Quantum Computer Gates

Pauli I Gate: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with eigenvectors $|0> = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1> = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and eigenvalues 1,1.

Pauli X Gate: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvectors $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ and eigenvalues 1,-1.

Pauli Y Gate: $\begin{pmatrix} 0 & \text{-i} \\ \text{i} & 0 \end{pmatrix}$ with eigenvectors $\begin{bmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}$ and eigenvalues 1,-1.

Pauli Z Gate: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and eigenvalues 1,-1.

Phase Gate (S Gate): $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and eigenvalues 1,i.

$\frac{\pi}{8}$ Gate (T Gate): $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = e^{i\frac{\pi}{8}} \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix} =$ with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and eigenvalues 1,$e^{i\frac{\pi}{4}}$.
(Note that TT=S.)

Hadamard Gate (sometimes called the square root of NOT gate but this is incorrect as H^2=I):

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with eigenvectors $\begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1-\sqrt{2}) \end{bmatrix}$ and eigenvalues 1,-1.

Note that if A is unitary so AA=I we have , if x is a real number, that:

$$e^{\mathrm{iAx}} = (\mathrm{iAx})^0 + \mathrm{iAx} + (\mathrm{iA}x)^2 + (\mathrm{iAx})^3 + (\mathrm{iAx})^4 + \ldots.$$

$$= (I - x^2 I + x^4 I - x^6 I \pm \ldots) + \ (\mathrm{ixA} - \mathrm{ix}^3 A + \mathrm{ix}^5 A \mp \ldots. = \cos(x) I + i \sin(x) A$$

Rotation Gate $R_X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\Theta}{2} \\ -i\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} \end{pmatrix} = \cos\frac{\Theta}{2}\ \mathrm{I}\ -i\sin\frac{\theta}{2}\ \mathrm{X} = e^{-i\theta X/2}$

Rotation Gate $R_Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\Theta}{2} \\ \sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} \end{pmatrix} = \cos\frac{\Theta}{2}\ \mathrm{I}\ -i\sin\frac{\theta}{2}\ \mathrm{Y} = e^{-i\theta Y/2}$

Rotation Gate $R_Z = \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\Theta}{2} \end{pmatrix} = \cos\frac{\Theta}{2}\ \mathrm{I}\ -i\sin\frac{\theta}{2}\ \mathrm{Z} = e^{-i\theta Z/2}$

Qubit Two NOT Gate $\mathrm{NOT}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ so $\mathrm{NOT}_2(|00>) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01>$ and

$\mathrm{NOT2}(|01>) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00>$ and $\mathrm{NOT}_2(|10>) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11>$ and

$\mathrm{NOT}_2(|11>) = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |10>$ so we see that NOT_2 NOT's the second bit.

Note that $\mathrm{NOT}_2 = I\otimes X = \begin{pmatrix} 10\\ 01 \end{pmatrix} \otimes \begin{pmatrix} 01\\ 10 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 01\\ 10 \end{pmatrix} & 0\begin{pmatrix} 01\\ 10 \end{pmatrix} \\ 0\begin{pmatrix} 01\\ 10 \end{pmatrix} & 1\begin{pmatrix} 01\\ 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}$

The Controlled Not or CNOT gate is similar except it only changes the second bit if the first is a 1. It looks like $\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{bmatrix}$ so $\mathrm{CNOT}(|00>) = \mathrm{CNOT}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00>$ and

$$\mathrm{CNOT}(|01>) = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01> \text{ and}$$

$$\mathrm{CNOT}(|10>) = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = |11> \text{ and}$$

$$\mathrm{CNOT}(|11>) = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |10> \text{ so it}$$

does what we said - if the first qubit is 0 it does nothing. If the first cubit is 1 it flips the second. Another way of describing this is that $|\mathrm{ab}> \to |a(a\oplus b)>$ since $0\oplus x = x$ and $1\oplus x = \mathrm{NOT}\,x$.

Toffoli Gate T_F works on three qubits and has the property that $T_F(a,b,c) = (a,b,c\oplus \mathrm{ab})$ so

$T_F(a,b,1) = (a,b,a\,\mathrm{NAND}\,b)$ and $T_F(a,b,0) = (a,b,a\,\mathrm{AND}\,b)$ and $T_F(a,1,0) = (a,1,a) = (a,1,\mathrm{FANOUT})$.

(FANOUT is classical bit copying but it doesn't work for superposed values.)

Fredkin Gate F_R also works on three qubits and has the property that $F_R(x,0,y) = (x, x\,\mathrm{AND}\,y, y)$ and

$F_R(1,x,y) = (1,y,x)$ and $F_R(x,1,0) = (x,\bar{x},x)$.

$|000> = |00>\otimes|0> = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$ and $|001> = |00>\otimes|1> = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}$

$|010> = |01>\otimes|0> = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}$ and $|011> = |01>\otimes|1> = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}$

$$|100>=|10>\otimes|0>=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix} \text{ and } |101>=|10>\otimes|1>=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}$$

$$|110>=|11>\otimes|0>=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix} \text{ and } |111>=|11>\otimes|1>=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}$$

This means that the Fredkin Gate has matrix $F_R=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\end{pmatrix}$ and the Toffoli gate has matrix $T_F=F_R=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{pmatrix}$