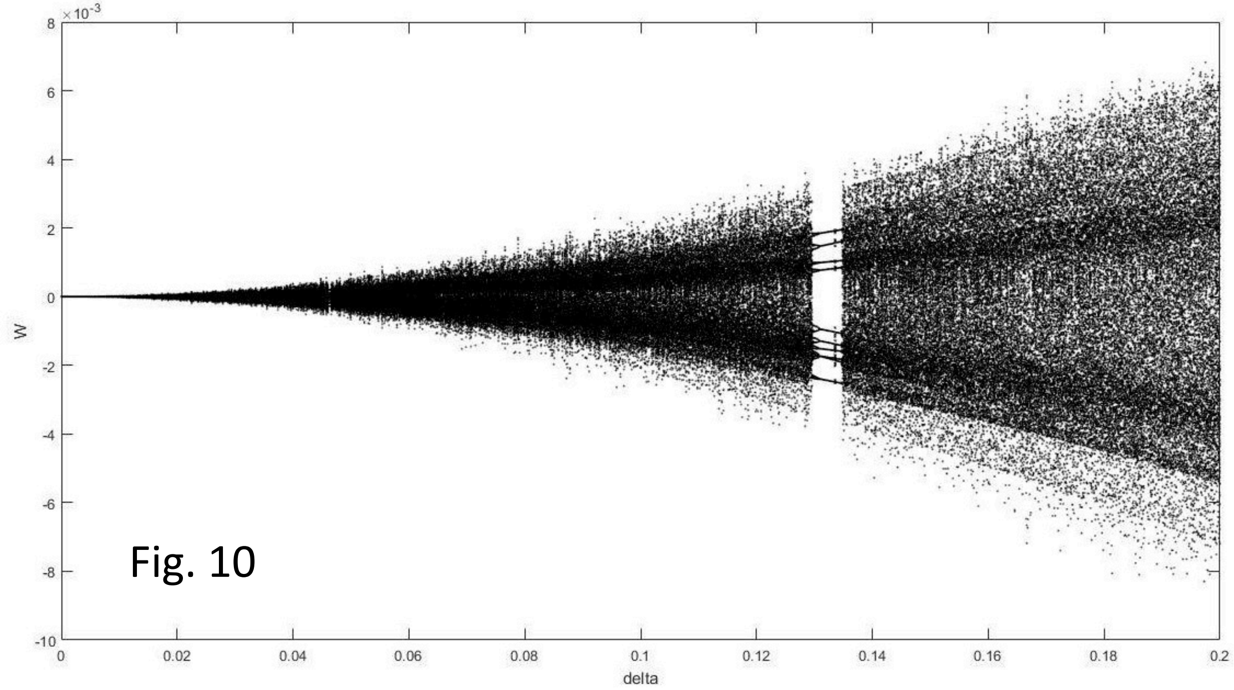


Fig. 10

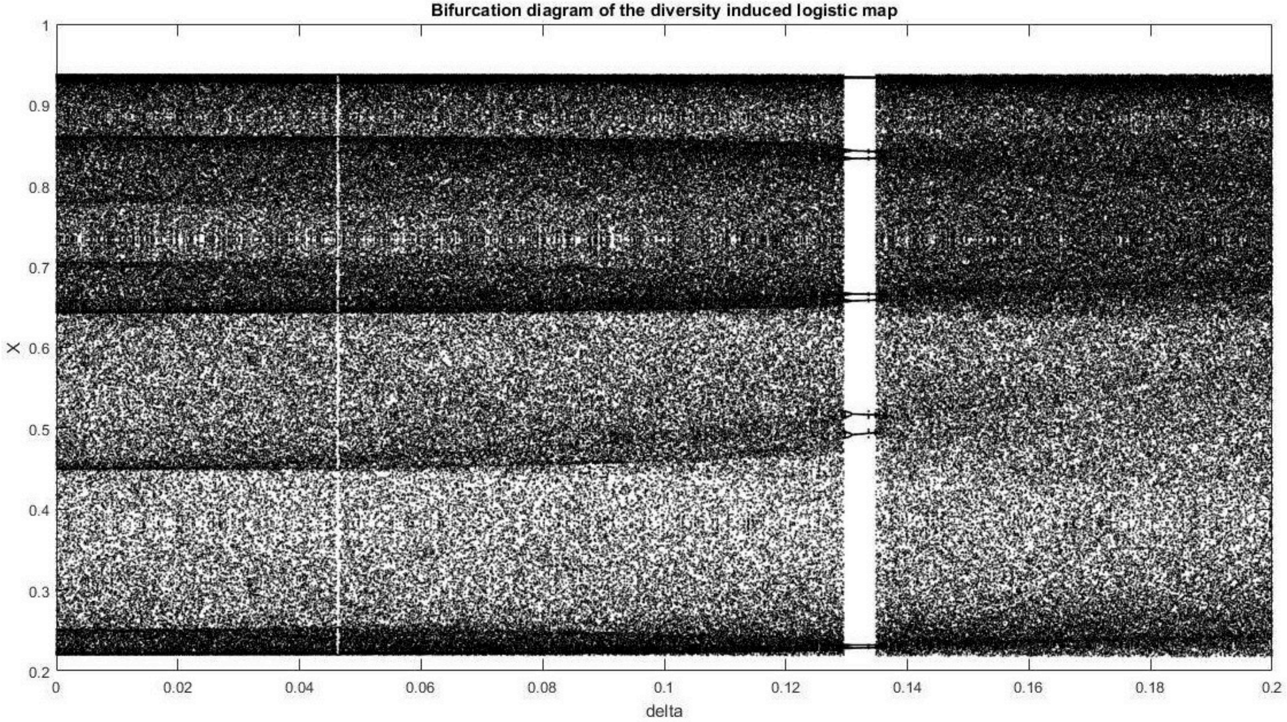


Fig. 12

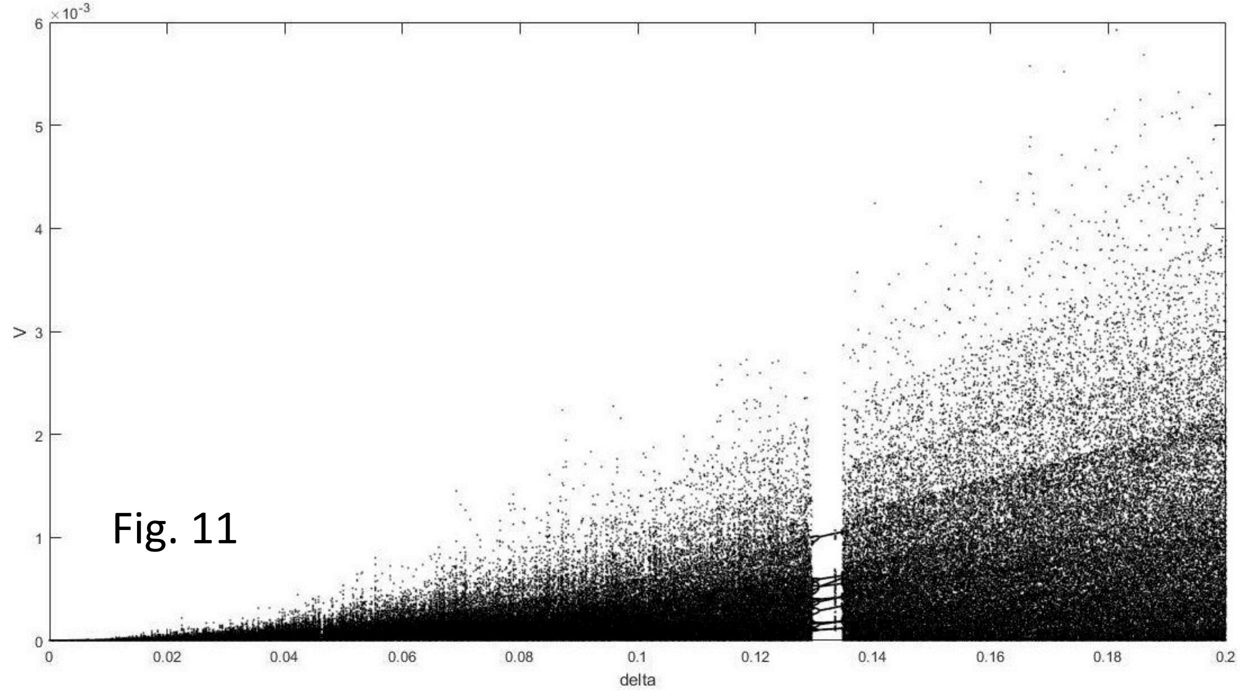


Fig. 11

These figures show the bifurcation diagrams of the reduced model with diversity for $\hat{a} = 3.75$ and $\epsilon = 0.4$. All the three macroscopic variables, W **(Fig. 10)**, V **(Fig. 11)**, and X **(Fig. 12)** are plotted. We saw that regularity is maximum at an optimum diversity level. Here, we see a period 5 window around $\delta \sim 0.13$. This corresponds to the peak in the resonance curve that we saw in Sampling Method II **(Fig. 7).**

