Supplemental Information for Control over Dynamics of Quantum Thermal Machines

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DERIVING THE OPEN SYSTEM EVOLUTION IN THE CONTROL AND INTERACTION PICTURE

In this section, we derive the correct equation of motion used to prove the uniform convergence of the open quantum systems dynamics, namely

$$\frac{d\varsigma(t)}{dt} = -i\lambda_1[C_I(t), \varsigma(t)] + \lambda_2^2 \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \varsigma(t)\} \right). \tag{1}$$

We note that this derivation is similar to the standard derivations of master equations and is presented here for completeness. Let us begin with the system-environmental Hamiltonian $H = H_0 + W$, where $H_0 = H_S + H_E$ is the bare Hamiltonian and $W = \lambda_1 C(t) \otimes I + \lambda_2 V$, with $V = \sum_k A_k \otimes B_k$ is the sum of the Hamiltonian control on the system and the system-environment interaction term. Here $C(t) = C[\xi(t)]$ is the control Hamiltonian which is a functional of the control field $\xi(t)$ (We consider the case of one control field, but the generalization to multiple fields is straightforward). Constants $\lambda_{1,2}$ moderate the strength of the Hamiltonian terms and are taken to be small. Now, we start with $\dot{\rho}_{SE} = -i[H, \rho_{SE}]$. Let us write $\rho_I = U_0 \rho_{SE} U_0^\dagger$. Here $\dot{U}_0 = iH_0 U_0$. Note that $U_0 = U_S \otimes U_E$. This means that $\dot{\rho}_I = -i[W_I, \rho_I]$. Now, $W_I = \lambda_1 U_0 \left(C(t) \otimes I\right) U_0^\dagger + \lambda_2 \sum_k A_k^I(t) \otimes B_k^I(t) \equiv \lambda_1 C_I(t) + \lambda_2 V_I$. Here $V_I(t) = \sum_k A_k^I(t) \otimes B_k^I(t)$, $A_I(t) = U_S A U_S^\dagger$, $B_I(t) = U_E B U_E^\dagger$ and $C_I(t) = U_S C(t) U_S^\dagger$. This transformation defines the control and interaction picture (CIP). Note that though all terms in the interaction and control Hamiltonian are time dependant, the time dependance of $C_I(t)$ arises from the time dependance of $C(t)$ and U_S, whereas the other operators are time dependant because all the unitaries are parametrized by time in the exponant.

Now, the formal solution of this differential equation is $\rho_I(t) = \rho_I(0) - i\int_0^t du[W_I(u), \rho_I(u)]$. Let us consider this to first order in the control term λ_1 and second order in the interaction term λ_2. This implies that we first expand the equation of motion to get

$$\rho_I(t) = \rho_I(0) - i\lambda_1 \int_0^t du[C_I(u), \rho_I(u)] - i\lambda_2 \int_0^t du[V_I(u), \rho_I(u)] \tag{2}$$

We substitute the entire equation into the second term to get

$$\rho_I(t) = \rho_I(0) - i\lambda_1 \int_0^t du[C_I(u), \rho_I(u)] - i\lambda_2 \int_0^t du[V_I(u), \left(\rho_I(0) - i\lambda_1 \int_0^u ds[C_I(s), \rho_I(s)] - i\lambda_2 \int_0^u ds[V_I(s), \rho_I(s)]\right)]. \tag{3}$$

We make the assumption that only terms of the order $\lambda_1 \approx \lambda_2^2$ are to be retained. All terms of higher orders can be dropped. This yields

$$\rho_I(t) = \rho_I(0) - i\lambda_1 \int_0^t du \left[C_I(u), \rho_I(u)\right] - i\lambda_2 \int_0^t du \left[V_I(u), \left(\rho_I(0) - i\lambda_2 \int_0^u ds[V_I(s), \rho_I(s)]\right)\right]. \tag{4}$$

Now, we can consider the equation of motion for $\varsigma(t)$, the reduced system state in this control and interaction picture we have defined. This equation is given by

$$\varsigma(t) = \varsigma(0) - i\lambda_1 \int_0^t du\, \mathrm{Tr_E}\left[C_I(u), \rho_I(u)\right] - i\lambda_2 \int_0^t du\, \mathrm{Tr_E}\left[V_I(u), \left(\rho_I(0) - i\lambda_2 \int_0^u ds\, [V_I(s), \rho_I(s)]\right)\right]. \tag{5}$$

