Adiabatic elimination for open quantum systems

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Résumé

Cette thèse traite du problème de la réduction de modèle pour les systèmes quantiques ouverts possédant différentes échelles de temps, également connu sous le nom d'élimination adiabatique. L'objectif est d'obtenir une méthode générale d'élimination adiabatique assurant la structure quantique du modèle réduit.

On considère un système quantique ouvert, décrit par une équation maîtresse de Lindblad possédant deux échelles de temps, la dynamique rapide faisant converger le système vers un état d'équilibre. Les systèmes associés à un état d'équilibre unique ou une variété d'états d'équilibre ("decoherence-free space") sont considérés. La dynamique lente est traitée comme une perturbation. En utilisant la séparation des échelles de temps, on développe une nouvelle technique d'élimination adiabatique pour obtenir, à n'importe quel ordre, le modèle réduit décrivant les variables lentes. Cette méthode, basée sur un développement asymptotique et la théorie géométrique des perturbations singulières, assure une bonne interprétation physique du modèle réduit au second ordre en exprimant la dynamique réduite sous une forme de Lindblad et la paramétrisation définissant la variété lente dans une forme de Kraus (préservant la trace et complètement positif). On obtient ainsi des formules explicites, pour calculer le modèle réduit jusqu'au second ordre, dans le cas des systèmes composites faiblement couplés, de façon Hamiltonienne ou en cascade; des premiers résultats au troisième ordre sont présentés. Pour les systèmes possédant une variété d'états d'équilibre, des formules explicites pour calculer le modèle réduit jusqu'au second ordre sont également obtenues.

Mots-clés

Élimination adiabatique ; Perturbations singulières ; Systèmes quantiques ouverts ; Systèmes quantiques composites ; Systèmes multi-échelles ; Réduction de modèle ; Équation maîtresse de Lindblad

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Abstract

This thesis addresses the model reduction problem for open quantum systems with different time-scales, also called adiabatic elimination. The objective is to derive a generic adiabatic elimination technique preserving the quantum structure for the reduced model.

We consider an open quantum system, described by a Lindblad master equation with two time-scales, where the fast time-scale drives the system towards an equilibrium state. The cases of a unique steady state and a manifold of steady states (decoherence-free space) are considered. The slow dynamics is treated as a perturbation. Using the time-scale separation, we developed a new adiabatic elimination technique to derive at any order the reduced model describing the slow variables. The method, based on an asymptotic expansion and geometric singular perturbation theory, ensures the physical interpretation of the reduced second-order model by giving the reduced dynamics in a Lindblad form and the mapping defining the slow manifold as a completely positive trace-preserving map (Kraus map) form. We give explicit second-order formulas, to compute the reduced model, for composite systems with weak - Hamiltonian or cascade - coupling between the two subsystems and preliminary results on the third order. For systems with decoherence-free space, explicit second order formulas are as well derived.

Keywords

Adiabatic elimination ; Singular perturbations ; Open quantum systems ; Quantum composite systems ; Different time-scales ; Model reduction ; Lindblad master equation

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Contents

Chapter 0

Introduction (version française)

Présentation du problème

Du fait que tout système quantique interagit avec son environnement, la façon rigoureuse pour décrire son évolution serait d'utiliser l'équation de Schrödinger sur l'ensemble système et environnement. Étudier cet ensemble n'est généralement pas possible en raison de la complexité de l'environnement. Ainsi, en utilisant des hypothèses classiques sur l'environnement (notamment des approximations de Markov), il est possible d'obtenir une équation maîtresse de Lindblad [BP06] concernant seulement le système d'intérêt et donc n'incluant pas l'environnement. Une situation similaire apparaît naturellement au sein d'un système quantique composé de plusieurs sous-systèmes interagissant: on peut souhaiter obtenir la dynamique d'un sous-système particulier d'intérêt en utilisant certaines hypothèses pour ne pas prendre en compte les autres sous-systèmes. Lorsque le sous-système d'intérêt a des taux de dissipation (c'est-à-dire des échelles de temps) bien plus lents que les autres sous-systèmes et est faiblement couplé à ceux-ci, une telle procédure est appelée élimination adiabatique (correspondant à la théorie des perturbations en théorie des systèmes dynamiques). Ce problème de réduction de modèle apparaît dans de nombreux champs de la physique quantique pour différentes raisons :

- Cette question est étroitement liée à l'ingénierie de réservoir (reservoir engineering) [PCZ96] où le but est de choisir la dynamique d'un système cible en choisissant de façon appropriée ses interactions avec d'autres sous-systèmes. L'interaction est conçue de telle manière que la dynamique du système cible est, après élimination adiabatique des autres sous-systèmes, celle désirée. Calculer rigoureusement la dynamique réduite associée au sous-système d'intérêt est une tâche difficile. Ceci souligne le besoin de développer des procédures systématiques d'élimination adiabatique. Ces procédures doivent préserver la structure quantique afin d'assurer une interprétation physique du modèle réduit.

- Lorsque l'on connecte plusieurs systèmes quantiques, l'espace de Hilbert du système complet est le produit tensoriel des espaces des systèmes individuels. Les dimensions se multiplient, amenant rapidement à un système difficile à étudier en raison du grand nombre de variables. Dans le but d'obtenir une meilleure compréhension physique d'un phénomène particulier dans un ensemble de systèmes, ou simplement afin de pouvoir simuler numériquement un tel système, il faut développer des méthodes génériques de réduction de modèles.

État de l'art

Les techniques d'élimination adiabatique pour les systèmes quantiques fermés, gouvernés par l'équation de Schrödinger sont standard. Dans de tels cas, l'évolution quantique est unitaire et la théorie des perturbations régulières est utilisée ([Sak94]): suivre l'évolution des valeurs propres et vecteurs propres du Hamiltonien suite à une faible perturbation est bien connu. Le cas des systèmes quantiques ouverts, décrits par une équation maîtresse de Lindblad ([BP06]) est cependant beaucoup plus complexe et implique d'utiliser la théorie des perturbations singulières.

Dans la littérature, de nombreux exemples particuliers de systèmes quantiques ouverts ont été traités avec succès. Dans [BPM07], l'élimination adiabatique pour des systèmes en lambda avec un état excité et deux états fondamentaux est étudiée. Pour de tels systèmes, les auteurs illustrent clairement comment certaines techniques standard d'élimination adiabatique de l'optique quantique peuvent amener à des résultats ambiguës. Cela souligne la nécessité de développer des approches rigoureuses. Ils proposent ensuite deux techniques d'élimination adiabatique pour ces systèmes à trois dimensions en considérant des perturbations Hamiltoniennes. La première technique est basée sur une solution explicite de l'amplitude de l'état excité; la seconde utilise le formalisme des fonctions de Green pour obtenir une expression du propagateur associé à la dynamique. Des systèmes en lambda où un état excité se dissipe rapidement vers un nombre arbitraire d'états fondamentaux sont considérés dans [MR09]. A partir de l'équation maîtresse de Lindblad du système complet décrit par l'opérateur densité ρ, les auteurs présentent un changement de variable $\rho \to (\rho_f, \rho_s)$, correspondant à une séparation état excité/fondamentaux et permettant d'obtenir une dynamique dans la forme normale de Tikhonov. Ensuite, en utilisant le théorème de Tikhonov, ils obtiennent une dynamique réduite au premier ordre pour tout type de perturbation. L'extension aux systèmes en lambda avec plusieurs états excités est considérée dans [RS12]. Les auteurs proposent une approche similaire, en décomposant l'espace de Hilbert en deux sous-espaces, un pour les états fondamentaux et un pour les états excités. En utilisant un développement asymptotique, en puissance du paramètre séparant les deux échelles de temps, ils obtiennent une équation maîtresse de Lindblad effective pour les états fondamentaux jusqu'au second ordre. Différents types de rétroactions (cohérentes et incohérentes) entre un système d'intérêt et un système auxiliaire (un qubit ou une cavité) sont étudiés dans [WW00]. Lorsque le système auxiliaire est sur une échelle de temps rapide, une technique d'élimination adiabatique similaire à l'approche de [Car93a] est utilisée pour obtenir la dynamique réduite au premier ordre du système d'intérêt et ainsi étudier l'effet de la rétroaction. Pour les systèmes composés de deux sous-systèmes distincts, sujets à une mesure continue et dans lesquels le sous-système rapide a une dynamique Gaussienne et le sous-système lent est dépourvu de dynamique interne, une technique d'élimination adiabatique est présentée dans [ČVH15]. En utilisant le fait que la dynamique rapide Gaussienne peut être décrite seulement par ses moments statistiques d'ordre un et deux, les auteurs obtiennent une équation maîtresse stochastique pour le sous-système lent en calculant explicitement la trace partielle par rapport au sous-système rapide de la dynamique complète.

En revanche, le développement de techniques génériques d'élimination adiabatique est bien plus complexe et a attiré moins d'attention. Une généralisation aux systèmes quantiques ouverts du formalisme de Schrieffer-Wolff est développée dans [Kes12]. L'équation maîtresse de Lindblad est traitée comme un système linéaire standard et un changement de variables général est proposé dans le but d'exprimer la dynamique dans la forme normale

de Tikhonov. La dynamique réduite est exprimée sous forme de Lindblad, au second ordre, dans le cas des systèmes composites avec perturbations Hamiltoniennes. Les systèmes fortement dissipatifs possédant une variété d'états d'équilibre (decoherence-free space) sont étudiés dans [ZV14]. En utilisant la théorie des perturbations pour les opérateurs linéaires présentée dans [Kat66], les auteurs obtiennent la dynamique réduite au premier ordre sur l'espace sans décohérence causée par une faible perturbation. Des bornes sur l'erreur entre le modèle complet et réduit sont également données. Une approche similaire est utilisée dans [MGLG16] pour traiter les systèmes avec des états métastables, c'est-à-dire des états à grande durée de vie mais distincts des états d'équilibre. Leur spécificité est d'obtenir la dynamique réduite dans le régime métastable. Dans le cadre général du formalisme des modèles quantiques stochastiques introduit par Hudson et Parthasarathy, une méthode de calcul de la dynamique réduite lente est proposée dans [GvH07, BS08] puis généralisée aux opérateurs non bornés dans [BvHS08]. Il y est prouvé que la dynamique d'origine du système complet converge vers la dynamique réduite lente lorsque la vitesse du système rapide tend vers l'infini.

Contributions

Dans cette thèse, nous proposons une nouvelle approche géométrique d'élimination adiabatique pour n'importe quel système quantique ouvert possédant des échelles de temps différentes: une rapide et une lente. En utilisant cette approche géométrique, nous traitons la dynamique lente comme une perturbation de la dynamique rapide. Voici nos principales contributions au problème. Premièrement, nous utilisons un développement asymptotique permettant de choisir l'ordre d'approximation, pour la dynamique réduite et la caractérisation de la variété lente, en fonction de la séparation des échelles de temps entre la dynamique rapide et la perturbation lente. Ceci est motivé par le fait que dans certaines expériences quantiques récentes, visant grâce à l'ingénierie de réservoir la stabilisation indirecte de systèmes quantiques [LTP+15], une bonne connaissance de l'ordre de validité des approximations devient nécessaire: l'augmentation de la précision des expériences implique un besoin de calculer des modèles d'ordre élevé afin de décrire correctement la dynamique lente. Deuxièmement, contrairement à l'approche standard utilisant la théorie des perturbations, nous proposons une méthode préservant les propriétés structurelles des systèmes quantiques ouverts. Ce point est crucial afin de permettre une interprétation physique du modèle réduit. Nous imposons donc les contraintes suivantes pour ce problème de réduction de modèle :

- La dynamique réduite, paramétrisée par un opérateur densité réduit, est décrite par une équation maîtresse de Lindblad [BP06].

