Effective Hamiltonians in quantum optics: a systematic approach

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We discuss a general and systematic method for obtaining effective Hamiltonians that describe different nonlinear optical processes. The method exploits the existence of a nonlinear deformation of the usual su(2) algebra that arises as the dynamical symmetry of the original model. When some physical parameter, dictated by the process under consideration, becomes small, we immediately get a diagonal effective Hamiltonian that correctly represents the dynamics for arbitrary states and long times. We extend the technique to su(3) and su(N), finding the corresponding effective Hamiltonians when some resonance conditions are fulfilled.

I. INTRODUCTION

Quantum optics provides the ideal arena to deal with the interaction of radiation and matter. Indeed, by using standard techniques one easily gets a system of nonlinear coupled equations that govern, under the approximation of negligible damping, the interaction of a discrete set of field modes with an ensemble of atoms [1, 2, 3, 4]. This is a very general description, but too involved to draw any immediate physical conclusion. Nevertheless, as soon as one realizes the existence of fast and slow variables with a large difference in time scales, it is rather obvious that a simplified formulation can be derived from first principles, at least for some limiting cases.

In most nonlinear optical phenomena the fields and the atomic transitions are usually far from resonance [5]. In fact, no appreciable population redistribution is brought about the irradiation even of intense fields, and only minute fractions of light intensities are absorbed. In the reference frame of the external field oscillations, the atomic polarization is then a fast variable controlled by the slow motion of the electromagnetic field amplitudes and follows its evolution adiabatically. Under this assumption we can eliminate the atomic degrees of freedom and are left with a small number of equations for the fields alone. Due to the elimination procedure, these equations appear as suffering a nonlinear field-field interaction and can consequently be reinterpreted as the Heisenberg equations of motion for the field operators under the dynamics of an effective nonlinear Hamiltonian. In other words, such an effective Hamiltonian acts only on the slow degrees of freedom and, because it correctly describes the slow motion, it incorporates the effect of the coupling of these slow degrees of freedom with the fast ones.

However, this adiabatic elimination (that was first used to study the parametric oscillator [6, 7]) presents some drawbacks. First, it does not provide a general prescription for finding effective Hamiltonians, since the particular details strongly depend on the model considered. Second, it could become very cumbersome, and explicit but enormously complicated expressions for the different orders of approximation can be found in many original publications [8, 9]. Third, and even worse, the procedure is not uniquely defined: depending on the term eliminated, the outcome of the final Hamiltonian could be different [10, 11, 12].

For these reasons, other methods of deriving effective Hamiltonians exist. For the field of quantum optics, the ones devised in Refs. [13] and [14] are especially germane. A thorough review of the different ways of formally constructing effective Hamiltonians may be found in Ref. [15]. Roughly speaking, all the methods have in common that at some point in their implementation one applies a unitary (or canonical) transformation to the total Hamiltonian and keeps only terms up to some fixed order. It is worth emphasizing that this technique is standard in condensed matter physics [16] and important examples of its application may be given [17].

In consequence, it seems pertinent to find a setting to support this usual approach of effective Hamiltonians: it is the main goal of this paper to provide such a setting by resorting to some elementary notions of group theory. To this end, we first note that most of the effective Hamiltonians in quantum optics contain cubic or higher terms in creation and annihilation operators. Among others, typical examples are kth harmonic generation, k-wave mixing, and generalized Dicke models. The key point for our purposes is the recent observation that the common mathematical structure underlying all these cases is a nonlinear or polynomial deformation of su(2), which arises as the dynamical symmetry algebra of the corresponding Hamiltonian. This nonlinear algebra has recently found an important place in quantum optics [18, 19, 20, 21, 22] because it allows us to handle the problems in very close analogy with the usual treatment of an angular momentum. In particular, we get a decomposition of the Hilbert space into direct sums of invariant subspaces and the dynamical problem generated by the corresponding Hamiltonian can be reduced to the diagonalization of a finite-dimensional matrix.

This is a considerable achievement by itself, but unfortunately it is impossible to obtain analytic expressions for the eigenvalues and eigenstates of those matrices. For this reason, in Ref. [23] we have devised a Lie-like method [24] that allows one to get approximately effective Hamiltonians that can be diagonalized in an exact form. The idea is to apply a small nonlinear "rotation" to the original Hamiltonian: although, in general, its action is rather involved, when some physical parameter (dictated by the particular model under consideration) becomes small, it generates an effective Hamiltonian that is diagonal and represents correctly the dynamics for arbitrary states and even for long times. The method appears then as the natural and systematic tool for finding effective Hamiltonians, provided the existence of this nonlinear su(2) dynamical algebra.

Furthermore, it is worth noting that the su(3) algebra is the natural extension of su(2) to study the dynamical evolution of three-level systems. Far from being an exotic curiosity, this evolution is central to the discussion of many fascinating problems, such as two-photon coherence [25, 26], resonant Raman scattering [27, 28], superradiance [29] and three-level echoes [30, 31]. As one would expect, when these three-level systems interact with quantum fields, a nonlinear deformation of su(3) naturally emerges. Much in the same way, a su(N) deformed structure arises when considering the interaction of N-level systems with quantized fields.

In this paper we apply our method to some nontrivial examples: a collection of two-level atoms dispersively interacting with a quantum field in a cavity; and Ξ and Λ configurations of three-level systems interacting with fields under different resonance conditions. We also obtain effective Hamiltonians for multilevel systems under N-photon resonance conditions. Finally, we present how our method could work for more general situations, which would greatly facilitate the physical applications of this technique to many quantum optical problems.