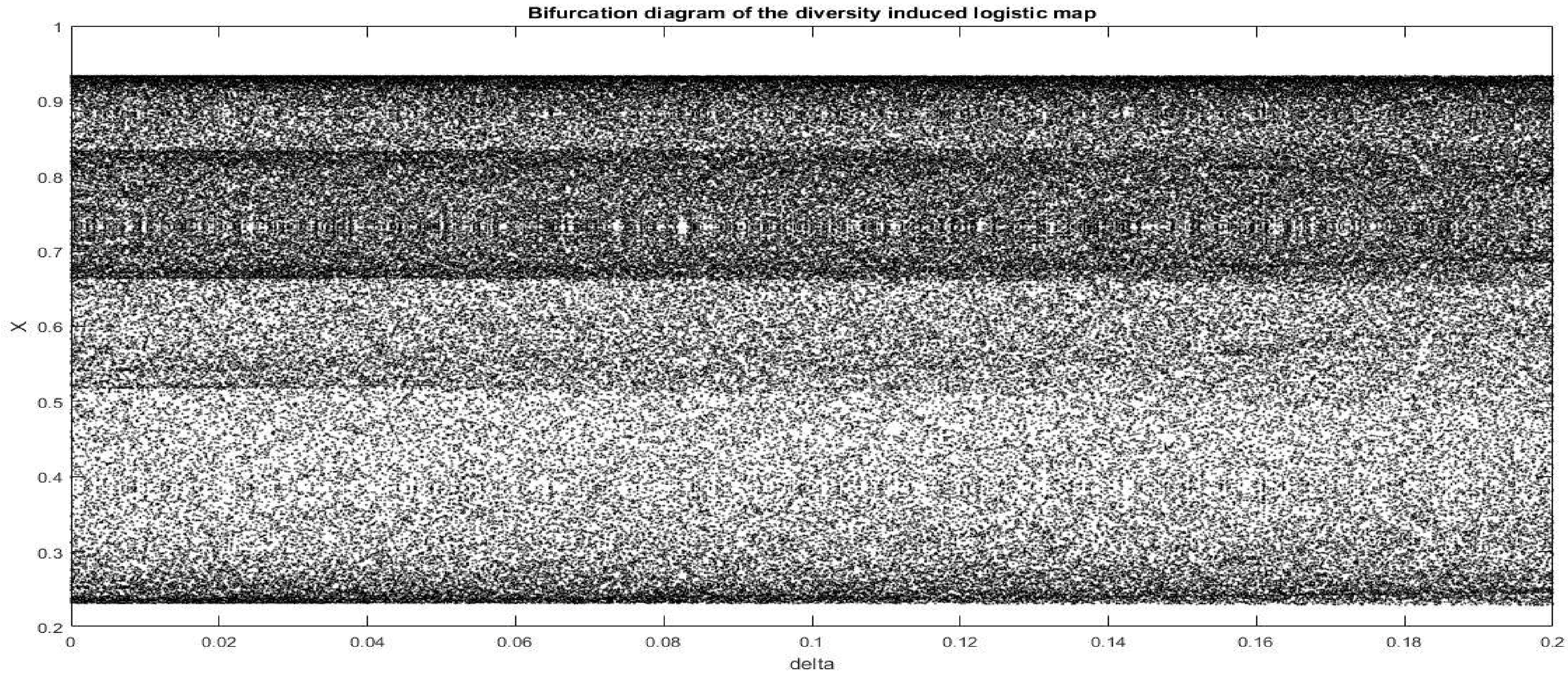
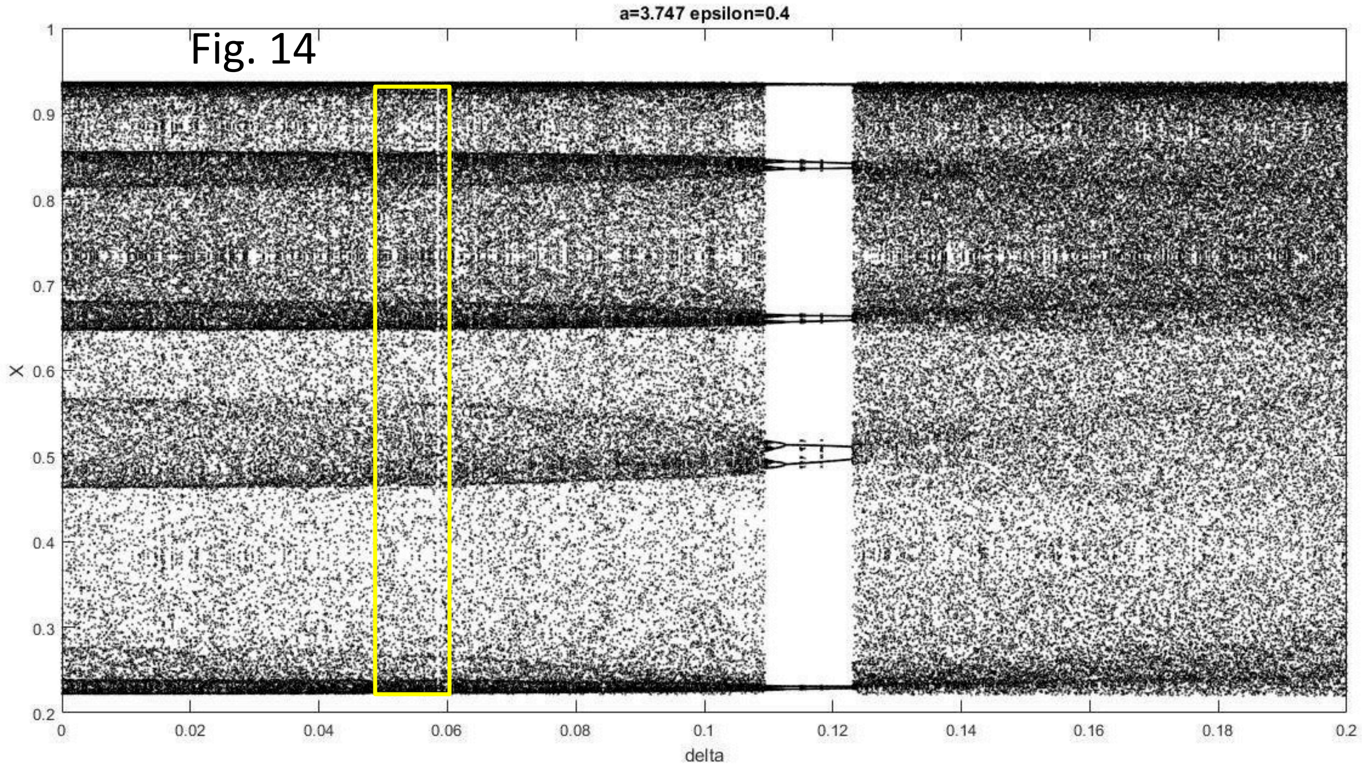
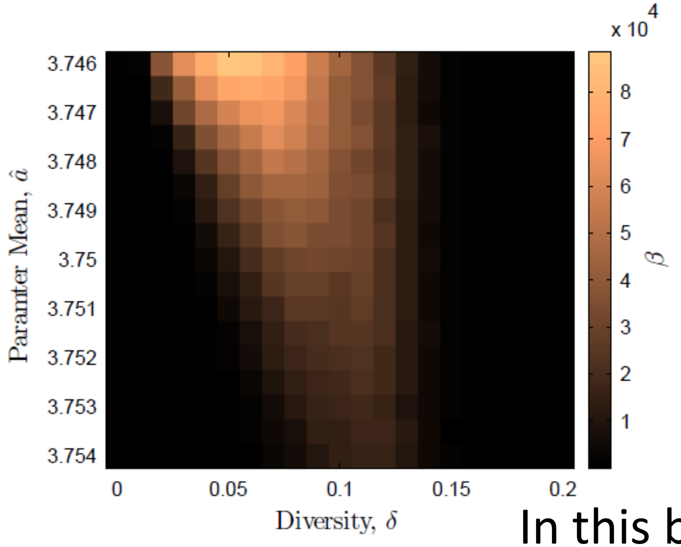


Fig. 13

Here in **Fig. 13**, the bifurcation diagram for $\hat{a} = 3.735$ is presented. We saw no resonance peak for this value in Sampling Method II **(Fig. 7).** This is because the bifurcation diagram is devoid of any periodic windows.





From paper
Fig. 15

We had seen earlier **(Fig. 15)** that for values of the parameter mean $\hat{a}$ closer to the bifurcation point, the resonance was stronger and at lower diversity levels.

In this bifurcation diagram **(Fig. 14),** we see that the period 5 window has shifted to lower diversity level and is also wider.

We also saw a double resonance for $\hat{a} = [3.746, 3.748]$ **(Fig. 17).** Correspondingly, we see two period 5 windows in the bifurcation diagram above **(Fig. 14).** The window for the lower diversity has been magnified beside **(Fig. 16).**

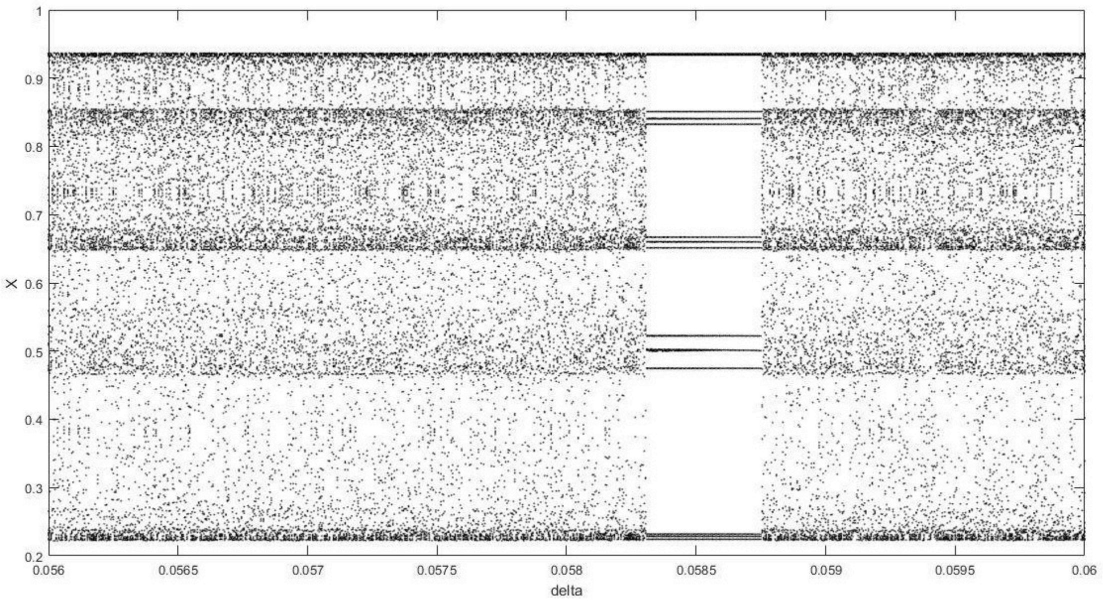
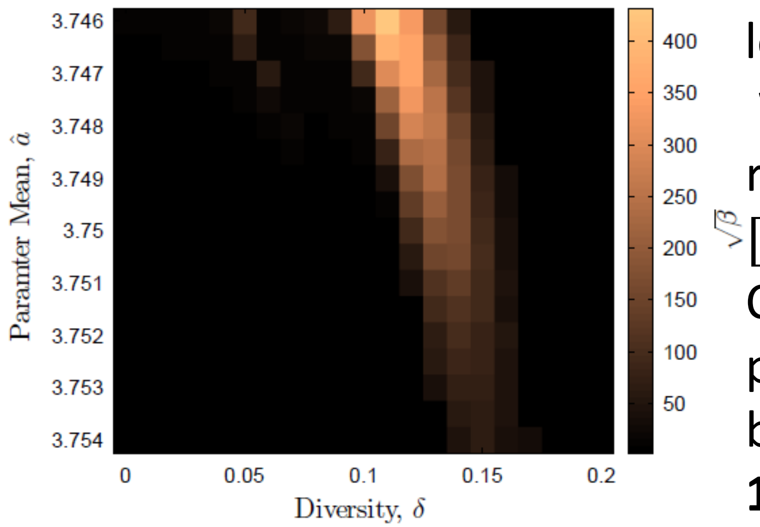