- La paramétrisation de la variété lente est explicitement donnée sous forme d'une application complètement positive et préservant la trace, aussi appelé Kraus map [Cho75].

La paramétrisation peut être vue comme une application de l'opérateur densité réduit vers l'opérateur densité du système complet. La forme de Kraus assure la préservation des propriétés de l'opérateur densité. De plus, le calcul de cette forme de Kraus permet par exemple d'estimer l'intrication résiduelle entre deux sous-systèmes, ce qui peut avoir d'importantes conséquences pratiques.

Une telle attention sur l'application liant le système complet et réduit est une nouveauté dans les techniques d'élimination adiabatique. La combinaison d'un développement asymptotique, avec à la fois une expression sous forme de Lindblad pour la dynamique et une application sous forme de Kraus, n'a jamais été abordée précédemment. Nous proposons des formules directement applicables, avec contrôle de l'ordre de l'approximation et une expression explicite de la façon dont le système réduit est intégré dans l'espace de Hilbert de l'ensemble. Enfin, nous soulignons que notre approche s'appuie sur le cadre de la théorie des variétés centres [Car81] et de la théorie géométrique des perturbations singulières [Fen79] pour obtenir des équations de récurrence entre les approximations à différents ordres. Ces équations de récurrence ouvrent ainsi une voie claire pour calculer les termes d'ordre supérieur à partir de ceux d'ordre inférieur. Dans ce manuscrit, nous obtenons des résultats généraux, concernant une large gamme de systèmes, pour calculer le modèle réduit jusqu'au second ordre et des résultats préliminaires sur le troisième ordre. La généralisation à d'autres cas et pour des ordres plus élevés nécessite des développements complémentaires importants, mathématiquement intéressants et non usuels pour aborder ce problème de réduction de modèle tout en préservant la structure non commutative de l'équation maîtresse de Lindblad.

Dans le chapitre 2, nous décrivons le cadre mathématique de cette thèse. Nous présentons des notions standard sur les systèmes quantiques et sur la théorie des perturbations singulières. Ensuite, nous introduisons la structure des systèmes quantiques ouverts de dimension fini composés de deux échelles de temps qui sera étudiée tout au long de ce manuscrit, ainsi que la formulation mathématique des contraintes imposées sur le modèle réduit afin de lui assurer une signification physique; à savoir une dynamique réduite sous forme de Lindblad et une paramétrisation de la variété lente sous une forme de Kraus. Pour résoudre ce problème, nous traitons la dynamique lente comme une perturbation. Nous effectuons un développement asymptotique en fonction des puissances du petit paramètre décrivant la différence des échelles de temps puis utilisons la théorie des variétés centres et la théorie géométrique des perturbations singulières. Nous obtenons ainsi des relations de récurrence liant le modèle complet et réduit qui doivent être résolues.

Le chapitre 3 présente la solution de ces relations de récurrences jusqu'au second ordre. Nous considérons un modèle quantique standard dans lequel les variables lentes et rapides correspondent à deux différents sous-systèmes quantiques interagissant. Les variétés lente et rapide sont donc factorisées sous forme d'un produit tensoriel et non pas d'un produit cartésien, qui est standard dans les systèmes dynamiques classiques. Dans ce cas, la dynamique réduite peut être interprétée comme la dynamique du sous-système quantique lent. Au premier ordre, nous obtenons des formules explicites pour calculer la dynamique réduite sous forme de Lindblad. Nous retrouvons immédiatement la dynamique Zeno pour n'importe quel type de couplage entre le système lent et le système rapide. Nous obtenons également une formule explicite pour la paramétrisation de la variété lente sous forme de Kraus. Elle montre que l'intrication entre le système lent et rapide apparait déjà à cet ordre; à notre connaissance c'est la première fois qu'une telle intrication au premier ordre est systématiquement montrée dans une technique d'élimination adiabatique. Ensuite, en se concentrant sur les deux formes usuelles de couplage, à savoir une interaction Hamiltonienne et une interaction en cascade, nous obtenons des formules explicites pour l'approximation au second ordre tout en assurant une dynamique réduite sous forme de Lindblad et une paramétrisation sous forme de Kraus. Les formules montrent clairement qu'une interaction Hamiltonienne avec un système rapide dissipatif implique de la décohérence au second ordre. Ces formules explicites nous permettent de plus d'établir plusieurs résultats structurels

généraux, sur le nombre de canaux de décohérence et sur la structure des opérateurs de décohérence associés.

Dans le chapitre 4, nous nous concentrons sur un système quantique standard : l'oscillateur harmonique quantique. Après une brève présentation d'un tel système, nous présentons plusieurs techniques permettant de calculer le modèle réduit lorsque le système rapide est un oscillateur harmonique amorti. Premièrement, nous montrons comment utiliser le point de vue d'Heisenberg afin d'effectuer les calculs pour des oscillateurs harmoniques quantiques linéaires. Ensuite, nous utilisons une formulation explicite du propagateur pour les oscillateurs non-linéaires à condition d'effectuer des hypothèses supplémentaires sur la dynamique rapide. Les systèmes considérés étant de dimension infinie, cela illustre également comment notre méthode, développée rigoureusement pour les systèmes de dimension finie dans les chapitres précédents, peut en principe être appliquée également aux systèmes de dimension infinie. Nous traitons plusieurs exemples, permettant d'apprécier la relative simplicité calculatoire de l'application de nos formules.

Dans le dernier chapitre 5, nous présentons une extension de l'approximation au troisième ordre pour les systèmes dont la dynamique correspond à celle décrite dans le chapitre 3 et en supposant que l'interaction Hamiltonienne ne contient qu'un seul terme sous forme de produit tensoriel. Nous obtenons une formule explicite pour calculer le troisième ordre, qui nous permet déjà de donner des résultats structurels sur le modèle réduit au troisième ordre: il n'y a pas de nouveau canal de décohérence au troisième ordre, seulement de légères modifications de ceux pré-existants. Nous considérons ensuite un type de système complètement différent, pas nécessairement composite, où la dynamique rapide fait converger le système vers un sous-espace de l'espace de Hilbert complet (decoherence-free space). De tels systèmes sont amenés à jouer un rôle important dans le traitement quantique de l'information [LCW98]. Pour une dynamique rapide générique, nous obtenons des formules explicites des opérateurs de Lindblad décrivant le développement au premier ordre. Dans le cas particulier d'une perturbation Hamiltonienne, nous retrouvons l'effet Zeno. De plus, pour une dynamique rapide décrite par un unique opérateur de décohérence et sujette à une perturbation Hamiltonienne, nous présentons des formules explicites pour calculer la paramétrisation de la variété lente sous forme de Kraus au premier ordre et la dynamique réduite au second ordre. Cela souligne que dans ce cas également, un effet Zeno au premier ordre est associé à une décohérence au second ordre.

Une conclusion et des perspectives sont discutées dans le chapitre 6

Note. Les travaux présentés dans cette thèse ont fait l'objet des publications suivantes :

- [ASR15] R. Azouit, A. Sarlette and P. Rouchon. Convergence and adiabatic elimination for a driven dissipative quantum harmonic oscillator. *In Proceeding of the 54th IEEE Conference on Decision and Control, CDC*, 2015.

- [ASR16a] R. Azouit, A. Sarlette and P. Rouchon. Adiabatic elimination for open quantum systems with effective Lindblad master equations. *In Proceeding of the 55th IEEE Conference on Decision and Control, CDC*, 2016.

- [ACSR17a] R. Azouit, F. Chittaro, A. Sarlette and P. Rouchon. Structure-preserving adiabatic elimination for open bipartite quantum systems. *In Proceeding of the IFAC World Congress,* 2017.

- [ACSR17b] R. Azouit, F. Chittaro, A. Sarlette and P. Rouchon. Towards generic Adiabatic elimination for composite open quantum systems. *IOP-Quantum Science*

and Technology, 2017.

Au début de ma thèse, j'ai également participé dans une moindre mesure à la publication suivante. J'ai été principalement impliqué dans la caractérisation du taux de convergence via une fonction Lyapunov:

- [ASR16b] R. Azouit, A. Sarlette and P. Rouchon. Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. *In ESAIM: Control, Optimisation and Calculus of Variations*, 2016.

J'ai également eu une participation mineure, dans l'analyse des données pour :

- [CJB+17] N. Cottet, S. Jezouin, L. Bretheau, P. Campagne-Ibarcq, Q. Ficheux, J. Anders, A. Auffèves, R. Azouit, P. Rouchon and B. Huard. Observing a quantum Maxwell demon at work. *Proceedings of the National Academy of Sciences*, 2017.

Notations

Nous présentons ici les différentes notations utilisées tout au long de ce manuscrit.

Les espaces de Hilbert sont désignés par $\mathcal{H}$, éventuellement avec un indice lorsque qu'il y en a plusieurs. $\mathcal{L}(\mathcal{H})$ dénote l'ensemble des opérateurs linéaires sur $\mathcal{H}$. Les opérateurs sur un espace de Hilbert sont désignés avec des lettres latines en gras, majuscules ou minuscules (par exemple $\boldsymbol{a}$ ou $\boldsymbol{X}$). Les super-operateurs, c'est-à-dire des fonctions de $\mathcal{L}(\mathcal{H})$ vers $\mathcal{L}(\mathcal{H})$ sont désignés avec des lettres latines capitales et calligraphiques (comme $\mathcal{D}$ ou $\mathcal{K}$). $\bullet^\dagger$ est le dual de $\bullet$. Pour n'importe quel opérateur $\boldsymbol{X}$, $\mathcal{D}_{\boldsymbol{X}}(\bullet) = \boldsymbol{X}\bullet\boldsymbol{X}^\dagger - \frac{1}{2}\left(\boldsymbol{X}^\dagger\boldsymbol{X}\bullet + \bullet\,\boldsymbol{X}^\dagger\boldsymbol{X}\right)$. $\mathbf{1}_{\mathcal{H}}$ est l'opérateur identité sur l'espace de Hilbert $\mathcal{H}$. Lorsque le contexte est clair, l'indice sera omis. Le commutateur entre deux opérateurs sur $\mathcal{H}$ est désigné par $[\boldsymbol{X}, \boldsymbol{Y}] = \boldsymbol{X}\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{X}$.

Chapter 1

Introduction

1.1 Presentation of the problem

As any quantum system interacts with an environment, the rigorous way to describe its evolution would be a Schrödinger equation including the environment. Studying this whole system is usually not possible due to the complexity of the environment. Therefore, using typical assumptions on the environment one can get rid of it by some Markov approximations and obtain a Lindblad master equation [BP06] governing only the system of interest. A similar situation arises naturally within a quantum system composed of several interacting subsystems: one may want to get the dynamics of a particular subsystem of interest by using some assumptions to get rid of the other ones. When the subsystem of interest has much slower dissipation rates (i.e. time scales) than the other ones and is weakly coupled to the other ones, such procedure is known as adiabatic elimination (corresponding to perturbation theory in the field of system-theory). This model reduction problem arises naturally within different fields of quantum physics for various reasons :

- It is closely related to reservoir engineering [PCZ96] where the goal is to design the dynamics of a target subsystem by properly choosing its interaction with the other subsystems. The interaction is engineered in such a way that the dynamics of the target subsystem is, after adiabatic elimination of the other subsystems, the desired one. Rigorously computing the reduced dynamics associated with the subsystem of interest is not straightforward emphasizing the need to develop systematic adiabatic elimination procedure. These procedures must preserve the quantum structure in order to ensure a physical interpretation of the reduced model.
- When connecting several quantum systems, the Hilbert space of the complete system is the tensor product of the spaces of the individual systems. Hence the dimensions are multiplied, quickly leading to an intractable system due to the number of variables. In order to get better physical insight on some particular phenomenon within the ensemble of systems, or simply in order to be able to simulate numerically such system, we need to be able to develop generic model reduction methods.