II. EFFECTIVE HAMILTONIANS FOR NONLINEAR SU(2) DYNAMICS

A. Motivation of the method

To keep the discussion as self-contained as possible and to introduce the physical ideas underlying the method, let us start with the very simple example of a particle of spin j in a magnetic field. The Hamiltonian for this system has the following form (in units $\hbar = 1$, which will be used throughout all this paper)

$$H = \omega S_3 + g(S_+ + S_-), \tag{2.1}$$

where g is the coupling constant and the operators $S_3, S_+,$ and S_- constitute a $(2j + 1)$-dimensional representation of the su(2) algebra, obeying the usual commutation relations

$$[S_3, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = 2S_3. \tag{2.2}$$

In the traditional angular momentum basis $|j, m\rangle$ $(m = -j, -j + 1, \ldots, j - 1, j)$ the operator S_3 is diagonal

$$S_3|j, m\rangle = m|j, m\rangle, \tag{2.3}$$

while the action of the ladder operators $S_\pm$ is given by

$$S_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle. \tag{2.4}$$

The Hamiltonian (2.1) belongs to the class of the so-called linear Hamiltonians [32] and admits an exact solution. A convenient way of working out the solution is to apply the rotation

$$U = \exp\left[\alpha(S_+ - S_-)\right], \tag{2.5}$$

and recalling that $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$, the rotated Hamiltonian, which is unitarily equivalent to the original one, becomes

$$\tilde{H} = UHU^\dagger = [\omega \cos(2\alpha) + 2g \sin(2\alpha)]S_3 + \frac{1}{2}[2g \cos(2\alpha) - \omega \sin(2\alpha)](S_+ + S_-). \tag{2.6}$$

Now, the central idea is to choose the parameter α so as to cancel the nondiagonal terms appearing in Eq. (2.6). This can be accomplished by taking

$$\tan(2\alpha) = \frac{2g}{\omega}, \tag{2.7}$$

and the transformed Hamiltonian reduces then to

$$H_{\text{eff}} = \omega\sqrt{1 + \frac{4g^2}{\omega^2}}\ S_3. \tag{2.8}$$

Since this effective Hamiltonian is diagonal in the angular momentum basis, the dynamical problem is completely solved. The crucial observation is that when $g \ll \omega$ we can approximate Eq. (2.7) by $\alpha \simeq g/\omega$, and then Eq. (2.5) can be substituted by

$$U \simeq \exp\left[\frac{g}{\omega}(S_+ - S_-)\right]. \tag{2.9}$$

This small rotation approximately (i.e., up to second-order terms in g/ω) diagonalizes the Hamiltonian (2.1), giving rise to the effective Hamiltonian

$$H_{\text{eff}} = UHU^\dagger \simeq \left(\omega + 2\frac{g^2}{\omega}\right)S_3 \tag{2.10}$$

which obviously coincides with the exact solution (2.8) after expanding in a series of g^2/ω^2. A direct application of the standard time-independent perturbation theory [33] to Eq. (2.1) leads immediately to the same eigenvalues and eigenstates that the Hamiltonian (2.10) gives in the same order of approximation. However, we stress that our method is fully operatorial and avoids the tedious work of computing the successive corrections as sums over all the accessible states.

B. Nonlinear small rotations

Having in mind the previous simple example, let us go one step further by treating the more general case of a system that admits some integrals of motion N_j and whose interaction Hamiltonian can be written as

$$H_{\text{int}} = \Delta\ X_3 + g(X_+ + X_-), \tag{2.11}$$

where g is a coupling constant and Δ is a parameter usually representing the detuning between frequencies of different subsystems (although it is not necessary). The operators $X_\pm$ and X_3 maintain the first commutation relation of su(2), $[X_3, X_\pm] = \pm X_\pm$, but the second one is modified in the following way

$$[X_+, X_-] = P(X_3), \tag{2.12}$$

where $P(X_3)$ is an arbitrary polynomial function of the diagonal operator X_3 with coefficients perhaps depending on the integrals of motion N_j. These commutation relations correspond to the so-called polynomial deformation of su(2). Such nonlinear algebras were discovered by Sklyanin [34] and Higgs [35] and have already played an important role in quantum mechanics [36, 37, 38, 39].

Now suppose that for some physical reasons (depending on the particular model under consideration) the condition

$$\varepsilon = \frac{g}{\Delta} \ll 1 \tag{2.13}$$

is fulfilled. Then, it is clear that (2.11) is *almost* diagonal in the basis that diagonalizes X_3. In fact, a standard perturbation analysis immediately shows that the first-order corrections introduced by the nondiagonal part $g(X_+ + X_-)$ to the eigenvalues of X_3 vanish and those of second order are proportional to ε. Thus, we apply the following unitary transformation to (2.11) (which, in fact, is a *small* nonlinear rotation)

$$U = \exp[\varepsilon(X_+ - X_-)]. \tag{2.14}$$

After some calculations we get, up to order ε^2, the effective Hamiltonian we are looking for

$$H_{\text{eff}} = \Delta\ X_3 + \frac{g^2}{\Delta}P(X_3), \tag{2.15}$$

which is diagonal in the basis of eigenstates of X_3. Then, the evolution (as well as the spectral) problem is completely solved in this approximation. Besides the advantage of having the effective Hamiltonian expressed in an operatorial

form, the method has the virtue of generality, since it is valid for any model whose Hamiltonian could be written down in terms of the generators of a polynomial deformation of su(2).