Fig. 16



From paper
Fig. 17

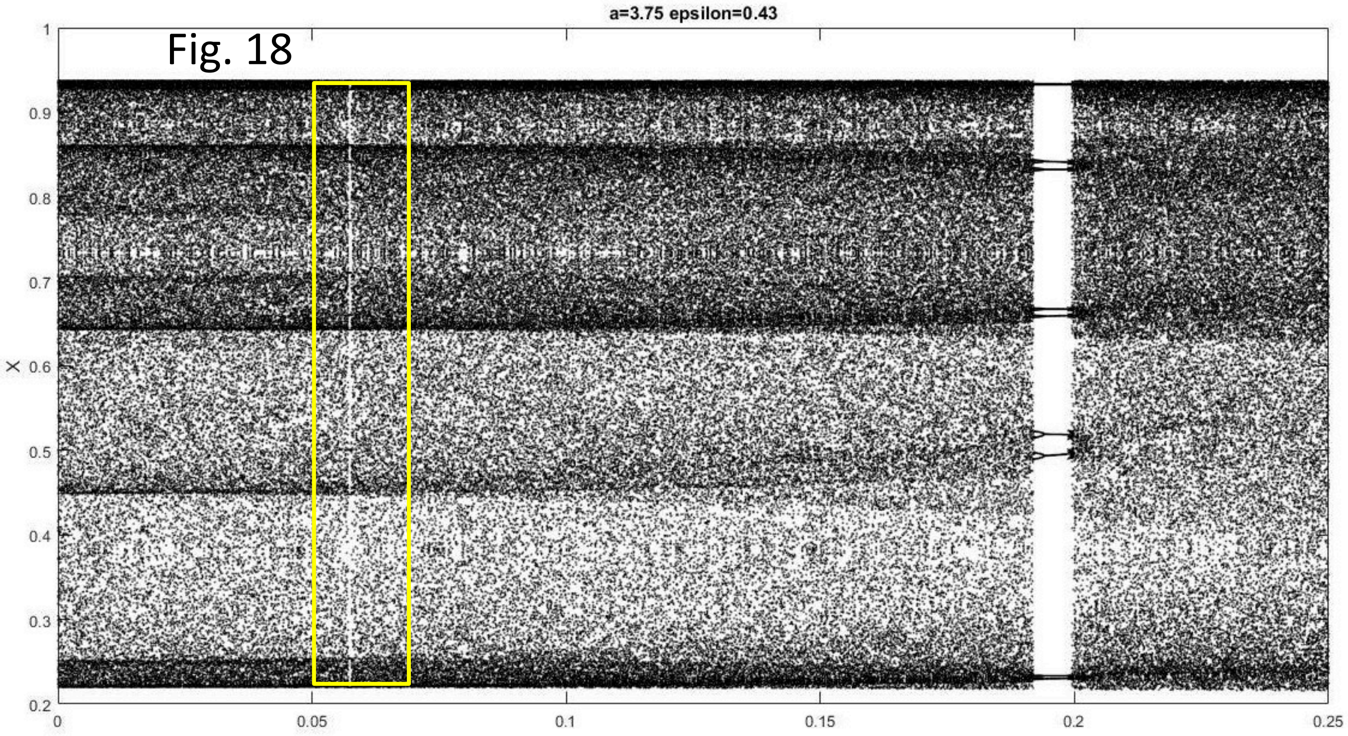


Fig. 18

As we increased the coupling strength, we saw that the resonance peak shifted to higher diversity levels and got broader. We also saw two resonance peaks earlier for $\epsilon = 0.43$: one at $\delta \sim 0.055$ and other at $\delta \sim 0.2$ **(Fig. 8).** A period 5 window for each of the diversity levels can be seen in the bifurcation diagram **(Fig. 18).** The window at lower diversity has been magnified below **(Fig. 20).**

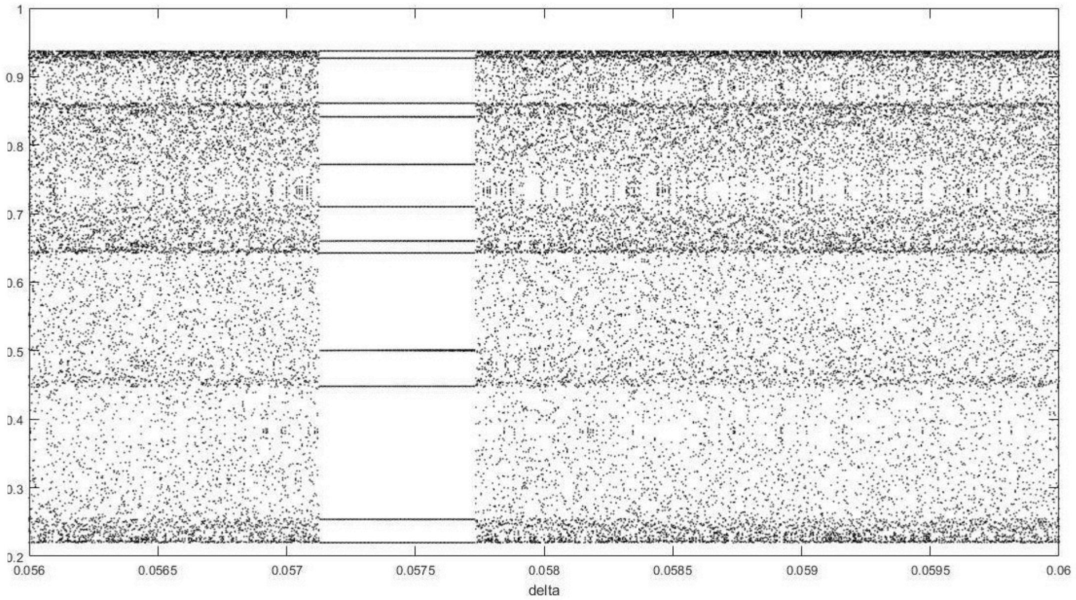


Fig. 20

A third period 5 window can also be seen around $\delta \sim 0.118$. This has been magnified beside **(Fig. 19).** However, the corresponding peak is missing due to poor resolution.