1.2 State of the art

Adiabatic elimination in the case of closed quantum systems, described by the Schrödinger equation is standard. In such case, the quantum evolution is unitary and regular perturbation theory is routinely applied ([Sak94]): it is well known how to follow the evolution of

the eigenvalues and eigenvectors of the Hamiltonian due to a small perturbation. The case of open quantum systems, described by a Lindblad master equation ([BP06]), is much more complicated and involves singular perturbation theory.

A lot of particular examples of open quantum systems have been successfully treated in the literature. In [BPM07], adiabatic elimination for lambda systems with an excited state and two ground states is investigated. For such systems, the authors clearly illustrate how some standard adiabatic elimination techniques from quantum optics may lead to ambiguous results and thus the need to use rigorous approach. They propose two adiabatic elimination techniques for these three dimension systems considering Hamiltonian perturbations. The first one is based on an explicit solution for the amplitude of the excited state; the second one uses the Green's function formalism to derive an expression of the propagator of the dynamics. Lambda systems where an excited state is decaying fast towards an arbitrary number of ground states is considered in [MR09]. From the Lindblad master equation of the complete system described by the density operator ρ, the authors derived a suitable change of variables $\rho \to (\rho_f, \rho_s)$, corresponding to the separation excited/ground states and leading to a dynamics in the Tikhonov normal form. Then, by applying the Tikhonov's theorem, they obtained the first order reduced dynamics for Hamiltonian and/or Lindblad perturbations. The extension to lambda systems with several excited states is considered in [RS12]. The authors have a similar approach by decomposing the Hilbert space into two subspaces, one for the ground states and one for the excited states. Using an asymptotic expansion in power of the small parameter, they derive an effective Lindblad master equation for the ground states up to second order. Different type of feedback (coherent and incoherent) between a system of interest and an ancilla (either a qubit or a cavity) are investigated in [WW00]. When the ancilla is on a fast time-scale, an adiabatic elimination technique similar to the standard approach of [Car93a] is performed to derive the first order reduced dynamics concerning the system of interest only and study the effect of the feedback. For bipartite systems under continuous measurement - where the fast subsystem has a Gaussian dynamics and the slow subsystem is assumed without internal dynamics - an adiabatic elimination technique is presented in [ČVH15]. Using the fact that the fast Gaussian dynamics can be described only by means of the first and second order statistical moments, they derived a quantum stochastic master equation for the slow subsystem by computing the partial trace with respect to the fast subsystem of the complete dynamics.

In contrast, the development of generic techniques for adiabatic elimination is much more complicated and have attract less attention. A generalisation to open quantum systems of the Schrieffer-Wolff formalism is developed in [Kes12]. The Lindblad master equation is treated as a standard linear system and a general change of variables is derived in order to express the dynamics in the Tikhonov normal form. The reduced dynamics is ensured to be in a Lindblad form for composite systems with Hamiltonian perturbations up to second order. Strongly dissipative systems admitting a decoherence-free space i.e. a manifold of steady states are investigated in [ZV14]. Using perturbation theory for linear operators from [Kat66], they derived the first order reduced dynamics on the decoherence-free space induced by a small perturbation. Error bounds between the complete and reduced system are given. A similar approach is used in [MGLG16] to tackle systems with metastable states i.e. long-lived states that are distinct from the steady states and derive a reduced dynamics in the metastable regime. The slow dynamics has been derived for the formalism of quantum stochastic models introduced by Hudson and Parthasarathy, first in [GvH07, BS08] then generalized for unbounded operators in [BvHS08]. The original

dynamics, of the full system, is proven to converge to the reduced slow dynamics as the speed of the fast dynamics tends to infinity.

1.3 Contributions

In this thesis, we propose a new geometric approach to perform adiabatic elimination for any open quantum system featuring different time-scales: a fast and a slow one. Using this geometric approach, we treat the slow dynamics as a perturbation of the fast one. Our main contributions are the following. First, we provide an asymptotic expansion that allows to choose the order of approximation, both for the reduced dynamics and for the characterization of the slow manifold, as a function of the time-scale separation between the fast dynamics and its perturbation. This is motivated by the fact that in some recent quantum experiments aiming via reservoir engineering at strong indirect stabilization of quantum systems [LTP+15], a good knowledge of the order of validity of such approximations is becoming necessary : the increase in the accuracy of experiments implies the need to compute higher order model in order to properly describe the slow dynamics. Second, in contrary to standard perturbation theory from system theory, we preserve the structural properties of open quantum systems. This is crutial in order to allow a physical interpretation of the reduced model. Therefore, we impose the following constraints on the model reduction problem :

- The reduced dynamics, parametrised by a reduced density operator follows a Lindblad master equation [BP06].
- The parametrisation of the slow manifold is explicitly given as a trace preserving completely positive map, also called Kraus map [Cho75].

The parametrisation may be seen as a mapping from the reduced density operator to the density operator of the entire system. The Kraus map form ensures the preservation of the density operator properties. Moreover, the computation of the Kraus map allows to estimate for instance the residual entanglement, between the two subsystems of a reservoir engineering setup, which may have important practical consequences.

Such care about the mapping between the complete and reduced systems is new in adiabatic elimination techniques. Combining asymptotic expansion with both completely positive map and Lindbladian formulation has never been addressed before. We provide formulas with direct applicability, control on the order of approximation, and explicit expression of how the reduced system is embedded in the full Hilbert space. Last, we emphasize that our approach builds on the framework of center manifold theory [Car81] and geometric singular perturbations theory [Fen79] to obtain recurrence relations between the approximations at different orders. These recurrence relations thus open a clear path to compute higher order terms from lower ones. In this manuscript, we derive general results, considering a large class of systems, for the reduced model up to second order and partial results for the third order. The extension to more general cases and arbitrary order requires further investigations and to develop non-usual and mathematically interesting tools in order to tackle the model reduction problem preserving the non-commutative structure of the Lindblad master equation.

In chapter 2, we begin by a mathematical presentation of the framework of this thesis. We present standard textbook material on quantum systems and singular perturbation theory. Then, we introduce the two-time scales structure of considered finite-dimension

open quantum, as well as the mathematical formulation of the constrains we impose on the reduced model in order to ensure a physical significance; namely a reduced dynamics in a Lindblad form and a parametrisation of the slow manifold in a Kraus map form. To solve this problem, we treat the slow dynamics as a perturbation. We perform an asymptotic expansion in terms of powers of the small parameter describing the time-scales and use center manifold theory and geometric singular perturbation theory. It yields recurrence relations between the complete and the reduced model that have to be solved.

The chapter 3 derives the solution of these recurrence relations up to second order. We consider a standard quantum model, where the slow and fast variables correspond to two different interacting quantum subsystems. The slow and fast manifolds are hence factored in tensor product form, rather than the Cartesian product which is standard in classical dynamical systems. We can then interpret the reduced dynamics as the dynamics of the slow quantum subsystem. At the first order, we derive explicit formulas to compute the reduced dynamics in a Lindblad form. We readily retrieve the so-called Zeno dynamics for any type of coupling between the fast and slow subsystems. We also derive explicit formulas for the parametrisation of the slow manifold in a Kraus map form. It shows that entanglement already appears at this order; to our knowledge, this is the first time that such systematic first order entanglement is shown in adiabatic elimination. Then, focusing on two standard forms of coupling, namely Hamiltonian interaction and cascade interaction, we derive formulas for the second order approximations while ensuring a Lindblad form for the reduced dynamics and a mapping in Kraus map form. The formulas show that the Hamiltonian interaction with a decohering subsystem leads to decoherence at the second order. In addition to the explicit formulas, it establishes several general structural result; on the number of decoherence channels and on the structure of the associated decoherence operators.

In chapter 4 we focus on a standard quantum system: the quantum harmonic oscillator. After the presentation of brief notion on such systems, we present several techniques allowing to compute the reduced model when the fast relaxing part corresponds to a damped harmonic oscillator. First, we show how using the Heisenberg picture allow to perform the computation on linear quantum harmonic oscillators. Second, we present and use an explicit formula of the propagator for non-linear harmonic oscillators provided some assumptions on the dynamics of the fast subsystem. As these systems are infinite-dimension systems, it illustrates how our method, rigorously developed in the previous chapter for finite dimension systems, can in principle, be applied to infinite dimension ones. We treat several examples, leading one to appreciate the computational simplicity of applying our formulas.

In the last chapter 5, we present an extension to the third order approximation of systems with dynamics corresponding to chapter 3 provided that the Hamiltonian interaction includes only one tensor-product term. We obtain a formula in order to compute the third order, which already allows to make structural result on the third order reduced model: at the third order, there is no new decoherence channel, only slight modification of the second order ones. We consider then a completely different type of system, not necessarily bipartite, where the fast Lindbladian makes the system converge to a decoherence-free subspace of the overall Hilbert space. Such systems are believed to play an important role in quantum information processing [LCW98]. For general fast dynamics satisfying this setting, we get explicit formulas for the Lindblad operators describing the first order expansion. In the particular case of a Hamiltonian perturbation, we retrieve the well known Zeno effect. Furthermore, for a fast Lindbladian described by a

single decoherence operator and subject to a Hamiltonian perturbation, we derive explicit formulas for the first-order effect on the location of the center manifold and for Lindblad operators describing the second order expansion of the dynamics. This allows to highlight that in this type of structure, a first-order Zeno effect is also associated to second-order decoherence.

Perspectives and conclusions are discussed in chapter 6.

Note. The works presented in this thesis have been the subject of the following publications :

- [ASR15] R. Azouit, A. Sarlette and P. Rouchon. Convergence and adiabatic elimination for a driven dissipative quantum harmonic oscillator. *In Proceeding of the 54th IEEE Conference on Decision and Control, CDC*, 2015.
- [ASR16a] R. Azouit, A. Sarlette and P. Rouchon. Adiabatic elimination for open quantum systems with effective Lindblad master equations. *In Proceeding of the 55th IEEE Conference on Decision and Control, CDC*, 2016.
- [ACSR17a] R. Azouit, F. Chittaro, A. Sarlette and P. Rouchon. Structure-preserving adiabatic elimination for open bipartite quantum systems. *In Proceeding of the IFAC World Congress,* 2017.
- [ACSR17b] R. Azouit, F. Chittaro, A. Sarlette and P. Rouchon. Towards generic Adiabatic elimination for composite open quantum systems. *IOP-Quantum Science and Technology,* 2017.

In the beginning of my thesis, I participated with a lesser contribution to the following publication. I was mainly involved in the characterisation of the convergence rate via a Lyapunov function:

- [ASR16b] R. Azouit, A. Sarlette and P. Rouchon. Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. *In ESAIM: Control, Optimisation and Calculus of Variations*, 2016.