The equation of motion for $\varsigma(t)$ can hence be written as the first derivative of the equation above namely

$$\frac{d\varsigma(t)}{dt} = -i\lambda_1 \mathrm{Tr}_{\mathrm{E}}[C_I(t), \rho_I(t)] - i\lambda_2 \mathrm{Tr}_{\mathrm{E}}[V_I(u), \rho_I(0)] - \lambda_2^2 \int_0^t ds\, \mathrm{Tr}_{\mathrm{E}}[V_I(t), [V_I(s), \rho_I(s)]]. \tag{6}$$

All terms on the RHS can be simplified as follows. The first term $\mathrm{Tr}_{\mathrm{E}}[C_I(u), \rho_I(u)] = \mathrm{Tr}_{\mathrm{E}}(\rho_B)[C_I(t), \varsigma(t)] = [C_I(t), \varsigma(t)]$. This uses two approximations: the Born approximation namely $\rho_I(t) = \varsigma(t) \otimes \rho_B$ and the assumption that the bath state is the Gibbs state of the bath Hamiltonian (so bath state is time independent in the ICP). The second term is zero, since we assume that the bath operators are zero mean, namely $\mathrm{Tr}_{\mathrm{E}}(B_k(0)\rho_B) = 0$. This means that we drop this term. The final term is simplified by defining $\Gamma_{kl}(t,s) = \mathrm{Tr}_{\mathrm{E}}(B_k(t)B_l(s)\rho_B)$. Note that $\Gamma_{kl}(t,s) = \mathrm{Tr}_{\mathrm{E}}(B_k(t)B_l(s)\rho_B) = \mathrm{Tr}_{\mathrm{E}}(e^{iH_E t}B_k e^{-iH_E t}e^{iH_E s}B_l e^{-iH_E s}\rho_B) = \mathrm{Tr}_{\mathrm{E}}(e^{iH_E t}B_k e^{-iH_E t}e^{iH_E s}B_l \rho_B e^{-iH_E s}) = \mathrm{Tr}_{\mathrm{E}}(e^{iH_E(t-s)}B_k e^{-iH_E(t-s)}B_l\rho_B) = \Gamma_{kl}(t-s)$, proving stationarity for the set of assumptions. With all these assumptions, we get

$$\frac{d\varsigma(t)}{dt} = -i\lambda_1[C_I(t), \varsigma(t)] - \lambda_2^2 \int_0^t ds \sum_{kl} \Gamma_{kl}(t-s)[A_k^I(t), A_l^I(s)\varsigma(s)] + \Gamma_{lk}(s-t)[A_k^I(t), \varsigma(s)A_l^I(s)]. \tag{7}$$

Note that $\Gamma_{lk}^*(\tau) = \Gamma_{kl}(\tau)$. We make two assumptions: (1) that we can extend the integral to ∞ and (2) the Markov assumption, where we replace $\varsigma(s) \approx \varsigma(t)$. With these assumptions, we get

$$\frac{d\varsigma(t)}{dt} = -i\lambda_1[C_I(t), \varsigma(t)] - \lambda_2^2 \int_0^\infty ds \sum_{kl} \Gamma_{kl}(t-s)[A_k^I(t), A_l^I(s)\varsigma(t)] + \Gamma_{lk}(s-t)[A_k^I(t), \varsigma(t)A_l^I(s)]. \tag{8}$$

Now, lets Fourier transform the system operators of V_I as $A_k^I(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\tilde{A}_k^I(\omega)$. Using this we can write

$$\frac{d\varsigma(t)}{dt} = -i\lambda_1[C_I(t), \varsigma(t)] - \lambda_2^2 \int_0^\infty ds \sum_{kl} \Gamma_{kl}(t-s)e^{i(\omega t+\Omega s)}[\tilde{A}_k^I(\omega), \tilde{A}_l^I(\Omega)\varsigma(t)] + \Gamma_{lk}(s-t)e^{i(\omega t+\Omega s)}[\tilde{A}_k^I(\omega), \varsigma(t)\tilde{A}_l^I(\Omega)]. \tag{9}$$