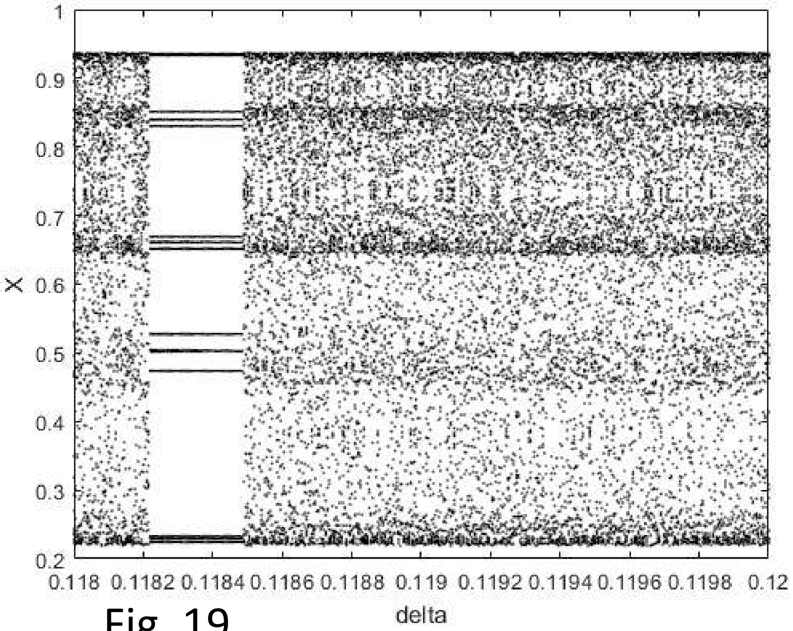
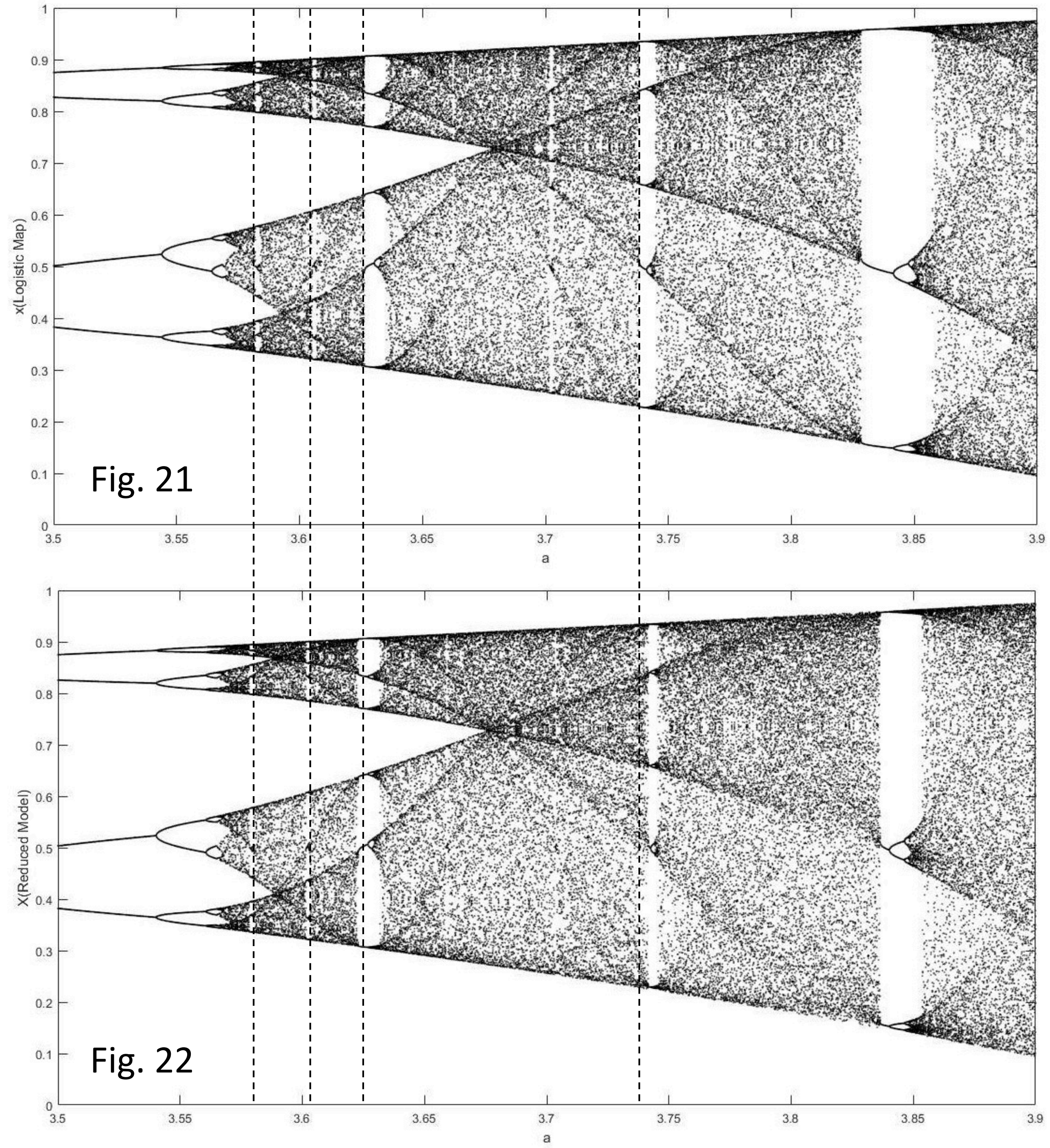


Fig. 19



Comparing the bifurcation diagram for the logistic map **(Fig. 21)** and the reduced model **(Fig. 22),** three types of shifts can be seen.

1. **Left shift**: For $\hat{a} < 3.65$, we see that the periodic windows shift left. This implies resonance if $\hat{a}$ is set to the left of periodic window.
2. **Right shift**: For $\hat{a} \sim 3.745$, the Period 5 window shrinks in size and shifts to the right. Hence, resonance will be observed if $\hat{a}$ is set to the right side of the Period 5 window.
3. **No shift**: For $\hat{a} \sim 3.85$, we see that the Period 3 window only shrinks in size. It does not shift. Thus, no resonance would be observed for $\hat{a}$ around the Period 3 window.

Checking The Robustness of The System

In the following figures, the same simulations as before were carried out for 18 oscillators (N=18). Though the quality of the curves may not be as sharp, we observe peaks as before in all the cases, for both the sampling methods.