I also had a minor participation, in the analysis of the data in :

- [CJB+17] N. Cottet, S. Jezouin, L. Bretheau, P. Campagne-Ibarcq, Q. Ficheux, J. Anders, A. Auffèves, R. Azouit, P. Rouchon and B. Huard. Observing a quantum Maxwell demon at work. *Proceedings of the National Academy of Sciences*, 2017.

1.4 Notations

We present here the different notations used through this manuscript.

Hilbert spaces are denoted by $\mathcal{H}$, eventually with a subscript when different ones are considered. $\mathcal{L}(\mathcal{H})$ denotes the set of linear operators on $\mathcal{H}$. Operators on Hilbert spaces are denoted with bold Latin letters either uppercase or lowercase (e.g. $\boldsymbol{a}$ or $\boldsymbol{X}$). Super-operators i.e. functions from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$ are denote with calligraphic capital Latin letters (e.g. $\mathcal{D}$ or $\mathcal{K}$). $\bullet^\dagger$ is the dual of $\bullet$. For any operator $\boldsymbol{X}$, $\mathcal{D}_{\boldsymbol{X}}(\bullet) = \boldsymbol{X} \bullet \boldsymbol{X}^\dagger - \frac{1}{2}\left(\boldsymbol{X}^\dagger \boldsymbol{X} \bullet + \bullet \boldsymbol{X}^\dagger \boldsymbol{X}\right)$. $\mathbf{1}_{\mathcal{H}}$ is the identity operator on the Hilbert space $\mathcal{H}$. When the context is clear we omit the subscript dependence. The commutator between two operators on $\mathcal{H}$ is denoted $[\boldsymbol{X}, \boldsymbol{Y}] = \boldsymbol{X}\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{X}$.

Chapter 2

Adiabatic elimination for open quantum systems

The goal of this chapter is to derive our method to perform adiabatic elimination on open quantum systems with two time-scales. We begin in Section 2.1 with some textbook notions on quantum systems. In Section 2.2, after an introduction on model reduction we present the two time-scales structure of quantum systems considered throughout this manuscript. Section 2.3 introduces some mathematical theory for model reduction, mainly geometric singular perturbation theory in the classical context of finite dimensional system theory. We illustrate then the main difficulties to perform adiabatic elimination (model reduction for systems with different time-scales) on open quantum systems. Readers familiar with quantum systems and singular perturbation theory may skip these sections as all crucial informations for the model reduction problem on open quantum systems are reminded in the beginning of Section 2.4. We present our new method, based on the mathematical tools previously introduced, to tackle such problem of adiabatic elimination for open quantum systems in Section 2.4 where the quantum structure is preserved (dynamics of Lindblad form and mapping of Kraus form).

2.1 Structure of quantum systems

We present in this section standard textbook material on quantum systems. Our goal is to introduce some notions of quantum physics used in this manuscript for readers not familiar with quantum mechanics. For a complete and detailed introduction to quantum physics see e.g. [BD02], [CTDL77]. See also [HR06] for a presentation including recent experiments and results, also [NC00], focusing on quantum information.

2.1.1 Closed quantum systems

An *isolated* or *closed* quantum system corresponds to a system which does not interact with its environment. The state of a closed quantum system, the *wave function*, is a vector of an Hilbert space $\mathcal{H}$ of finite or infinite dimension. It is usually denoted by the *ket* $|\psi(t)\rangle$ and may be seen as a column vector. Its adjoint, the *bra* $\langle\psi(t)|$ can be seen as a row vector. We adopt this notation introduced by Dirac as it simplifies some expressions and computations. For example, the scalar product between two states $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by the compact notation $\langle\psi_1|\,\psi_2\rangle$.

The notion of quantum state is closely related to probability. For each dimension of the Hilbert space $\mathcal{H}$ we can associate a quantum state of the physical system. Then, the components of the wave function can be seen as the *probability amplitude* for the system to be in the corresponding state. We use the term probability amplitude to emphasize that it may include a phase in the complex plane. This relationship with probability implies that for any wave function $|\psi\rangle$, we have

$$\langle\psi|\,\psi\rangle = 1 \tag{2.1}$$

The dynamics of a closed quantum system is governed by the Schrödinger equation :

$$\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar}\boldsymbol{H}(t)\,|\psi(t)\rangle \tag{2.2}$$

where $\boldsymbol{H}$ is a Hermitian operator on $\mathcal{H}$. $\hbar$ is the Planck constant, in the following we will consider $\hbar = 1$. One easily verifies that property (2.1) is conserved by an evolution through (2.2) using standard derivation rules:

$$\begin{aligned}\frac{d}{dt}\left(\langle\psi(t)|\,\psi(t)\rangle\right) &= \langle\psi(t)|\,\frac{d}{dt}(|\psi(t)\rangle) + \frac{d}{dt}(\langle\psi(t)|)\,|\psi(t)\rangle \\ &= 0\end{aligned}$$

2.1.2 The density operator

The wave function formalism is unfortunately not suitable to model every possible quantum state. In the presence of uncertainty or partial information on the system, we have to use the density operator formalism as illustrated in the example below.

Consider a two level system (or so called *qubit* for "quantum bit" with analogy to classical bits) on a Hilbert space $\mathcal{H}$ isomorphic to $\mathbb{C}^2$. This corresponds for example to the spin of an electron which can be either "up", denoted with the ket $|\uparrow\rangle$ or "down", denoted with $|\downarrow\rangle$. By using wave functions we can't describe an electron which is up or down with a classical probability of 1/2: the ket $|\uparrow + \downarrow\rangle = |\uparrow\rangle + |\downarrow\rangle$ corresponds to a quantum superposition of these states and not a classical uncertainty.

Generally speaking, a quantum system may be in a state $|\psi_1\rangle$ with some probability p_1, in a state $|\psi_2\rangle$ with some probability p_2 etc... Such states are very common in quantum experiments for example when one wants to design several times the same state. Due to physical imperfections one will never obtain rigorously the exact same state but rather a probability distribution of states. As it can't be described by a single ket, we need to introduce the formalism of density operators.

The density operator (or density matrix) is a non-negative trace-class operator on a Hilbert space $\mathcal{H}$ (see e.g. [Tar08]) . For a statistical mixture of states $|\psi_1\rangle, |\psi_2\rangle, \dots$ associated with the probabilities $p_1, p_2, \dots$ it is defined as such :

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$

where $\sum_i p_i = 1$. When ρ is of rank one, corresponding to a state described by a single ket, the state is said to be **pure**. Otherwise, it is a **mixted state**. Any quantum state can be described by an appropriate density operator. It has the following properties :

- Hermiticity : $\rho = \rho^\dagger$
- Positivity : $\rho \geq 0$
- Normalized : $\mathrm{Tr}\,(\rho) = 1$

Moreover, it verifies $\mathrm{Tr}\,(\rho^2) \leq 1$, with equality if and only if ρ is a pure state. We denote by $\mathcal{D}$ the closed convex set of density operators on the Hilbert space $\mathcal{H}$ equipped with the nuclear norm (see [Tar08]): $||\boldsymbol{A}||_{\mathrm{nucl}} = \mathrm{Tr}\left(\sqrt{\boldsymbol{A}^\dagger \boldsymbol{A}}\right)$.

Using density operators, the Schrödinger equation (2.2) becomes :

$$\begin{aligned}
\frac{d}{dt}\rho &= \frac{d}{dt}\sum_i p_i \, |\psi_i\rangle \, \langle\psi_i| \\
&= \sum_i p_i \left(\frac{d}{dt}\left(|\psi_i\rangle\right) \langle\psi_i| + |\psi_i\rangle \, \frac{d}{dt}\left(\langle\psi_i|\right) \right) \\
&= -i \sum_i p_i \left(\boldsymbol{H}(t) \, |\psi_i\rangle \, \langle\psi_i| - |\psi_i\rangle \, \langle\psi_i| \, \boldsymbol{H}(t) \right) \\
&= -i[\boldsymbol{H}(t), \rho]
\end{aligned}$$

Resulting in the Liouville/Von Newmann master equation:

$$\frac{d}{dt}\rho = -i[\boldsymbol{H}(t), \rho] \tag{2.3}$$

For such master equation, we can define an **evolution operator** or **propagator**. Consider a density operator $\rho(t_0)$ at time t_0 evolving according to (2.3) up to time t_1. Then the propagator $U(t_0, t_1)$ is defined by :

$$\rho(t_1) = \boldsymbol{U}(t_0, t_1)\rho(t_0)\boldsymbol{U}(t_0, t_1)^\dagger \tag{2.4}$$

formally given by the integral equation

$$\boldsymbol{U}(t_1, t_0) = \mathbf{1} - i \int_{t_0}^{t_1} \boldsymbol{H}(t)\boldsymbol{U}(t, t_0)dt$$

This operator has many important properties [CTDL77]. Among them, for any time t_2, t_1, t_0 we have :

$$\boldsymbol{U}(t_2, t_1)\boldsymbol{U}(t_1, t_0) = \boldsymbol{U}(t_2, t_0)$$

In particular for $t_2 = t_0$, $\boldsymbol{U}(t_0, t_1)\boldsymbol{U}(t_1, t_0) = \mathbf{1}$ which implies the reversibility of the evolution. This operator is unitary: $\boldsymbol{U}^{-1}(t_1, t_0) = \boldsymbol{U}^{\dagger}(t_1, t_0)$. In the particular case of time-independent Hamiltonian, it is readily given by $\boldsymbol{U}(t_1, t_0) = e^{-i\boldsymbol{H}(t_1 - t_0)}$. This reversibility of the evolution highlights the fact that Hamiltonian evolutions associated with closed quantum systems are processes without loss of information.

We present in the next section how to model systems with loss of information known as open quantum systems.

2.1.3 The Lindblad master equation

Until now, we considered perfectly closed quantum systems without any interaction with the environment. In quantum experiments, this is never true. Due to inevitable coupling with the environment, a part of the information of the system of interest leaks into the environment, resulting in a loss of information. Moreover, to perform any measurement on the system, one needs to interact with it and therefore we cannot consider the system as closed. One possible way to circumvent this problem would be to consider not only the evolution of the system of interest but the evolution of the system and its environment. As the ensemble {system+environment} is closed, its dynamics is given by the Schrödinger equation (2.2) on a Hilbert space $\mathcal{H} = \mathcal{H}_{syst} \otimes \mathcal{H}_{environment}$ [1]. As the environment is usually a complex and large dimensional system, studying this whole dynamics is impossible in practice.

It is possible, with some commonly verified assumptions, to derive a master equation governing the system of interest only while taking into consideration its coupling with the environment. The system will therefore be considered as an **open** quantum system [BP06].

Using the Born-Markov approximation (corresponding to the assumption that any information leaked into the environment is lost and never comes back into the system), the dynamics of an open quantum system is given by the Lindblad master equation first introduced in [Lin76] and independently in [GKS76] (see e.g. [CTDRG92], [Car93a] for its rigorous derivation from the Schrödinger equation or [HR06] for a more deductive approach focusing on the physical interpretation):

$$\frac{d}{dt}\rho = -i\left[\boldsymbol{H}, \rho\right] + \sum_k \boldsymbol{L}_k \rho \boldsymbol{L}_k^{\dagger} - \frac{1}{2}\left(\boldsymbol{L}_k^{\dagger}\boldsymbol{L}_k\rho + \rho\boldsymbol{L}_k^{\dagger}\boldsymbol{L}_k\right) \tag{2.5}$$

where $\boldsymbol{H}$ is a Hermitian operator on $\mathcal{H}$, $\boldsymbol{L}_k$ are some operators on $\mathcal{H}$, not necessarily hermitian and they might be time-dependant, ρ is a density operator. The operators $\boldsymbol{L}_k$ correspond to the different dissipation channels through which the information in the system leaks into the environment. They are named **dissipation** or **decoherence operators**.