Our technique also provides a valuable tool for obtaining corrections to the eigenstates of (2.11). Indeed, it is easy to realize that these eigenstates can be approximated by

$$|\Psi_m\rangle = U^\dagger|m\rangle, \tag{2.16}$$

where $|m\rangle$ denotes an eigenstate of X_3 and U is the corresponding small rotation. Since U and $|m\rangle$ do not depend on time, the operator U can be applied to $|m\rangle$ as an expansion in ε. For example, the eigenstate $|\Psi_m\rangle$ up to order ε^2 takes on the form

$$|\Psi_m\rangle = \left[1 - \varepsilon\ (X_+ - X_-) - \frac{\varepsilon^2}{2}\ (1 + 2X_+X_- - X_+^2 - X_-^2)\right]|m\rangle. \tag{2.17}$$

This representation is especially advantageous when the state space of the model is a representation space of the deformed su(2) algebra that is constructed in the usual way by the action of the raising operator X_+; i.e., $|m\rangle \sim X_+^m|0\rangle$, where $|0\rangle$ is a lowest weight vector fulfilling the standard condition $X_-|0\rangle = 0$.

This general procedure shows that can always adiabatically eliminate all nonresonant transitions and work with an effective Hamiltonian containing only (quasi) resonant transitions. The effect of nonresonant terms reduces to a dynamical Stark shift (which can have a quite complicated form). Obviously, transformations generating effective Hamiltonians also change eigenfunctions, but the corrections are of order ε and do not depend on time. Such corrections correspond to low-amplitude transitions that take place in the case of nonresonant interactions. We shall elaborate on these topics in the next Sections.

C. An example: dispersive limit of the Dicke model

As a relevant example, let us apply our method to the well-known Dicke model that describes the interaction of a single-mode field of frequency $\omega_{\rm f}$ with a collection of A identical two-level atoms with transition frequency ω_0. Making the standard dipole and rotating-wave approximations, the model Hamiltonian reads as [40]

$$H = \omega_{\rm f}a^\dagger a + \omega_0 S_3 + g(aS_+ + a^\dagger S_-), \tag{2.18}$$

where $a(a^\dagger)$ are the annihilation (creation) operators for the field mode, and $(S_\pm, S_3)$ are collective atomic operators forming an $(A+1)$-dimensional representation of su(2).

By introducing the excitation number $N = a^\dagger a + S_3$, which is an integral of motion, we can recast (2.18) as $H = H_0 + H_{\rm int}$ with

$$H_0 \ = \ \omega_{\rm f}N,$$

$$H_{\rm int} \ = \ \Delta\ S_3 + g(aS_+ + a^\dagger S_-), \tag{2.19}$$

the detuning being $\Delta = \omega_0 - \omega_{\rm f}$.

We assume now that the dispersive limit holds [41]; i.e.,

$$|\Delta| \gg Ag\sqrt{\bar{n}+1}, \tag{2.20}$$

where $\bar{n}$ is the average number of photons in the field. If, after our previous discussion, we introduce the deformed su(2) operators as

$$X_+ = aS_+, \qquad X_- = a^\dagger S_-, \qquad X_3 = S_3, \tag{2.21}$$

then we immediately get that the small nonlinear rotation (2.14) transforms the interaction Hamiltonian (2.19) into

$$H_{\rm eff} = \Delta\ S_3 + \frac{g^2}{\Delta}[S_3^2 - 2(a^\dagger a + 1)S_3 - C_2], \tag{2.22}$$

where $C_2 = A/2(A/2 + 1)$ is the value of the Casimir operator for su(2). The effective Hamiltonian (2.22) was previously obtained in Ref. [42] by quite a different method (see also Ref. [43]) and, due to the presence of the nonlinear term S_3^2, has been considered as a candidate for the generation of squeezed atomic states [44].

III. EFFECTIVE HAMILTONIANS FOR NONLINEAR SU(3) DYNAMICS

A. Three-level systems interacting with quantum fields

The method of small rotations discussed in the previous Section can be applied not only to Hamiltonians having a nonlinear su(2) structure, but also to more complicated systems. In this Section we shall focus on Hamiltonians that can be represented in terms of the su(3) algebra. This structure naturally arises when dealing with systems with three relevant levels. It is well known that in this case three possible configurations (commonly called Ξ, V, and Λ) are admissible (see Ref. [45] for details).

For definiteness, we shall consider the interaction of A identical three-level systems in a cascade or Ξ configuration (i.e., with associated energies $E_1 < E_2 < E_3$ and allowed dipole transitions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$, but not the $1 \leftrightarrow 3$) interacting with a single-mode quantum field of frequency ω_{f}. The Hamiltonian of this model is

$$H_\Xi = H_{\mathrm{field}} + H_{\mathrm{atom}} + H_{\mathrm{int}}, \tag{3.1}$$

with

$$
\begin{aligned}
H_{\mathrm{field}} &= \omega_{\mathrm{f}} a^\dagger a, \\
H_{\mathrm{atom}} &= E_1 S^{11} + E_2 S^{22} + E_3 S^{33}, \\
H_{\mathrm{int}} &= g_{12}(a S_+^{12} + a^\dagger S_-^{12}) + g_{23}(a S_+^{23} + a^\dagger S_-^{23}).
\end{aligned}
\tag{3.2}
$$

As usual, the three diagonal observables $S^{ii} = |i\rangle\langle i|$ ($i = 1, 2, 3$) measure the population of the ith energy level, while the off-diagonal polarizations $S^{ij} = |j\rangle\langle i|$ generate transitions from level i to j. They satisfy the commutation relations

$$[S^{ij}, S^{kl}] = \delta_{jk} S^{il} - \delta_{il} S^{kj}, \tag{3.3}$$

which turn out to be those of the u(3) algebra. In fact, they form a $(A+1)(A+2)/2$-dimensional representation of the u(3) algebra.