We can write $e^{i\omega t+i\Omega s} = e^{i\Omega(s-t)}e^{i(\Omega+\omega)t}$ and write the last term on the RHS as

$$= \sum_{kl}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Omega e^{i(\Omega+\omega)t}\int_0^\infty ds \Gamma_{kl}(t-s)e^{i\Omega(s-t)}[\tilde{A}_k^I(\omega), \tilde{A}_l^I(\Omega)\varsigma(t)] + \Gamma_{lk}(s-t)e^{i\Omega(s-t)}[\tilde{A}_k^I(\omega), \varsigma(t)\tilde{A}_l^I(\Omega)]. \tag{10}$$

Change variables from $s \to (s-t) = \tau$ and write

$$= \sum_{kl}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Omega e^{i(\Omega+\omega)t}\tilde{\Gamma}_{kl}(\Omega)[\tilde{A}_k^I(\omega), \tilde{A}_l^I(\Omega)\varsigma(t)] + \tilde{\Gamma}_{lk}(-\omega)[\tilde{A}_k^I(\omega), \varsigma(t)\tilde{A}_l^I(\Omega)]. \tag{11}$$

Application of the standard secular approximation yields the desired master equation, namely Eq.(1).

PROOF OF UNIFORM CONVERGENCE OF OQS-KROTOV

Next, we restate and prove theorem 1 for completeness.

Theorem 1 *The open quantum system with weak control given by Eq.(XXX) is controllable through a local first order variational approach. Furthermore, the algorithm to control the open quantum system is uniformly convergent.*

Proof. We will begin by rewriting the Lindblad equation in the Liouville picture which involves the following prescription for vectorizing the differential equation namely

$$T_m\varsigma(t)T_n^\dagger \to \overline{T}_n \otimes T_m col(\varsigma(t)) := \overline{T}_n \otimes T_m|\psi\rangle\rangle, \tag{12}$$

in the column vector notion of the density matrix which we call as the Liouville space. Taking the derived equation:

$$\frac{d\varsigma(t)}{dt} = -i\lambda_1[C_I(t), \varsigma(t)] + \lambda_2^2 \left[\mathcal{L}[l_+^{\frac{1}{2}}\sigma_+]\varsigma(t) + (\mathcal{L}[l_-^{\frac{1}{2}}\sigma_-]\varsigma(t) \right], \tag{13}$$

which represents a thermal Lindbladian with the notation $\mathcal{L}[P]\varsigma := P\varsigma P^\dagger - \frac{1}{2}\{P^\dagger P, \varsigma\}$, $l_+ = \frac{1}{e^\beta - 1}$ and $l_- = \frac{e^\beta}{e^\beta - 1}$. We convert this equation into the Liouville space i.e. the columnized density matrix. We know that:

$$C_I(t) = e^{iH_0 t/\hbar} C_s(t) e^{-iH_0 t/\hbar} \tag{14}$$

For $H_0 = \frac{\hbar\omega}{2}\sigma_z$ and $C_s = -\xi(t)\sigma_x$ we find that

$$C_I = -\{cos(\omega t)\sigma_x - sin(\omega t)\sigma_y\}\xi(t) = -\xi(t)\nu(t) \tag{15}$$

where we define $\nu(t) = cos(\omega t)\sigma_x - sin(\omega t)\sigma_y$ as the control Hamiltonian operator σ_x in the CIP. Using this in the Lioville space representation of the master equation above, we get (xxx Nischay Replace all these Cs with the correct operators...)

$$\begin{aligned} \frac{d}{dt}|\psi\rangle\!\rangle &= -i\lambda_1\xi(t)(-I \otimes \nu(t) + \overline{\nu}(t) \otimes I)|\psi\rangle\!\rangle \\ &+ \lambda_2^2\{(\overline{C} \otimes C) - \frac{1}{2}(I \otimes (C^\dagger C) + \overline{(C^\dagger C)} \otimes I)\}|\Psi\rangle\!\rangle \Rightarrow \end{aligned} \tag{16}$$

$$\frac{d}{dt}|\psi\rangle\!\rangle = -i\lambda_1 H_0|\psi\rangle\!\rangle + \lambda_2^2(H_1 + A_1)|\psi\rangle\!\rangle \tag{17}$$