If the initial condition $\rho(0)$ belongs to $\mathcal{D}$, then the solution $\rho(t)$ of (2.5) remains in $\mathcal{D}$ and is defined for all $t \geq 0$.

[1] See section 2.1.5 for more details on this structure

It is convenient to introduce the Lindblad super-operator: for any operator $\boldsymbol{X}$ on $\mathcal{H}$ we define :

$$\mathcal{D}_{\boldsymbol{X}}(\bullet) = \boldsymbol{X} \bullet \boldsymbol{X}^\dagger - \frac{1}{2}\left(\boldsymbol{X}^\dagger \boldsymbol{X} \bullet + \bullet \boldsymbol{X}^\dagger \boldsymbol{X}\right)$$

thus leading to a compact notation for the Lindblad master equation (2.5) :

$$\frac{d}{dt}\rho = -i\left[\boldsymbol{H}, \rho\right] + \sum_k \mathcal{D}_{\boldsymbol{L}_k}(\rho)$$

Similarly to (2.4) for closed quantum systems, we are interested in the characterisation of the evolution of the density operator from a time t_0 to t_1. In the case of open quantum systems, it is not given by an unitary operator but rather by a set of operators $\{\boldsymbol{M}_k\}$ on $\mathcal{H}$:

$$\rho(t_1) = \sum_k \boldsymbol{M}_k(t_1, t_0)\rho(t_0)\boldsymbol{M}_k^\dagger(t_1, t_0)$$

and verifying the **completeness relation**

$$\sum_k \boldsymbol{M}_k^\dagger \boldsymbol{M}_k = \mathbf{1}$$

This structure is in fact much more general as any linear quantum process transforming a density operator into another one (through evolution, measurement, coupling to another system, etc...) can be expressed using this formalism [NC00]. More precisely, for any density operator $\rho \in \mathcal{D}$ and for any quantum transformation $\mathcal{E} : \mathcal{D} \to \mathcal{D}$, there exists a family of operator $\{\boldsymbol{M}_k\}$ verifying the completeness relation such that

$$\mathcal{E}(\rho) = \sum_k \boldsymbol{M}_k \rho \boldsymbol{M}_k^\dagger$$

This elegant representation is known as the **Kraus sum representation** or **Kraus map** [Cho75]. The operators $\boldsymbol{M}_k$ are known as **Kraus operators**. It is a completely positive trace preserving mapping and therefore preserves the properties of the density operator [Cho75].

2.1.4 Schrödinger versus Heisenberg picture

Measurement and observables

We saw previously that the introduction of the density operator formalism is essential in order to be able to completely describe any quantum system. If the system is closed, its dynamics is governed by the Liouville/Von Newmann equation, while if the system is open it is governed by the Lindblad master equation. We first make more precise the meaning of "completely describe the system" or "contains all the information needed on the systems" for the density operator.

As the nature of quantum systems is probabilistic, the perfect knowledge of the density operator does not mean being able to predict the outcome of any measurement. Instead, it means that the density operator contains all the information needed to predict *the statistics* of the outcome of an observation on the system. Consider an observable $\boldsymbol{O}_A$, an operator on $\mathcal{H}$, corresponding to some measurement A on the system. Assume that the

system is described by the density operator $\rho \in \mathcal{D}$. Then, the probability of finding the eigenvalue ϵ_j of $\boldsymbol{O}_A$ is given by

$$\pi_j = \mathrm{Tr}\left(\rho \boldsymbol{P}_j^A\right) \tag{2.6}$$

where $\boldsymbol{P}_j^A$ is the orthogonal projector on the corresponding eigenspace. The expectation value of $\boldsymbol{O}_A$ is given by

$$\langle \boldsymbol{O}_A \rangle_\rho = \sum_j \pi_j \epsilon_j = \mathrm{Tr}\left(\boldsymbol{O}_A \rho\right) \tag{2.7}$$

We can therefore explain the properties of the density operator in this context to give them more insight: the hermiticity and positivity correspond to the fact that a probability must always be positive (2.6). The normalisation ensures that all probabilities sum to one.

The Heisenberg picture

Until now, we have implicitly considered the evolution of a quantum system in the **Schrödinger picture**. This may be the most "natural" point of view because, in this picture, the state evolves through time while the observables are time independent, as usually considered in classical systems. In the dual approach, known as **Heisenberg picture**, the observables are evolving through some dynamics while the state remains unchanged:

Consider a density operator $\rho \in \mathcal{D}$ with dynamics given by a Lindblad master equation

$$\frac{d}{dt}\rho = -i\left[\boldsymbol{H}, \rho\right] + \sum_k \boldsymbol{L}_k \rho \boldsymbol{L}_k^\dagger - \frac{1}{2}\left(\boldsymbol{L}_k^\dagger \boldsymbol{L}_k \rho + \rho \boldsymbol{L}_k^\dagger \boldsymbol{L}_k\right)$$

For some time-independent operators $\boldsymbol{H}, \boldsymbol{L}_k$ on $\mathcal{H}$. We denote by $\rho(t)$ its solution at time t with initial condition $\rho(0) = \rho_0$. We saw on the previous section that the knowledge of $\rho(t)$ is sufficient to predict the expected value of the measurement of any observable $\boldsymbol{O}_A$: $\langle \boldsymbol{O}_A \rangle_t = \mathrm{Tr}\left(\boldsymbol{O}_A \rho(t)\right)$. Now we use the Kraus map representation $\rho(t) = \sum_k \boldsymbol{M}_k(t)\rho(0)\boldsymbol{M}_k^\dagger(t)$ for some operators $\boldsymbol{M}_k$ on $\mathcal{H}$. Therefore we get

$$\langle \boldsymbol{O}_A \rangle_t = \sum_k \mathrm{Tr}\left(\boldsymbol{O}_A \boldsymbol{M}_k(t)\rho(0)\boldsymbol{M}_k^\dagger(t)\right)$$

Then using the cyclic property of the trace,

$$\langle \boldsymbol{O}_A \rangle_t = \sum_k \mathrm{Tr}\left(\boldsymbol{M}_k^\dagger(t)\boldsymbol{O}_A \boldsymbol{M}_k(t)\rho(0)\right)$$

With this reformulation, one may see the operator $\sum_k \boldsymbol{M}_k^\dagger(t)\boldsymbol{O}_A\boldsymbol{M}_k(t)$ as the time evolution of the observable operator $\boldsymbol{O}_A$ while the density operator is given at any time by $\rho(0)$. This is the **Heisenberg picture**. By a small abuse of notation we denote by $\boldsymbol{O}_A(t)$ this operator. The dynamics of $\boldsymbol{O}_A(t)$ is given by the **adjoint** Lindblad master equation :

$$\frac{d}{dt}\boldsymbol{O}_A(t) = i\left[\boldsymbol{H}, \boldsymbol{O}_A(t)\right] + \sum_k \boldsymbol{L}_k^\dagger \boldsymbol{O}_A(t) \boldsymbol{L}_k - \frac{1}{2}\left(\boldsymbol{L}_k^\dagger \boldsymbol{L}_k \boldsymbol{O}_A(t) + \boldsymbol{O}_A(t)\boldsymbol{L}_k^\dagger \boldsymbol{L}_k\right)$$

with $\boldsymbol{O}_A(0) = \boldsymbol{O}_A$. Note that in the particular case $\boldsymbol{O}_A(0) = \mathbf{1}$ then we directly get $\boldsymbol{O}_A(t) = \boldsymbol{O}_A(0) = \mathbf{1}$ corresponding to the fact that the trace of the density operator is preserved through any evolution.

This point of view is particularly useful when one wants to compute the expectation values of some particular operators: while solving the Lindblad master equation for ρ - thus obtaining every possible information on the system - might be tedious, the adjoint Lindblad master equation only gives information about a particular observable but may be easier to solve. See chapter 4 for several applications.

2.1.5 Entanglement

Consider two distinct quantum systems A and B on some Hilbert space respectively $\mathcal{H}_A$ and $\mathcal{H}_B$. Then, the underlying Hilbert space $\mathcal{H}$ of the composite $\{A, B\}$ system is given by the tensor product: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. This important quantum feature has essential implications. Contrary to Cartesian product where dimensions add, we have $\dim(\mathcal{H}) = \dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B)$, expressing the quick growth in dimensions when interconnecting several quantum devices.

Assume that the density operator ρ of the composite system $\{A, B\}$ is given by :

$$\rho = \rho_A \otimes \rho_B \tag{2.8}$$

for some density operator ρ_A on $\mathcal{H}_A$ and ρ_B on $\mathcal{H}_B$. It is then possible from ρ to compute ρ_A and ρ_B from corresponding respectively to the density operator of the systems A and B by using the **partial trace** (with respect to A or B):

$$\begin{aligned} \mathrm{Tr}_A(\rho) &= \rho_B \\ \mathrm{Tr}_B(\rho) &= \rho_A \end{aligned}$$

Note that, in this case, $\rho = \mathrm{Tr}_B(\rho) \otimes \mathrm{Tr}_A(\rho)$.

The partial trace operation has the following properties. Denote by $\boldsymbol{O}_A$ ($\boldsymbol{O}_B$) any operator on $\mathcal{H}_A$ ($\mathcal{H}_B$) and $\boldsymbol{O}_{AB}$ any operator on $\mathcal{H}$. Then :

$$\begin{aligned} \mathrm{Tr}_B(\boldsymbol{O}_A \otimes \boldsymbol{O}_B) &= \mathrm{Tr}(\boldsymbol{O}_B)\,\boldsymbol{O}_A \\ \mathrm{Tr}_B((\boldsymbol{O}_A \otimes \mathbf{1}_{\mathcal{H}_B})\boldsymbol{O}_{AB}) &= \boldsymbol{O}_A \cdot \mathrm{Tr}_B(\boldsymbol{O}_{AB}) \end{aligned}$$

The density operator of the composite system $\{A + B\}$ is said to be separable if it can be expressed as

$$\rho = \sum_k p_k\, \rho_{kA} \otimes \rho_{kB}$$

for some set of density operators $\{\rho_{kA}\}$ on $\mathcal{H}_A$ and similarly for B, $\forall k\ p_k \geq 0$ and they verify $\sum_k p_k = 1$. If a state is **not separable** then it said to be an **entangled states**. Note that in this case $\rho \neq \mathrm{Tr}_B(\rho)\,\mathrm{Tr}_A(\rho)$: the state of the complete system is different from the state of each subsystem taken individually. An entangled state signifies the existence of quantum correlations between both subsystems is considered to be the most non-classical manifestations of quantum physics. The quantum correlations between two quantum systems are the core of advantage of quantum systems over classical systems, allowing e.g. quantum teleportation [BBC+93]. For this reason, entangled states have attracted a lot attention. However, their characterisation and quantification remain a difficult task. For more details on entanglement see e.g. [HHHH09] for a complete review, or [BT17] discussing more general type of quantum correlations.