Because the sum $S^{11} + S^{22} + S^{33} = A$ is an obvious integral of motion that determines the total number of atoms, only two of the populations can vary independently. For this reason it is customary to introduce the two traceless operators

$$S_3^{12} = \frac{1}{2}(S^{22} - S^{11}), \qquad S_3^{23} = \frac{1}{2}(S^{33} - S^{22}), \tag{3.4}$$

that represent population inversion between the corresponding levels. Furthermore, to emphasize the idea of transition between levels, it is usual to define the operators $S_+^{ij} = S^{ij}$ and $S_-^{ij} = S^{ji}$ (for $i < j$). Then the eight independent operators $(S_\pm^{ij}, S_3^{ij})$ satisfy the commutation relations of su(3). Moreover, $(S_\pm^{12}, S_3^{12})$ and $(S_\pm^{23}, S_3^{23})$ form two su(2) subalgebras (each one describing the transitions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$). Nevertheless, transitions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ are not physically independent, since we have

$$[S_+^{12}, S_+^{23}] = -S_+^{13}, \qquad [S_-^{12}, S_-^{23}] = S_-^{13}, \qquad [S_+^{12}, S_-^{23}] = 0. \tag{3.5}$$

It seems natural also to introduce the deformed su(3) algebra as

$$X^{11} = S^{11}, \qquad X^{22} = S^{22}, \qquad X^{33} = S^{33},$$

$$X_+^{12} = a S_+^{12}, \qquad X_+^{23} = a S_+^{23}. \tag{3.6}$$

Then the Hamiltonian (3.1) can be recast as $H_\Xi = H_0 + H_{\mathrm{int}}$, with

$$
\begin{aligned}
H_0 &= \omega_{\mathrm{f}} N_\Xi, \\
H_{\mathrm{int}} &= -\Delta_{12}\, X^{11} + \Delta_{23}\, X^{33} + g_{12}(X_+^{12} + X_-^{12}) + g_{23}(X_+^{23} + X_-^{23}),
\end{aligned}
\tag{3.7}
$$

where we have used the conserved excitation number $N_\Xi = a^\dagger a + S^{33} - S^{11}$ and we have introduced the detunings as

$$\Delta_{12} = E_2 - E_1 - \omega_{\mathrm{f}}, \qquad \Delta_{23} = E_3 - E_2 - \omega_{\mathrm{f}}. \tag{3.8}$$

The operators X^{ij} satisfy the usual su(3) commutation relations, with (3.5) recast as

$$[X_+^{12}, X_+^{23}] = -Y_+^{13}, \qquad [X_-^{12}, X_-^{23}] = Y_-^{13}, \qquad [X_+^{12}, X_-^{23}] = 0. \tag{3.9}$$

where

$$Y_+^{13} = a^2 S_+^{13}. \tag{3.10}$$

However, we have to modify some of them in the following way:

$$[X_+^{ij}, X_-^{ij}] = P(X^{ii}, X^{jj}), \qquad [Y_+^{ij}, Y_-^{ij}] = Q(X^{ii}, X^{jj}), \tag{3.11}$$

where $P(X^{ii}, X^{kk})$ and $Q(X^{ii}, X^{kk})$ are polynomials of the diagonal operators X^{ii} ($i = 1, 2, 3$) and define, then, a polynomial deformation of su(3).

1. Effect of a far-off resonant level

The dynamics generated by su(3) is obviously richer than that of su(2), since a greater number of physical degrees of freedom are now available. To see how our method works in this case, let us assume that one of the transitions, say the $2 \leftrightarrow 3$, is (quasi) resonant with the field; i.e.,

$$|\Delta_{23}| \ll A g_{23} \sqrt{\bar{n} + 1}, \tag{3.12}$$

while the transition $1 \leftrightarrow 2$ is far-off resonant

$$|\Delta_{12}| \gg A g_{12} \sqrt{\bar{n} + 1}. \tag{3.13}$$

It is clear from our previous analysis that the small nonlinear rotation

$$U_{12} = \exp[\varepsilon_{12}(X_+^{12} - X_-^{12})], \tag{3.14}$$

with $\varepsilon_{12} = g_{12}/\Delta_{12} \ll 1$, eliminates the interaction term $g_{12}(X_+^{12} + X_-^{12})$, representing the nonresonant transition $1 \leftrightarrow 2$. In such a way, we obtain the effective Hamiltonian

$$H_{\text{eff}} = \Delta_{12} X^{11} + \Delta_{23} X^{33} + g_{23}(X_+^{23} + X_-^{23}) - \frac{g_{12}g_{23}}{\Delta_{12}}(Y_+^{13} + Y_-^{13})$$
$$+ \frac{g_{12}^2}{\Delta_{12}} P(X^{11}, X^{22}) + \frac{g_{12}g_{23}}{\Delta_{12}}([X_+^{12}, X_-^{23}] + [X_+^{23}, X_-^{12}]). \tag{3.15}$$

The remarkable point is that by eliminating the transition $1 \leftrightarrow 2$, we have generated an effective transition $1 \leftrightarrow 3$ (represented by the operators $Y_\pm^{13}$), which was absent in the initial Hamiltonian. Nevertheless, this transition is also nonresonant due to conditions (3.12) and (3.13) and, accordingly, can be eliminated by the following transformation

$$U_{13} = \exp[\varepsilon_{13}(Y_+^{13} - Y_-^{13})], \tag{3.16}$$

where the parameter ε_{13} in the above equation must fulfill

$$\varepsilon_{13} = \frac{g_{12}g_{23}}{\Delta_{12}(\Delta_{12} + \Delta_{23})} \ll 1. \tag{3.17}$$

In this particular case the polynomial function $P(X^{ii}, X^{jj})$ is

$$P(X^{11}, X^{22}) = S_+^{12} S_-^{12} + a^\dagger a (S^{22} - S^{11}). \tag{3.18}$$

If initially level 1 is unpopulated we have $S^{11} = 0$ (which will be conserved, since there are no transitions to the level 1), and

$$P(X^{11}, X^{22}) = S^{22}(a^\dagger a + 1). \tag{3.19}$$

In consequence, the effective Hamiltonian, taking into account the existence of a far-off resonant level, has the form

$$H_{\text{eff}} = \Delta_{23} S^{33} + g_{23}(a S_+^{23} + a^\dagger S_-^{23}) + \frac{g_{12}^2}{\Delta_{12}} S^{22}(a^\dagger a + 1). \tag{3.20}$$

This means that the far-lying level produces a mark in the system in the form of a dynamical Stark-shift term. It is interesting to observe that due to this Stark shift, the initially nonresonant transition $2 \leftrightarrow 3$ becomes resonant in a subspace with some fixed photon number. This opens the possibility of separating n-photon field states from an initial coherent state interacting with a collection of three-level atoms just by choosing a suitable relation between the detunings and the interaction constants, and projecting to the second level at appropriate moments.