Sampling Method I

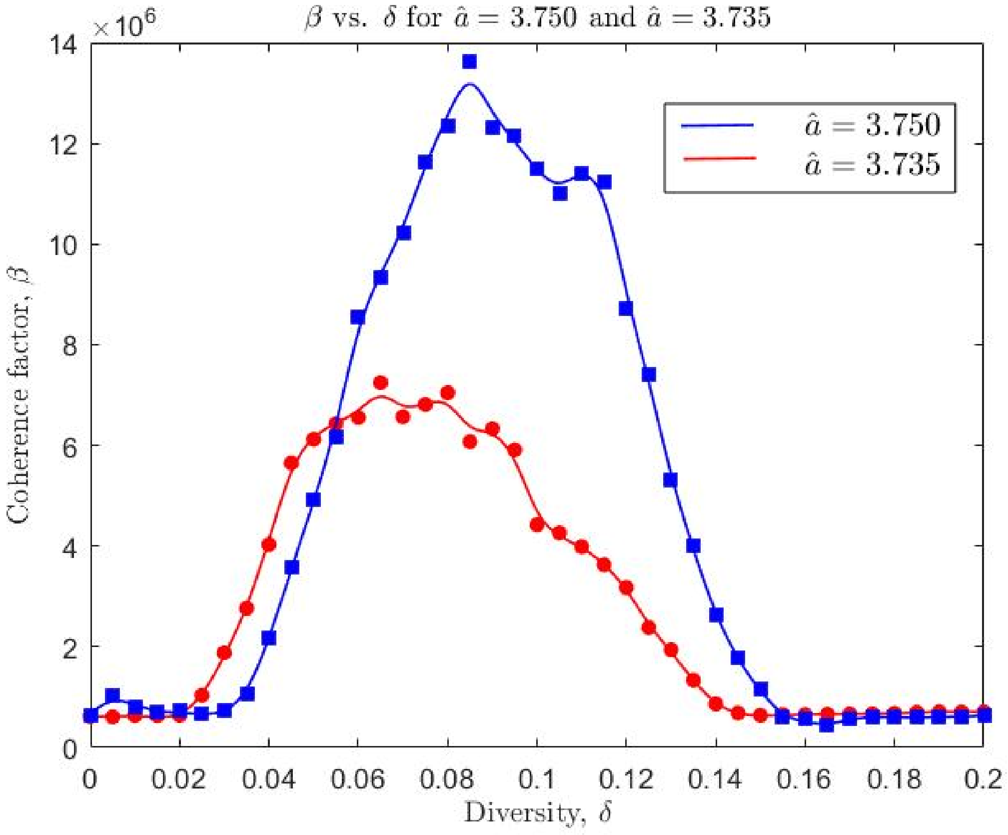


Fig. 5

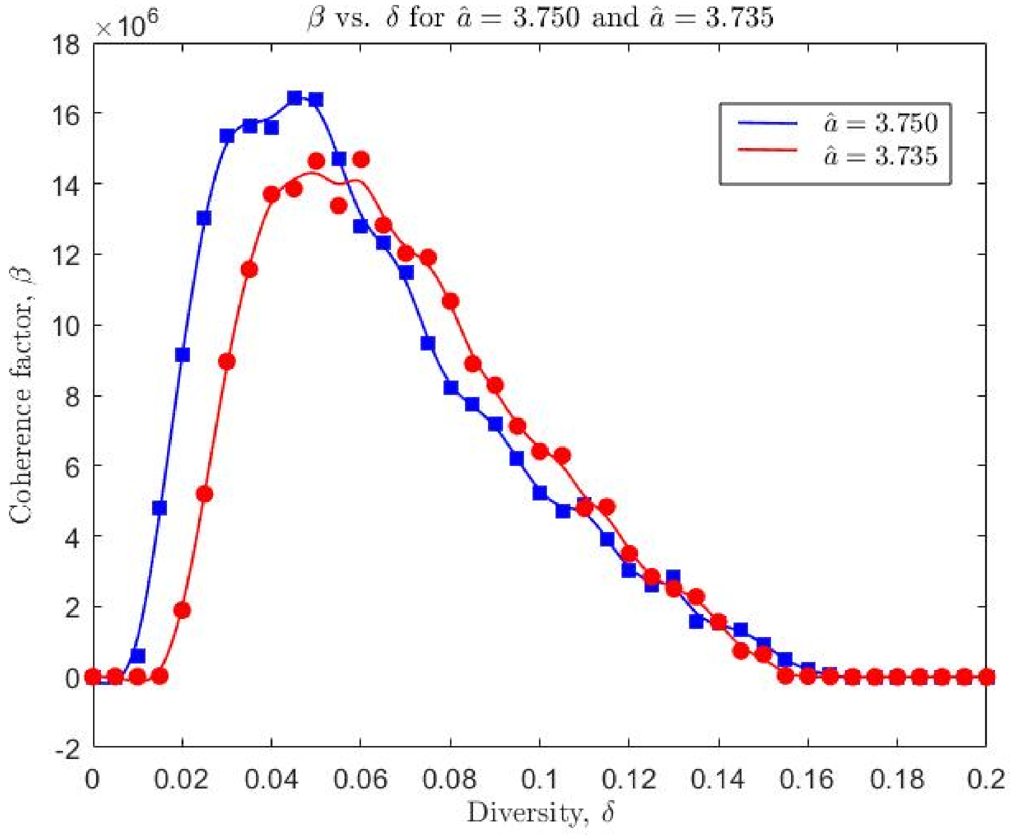


Fig. 24

Checking The Robustness of The System

Sampling Method I

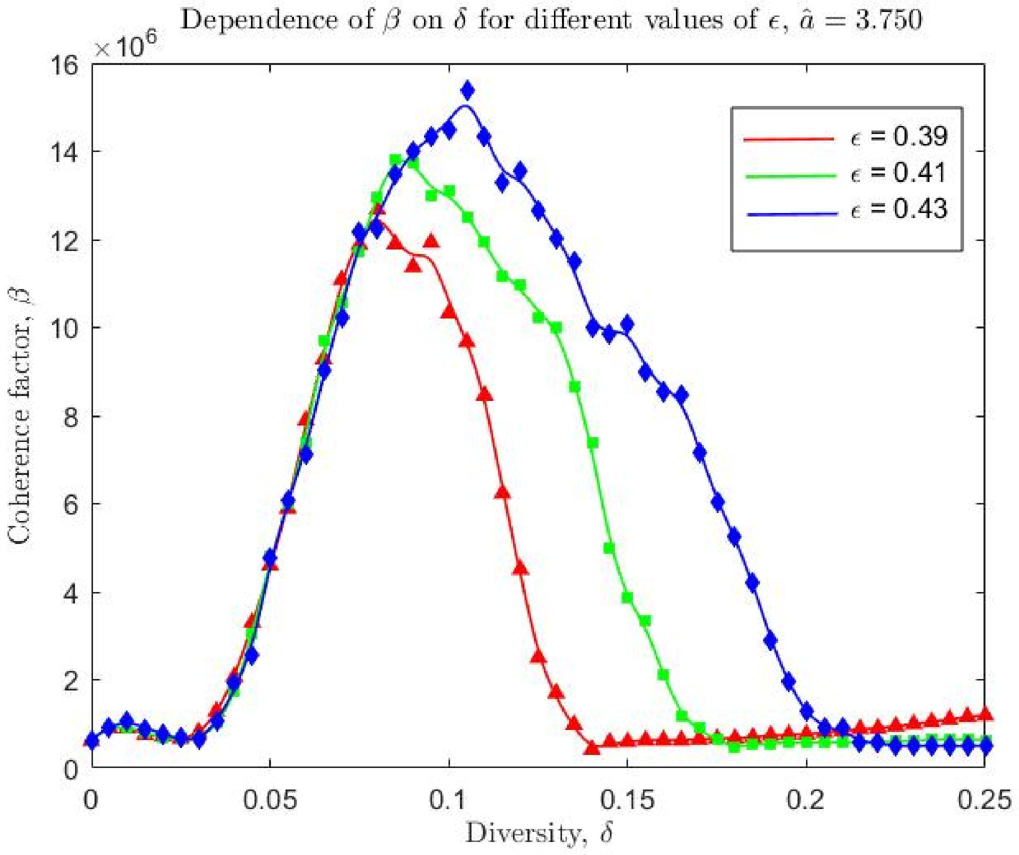