2.2 Model reduction in considered quantum systems

The goal of this section is to present the structure of considered quantum systems for model reduction. We begin by a general presentation of the notion of model reduction, emphasising why it is necessary to develop model reduction techniques in 2.2.1. The considered systems with different time-scales are introduced in 2.2.2 leading to the notion of adiabatic elimination. They are mathematically described in 2.2.3. In this manuscript, we will rigorously consider only finite-dimensional systems, for which definitions and theorems of section 2.3 are rigorously stated. In chapter 4 we show how we can also treat infinite dimension systems. We emphasize that this treatment is solely formal, in order to be rigorous, one need to take care of problems related to the infinite dimension, which is not the aim of this manuscript.

2.2.1 Notion of model reduction

Mathematical model of a system

In order to study the properties of a physical system the first task is to derive a mathematical model that describes the behaviour (*well enough*) of the system of interest. This system may come from a large variety of scientific domains (e.g. physics, biology, control theory, etc.) and the modelling could be a difficult task. It usually comes from basic physical principles and/or experiments. Commonly, the dynamics of the system is described by a set of differential equations [Kha92]. The number of first order differential equations used in the model is referred as the **order** of the model: it represents the "complexity" of the model. If the studied system includes some complex phenomenons, a high order model will be needed in order to correctly describe these complicated dynamics. This order may even be infinite for systems described by partial differential equations (corresponding to an infinity of ordinary differential equations) appearing in a large variety of fields.

The need of a reduced model

From the preceding paragraph, it emerges that it might be interesting to consider a very high order model in order to accurately describe a physical system. However, in practice, this is not always the case and frequently we are interested in a low order model describing the dynamics of the system. There are numerous reasons to this requirement, we will list several important ones:

- Studying the evolution of a high order system may be difficult, and an analytic solution could be hard to find. A reduced model therefore simplifies this demand.
- Some particular phenomenon or behaviour of the system may be clouded by a complex model with high order. A reduced model may thus highlight a dominant dynamics and give better physical insight on a specific process at work within the system of interest.
- The exact modelling of the system may imply to take into account several undesired systems coupled with the interesting one. A lower order model can approximate the dynamics of the system of interest only.
- For numerical simulations or any numerical procedure, a high order system is computationally expensive in resources and/or in time.

The idea of reduced model

This shows that we have to develop methods to compute a **reduced order model** from a complex high order model. As the reduced order model will be an approximation of the complete model, we will rely on some assumptions on the complete model in order to make this approximation valid. We emphasize that in practice we start from the assumptions and compute a reduced model accordingly. These assumptions are usually suggested by some physical properties or expected behaviour of the system.

Nevertheless, after the model reduction, some degree of details will inevitably be lost resulting in some errors between the dynamics of the reduced model and the exact dynamics. A key element of model reduction is then to find a way to characterise this error in order to track the amount of difference we can expect between the two models.

2.2.2 Time-scales separation

Using different time-scales

For all the reasons presented in the last paragraph, we want to compute a rigorous reduced model for quantum systems governed by Lindblad dynamics (2.5). We present here the type of assumption considered for the complete quantum system throughout this manuscript.

A first example of reduced model was presented in section 2.1 to derive the Lindblad master equation. From a complete high order model consisting of a system of interest coupled with an environment, the use of the Born-Markov approximation allows us to get rid of the environment and therefore get a reduced dynamics for the system of interest only. An analogous situation emerges naturally *within* a quantum system: we may be interested in the evolution of some part of the variables describing the system while the other ones are less relevant.

In typical systems, there is a time-scale separation between the interesting variables and the other ones : the variables of interest evolve on slow time-scale while the other ones evolve on a fast time-scale. In this case, one can derive a reduced model with guaranteed approximations, by eliminating the fast variables with so-called **adiabatic elimination** techniques. We essentially treat the slow dynamics of interest as a small perturbation to the fast dynamics, and solve for the effect of this perturbation. This is the assumption used throughout this manuscript to perform the model reduction. We will often refer to **adiabatic elimination** as it is the standard vocabulary in quantum physics for model reduction in the presence of different time-scales. We present more precisely the model of two time-scales quantum systems in the next section 2.2.3. The mathematical ideas behind methods using different time-scales to compute a reduced model are illustrated in 2.3.1.

Reservoir engineering

This two time-scales structure is common in quantum experiments where a system of interest is often coupled to a low quality (fast dissipative) harmonic oscillator that acts as a filter. Another important need for generic adiabatic elimination methods is in the upcoming field of reservoir engineering [SBRR11]. It comes from the non-intuitive idea from [PCZ96] that dissipation may help to stabilise a desired state (in analogy with Watt governor for classical systems) but also to preserve the quantum properties in a system.

The goal of reservoir engineering is to design the dynamics of a target subsystem by properly choosing its interaction with the other subsystems. The interaction is engineered

in such a way that the dynamics of the target subsystem is, after adiabatic elimination of the other subsystems, the desired one. It is therefore necessary to develop methods to rigorously compute the reduced dynamics associated with the subsystem of interest in the presence of different time-scales.

2.2.3 Considered quantum structure

We consider an open quantum system whose dynamics is governed by a Lindblad master equation. We introduce a small parameter ϵ in the dynamics leading to a separation between fast and slow variables. This small parameter will be treated as a perturbation of the initial ($\epsilon = 0$) system called the **unperturbed** system. The dynamics of the unperturbed system is assumed to be stable and converging towards a steady state.

As stated in the introduction of this section, we restrict ourself to finite dimension systems. This assumption is necessary in order to be able to rigorously use the mathematical tools presented in 2.4. Whereas several standard quantum systems (such as the quantum harmonic oscillator, see 4) are infinite dimension systems, they can be approximated by finite dimension ones by truncating the higher dimensions. This corresponds to neglecting high energy states which in practice are not populated. We refer to [ACS15], for a rigorous study of the effect of the truncation and the derivation of error bounds on the finite-dimensional approximation.

We will come back on this finite-dimension assumption in chapter 4 to show how our method may also leads to consider infinite dimension systems.

Formal description

Denote by $\mathcal{H}$ a Hilbert space of finite dimension, by $\mathcal{D}$ the compact convex set of density operators ρ on $\mathcal{H}$ (ρ is Hermitian, nonnegative and trace one). We consider a two time-scales dynamics on $\mathcal{D}$ described by the master differential equation

$$\frac{d}{dt}\rho = \mathcal{L}_0(\rho) + \epsilon\mathcal{L}_1(\rho) \tag{2.9}$$

where ϵ is a small positive parameter ($0 < \epsilon \ll 1$) giving rise to the two time-scales. The linear super-operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are of Lindbladian forms. That is, there exist two finite families of operators on $\mathcal{H}$, denoted by $(\boldsymbol{L}_{0,\nu})$ and $(\boldsymbol{L}_{1,\nu})$, and two Hermitian operators $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ (called Hamiltonians) such that, for $r = 0, 1$, we have

$$\mathcal{L}_r(\rho) = -i[\boldsymbol{H}_r, \rho] + \sum_\nu \boldsymbol{L}_{r,\nu}\rho\boldsymbol{L}_{r,\nu}^\dagger - \frac{1}{2}\left(\boldsymbol{L}_{r,\nu}^\dagger\boldsymbol{L}_{r,\nu}\rho + \rho\boldsymbol{L}_{r,\nu}^\dagger\boldsymbol{L}_{r,\nu}\right). \tag{2.10}$$

Note that all these operators are assumed time-independent. We assume that, for $\epsilon = 0$, the unperturbed master equation $\frac{d}{dt}\rho = \mathcal{L}_0(\rho)$ converges to a stationary regime (which may depend of the initial condition). More precisely, we assume that the unperturbed master equation admits a set of stationary operators coinciding with the Ω-limit set of its trajectories. We refer to [BNT08, BN08] for detailed informations on the existence and characterization of stationary states for Lindblad master equations. We denote by

$$\mathcal{D}_0 = \left\{\rho \in \mathcal{D} \;\middle|\; \mathcal{L}_0(\rho) = 0\right\}$$

the compact and convex set of stationary operators. We thus assume that, for all $\rho_0 \in \mathcal{D}$, the solution of $\dot{\rho} = \mathcal{L}_0(\rho)$ with $\rho(0) = \rho_0$ converges for t tending to $+\infty$ towards an element

of $\mathcal{D}_0$ denoted by $\overline{\mathcal{R}}(\rho_0)$ with

$$\overline{\mathcal{R}}(\rho_0) = \lim_{t\to+\infty} \rho(t) = \lim_{t\to+\infty} e^{t\mathcal{L}_0}\rho_0 \tag{2.11}$$

where $e^{t\mathcal{L}_0}$ is the propagator associated with $\mathcal{L}_0$ i.e. $e^{t\mathcal{L}_0}\rho_0$ is the solution at time t of the differential equation (2.9) when $\epsilon = 0$ for $\rho(0) = \rho_0$. In particular, it implies that all the eigenvalues of $\mathcal{L}_0$ are nonpositive. Since, for any $t \geq 0$, the propagator $e^{t\mathcal{L}_0}$ is a completely positive linear map [NC00, Chap.8], $\overline{\mathcal{R}}$ is also a completely positive map. By Choi's theorem [Cho75] there exists a finite set of Kraus operators on $\mathcal{H}$ denoted by $(\boldsymbol{M}_\mu)$ such that

$$\overline{\mathcal{R}}(\rho_0) = \sum_\mu \boldsymbol{M}_\mu \rho_0 \boldsymbol{M}_\mu^\dagger$$

with $\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \mathbf{1}$. We thus assume that

$$\mathcal{D}_0 = \left\{ \overline{\mathcal{R}}(\rho) \;\middle|\; \rho \in \mathcal{D} \right\} \text{ and } \forall \rho \in \mathcal{D}_0,\ \overline{\mathcal{R}}(\rho) = \rho.$$

This generic structure of quantum systems will be refined in chapter 3 to specifically address composite systems.

2.3 Model reduction problem

We present in this section the mathematical tools used in order to solve the model reduction problem for finite-dimension open quantum systems. In 2.3.1 we make more precise the notion of reduced model. This allows in 2.3.3 to specifically expose some constraints needed to ensure a physical relevance of the reduced model. Standard model reduction methods do not ensure the respect of these constraints, therefore, we present in 2.3.2 the results from geometric singular perturbation theory used to tackle this problem. We use the framework of system theory to present these methods, see 2.4 for their adaptation to quantum systems.

There exists a large variety of literature on model reduction and singular perturbation theory for finite-dimension systems. See e.g. [KKO99, chap. 9] on singular perturbation theory. Tikhonov's theorem was initially introduced in [Tik52], we refer to [Klo83] for a survey. An extension of this theorem for a class of infinite-dimension systems can be found in [TPG15]. Geometric singular perturbation theory was introduced by Fenichel [Fen79], see [Jon95] or [Kap99] for detailed surveys. We follow an approach inspired from [DR96].

2.3.1 Mathematical definition of reduced model

Consider a dynamical system of finite dimension whose smooth dynamics is given by :

$$\frac{d}{dt}x = v(x) \tag{2.12}$$

where $x \in \mathbb{R}^n$ and v is a regular function of x.