2. Two-photon resonance

Let us now envisage the different situation in which the two-photon resonance condition between levels 1 and 3 is fulfilled; i.e., $E_3 - E_1 = 2\omega_{\rm f}$. This means that $\Delta_{12} = -\Delta_{23}$ and the transition generated by the operators $Y_{\pm}^{13}$ cannot be removed. However, the term $g_{23}(X_+^{23} + X_-^{23})$, which generates (nonresonant) transitions between levels 2 and 3, can be eliminated by a transformation U_{23} analogous to (3.14) with $\varepsilon_{23} \ll 1$. The transformed Hamiltonian becomes then

$$
\begin{aligned}
H_{\rm eff} &= -\Delta_{12}\left(X^{11} + X^{33}\right) - \frac{g_{12}g_{23}}{\Delta_{12}}(Y_+^{13} + Y_-^{13}) \\
&+ \frac{g_{12}^2}{\Delta_{12}}P(X^{11}, X^{22}) - \frac{g_{23}^2}{\Delta_{12}}P(X^{22}, X^{33}),
\end{aligned}
\tag{3.21}
$$

where $P(X^{11}, X^{22})$ and $P(X^{22}, X^{33})$ are defined according to Eq. (3.18). Finally, if we further impose the absence of initial population in level 2, we obtain

$$
\begin{aligned}
H_{\rm eff} &= \frac{g_{12}g_{23}}{\Delta_{12}}(a^2 S_+^{13} + a^{\dagger 2} S_-^{13}) \\
&+ (S_3^{13} + A/2)\left[(g_{23}^2/\Delta_{12} - g_{12}^2/\Delta_{12})a^\dagger a + g_{23}^2/\Delta_{12}\right] + A\frac{g_{12}^2}{\Delta_{12}}a^\dagger a,
\end{aligned}
\tag{3.22}
$$

which is the effective two-photon Dicke Hamiltonian including the dynamical Stark shift obtained by Puri and Bullough [10] (see also Ref.[12]).

B. The dispersive limit of the Λ configuration

Let us consider for a moment the case of the Λ configuration, in which the allowed dipole transitions are now $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, but not the $1 \leftrightarrow 2$. The Hamiltonian governing the evolution is still of the form (3.1), but now with

$$
H_{\rm int} = g_{13}(aS_+^{13} + a^\dagger S_-^{13}) + g_{23}(aS_+^{23} + a^\dagger S_-^{23}).
\tag{3.23}
$$

By using the integral of motion $N_\Lambda = a^\dagger a + S^{33}$ we can rewrite

$$
H_{\rm int} = -\Delta_{31}X^{11} - \Delta_{32}X^{22} + g_{13}(X_+^{13} + X_-^{13}) + g_{23}(X_+^{23} + X_-^{23}),
\tag{3.24}
$$

where, as in Eq. (3.6), we have introduced the deformed su(3) operators as

$$
X_+^{13} = aS_+^{13}, \qquad X_+^{23} = aS_+^{23}.
\tag{3.25}
$$

These deformed generators satisfy a set of commutation relations similar to (3.9), but instead of (3.10) we must use

$$
Y_+^{12} = (S^{33} - a^\dagger a),
\tag{3.26}
$$

which is the mathematical reason for the well-known different behaviours exhibited by Ξ and Λ configurations.

Let us focus on the dispersive regime, when

$$
|\Delta_{13}| \gg Ag_{13}\sqrt{\bar{n}+1}, \qquad |\Delta_{23}| \gg Ag_{23}\sqrt{\bar{n}+1}.
\tag{3.27}
$$

Then a couple of small rotations eliminate the far-off resonant transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, obtaining

$$
\begin{aligned}
H_{\rm eff} &= -\Delta_{31}S^{11} - \Delta_{32}S^{22} \\
&+ \frac{g_{13}^2}{\Delta_{13}}[(S^{11}+1)S^{33} + a^\dagger a(S^{33} - S^{11})] + \frac{g_{23}^2}{\Delta_{23}}[(S^{22}+1)S^{33} + a^\dagger a(S^{33} - S^{22})] \\
&+ \frac{g_{13}g_{23}}{\Delta_{31}}(S_+^{12} + S_-^{12})(S^{33} - a^\dagger a).
\end{aligned}
\tag{3.28}
$$

Note that both $a^\dagger a$ and S^{33} are now integrals of motion. The two first terms correspond to trivial free atomic dynamics. The next two terms represent the standard dynamical Stark shift. Finally, the last term describes an effective interaction between levels 1 and 2. The remarkable point is that there is a population transfer (and not only phase transfer, as it could be expected from a dispersive interaction) between these two levels without exchange of photons. The intensity of the transition $1 \leftrightarrow 2$ depends on the difference between the population of level 3 and the photon number. Thus, no population transfer between levels 1 and 2 will occur in the sector where the number of photons is exactly equal to the initial population of the level 3. It is easy to observe that the (effective) transitions $1 \leftrightarrow 2$ are stronger when $\Delta_{31} = \Delta_{32}$; i.e., when the levels 1 and 2 have the same energy (Zeeman-like systems).