Fig. 6

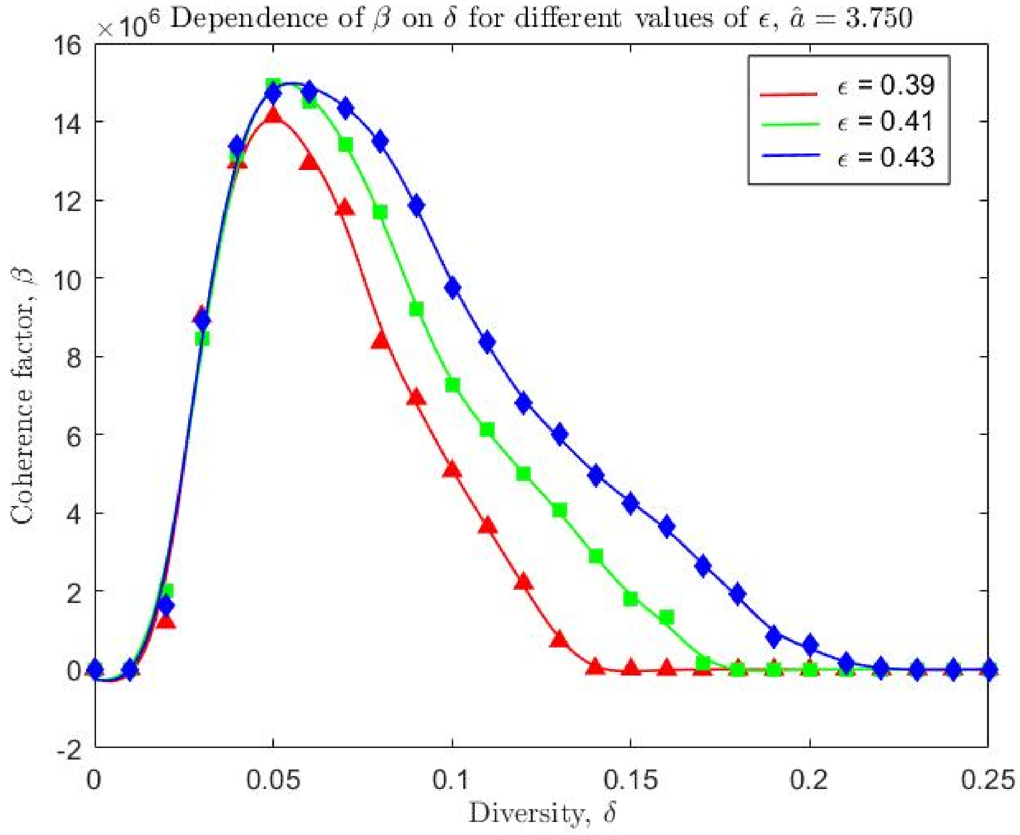


Fig. 25

Checking The Robustness of The System

Sampling Method II

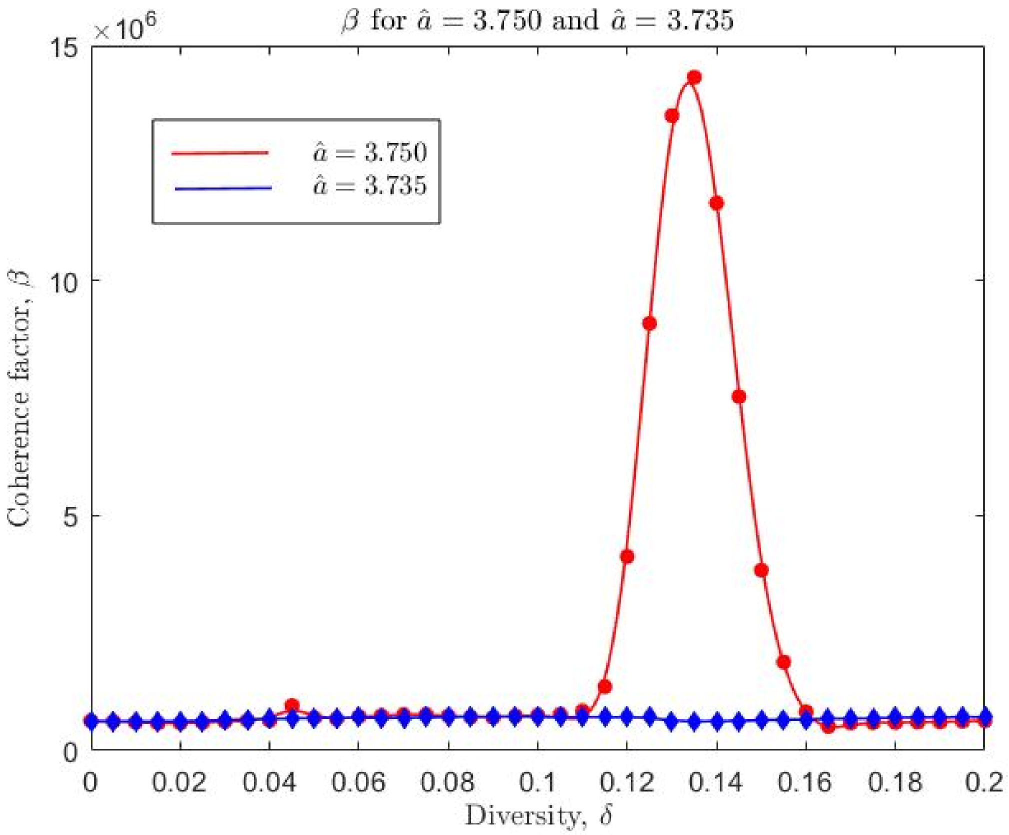


Fig. 7

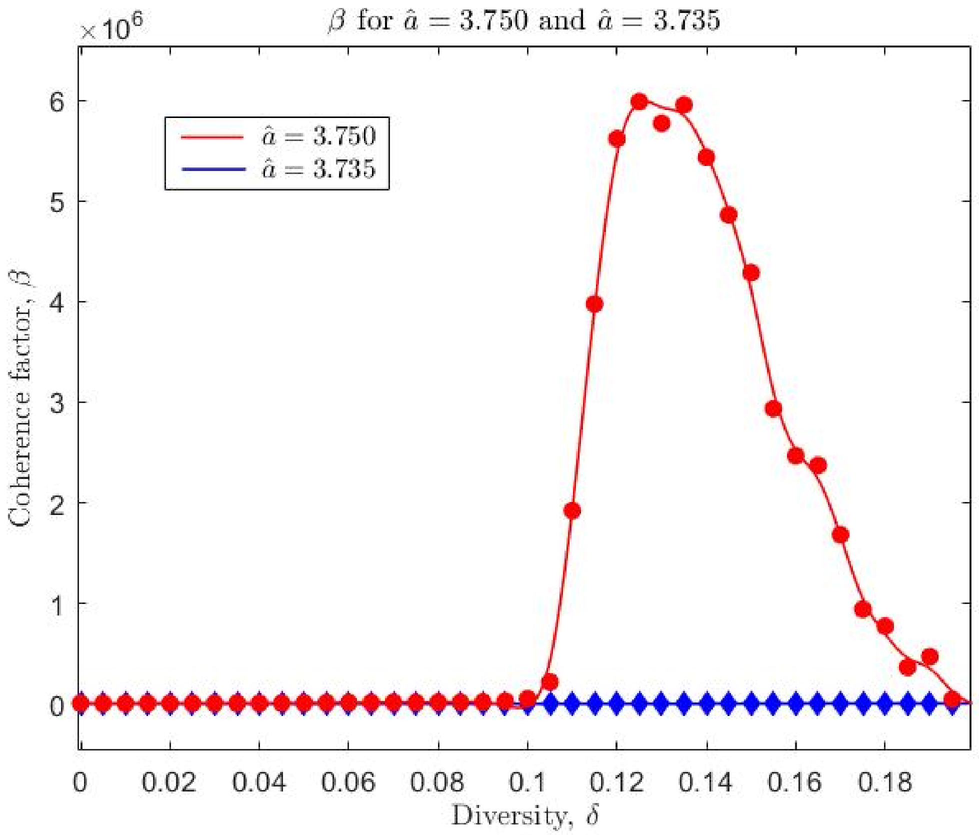


Fig. 26