In order to define the notion of reduced model for the system (2.12) we introduce two definitions :

Definition 1. A smooth manifold $\mathbf{\Sigma}$ is said to be a **locally invariant manifold** for (2.12) (or with respect to the vector field $v(x)$) if there exists $T > 0$ such that for any $x_0 \in \mathbf{\Sigma}$, the solution $x(t)$ of (2.12) initialised with $x(0) = x_0$ stays in $\mathbf{\Sigma}$ for all $t \in (-T, T)$. If t can be chosen arbitrarily, then $\mathbf{\Sigma}$ is said to be an **invariant manifold**.

Definition 2. An invariant manifold $\mathbf{\Sigma}$ is called **locally attractive** if it has a neighbourhood S such that for any $x_0 \in S$, the trajectory $t \to x(t)$ of (2.12) initialised with $x(0) = x_0$ is defined for all $t > 0$ and tends towards $\mathbf{\Sigma}$ when t goes to infinity. If $S = \mathbb{R}^n$ then $\mathbf{\Sigma}$ is an **attractive manifold**.

Then assume that there exists an attractive invariant manifold $\mathbf{\Sigma}$ for (2.12). By definition, all trajectories $x(t)$ will converge and then stay on $\mathbf{\Sigma}$. This is schematically illustrated on figure 2.1 taken from [DR96]. Therefore, we will approximate trajectories of the complete system with trajectories on $\mathbf{\Sigma}$. This approach leads us to the following definition of a reduced model :

Definition 3. Assume the existence of an attractive invariant manifold $\mathbf{\Sigma}$ associated with the system whose dynamics is given by (2.12). Denote by $n_{\mathbf{\Sigma}} < n$ the dimension of this manifold. Then the **reduced dynamics** of (2.12) corresponds to the restriction of the complete dynamics to the attractive invariant manifold. This reduced dynamics will be given by a set of $n_{\mathbf{\Sigma}} < n$ first order differential equations.

The main problem is therefore to rigorously characterise and define $\mathbf{\Sigma}$ which is unknown in practice. The goal of model reduction techniques is thus twofold :

- compute the reduced dynamics on $\mathbf{\Sigma}$.

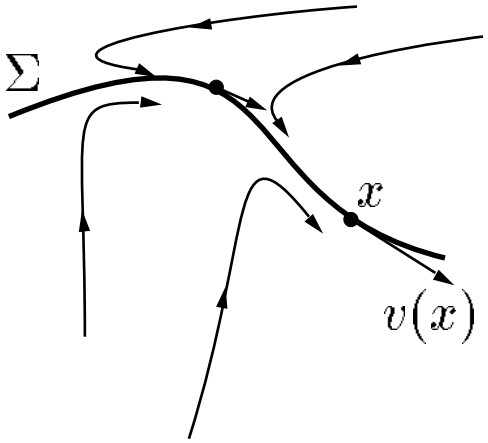

Figure 2.1: Representation of the attractive invariant manifold $\mathbf{\Sigma}$ for the dynamics $\dot{x} = v(x)$.

- derive the parametrisation of $\mathbf{\Sigma}$ in $\mathbb{R}^n$.

These objectives are difficult to achieve as only the vector field $v(x)$ is known. Finding the exact expression is generally as difficult as solving the initial complete problem (2.12). We will therefore look for approximations of them.

2.3.2 Singular perturbation theory

We present in this section different ways to approach the problem of model reduction. They are based on the two time-scales structure and known as singular perturbation theory. We begin with the first results from Tikhonov [Tik52] relying on a particular form for the complete dynamics and then introduce the geometric singular perturbation theory from Fenichel [Fen79] to deal with more general structures.

The need to use such theories for the problem of model reduction in open quantum systems is discussed in 2.3.3.

Tikhonov's theorem

We now focus more precisely on autonomous dynamical systems described by two time-scales and having an invariant attractive manifold. We introduce a small parameter $0 < \epsilon \ll 1$ describing the two time-scales. We consider a system described in the so-called *Tikhonov normal form* :

$$\begin{aligned} \frac{d}{dt}x_s &= \epsilon f(x_s, x_f, \epsilon) \quad x_s(0) = x_s^0 \\ \frac{d}{dt}x_f &= g(x_s, x_f, \epsilon) \quad x_f(0) = x_f^0 \end{aligned} \tag{2.13}$$

where $x_s \in \mathbb{R}^n$, $x_f \in \mathbb{R}^m$, the vector functions f and g are assumed to be sufficiently many times continuously differentiable functions of their arguments.

It is convenient to reformulate (2.13) with the change of variable $\epsilon t = \tau$, valid for any

$\epsilon \neq 0$:

$$\frac{d}{d\tau}x_s = f(x_s, x_f, \epsilon) \tag{2.14}$$

$$\epsilon\frac{d}{d\tau}x_f = g(x_s, x_f, \epsilon) \tag{2.15}$$

The terminology *singular* can be seen in (2.15) : in the limit $\epsilon = 0$ the differential equation $\epsilon\frac{d}{d\tau}x_f = g(x_s, x_f, \epsilon)$ degenerates into the algebraic equation $g(x_s, x_f, 0) = 0$.

An intuitive way to compute the reduced model is the following:

We begin by roughly set $\epsilon = 0$ in equations (2.14), (2.15). We obtain the *differential-algebraic system* :

$$\begin{aligned} \frac{d}{d\tau}x_s &= f(x_s, x_f, 0) \quad x_s(0) = x_s^0 \\ 0 &= g(x_s, x_f, 0) \end{aligned} \tag{2.16}$$

Assume, for any x_s, the existence of a root $x_f = \Phi(x_s)$ for the equation $0 = g(x_s, x_f, 0)$, where Φ is a smooth function of x_s. Then the reduced dynamics would be given by

$$\frac{d}{d\tau}x_s = f(x_s, \Phi(x_s), 0) \quad x_s(0) = x_s^0 \tag{2.17}$$

This intuitive approximation turns out to be true up to the first order in ϵ as stated by the Tikhonov's theorem (we give here the version from [KKO99]), under suitable assumptions :

Theorem 1. **(Tikhonov 1952)** Denote $x_f = \Phi(x_s)$, with Φ a smooth function of x_s, a root for the equation $g(x_s, x_f, 0) = 0$. Let $\bar{x}_s(t)$ be a solution of the reduced problem (2.17). Assume that the initial system (2.13) verifies the following assumptions :

- the eigenvalues of the linearisation of g evaluated for $\epsilon = 0$ along $x_f = \Phi(x_s)$ have a strictly negative real part i.e :

$$\left.\frac{\partial g}{\partial x_f}\right|_{(x_s, x_f = \Phi(x_s), 0)} \quad \text{has strictly negative eigenvalues}$$

- the initial value x_f^0 lies in the domain of attraction of the root $x_f = \Phi(x_s)$ for initial values $(x_s^0, 0)$ i.e. trajectories of

$$\frac{d}{dt}x_f = g(x_s, x_f, 0) \quad x_f(0) = x_f^0$$

  when x_s is considered as a fixed parameter, tend to $x_f = \Phi(x_s)$ when τ goes to infinity.

Then, when $\epsilon \to 0$, the solution of the initial system [(2.14),(2.15)] tends to solution of the degenerated system (2.16). More precisely, there exists $T > 0$, such that the solution $(x_s(t, \epsilon), x_f(t, \epsilon))$ of the initial system [(2.14),(2.15)] verifies :

$$\begin{aligned} \forall t \in [0, T] \qquad & \lim_{\epsilon \to 0^+} x_s(t, \epsilon) = \bar{x}_s(t) \\ \forall t \in]0, T] \qquad & \lim_{\epsilon \to 0^+} x_f(t, \epsilon) = \Phi(\bar{x}_s(t)) \end{aligned}$$

The convergence is uniform in the interval $0 \leq t \leq T$ for x_s and in any interval $0 < t_1 \leq t \leq T$ for x_f.

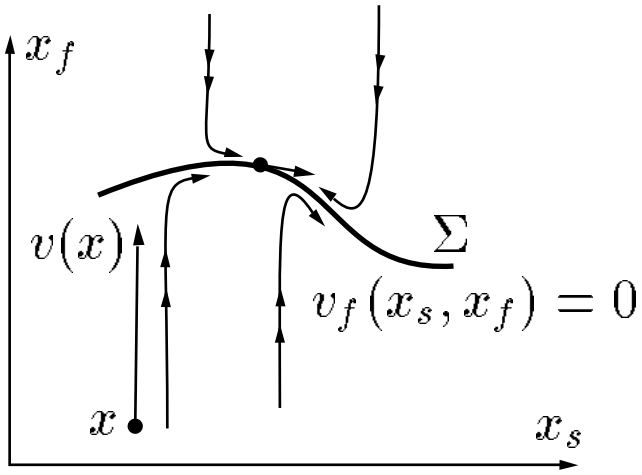

Figure 2.2: Evolution of a dynamical system in the Tikhonov normal form: during the transient dynamics, the evolution is quasi-vertical.

The schematic result of this theorem is illustrated in figure 2.2: in the phase space (x_s, x_f), due to the fast and stable dynamics v_f, the system nearly evolves along lines $x_s = constant$ before being caught by the slow manifold.

The theorem depicted here leads to a first order approximation in ϵ, however it can be refined, using asymptotic expansions, to be at the desired precision (see e.g. [Ver05]).

In order to apply the Tikhonov's theory, the dynamics of the system have to be in the particular form (2.13) with an explicit decomposition between the fast and slow variables. As the dynamics of the considered quantum systems (2.9) isn't in the Tikhonov normal form, a possible way to approach the model reduction for open quantum is to find an appropriate change of variable in order to decompose the quantum dynamics into fast and slow variables. We made a first attempt using this approach in [ASR15] (see also [MR09] for a similar approach with lambda systems). While we were able to perform model reduction up to first order in ϵ for some class of linear quantum systems, it seems difficult to extend this approach and find explicitly the appropriate change of coordinate for an arbitrary quantum system.

We are therefore interested in a coordinate-free theory in order to compute the reduced model.

Geometric singular perturbation theory

The geometric approach to singular perturbation theory leads to the desired coordinate-free method for model reduction. It was first introduced by Fenichel in [Fen79], see also a survey by Jones [Jon95].

Consider the system of differential equations

$$\frac{d}{dt}x = v(x, \epsilon) \tag{2.18}$$

where $x \in \mathbb{R}^n$, $0 < \epsilon \ll 1$ and v is a smooth function.

Then assume :

- (H_1) In the case $\epsilon = 0$, equation (2.18) has an l-dimensional manifold $\mathbf{\Sigma}_0$ which is contained in the set $\{v(x, 0) = 0\}$.

- (H_2) For $\epsilon = 0$ and any $x_0 \in \mathbf{\Sigma}_0$, the linearisation of (2.18) around x_0 has exactly $m = n - l$ eigenvalues with strictly negative real part[2].

Under these two assumptions, the system (2.18) is said to have a slow and an asymptotically stable fast time-scales. We can then state the following theorem from Fenichel :

Theorem 2. **(Fenichel's Invariant Manifold Theorem)**: Under the assumptions (H_1), (H_2), if $\epsilon > 0$, but sufficiently small, there exists a manifold $\mathbf{\Sigma}_\epsilon$ that lies within $O(\epsilon)$ of $\mathbf{\Sigma}_0$ and is diffeomorphic to $\mathbf{\Sigma}_0$. Moreover it is locally invariant and attractive under the flow of (2.18).

This theorem ensures the persistence of an invariant attractive manifold when ϵ is non zero but sufficiently small. We will call $\mathbf{\Sigma}_\epsilon$ the slow manifold. The goal is therefore to compute the equations defining this slow manifold and deduce the reduced dynamics.