IV. EFFECTIVE HAMILTONIANS FOR NONLINEAR SU(N) DYNAMICS

A. Multilevel systems interacting with a quantum field

Having demonstrated the role played by nonlinear algebras in the systematic construction of effective Hamiltonians, we would like to pursue here a natural extension to multilevel systems.

Specifically, we are interested in considering Hamiltonians that can be represented in terms of the su(N) algebra, which naturally arises when describing systems with N relevant levels. Obviously, the evolution of a collection of A identical N-level systems (for definiteness, we assume a cascade configuration, such that $E_i < E_j$ for $i < j$) interacting with a single-mode quantum field can be modeled by a Hamiltonian as in Eq. (3.1) with

$$
\begin{aligned}
H_{\text{field}} &= \omega_{\text{f}} a^\dagger a, \\
H_{\text{atom}} &= \sum_{i=1}^{N} E_i S^{ii}, \\
H_{\text{int}} &= \sum_{i=1}^{N-1} g_i (a S_+^{ii+1} + a^\dagger S_-^{ii+1}),
\end{aligned}
\tag{4.1}
$$

where S^{ii} $(i = 1, \ldots, N)$ are population operators of the ith energy level, and $S_\pm^{ij}$ describe transitions between levels i and j. The operators S^{ij} form the u(N) algebra and satisfy the commutation relations

$$
[S^{ij}, S^{kl}] = \delta_{jk} S^{il} - \delta_{il} S^{kj}.
\tag{4.2}
$$

By introducing inversion-like operators

$$
S_3^{ii+1} = \frac{1}{2}(S^{i+1i+1} - S^{ii}),
\tag{4.3}
$$

then $(S_\pm^{ij}, S_3^{ii+1})$ turn out to be the su(N) algebra.

The Hamiltonian (4.1) admits the integral of motion

$$
N = a^\dagger a + \sum_{i=1}^{N-1} \mu_i S_3^{ii+1},
\tag{4.4}
$$

with $\mu_i = i(N - i)$. We also introduce the detunings by

$$
\Delta_j = E_j - E_1 - (j-1)\omega_{\text{f}},
\tag{4.5}
$$

and we shall assume that Δ_j satisfy the following resonant condition

$$
\Delta_N = 0,
\tag{4.6}
$$

which means that the field in a $(N-1)$-photon resonance with the atomic system; i.e., $E_N - E_1 = (N-1)\omega_{\text{f}}$. Thus, the Hamiltonian (4.1) can be recast as $H = H_0 + H_{\text{int}}$, with

$$
\begin{aligned}
H_0 &= \omega_{\text{f}} N + EA \\[4pt]
H_{\text{int}} &= h_0 + V,
\end{aligned}
\tag{4.7}
$$

where $E = (E_N + E_1)/2$ and

$$
h_0 = \sum_{i=1}^{N} \Delta_i S^{ii}, \qquad V = \sum_{i=1}^{N-1} g_i (a S_+^{ii+1} + a^\dagger S_-^{ii+1}).
\tag{4.8}
$$

From our previous experience it seems rather obvious that the operators

$$
X^{ii} = S^{ii}, \qquad X_+^{ij} = a S_+^{ij}, \qquad X_-^{ij} = a^\dagger S_-^{ij},
\tag{4.9}
$$

form a polynomial deformation of su(N). In consequence, and according to our general scheme, we can introduce the transformation

$$U = \exp\left[\sum_{i=1}^{N-1} \varepsilon_i(X_+^{ii+1} - X_-^{ii+1})\right],$$

(4.10)

where

$$\varepsilon_i = \frac{g_i}{\Delta_{i+1} - \Delta_i}$$

(4.11)

will be supposed to be small numbers, $\varepsilon_i \ll 1$, which means that the transitions are far from the one-photon resonance $(\Delta_{i+1} - \Delta_i = E_{i+1} - E_i - \omega_{\rm f} \gg g_i)$. Thus, all one-photon transitions are eliminated by (4.10) and the transformed Hamiltonian takes the form

$$H_{\rm eff}^{(1)} = h_0 + h_{\rm d} + h_{\rm nd} + \sum_{n=1}^{N-2} \frac{n}{(n+1)!} \sum_{i=1}^{N-n-1} \lambda_i^{(n+1)} \left(a^{n+1} S_+^{ii+n+1} + a^{\dagger n+1} S_-^{ii+n+1}\right).$$

(4.12)

Here, the effective interaction constants λ_i^n can be obtained from the recurrence relation

$$\lambda_i^{(n+1)} = \varepsilon_{i+n}\lambda_i^{(n)} - \varepsilon_i\lambda_{i+1}^{(n)},$$

(4.13)

with the initial term $\lambda_i^{(1)} = g_i$. It is easy to see that $\lambda_i^{(n+1)} \ll \lambda_i^{(n)}$.

The piece $h_{\rm d}$ contains only diagonal terms in the atomic operators and depends on the integral of motion N (or, equivalently, depends only on the photon-number operator $a^\dagger a$). This operator $h_{\rm d}$ appears naturally represented as an expansion in the small parameter ε_i whose first term is

$$h_{\rm d} = \sum_{i=1}^{N-1} g_i\varepsilon_i[a^\dagger a(S^{i+1i+1} - S^{ii}) + (S^{ii} + 1)S^{i+1i+1}].$$

(4.14)

The essential point for our purposes is that, given its structure, this diagonal part cannot be removed from the effective Hamiltonian (4.12). On the contrary, the operator $h_{\rm nd}$ contains only nondiagonal terms that can be eliminated by rotations of the type (4.10) unless some specific resonance conditions are fulfilled. In this respect, let us note that the price we pay for eliminating one-photon transitions is the generation of all possible k-photon transitions $(k = 2, ..., N-1)$.