Here, $H_0 := -\xi(t)\mu = -\xi(t)(I \otimes \nu(t) - \overline{\nu}(t) \otimes I)$ and $(H_1 + A_1) = (\overline{C} \otimes C) - \frac{1}{2}(I \otimes (C^\dagger C) + \overline{(C^\dagger C)} \otimes I)$. Note that $H_0 = H_0^\dagger$, $H_1 = H_1^\dagger$ and $A_1 = -A_1^\dagger$. (xxx NISCHAY CLARIFY)

With this, we can finally write the cost function which we wish to optimize, namely

$$J = \langle\!\langle\Psi|\hat{O}|\Psi\rangle\!\rangle - 2Re\int_0^T dt\langle\!\langle\chi(t)|\partial_t - L|\psi(t)\rangle\!\rangle - \alpha\int_0^T dt|\xi(t)|^2. \tag{18}$$

The first term enforces that the state overlaps with the target state at the final time, whereas the second term enforces the Lindblad evolution at each time via the Lagrange multiplier (often called the costate) $|\chi(t)\rangle\!\rangle$. The last term controls the fluence, which is a measure of the integrate power being spent on the control protocol and $L = -i\lambda_1 H_0|\psi\rangle\!\rangle - \lambda_2^2(H_1 + A_1)$. The importance of the fluence term is moderated by changing the parameter α, where a larger α produces pulse sequences that minimize the fluence at the cost of not attaining the final state. Note that $|\Psi\rangle\!\rangle = |\psi(T)\rangle\!\rangle$ is the Liouville state at the final time.

$$J = \langle\!\langle\Psi|\hat{O}|\Psi\rangle\!\rangle - 2Re\langle\!\langle\chi(t)|\psi\rangle\!\rangle\Big|_0^T \tag{19}$$

$$- 2Re\int_0^T dt\Big\{\langle\!\langle\chi(t)|L|\psi(t)\rangle\!\rangle + \langle\!\langle\partial_t\chi|\psi\rangle\!\rangle\Big\} - \alpha\int_0^T dt|\xi(t)|^2 \tag{20}$$

$$\frac{\partial J}{\partial|\psi(t)\rangle\!\rangle} = \int_0^T dt\; 2Re\Big\{\langle\!\langle\chi|L + \frac{\partial\langle\!\langle\chi|}{\partial t}\Big\} = 0$$

$$\Rightarrow \frac{\partial|\chi\rangle\!\rangle}{\partial t} = -L^\dagger\;|\chi\rangle\!\rangle \tag{21}$$

$$\frac{\partial J}{\partial|\psi(T)\rangle\!\rangle} = \langle\!\langle\psi(T)|\hat{O} - \langle\!\langle\chi(T)| = 0$$

$$\Rightarrow |\chi(T)\rangle\!\rangle = \hat{O}\;|\psi(T)\rangle\!\rangle \tag{22}$$

$$\frac{\partial J}{\partial \xi(t)} = -\alpha \int_0^T dt \xi(t) + \int_0^T dt\ Re\Big\{ \langle\!\langle \psi | i\lambda_1 \nu | \psi \rangle\!\rangle \Big\} = 0$$

$$\Rightarrow \xi(t) = \frac{1}{\alpha} Im \langle\!\langle \chi | \lambda_1 \nu(t) | \psi \rangle\!\rangle \tag{23}$$

We summarize these equations below. (XXX Nischay, we are missing the motivation for making both the state and costate equations nonlinear....)

$$\partial_t |\psi^k\rangle\!\rangle = (-i\lambda_1 H_0^k + \lambda_2^2 (H_1 + A_1)) |\psi^k\rangle\!\rangle \tag{24}$$
$$\partial_t |\chi^k\rangle\!\rangle = (-i\lambda_1 \tilde{H}_0^k - \lambda_2^2 (H_1 - A_1)) |\psi^k\rangle\!\rangle \tag{25}$$
$$H_0^k = \mu \xi^k \tag{26}$$
$$\tilde{H}_0^k = \mu \tilde{\xi}^k \tag{27}$$
$$\tilde{\xi}^k(t) = (1-\eta)\xi^k(t) - \frac{\eta}{\alpha}\ \ Im \langle\!\langle \chi^k(t) | \mu\lambda_1 | \psi^k(t) \rangle\!\rangle \tag{28}$$
$$\xi^k(t) = (1-\delta)\tilde{\xi}^{k-1}(t) - \frac{\delta}{\alpha}\ \ Im \langle\!\langle \chi^{k-1}(t) | \mu\lambda_1 | \psi^k(t) \rangle\!\rangle \tag{29}$$