Checking The Robustness of The System

Sampling Method II

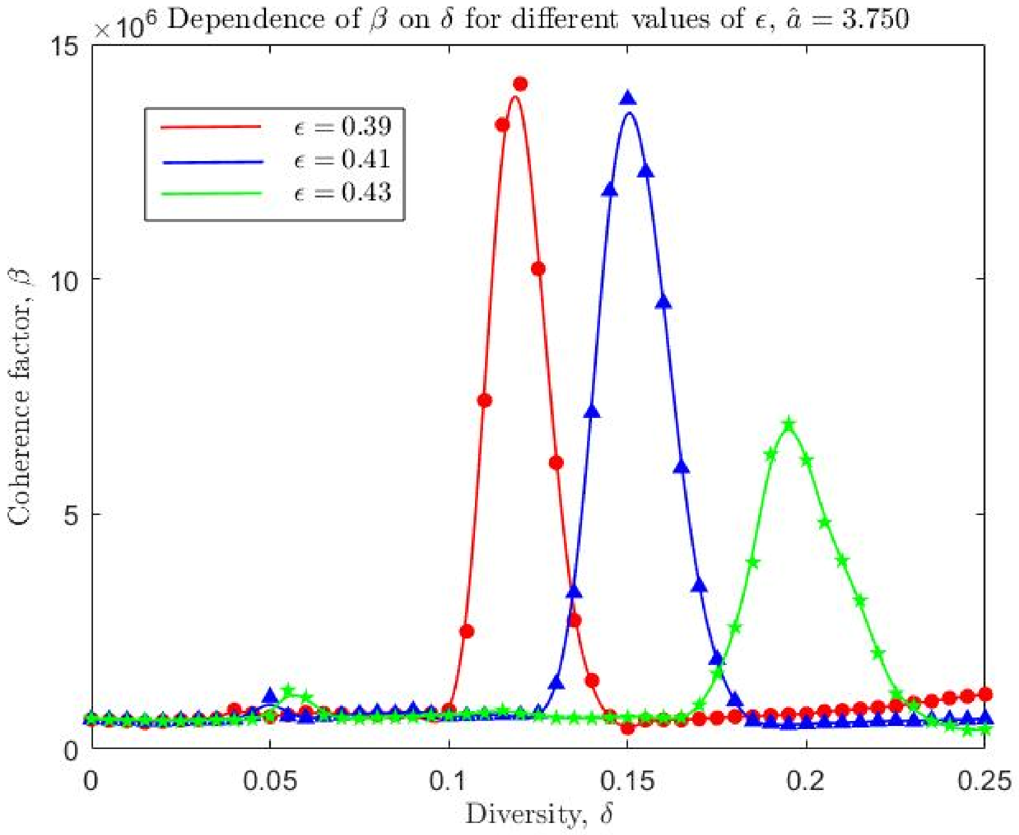


Fig. 8

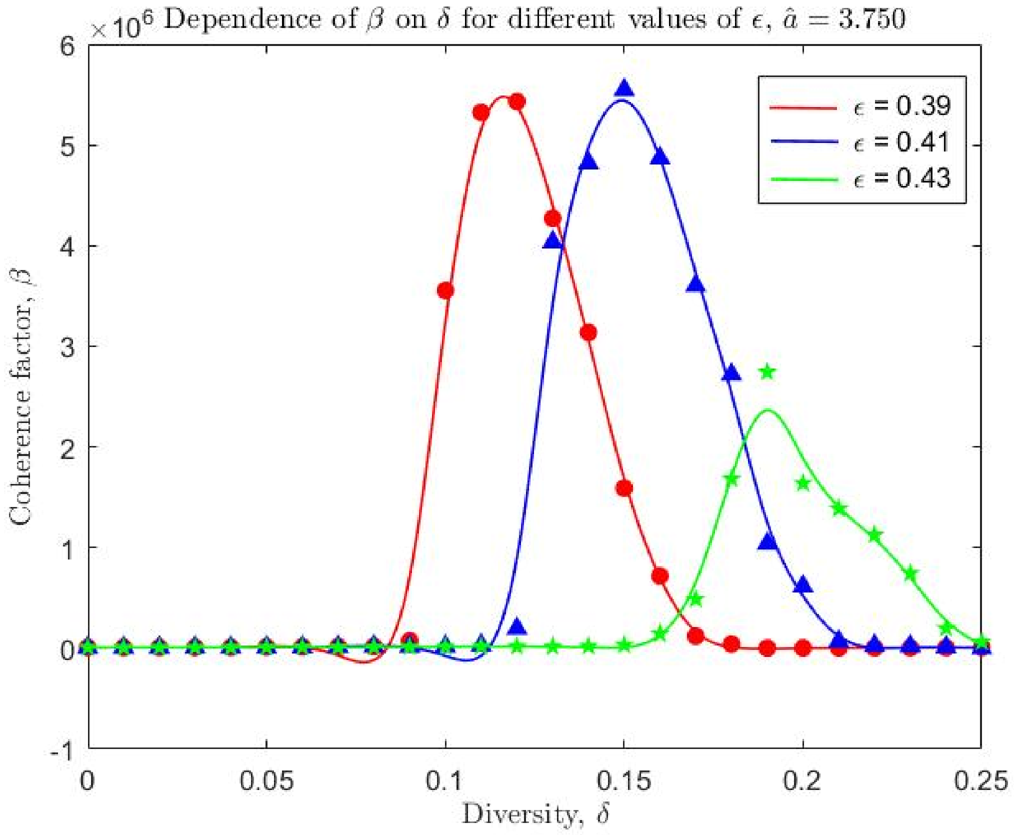
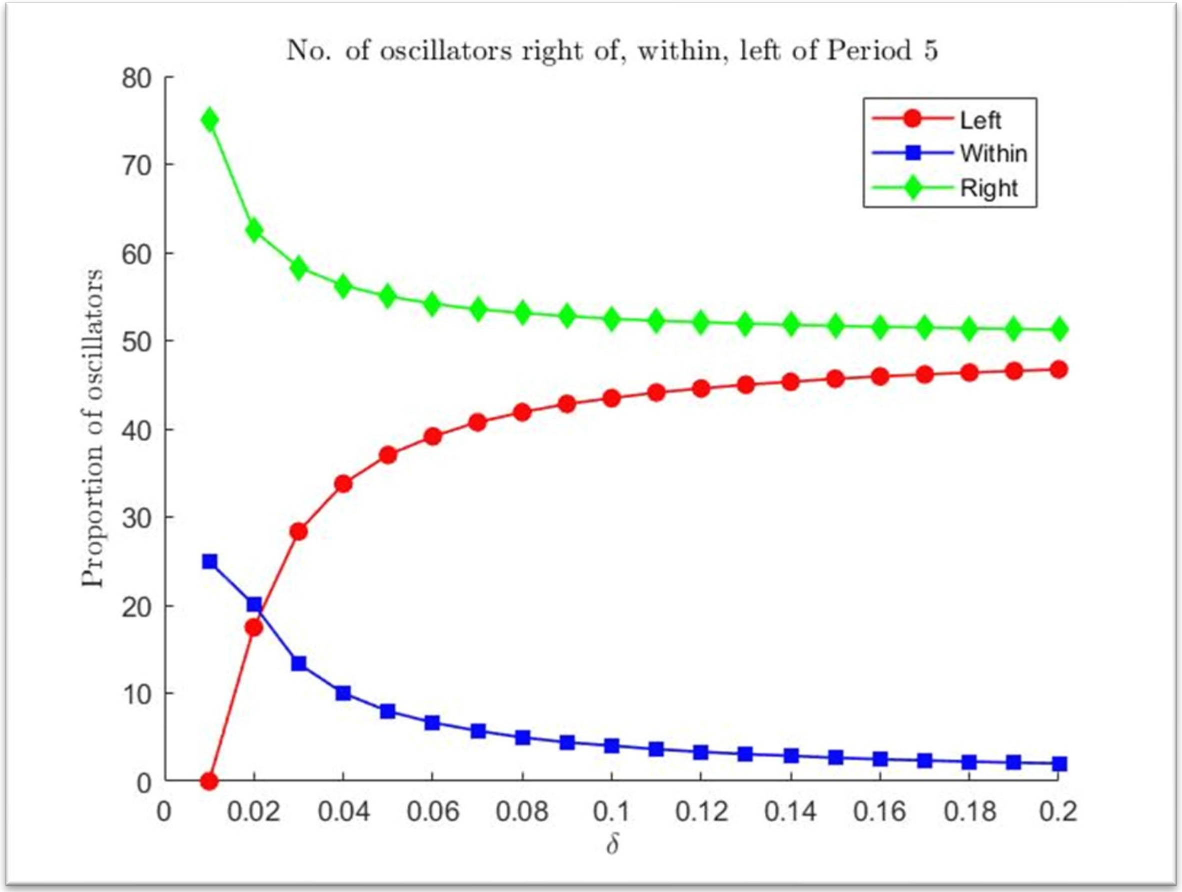
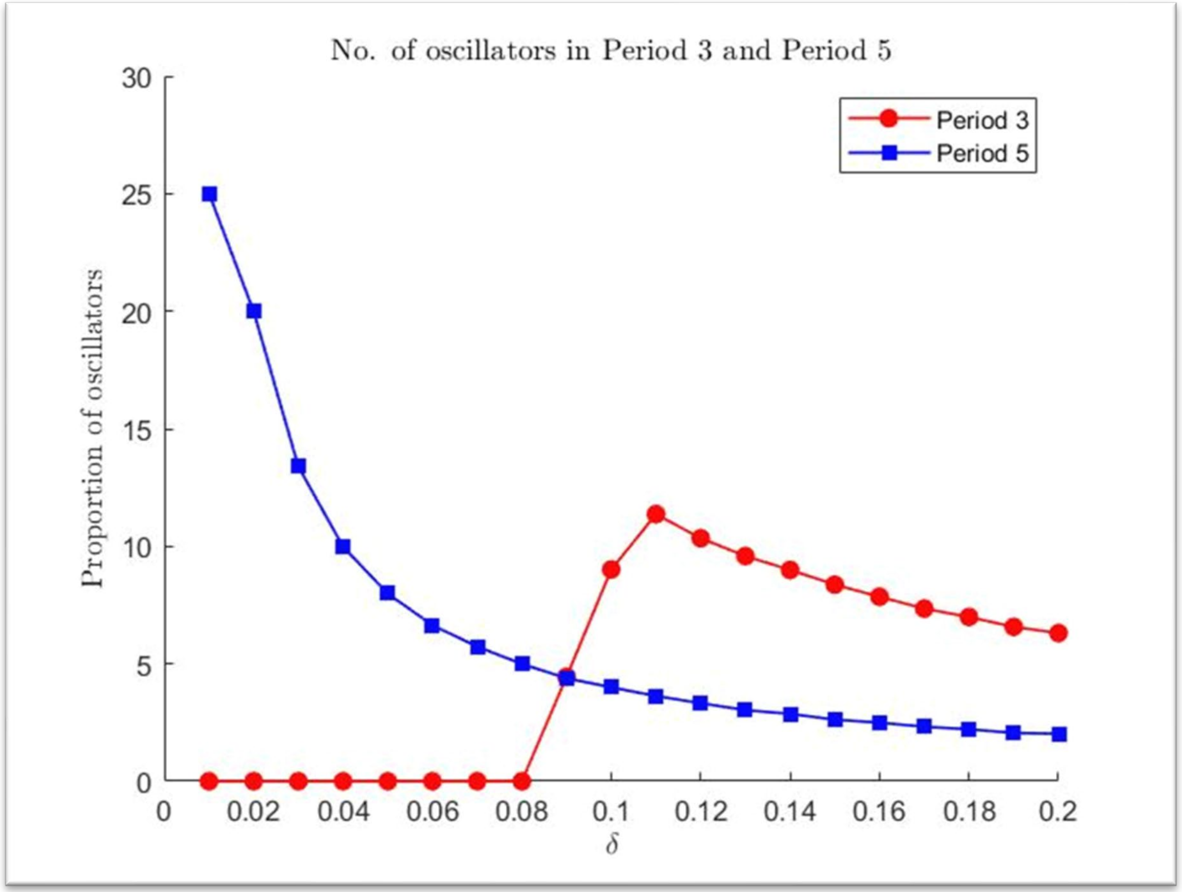