B.   Three-photon resonance

Let us consider the particular case of four-level systems $(N = 4)$ and suppose that there are no transitions in one- and two-photon resonance with the field. After eliminating one-photon transitions the transformed Hamiltonian (4.12) has the form

$$H_{\rm eff}^{(1)} = h_0 + h_{\rm d} + h_{\rm nd} + \frac{1}{3}\lambda_1^{(3)}\left(a^3 S_+^{14} + a^{\dagger 3} S_-^{14}\right) + \frac{1}{2}\sum_{i=1}^{2} \lambda_i^{(2)}\left(a^2 S_+^{ii+2} + a^{\dagger 2} S_-^{ii+2}\right),$$

(4.15)

where the interaction constants are defined, according to (4.11) and (4.13), as

$$\varepsilon_1 = \frac{g_1}{\Delta_2}, \qquad \varepsilon_2 = \frac{g_2}{\Delta_3 - \Delta_2}, \qquad \varepsilon_3 = -\frac{g_3}{\Delta_3},$$

$$\lambda_1^{(2)} = g_1 g_2 \frac{2\Delta_2 - \Delta_3}{\Delta_2(\Delta_3 - \Delta_2)}, \qquad \lambda_2^{(2)} = g_2 g_3 \frac{2\Delta_3 - \Delta_2}{\Delta_3(\Delta_2 - \Delta_3)}, \qquad \lambda_1^{(3)} = \frac{3g_1 g_2 g_3}{\Delta_3 \Delta_2},$$

(4.16)

and the resonance condition $\Delta_4 = 0$ (that is, three-photon resonance $E_4 - E_1 = 3\omega_{\rm f}$) has been imposed.

According to the general scheme, the term representing two-photon transitions in (4.15) can be removed using a transformation analogous to (4.10):

$$U_2 = \exp\left[\frac{1}{2}\sum_{i=1}^{2} \alpha_i^{(2)}(a^2 S_+^{ii+2} - a^{\dagger 2} S_-^{ii+2})\right],$$

(4.17)

where

$$\alpha_i^{(2)} = \frac{\lambda_i^2}{\Delta_{i+2} - \Delta_i} \tag{4.18}$$

is a small parameter because there are no resonant two-photon transitions ($\Delta_{i+2} - \Delta_i = E_{i+2} - E_i - 2\omega_{\rm f} \gg \lambda_i^{(2)}$) and thus $\alpha_j^{(2)} \ll \varepsilon_j$. It is worth noting that the transformation (4.17) does not introduce new terms of order ε^2 to the effective Hamiltonian.

The diagonal part in (4.15) is

$$h_{\rm d} = \sum_{i=1}^{3} g_i \varepsilon_i \left[a^\dagger a (S^{i+1i+1} - S^{ii}) + (S^{ii} + 1) S^{i+1i+1} \right], \tag{4.19}$$

while the nondiagonal term deserves a more careful analysis. Its explicit form is

$$h_{\rm nd} = \frac{1}{2}(\varepsilon_1 g_3 + \varepsilon_3 g_1)(S_+^{12} S_-^{34} + S_+^{34} S_-^{12}) + \frac{1}{2}(\varepsilon_1 g_2 + \varepsilon_2 g_1)(S_+^{12} S_-^{23} + S_+^{23} S_-^{12}). \tag{4.20}$$

It is clear that the first term in the above expression describes a resonant dipole-dipole interaction, under the condition $\Delta_2 = -\Delta_3$. On the other hand, the second term describes a resonant interaction whenever $2\Delta_2 = \Delta_3$, which is incompatible with the pervious one and the absence of one- and two-photon resonances. If no one of these conditions are fulfilled, the term $h_{\rm nd}$ can be eliminated by the transformation

$$U_2^{(2)} = \exp\left[\frac{1}{2} \sum_{i,j=1}^{3} \beta_{ij} (S_+^{ii+1} S_-^{jj+1} - S_+^{jj+1} S_-^{ii+1}) \right], \tag{4.21}$$

where

$$\beta_{ij} = \frac{\varepsilon_i g_j}{\Delta_{i+1} - \Delta_i + \Delta_j - \Delta_{j+1}}. \tag{4.22}$$

Then, since $S^{11} + S^{22} + S^{33} + S^{44} = A$ and imposing the condition of the absence of initial population in levels 2 and 3, we obtain the effective Hamiltonian describing three-photon resonant transitions

$$H_{\rm eff}^{(2)} = \frac{g_1 g_2 g_3}{\Delta_2 \Delta_3}(a^3 S_+^{14} + a^{\dagger 3} S_-^{14}) - (S_3^{14} + A/2)[a^\dagger a(g_1^2/\Delta_2 - g_3^2/\Delta_3) + g_3^2/\Delta_3] + A \frac{g_1^2}{\Delta_2} a^\dagger a. \tag{4.23}$$

The effective three-photon Hamiltonian (4.23) contains a dynamical Stark shift similar to that appearing in the two-photon case (3.22). Nevertheless, the essential difference consists in that in the two-photon case the interaction term and the Stark shift are of the same order of magnitude, while in the three-photon case the interaction term is one order of magnitude less than the Stark shift. This would lead to essential differences in the evolution of some observables.