Proof:

$$\begin{aligned} J(\xi^{k+1}) - J(\xi^k) = &\Big\{ \langle\!\langle \psi^{k+1}(T) | \hat{O} | \psi^{k+1}(T) \rangle\!\rangle - \alpha \int_0^T dt |\xi^{k+1}(t)|^2 \Big\} \\ &- \Big\{ \langle\!\langle \psi^{k}(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle - \alpha \int_0^T dt |\xi^{k}(t)|^2 \Big\} \end{aligned} \tag{30}$$

Consider the inner product $\langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \hat{O} | \psi^{k+1}(T) - \psi^k(T) \rangle\!\rangle$

$$\begin{aligned} &= \langle\!\langle \psi^{k+1}(T) | \hat{O} | \psi^{k+1}(T) \rangle\!\rangle + \langle\!\langle \psi^{k}(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle - \langle\!\langle \psi^{k}(T) | \hat{O} | \psi^{k+1}(T) \rangle\!\rangle - \langle\!\langle \psi^{k+1}(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle, \\ &= \langle\!\langle \psi^{k+1}(T) | \hat{O} | \psi^{k+1}(T) \rangle\!\rangle - \langle\!\langle \psi^{k}(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle - \langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle \\ &\quad - \langle\!\langle \psi^{k}(T) | \hat{O} | \psi^{k+1}(T) - \psi^k(T) \rangle\!\rangle. \end{aligned} \tag{31}$$

$$\begin{aligned} \Rightarrow \langle\!\langle \psi^{k+1}(T) | \hat{O} | \psi^{k+1}(T) \rangle\!\rangle - \langle\!\langle \psi^{k}(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle = &\langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \hat{O} | \psi^{k+1}(T) - \psi^k(T) \rangle\!\rangle \\ &+ 2Re \langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle. \end{aligned} \tag{32}$$

We put equation 60 in 58 and we get (xxx Nichay fix using label and reference command.),

$$\begin{aligned} = &\langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \hat{O} | \psi^{k+1}(T) - \psi^k(T) \rangle\!\rangle + 2Re \langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle \\ &- \alpha \int_0^T dt \Big\{ |\xi^{k+1}(t)|^2 - |\xi^k(t)|^2 \Big\} \end{aligned} \tag{33}$$

Now we take the part,

$$2Re \langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \hat{O} | \psi^{k}(T) \rangle\!\rangle = 2Re \langle\!\langle \psi^{k+1}(T) - \psi^k(T) | \chi^k(T) \rangle\!\rangle \tag{34}$$
$$= 2Re \int_0^T dt \langle\!\langle \frac{\partial(\psi^{k+1}(t) - \psi^k(t))}{\partial t}, \chi^k(t) \rangle\!\rangle + \langle\!\langle \psi^{k+1}(t) - \psi^k(t), \frac{\partial \chi^k(t)}{\partial t} \rangle\!\rangle \tag{35}$$
$$= \langle\!\langle \psi^{k+1} | i\lambda_1 H_0^{k+1} + \lambda_2^2 (H_1 - A_1) |\, \chi^k \rangle\!\rangle \tag{36}$$
$$+ \langle\!\langle \psi^{k} | - i\lambda_1 H_0^{k+1} - \lambda_2^2 (H_1 - A_1) |\, \chi^k \rangle\!\rangle \tag{37}$$
$$+ \langle\!\langle \psi^{k+1} | - i\lambda_1 \tilde{H_0}^{k+1} - \lambda_2^2 (H_1 - A_1) |\, \chi^k \rangle\!\rangle \tag{38}$$
$$- \langle\!\langle \psi^{k} | - i\lambda_1 \tilde{H_0}^{k} + \lambda_2^2 (H_1 - A_1) |\, \chi^k \rangle\!\rangle \tag{39}$$