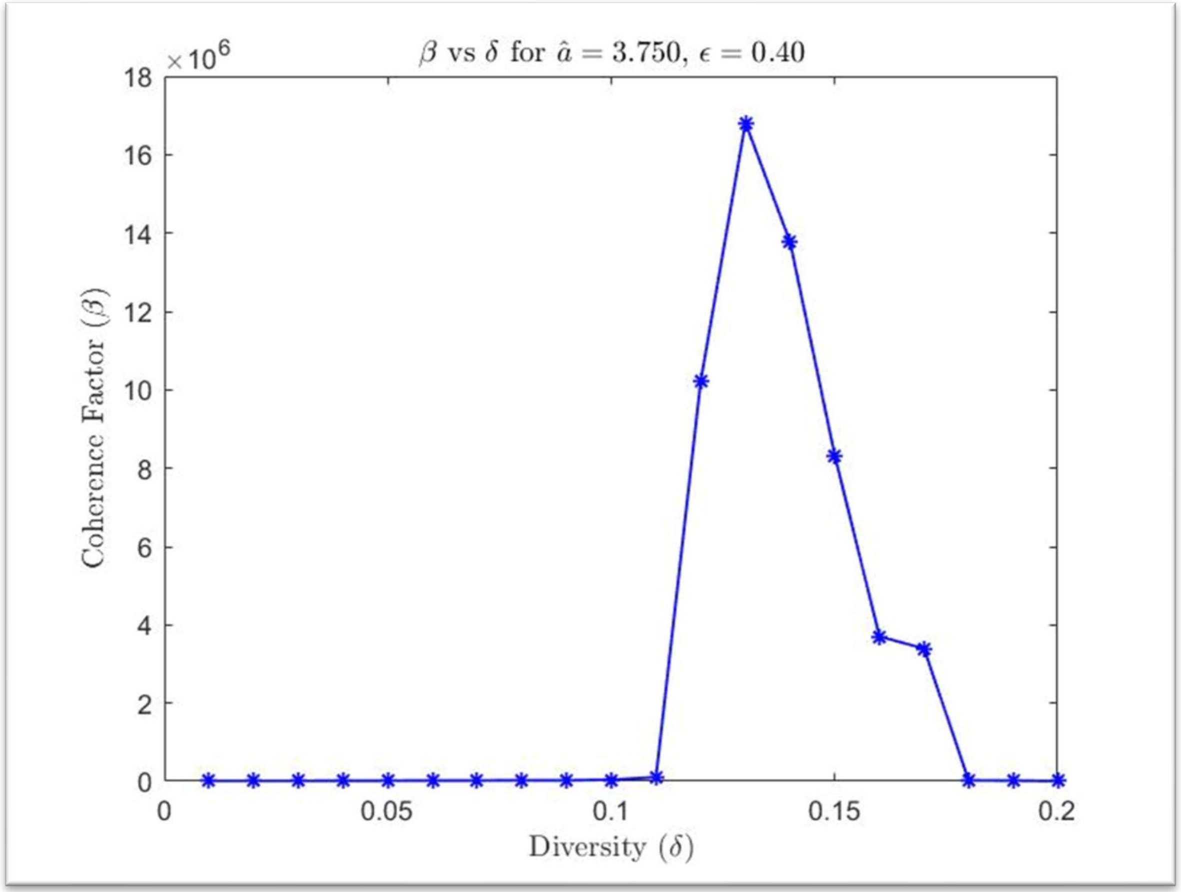
Fig. 27

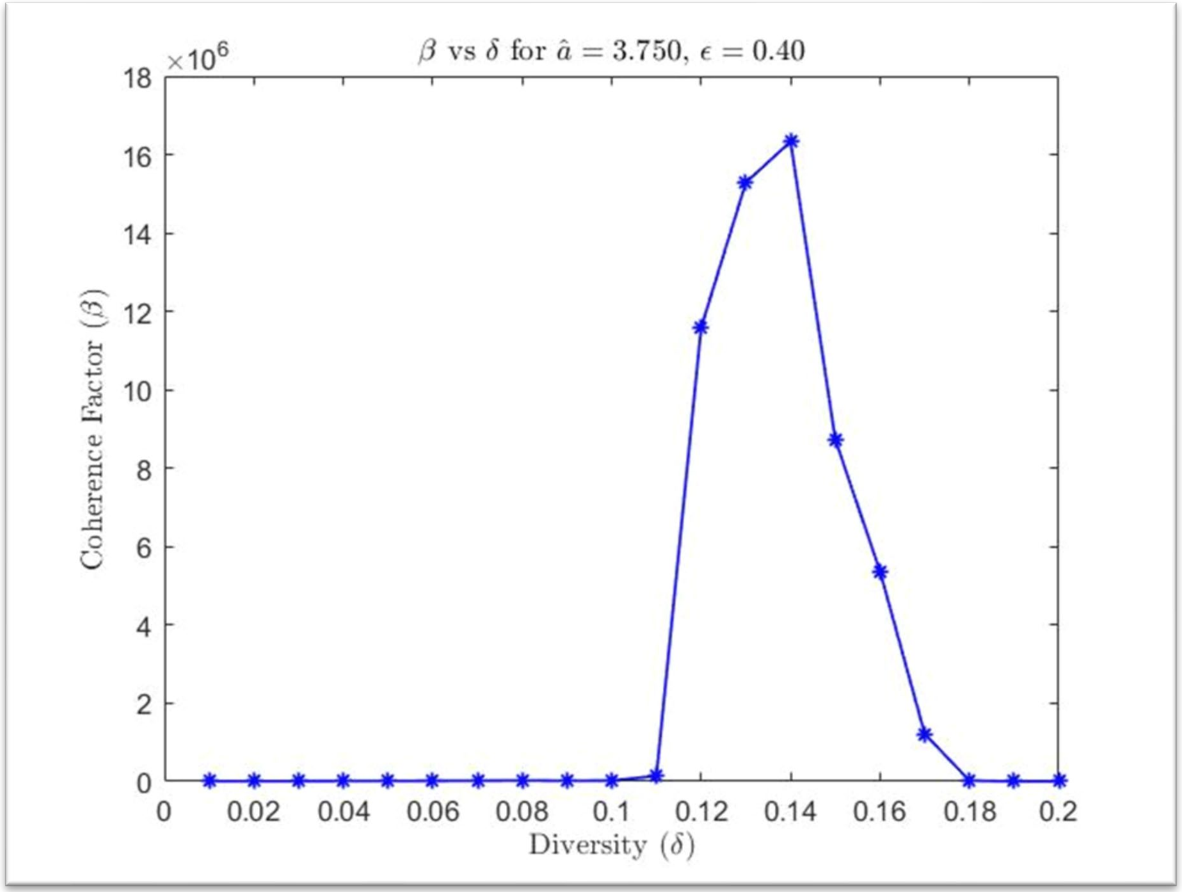
Number of Oscillators In Period 5



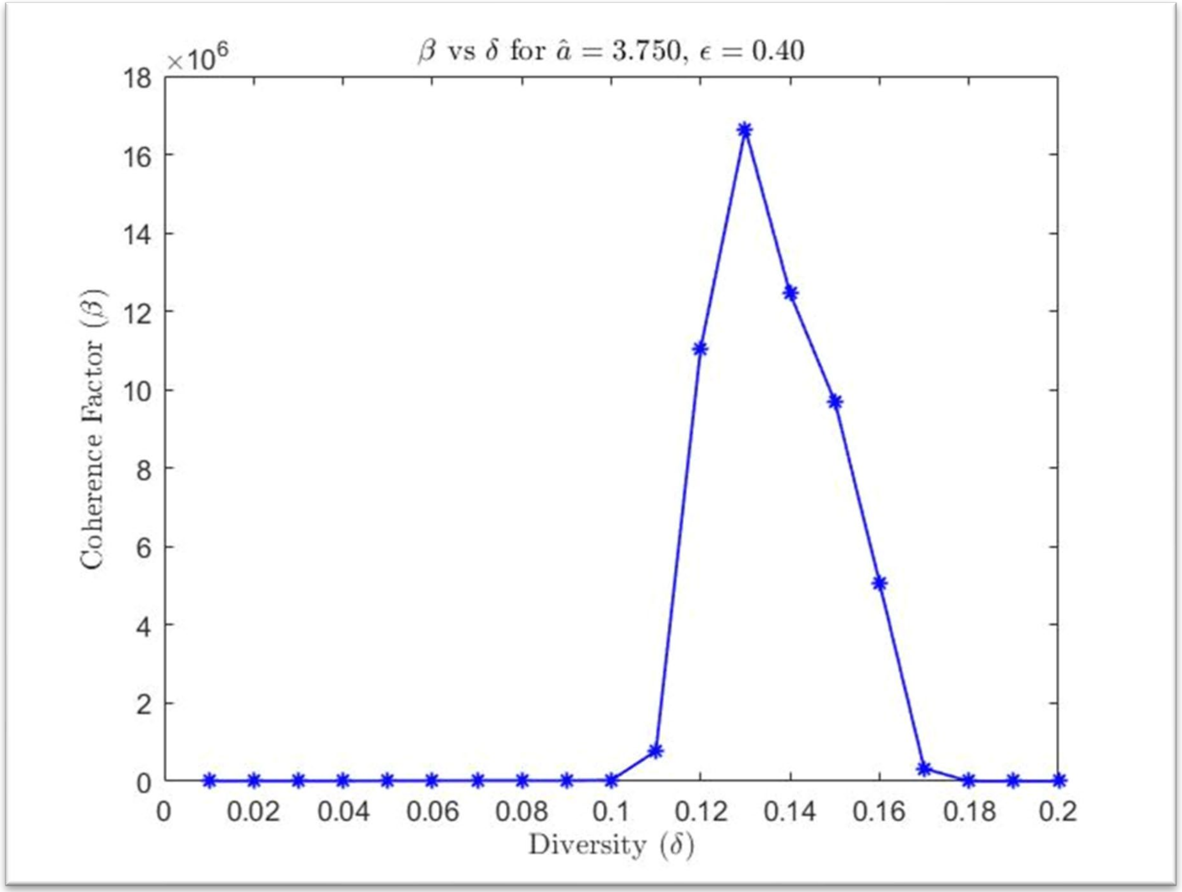


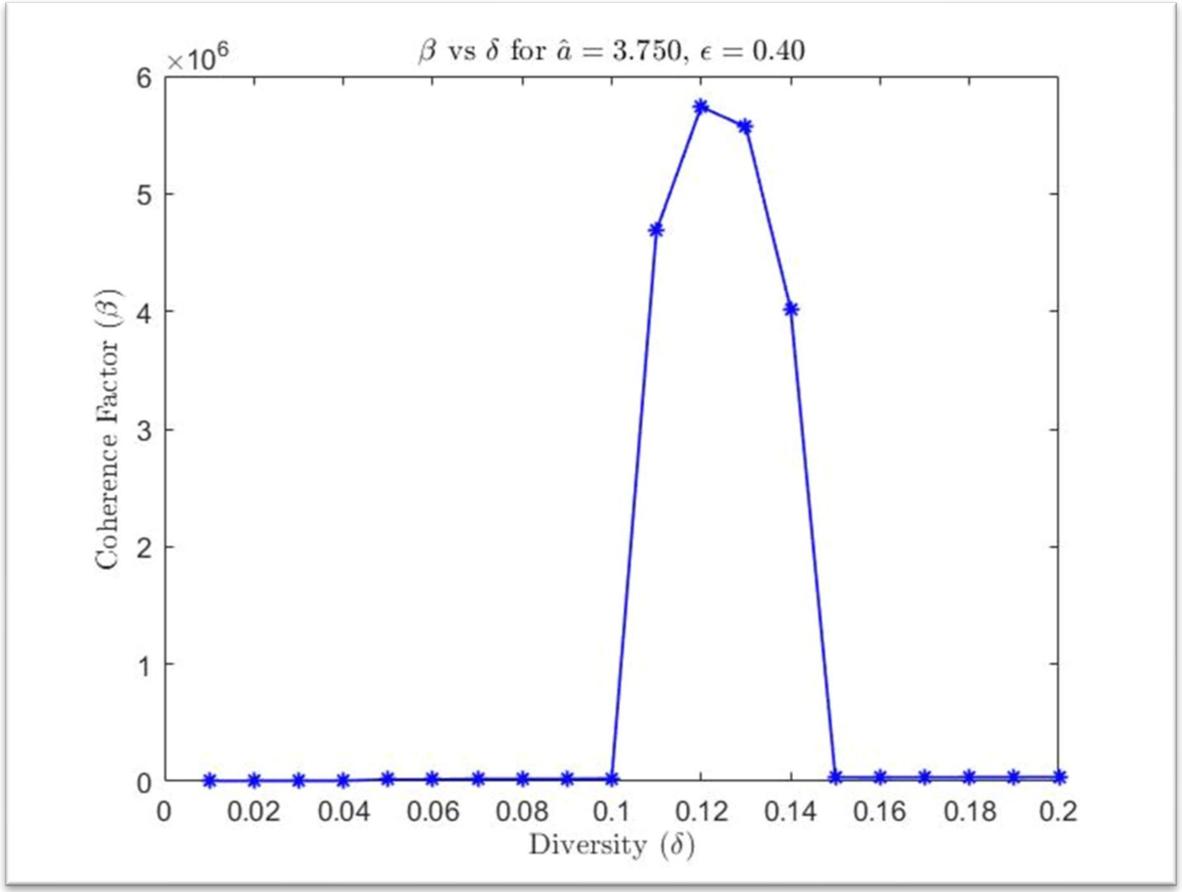
Local Coupling
6x6 Oscillator Array, Range = 8, 7



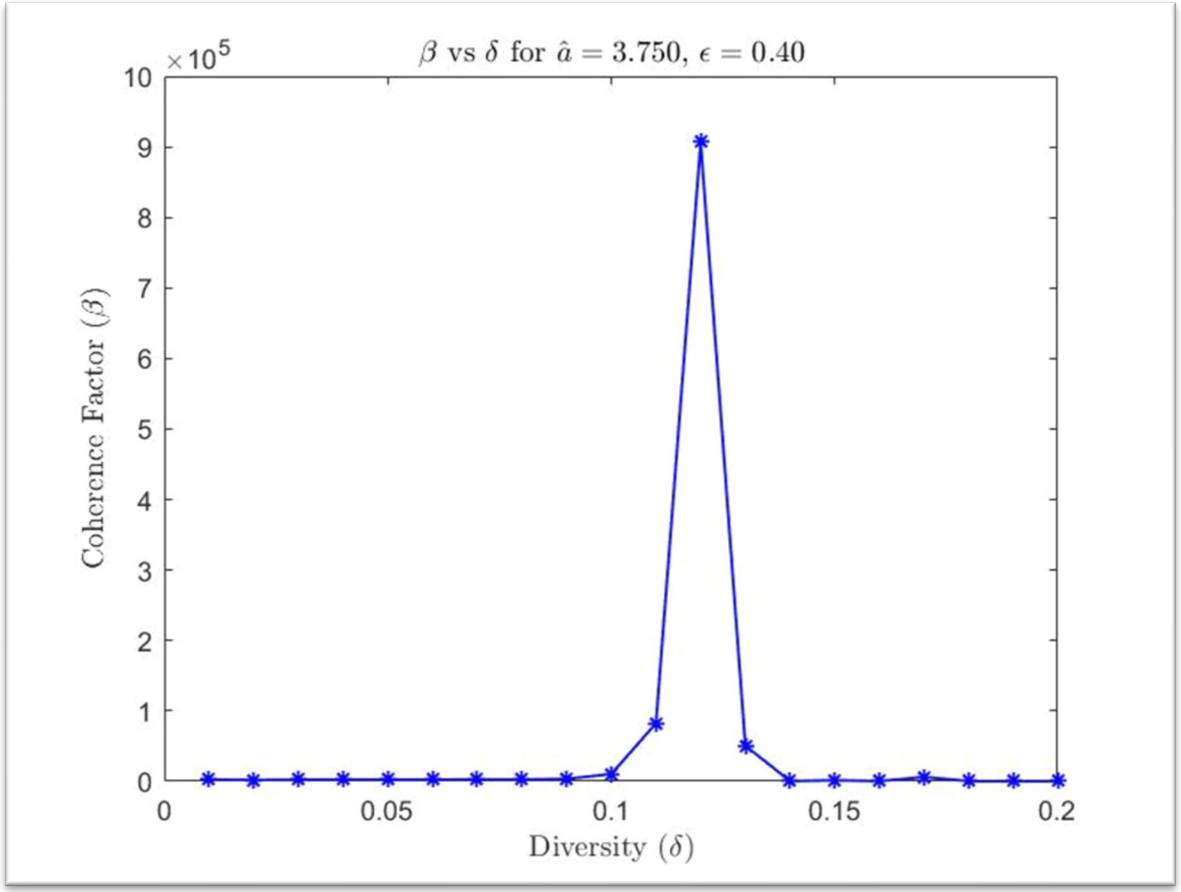


Local Coupling
6x6 Oscillator Array, Range = 6, 5





Local Coupling
6x6 Oscillator Array, Range = 4



Local Coupling
Variation of Maximum Coherence and Amount of Coherence with Range