C. Multimode fields and resonance conditions

The formalism developed can also be used to treat the interaction of multifrequency fields with atomic systems. This type of interaction is much richer because some additional resonance conditions can be satisfied. To this end, let us consider a four-level system in a Ξ configuration interacting with two field modes of frequencies ω_a and ω_b, and annihilation operators a and b, respectively. The Hamiltonian describing this interaction is still of the form (3.1) with

$$\begin{aligned}
H_{\rm field} &= \omega_a a^\dagger a + \omega_b b^\dagger b, \\
H_{\rm atom} &= \sum_{i=1}^{4} E_i S^{ii}, \\
H_{\rm int} &= \sum_{i=1}^{3} \left[(g_{ai} a + g_{bi} b) S_+^{ii+1} + (g_{ai} a^\dagger + g_{bi} b^\dagger) S_-^{ii+1} \right],
\end{aligned} \tag{4.24}$$

where g_{ai} and g_{bi} ate the corresponding coupling constants. We assume that the following resonance conditions are satisfied:

$$E_4 - E_1 = 3\omega_b, \qquad E_3 - E_1 = 2\omega_b, \tag{4.25}$$

while all one-photon transitions are out of resonance. In such a case, the Hamiltonian (4.24) may describe, for example, a fifth-order process involving the absorption of three photons from one field and the stimulated emission of two photons of different frequency. The effective Hamiltonian describing *explicitly* these transitions can be obtained according to the general method. To this end, we note that now the integral of motion (4.4) takes now the form

$$N = a^\dagger a + b^\dagger b + \sum_{i=1}^{3} \mu_i S_3^{ii+1}, \tag{4.26}$$

where $\mu_i = i(4 - i)$, and the Hamiltonian can be recast also as in (4.7) with $H_{\text{int}} = h_0 + V_a + V_b$, where

$$h_0 = \delta\, b^\dagger b + \sum_{i=1}^{4} \Delta_i S^{ii}, \tag{4.27}$$

$$V_a = \sum_{i=1}^{3} g_{ai}(X_{a+}^{ii+1} + X_{a-}^{ii+1}) \qquad V_b = \sum_{i=1}^{3} g_{bi}(X_{b+}^{ii+1} + X_{b-}^{ii+1}).$$

Here the polynomial deformation at hand is defined by

$$X_{a+}^{ij} = a S_+^{ij}, \quad X_{a-}^{ij} = a^\dagger S_-^{ij}; \qquad X_{b+}^{ij} = b S_+^{ij}, \quad X_{b-}^{ij} = b^\dagger S_-^{ij}, \tag{4.28}$$

and $\delta = \omega_b - \omega_a$ will be taken, for definiteness, as positive.

Now we can eliminate all the one-photon nonresonant transitions generated by means of a couple of transformations (one for mode a the other for mode b) identical to (4.10). The transformed Hamiltonian (up to order $1/\Delta^2$) is

$$
\begin{aligned}
H_{\text{eff}}^{(1)} =\ & h_0 + h_{\text{d}}^{(a)} + h_{\text{d}}^{(b)} \\
& + \frac{1}{2}\lambda_1^{(2)}\left(a^2 S_+^{13} + a^{\dagger 2} S_-^{13}\right) + \frac{1}{3}\lambda_1^{(3)}\left(b^3 S_+^{14} + b^{\dagger 3} S_-^{14}\right) + \xi_2^{(ab)}\left(ab S_+^{24} + a^\dagger b^\dagger S_-^{24}\right),
\end{aligned} \tag{4.29}
$$

where $h_{\text{d}}^{(a)}$ and $h_{\text{d}}^{(b)}$ are defined according to (4.14) for both modes and the coupling constants $\lambda_1^{(2)}$ and $\lambda_1^{(3)}$ are given by (4.16). The constant $\xi_2^{(ab)}$ is

$$\xi_2^{(ab)} = \frac{g_{a3}g_{b2}}{\Delta_4 - \Delta_3} - \frac{g_{b3}g_{a2}}{\Delta_3 - \Delta_2}. \tag{4.30}$$

This effective Hamiltonian deserves some comments. First of all, when the resonance condition

$$E_4 - E_2 = \omega_a + \omega_b \tag{4.31}$$

is fulfilled (which is compatible with the two- and three-photon resonance conditions), then the last term in the Hamiltonian describes resonant transitions between levels 2 and 4, as a result of the simultaneous absorption and emission of quanta from both modes, and take place only if levels 2 and 4 are populated. In absence of such an additional resonance condition, the Hamiltonian (4.29) describes two simultaneous and competing processes: a first-order process ($\lambda_1^{(2)} \sim 1/\Delta$) involving two-photon transitions between levels 1 and 3, and a second-order process ($\lambda_1^{(3)} \sim 1/\Delta^2$) of three-photon transitions between levels 1 and 4.

V. FURTHER EXTENSIONS AND CONCLUDING REMARKS

We have developed our method putting special emphasis in polynomial deformations of algebras su(N), due to the outstanding role that they play in describing the interaction of N-level systems with quantum fields. However, by no means the method is restricted to such kind of algebras. In fact, many other phenomena can be modeled by Hamiltonians quite similar to (2.11), namely

$$H_{\text{int}} = a\, Y_0 + g(Y_+ + Y_-) + C, \tag{5.1}$$

where C is some integral of motion and a a constant. In these theories, the polynomial deformation is defined in the following fashion in the Cartan-Weyl basis:

$$[Y_0, Y_\pm] = \pm Y_\pm, \qquad [Y_-, Y_+] = \Psi(Y_0) = \Phi(Y_0 + 1) - \Phi(Y_0), \tag{5.2}$$

where $\Phi(Y_0)$ are appropriate structure polynomials. A detailed study of the applications of these algebras for solving evolution problems in nonlinear quantum models may be found in Ref. [19]. For these model, the machinery of small rotations works well and constitutes the most systematic ways of constructing effective Hamiltonians.

In summary, what we expect to have accomplished in this paper is to develop an appropriate and systematic tool for obtaining effective Hamiltonians that describe nonlinear optical phenomena. Our approach exploits the existence of deformed algebras (arising as dynamical symmetries of the corresponding process) to construct small nonlinear rotations that perform the task.

This method constitutes a firm algebraic setting free from some inconsistencies found in the traditional adiabatic elimination of variables and provides unambiguous recipes for proceeding with any model Hamiltonian.

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