It simplifies to :

$$= 2Re\int_0^T dt \langle\!\langle \frac{\partial(\psi^{k+1}(t)-\psi^k(t))}{\partial t}, \chi^k(t)\rangle\!\rangle + \langle\!\langle \psi^{k+1}(t)-\psi^k(t), \frac{\partial \chi^k(t)}{\partial t}\rangle\!\rangle \tag{40}$$

$$= 2Re\int_0^T dt\Big\{\langle\!\langle \psi^{k+1}|i\lambda_1 H_0^{k+1}\chi^k\rangle\!\rangle \tag{41}$$

$$+ \langle\!\langle \psi^{k}| - i\lambda_1 H_0^{k+1}\chi^k\rangle\!\rangle \tag{42}$$

$$+ \langle\!\langle \psi^{k+1}| - i\lambda_1 \tilde{H_0}^{k+1}\chi^k\rangle\!\rangle \tag{43}$$

$$- \langle\!\langle \psi^{k}| - i\lambda_1 \tilde{H_0}^{k}\chi^k\rangle\!\rangle\Big\} \tag{44}$$

Using the above equations and substituting for the inner products we get :

$$= 2\int_0^T dt\ \frac{\alpha}{\delta}\xi^{k+1}(\xi^{k+1}-(1-\delta)\tilde{\xi}^k) - \frac{\alpha}{\eta}\xi^k(\tilde{\xi}^k-(1-\eta)\xi^k) - \frac{\alpha}{\delta}\tilde{\xi}^k(\xi^{k+1}-(1-\delta)\tilde{\xi}^k) + \frac{\alpha}{\eta}\tilde{\xi}(\tilde{\xi}^k-(1-\eta)\xi^k) \tag{45}$$

Finally we get

$$J(\xi^{k+1}) - J(\xi^k) = \langle\!\langle \psi^{k+1}(T)-\psi^k(T)|O|\psi^{k+1}(T)-\psi^k(T)\rangle\!\rangle + \alpha\int_0^T dt\Big(\frac{2}{\delta}-1\Big)(\xi^{k+1}-\tilde{\xi}^k)^2 + \Big(\frac{2}{\eta}-1\Big)(\tilde{\xi}^k-\xi^k)^2 \tag{46}$$

Therefore, we see that $J(\xi^{k+1}) - J(\xi^k) \geq 0$ always for $\eta,\delta \in [0,2]$. This concludes the proof that a generalized Krotov algorithm is uniformally convergent to open quantum systems. This is a generalization of a standard result to open quantum systems.

TWO-STROKE ENGINE, WORK & HEAT

Consider the two-stroke engine whose strokes are described as:

Work Stroke: The work stroke involves a SWAP operation. It is assumed that a Born-Oppenheimer-like degree of freedom carries away the work when it implements the unitary stroke. The work here is given by $W = \mathrm{tr}[H_0(\rho - U\rho U^\dagger)]$, where $\rho = \rho_L \otimes \rho_R$ with ρ_L is the state of the left bath at the end of the "Krotov thermalization" (and likewise for ρ_R and the right bath). Notice that this work is modified by a term that is proportional to the difference in the state we want to target (the thermal state of the hot bath for the left qubit and the thermal state of the cold bath for the right qubit). This modification can always be written as $W_{actual} = W_{id} - \delta$, though whether $\delta \geq 0$ has to be established (see below for conditions).

Heat Stroke: This stroke involves the left qubit thermalizing with the hot bath and the right qubit thermalizing with the cold bath. Now, if the control field is not applied to thermalize, it is assumed that the thermalization takes time $T_{free}^{(\varepsilon)}$ calculated below. We wish to speed this up with a control Hamiltonian $C(t)$. All of the energy exchanged with the qubit in this stroke is considered to be heat, since we are not connecting the qubit to the Born-Oppenheimer DOF that carry work away. The energy for the left qubit interacting with the hot bath, say, is given by

$$\frac{dQ}{dt} = \mathrm{tr}[\{H_0 + C(t)\}\frac{d\rho(t)}{dt}] \Rightarrow \tag{47}$$

$$Q = Q^{(uc)} + Q^{(c)}. \tag{48}$$

Here $\rho(t)$ is the system density matrix in the Schrodinger picture and $Q = \int_0^T \frac{dQ^+}{dt}$ is the total positive part of the heat. Note that this heat is split into two parts, one which is the uncontrolled heat $Q^{(uc)} = \mathrm{tr}(H_0[\rho_f - \rho_i])$ thats constant to us and Michele's paper, and the control heat which is only true for our time-dependant driving. There is another term that contributes, namely the positive part of the $\frac{dQ}{dt}$ integral.

So we can write the efficiency under our modified control scheme the following efficiency

$$\eta = \frac{W_{id} - \delta}{Q^{(uc)} + Q^{(c)}}. \tag{49}$$

To calculate δ, consider $W = \mathrm{tr}[H_0(\rho - U\rho U^\dagger)]$, which is the actual work. Now, you can add and subtract the ideal work $W_{id} = \mathrm{tr}[H_0(\rho_{id} - U\rho_{id}U^\dagger)]$. Here $\rho_{id} = \mathcal{G}_h \otimes \mathcal{G}_c$, the tensor product of the Gibbs state of the hot bath and the Gibbs state of the cold bath (the IDEAL states of the two baths at the end of thermalization, which of course is not what we have in ρ). So you can write $W := W_{id} - \delta = W_{id} - (W_{id} - W)$. This can be used to define $\delta := W_{id} - W$.

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Conditions for $\delta \geq 0$.— Consider $\delta = W_{id} - W$. This can be rewritten as follows

$$\delta = \langle H_0 \rangle_{id} - \langle U^\dagger H_0 U \rangle_{id} - \langle H_0 \rangle_{Nid} + \langle U^\dagger H_0 U \rangle_{Nid} \tag{50}$$

$$\delta = \delta E_1 - \delta E_2 = \delta E_1 - \delta E_3 + \delta E_4 \tag{51}$$

See Fig.(1) for an intuitive diagram.

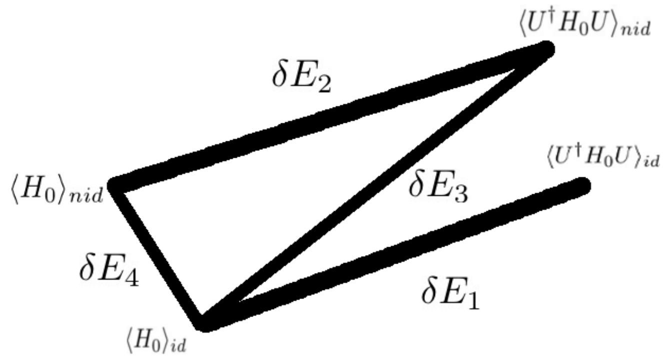

Figure 1. This diagram goes along with the proof that $\delta \geq 0$.

The difference of the first two terms $\delta E_1 - \delta E_3 \geq 0$ because the unitary U is optimal in extracting work. This means that if you start at ρ_{id} and transform to any other state that is not $U \rho_{id} U^\dagger$, like say $U \rho_{nid} U^\dagger$, then the corresponding energy you extract is suboptimal (only true if non-ideal transformation was unitary, not if its CPTP...). The last term $\delta E_4 \geq 0$ because the passive state ρ_{id} is the least energy state for a given spectrum. Now, we know that if we start at ρ_{neq} and attach the thermal bath, the state will transform to ρ_{id}. This means that the non-ideal state thermomajorizes the ideal state. If furthermore, they are energetically ordered (which is to simply say that $\delta E_4 \geq 0$) then $\delta \geq 0$. Otherwise you have to check.

ε-FREE TIME FOR SINGLE QUBIT THERMALIZATION

Once again, consider the Lindblad equation

$$\frac{d\rho(t)}{dt} = -i[H(t), \rho(t)] + \sum_k L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\} \tag{52}$$

Following arXiv:1307.7964v1, we can always write the Lindblad operators as traceless operators $L_k = \gamma_k^{\frac{1}{2}} \mathbf{l}_k \cdot \sigma$ and the Lindblad equation as

$$\dot{\mathbf{r}} = 2\mathbf{h} \times \mathbf{r} + 2\sum_k \gamma_k \left(\mathrm{Re}[\mathbf{l}_k . \mathbf{r} \mathbf{l}_k^*] - \mathbf{r} + i(\mathbf{l}_k \times \mathbf{l}_k^*) \right) \tag{53}$$

For generalized amplitude damping map (thermal maps), following Eq.(10) of arXiv:1307.7964v1, we can write

$$\dot{\mathbf{r}} = (0, 0, \omega)^T \times \mathbf{r} - \gamma_1 (r_x(t), r_y(t), 0)^T - \gamma_2 (0, 0, r_z(t))^T - \gamma_3 (0, 0, 1)^T. \tag{54}$$

Here $\gamma_1 = \frac{\gamma}{2 r_{fp}}$, $\gamma_2 = 2\gamma_1$ and $\gamma_3 = \gamma$. The solution to this set of differential equations is

$$r_x(t) = e^{-\gamma_1 t}[r_x(0)\cos(\omega t) - r_y(0)\sin(\omega t)] \tag{55}$$

$$r_y(t) = e^{-\gamma_1 t}[r_y(0)\cos(\omega t) + r_x(0)\sin(\omega t)] \tag{56}$$

$$r_z(t) = -\frac{\gamma_3}{\gamma_2} + e^{-\gamma_2 t}\left(r_z(0) + \frac{\gamma_3}{\gamma_2} \right) \Rightarrow \tag{57}$$

$$r_z(t) = -r_{fp} + e^{-\gamma_2 t}\left(r_z(0) + r_{fp} \right) \tag{58}$$

Now, we also note that if there are two density matrices $\rho = \frac{1}{2}(I + \mathbf{r}.\sigma)$ and $\varsigma = \frac{1}{2}(I + \mathbf{s}.\sigma)$, then the trace distance D_1 is given by

$$D_1(\rho, \varsigma) := \frac{1}{2}\|\mathbf{r} - \mathbf{s}\| = \frac{1}{2}[(r_1 - s_1)^2 + (r_2 - s_2)^2 + (r_3 - s_3)^2]^{\frac{1}{2}} \tag{59}$$

The state of the qubit at time t is given by $\mathbf{r}(t) = (r_x(t), r_y(t), r_z(t))^T$ with the elements of the vector defined above. We wish to estimate the trace distance from the target state, which is written in terms of the Bloch vector as $\mathbf{r}_* = (0, 0, -r_{fp})^T$. Here r_{fp}, the non-zero z-component of the Bloch sphere can be established by noticing that the way we write the thermal state is

$$\rho = \frac{1}{2n_{Th} + 1}\begin{pmatrix} n_{Th} & 0 \\ 0 & n_{Th} + 1 \end{pmatrix} \tag{60}$$

We compare this to the fixed point given by $\mathbf{r}_*$, namely

$$\rho_* = \frac{1}{2}\begin{pmatrix} 1 - r_{fp} & 0 \\ 0 & 1 + r_{fp} \end{pmatrix} \tag{61}$$

Comparing gives us the relationship

$$r_{fp} = \frac{e^{\beta} - 1}{e^{\beta} + 1} = \frac{1}{2n_{Th} + 1}. \tag{62}$$

The β relationship above is also from the paper cited. To check self consistency, we invert the last two parts of the equation above to write

$$e^{-\beta} = \frac{n_{Th}}{n_{Th} + 1}, \tag{63}$$

which seems right. Notice that this is the time we have to beat with quantum control. We can compute the trace distance analytically at any time between $\mathbf{r}$ and $\mathbf{r}_*$, namely

$$D_1(\mathbf{r}(0), r_{fp}, \gamma, t) = \frac{e^{-\gamma_1 t}}{2}\left(r_x^2(0) + r_y^2(0) + e^{-2\gamma_1 t}(r_z(0) + r_{fp})^2\right)^{\frac{1}{2}}. \tag{64}$$

Now, consider the trace distance $D_1(\rho, \rho_*) := \|\mathbf{r}(t) - \mathbf{r_{fp}}\|$. We can now contemplate the ε-free time, defined as the minimum time needed to get to within an ϵ distance of the fixed point. This is formally defined as

$$T^{\varepsilon}_{free} := \arg\min_t(\|\mathbf{r}(t) - \mathbf{r_{fp}}\| \leq \varepsilon) \tag{65}$$

This can be computed from the analytic formula above to be

$$T^{\varepsilon}_{free} = -\frac{1}{\gamma_2}\log\left(\frac{-\alpha^2 + \sqrt{\alpha^4 - 16\varepsilon\beta^2}}{2\beta^2}\right) \tag{66}$$

$$T^{\varepsilon}_{free} = -\frac{1}{2\gamma_1}\log\left(\frac{-\alpha + \sqrt{\alpha^2 + 16\varepsilon^2\beta^2}}{2\beta^2}\right) \tag{67}$$

Here $\alpha = r_x^2(0) + r_y^2(0)$ and $\beta = r_z(0) + r_{fp}$. Notice that $\lim_{\varepsilon\to 0} T^{\varepsilon}_{free} \to \infty$, as expected.