Then for each $x_0 \in \mathbf{\Sigma}_0$ it is possible to locally decompose the state vector x into two components (x_s, x_f) with $\dim(x_s) = l = \dim(\mathbf{\Sigma}_0)$ and $\dim(x_f) = m = n - l$ such that the projection of the slow manifold $\mathbf{\Sigma}_\epsilon$ on the slow coordinates x_s is a local diffeomorphism around x_0.

The dynamics of system (2.18) is then given by

$$\begin{aligned} \frac{d}{dt}x_s &= f(x_s, x_f, \epsilon) \\ \frac{d}{dt}x_f &= g(x_s, x_f, \epsilon) \end{aligned}$$

where $x_s \in \mathbb{R}^l$, $x_f \in \mathbb{R}^m$, $0 < \epsilon \ll 1$.

The following theorem from Carr [Car81] allows to compute an approximation of $\mathbf{\Sigma}_\epsilon$ up to the desired accuracy.

Theorem 3. **(Carr 1981)** Assume that for $k \in \mathbb{N}$, one knows a function $\Phi_k(x_s, \epsilon)$ satisfying the following equation corresponding to the invariance of the slow manifold $\mathbf{\Sigma}_\epsilon$:

$$f(x_s, \Phi_k(x_s, \epsilon), \epsilon) = \left.\frac{\partial \Phi_k(x_s, \epsilon)}{\partial x_s}\right|_{(x_s, \epsilon)} g(x_s, \Phi_k(x_s, \epsilon), \epsilon) + O(\epsilon^{k+1}) \tag{2.19}$$

Then, the equation $x_f = \Phi_k(x_s, \epsilon)$ is an approximation of $\mathbf{\Sigma}_\epsilon$ up to order $k+1$ in ϵ. The reduced dynamics is given by

$$\frac{d}{dt}x_s = f(x_s, \Phi_k(x_s, \epsilon), \epsilon) + O(\epsilon^{k+1})$$

2.3.3 Specific quantum constraints

As presented in section 2.2, our goal is to compute a reduced model of the dynamics (2.9) by taking advantage of the difference between the fast and slow time-scales. We emphasize here that (2.9) is a linear ordinary differential equation : the operator ρ on $\mathcal{H}$ may be represented by a matrix of dimension $\dim(\mathcal{H}) \times \dim(\mathcal{H})$ and therefore, by concatenating

[2]the necessary assumption is : the set $\mathbf{\Sigma}_0$ is normally hyperbolic relative to (2.18) i.e. for any $x_0 \in \mathbf{\Sigma}_0$, the linearisation of (2.18) around x_0 has exactly l purely imaginary eigenvalues. However, since we are interested in stable manifold we consider our assumption.

all the variables of the density operator ρ into a single vector x (called the coherence vector) the dynamics can be reformulated as :

$$\frac{d}{dt}x = (\boldsymbol{A} + \epsilon \boldsymbol{B})x \tag{2.20}$$

for some time-independent matrix $\boldsymbol{A}$ and $\boldsymbol{B}$. It can easily be solved analytically, moreover, following the perturbations of the eigenvalues and eigenstates of $\boldsymbol{A} + \epsilon \boldsymbol{B}$ is well known when the spectrum of $\boldsymbol{A}$ is non-degenerate and the construction of the reduced model to various order of approximation is standard [Sak94]. However, in considered quantum systems (see e.g. section 3.1) the spectrum of $\boldsymbol{A}$ is degenerated, leading to complications for applying similar perturbative methods.

Note that if the considered system is closed then (2.9) can be transformed into (2.2) which is already in the form (2.20) (where x is replaced with the wave function $|\psi\rangle$ and $\boldsymbol{A}$ is the Hamiltonian usually denoted $\boldsymbol{H}$) and therefore the computation of the reduced model is standard (if the spectrum of $\boldsymbol{H}$ is non-degenerate). However, the case of open quantum systems is much more complicated as presented below.

The main problem of performing a similar approach for open quantum systems described by Lindblad master equation is how to interpret the reduced model:

- What does physically represent the reduced variables obtained after a model reduction on the coherence vector ?
- Is it possible to give some physical meaning to the reduced dynamics ?

Both these questions are crucial and the standard approach of model reduction for linear differential equations does not give satisfactory answers. For example, the reduced dynamics would not be in a Lindblad form, preventing any quantum interpretation of the reduced dynamics, and thus the reduced model does not convey a physical meaning.

In order to ensure a physical interpretation of the reduced model, we therefore impose several conditions to **preserve the quantum structure**. Firstly, we are looking for reduced dynamics in a Lindblad form as only this type of dynamics conveys a physical sense. Secondly, the mapping from the reduced space to the complete space, i.e. the equations defining the slow manifold, must be in Kraus map form. This property ensures that we map density operators to density operators or equivalently that there exists a quantum process linking the reduced and the complete model. In the spirit of the Carr approximation lemma [Car81] we will perform an asymptotic expansion in power of the small parameter ϵ describing the two time-scales of the researched functions.

More precisely, if we parametrise the slow invariant attractive manifold $\boldsymbol{\Sigma}_\epsilon$ corresponding to the perturbation of $\boldsymbol{\Sigma}_0$ by a density operator ρ_s of the same dimension as $\boldsymbol{\Sigma}_0$, then we are searching for a reduced dynamics in the form of a linear function of ρ_s given by a linear super-operator $\mathcal{L}_s$. It will be developed in power of ϵ in order to obtain an asymptotic expansion of $\frac{d}{dt}\rho_s$:

$$\frac{d}{dt}\rho_s = \mathcal{L}_s(\rho_s) = \sum_{k\geq 0} \epsilon^k \mathcal{L}_{s,k} \tag{2.21}$$

For any finite $\bar{k} \in \mathbb{N}$, the reduced dynamics resulting from the truncation of the series up to $\bar{k}$ must be expressed in a Lindblad form up to higher order terms :

$$\sum_{k=0}^{\bar{k}} \epsilon^k \mathcal{L}_{s,k}(\rho_s) = -i[\boldsymbol{H}_{\bar{k}}, \rho_s] + \sum_{\nu} \boldsymbol{L}_{\bar{k},\nu}\rho_s \boldsymbol{L}_{\bar{k},\nu}^\dagger - \frac{1}{2}\Big(\boldsymbol{L}_{\bar{k},\nu}^\dagger \boldsymbol{L}_{\bar{k},\nu}\rho_s + \rho_s \boldsymbol{L}_{\bar{k},\nu}^\dagger \boldsymbol{L}_{\bar{k},\nu}\Big) + o(\epsilon^{\bar{k}})$$

for some hermitian operators $\boldsymbol{H}_k$ and some non necessarily hermitian operators $\boldsymbol{L}_{k,\nu}$. They both are time-independent but might depend on ϵ. The equations defining $\boldsymbol{\Sigma}_\epsilon$ are linear in ρ_s and given by the super-operator $\mathcal{K}$, leading after a development in power of ϵ to :

$$\rho(t) = \mathcal{K}(\rho_s(t)) = \sum_{k\geq 0} \epsilon^k \mathcal{K}_k(\rho_s(t)) \tag{2.22}$$

The truncation of the series up to $\bar{k} \in \mathbb{N}$ leads to a Kraus map expression of the equations (up to higher order terms) :

$$\sum_{k=0}^{\bar{k}} \epsilon^k \mathcal{K}_k(\rho_s(t)) = \sum_l \boldsymbol{M}_l(\epsilon)\rho_s(t)\boldsymbol{M}_l^\dagger(\epsilon) + o(\epsilon^{\bar{k}})$$

where $\{\boldsymbol{M}_l(\epsilon)\}$ are Kraus operators, they are time-independent functions of ϵ. Note that this particular linear form in ρ_s is valid because we are considering linear systems. Therefore, the solution ρ_s of the reduced master equation (2.21) yields via the completely positive mapping $\mathcal{K}$ a trajectory of the perturbed system (2.9).

The fact that the model reduction method for open quantum systems must preserve the quantum structure represented by the constraints (2.21) and (2.22) leads to a nontrivial problem even if the initial system is a linear differential equation. To tackle this problem, we will use the geometric singular perturbation theory presented in 2.3.2.

2.4 Presentation of our method

We begin this section by briefly describing the considered system with two-time scales on which we perform adiabatic elimination, as well as the constrains imposed on the reduced model in order to ensure a physical meaning of the computed reduced system. This has already been presented respectively in section 2.2.3 and 2.3.3 with more details but is given here in order to make this section self-consistent. Then, we present our adiabatic elimination method based on singular perturbation theory and asymptotic expansion to compute the reduced dynamics. Our goal, in this section, is to derive the recurrence relations that have to be solved in order to compute the reduced model. The resolution of these recurrence relations is developed in chapter 3.

The two time-scales dynamics of the considered quantum system is given by the master equation

$$\frac{d}{dt}\rho = \mathcal{L}_0(\rho) + \epsilon\mathcal{L}_1(\rho)$$

where ρ is the density operator on a finite dimension Hilbert space $\mathcal{H}$, $0 < \epsilon \ll 1$. For $\epsilon = 0$ and any initial condition $\rho_0 \in \mathcal{D}$ there exists an attractive invariant manifold of steady states $\mathcal{D}_0 = \left\{\rho \in \mathcal{D} \;\middle|\; \mathcal{L}_0(\rho) = 0\right\}$. We denote $\overline{\mathcal{R}}(\rho_0) = \lim_{t\to\infty} e^{t\mathcal{L}_0}(\rho_0)$.

The adiabatic elimination method must preserve the quantum structure by :

- ensuring a reduced dynamics in a Lindblad form for the reduced density operator ρ_s

$$\frac{d}{dt}\rho_s = \mathcal{L}_s(\rho_s) = \sum_{k\geq 0} \epsilon^k \mathcal{L}_{s,k}(\rho_s)$$

$$\forall \bar{k} \in \mathbb{N}, \quad \sum_{k=0}^{\bar{k}} \epsilon^k \mathcal{L}_{s,k}(\rho_s) = -i\left[\boldsymbol{H}_{\bar{k}}, \rho_s\right] + \sum_{\nu} \mathcal{D}_{\boldsymbol{L}_{\bar{k},\nu}}(\rho_s) + o(\epsilon^{\bar{k}})$$

- expressing the parametrisation of the slow manifold as a Kraus map

$$\rho = \mathcal{K}(\rho_s(t)) = \sum_{k\geq 0} \epsilon^k \mathcal{K}_k(\rho_s(t))$$

$$\forall \bar{k} \in \mathbb{N}, \quad \sum_{k=0}^{\bar{k}} \epsilon^k \mathcal{K}_k(\rho_s(t)) = \sum_{l} \boldsymbol{M}_l(\epsilon)\rho_s(t)\boldsymbol{M}_l^\dagger(\epsilon) + o(\epsilon^{\bar{k}})$$

This is summarised in the schematic figure 2.3.

By definition, the unperturbed system is linear and features an attractive invariant manifold $\mathcal{D}_0$, ensuring the existence of an attractive invariant manifold - the slow manifold - in the presence of a sufficiently small perturbation described the small parameter ϵ using the persistence Fenichel's theorem (2).

Noting that the mapping $\mathcal{K}$ is linear and time-invariant, an adaptation of the Carr's theorem 3 ($\rho \leftrightarrow (x_s, x_f)$) leads to the following equation corresponding to the invariance of the attractive manifold :

$$\frac{d}{dt}\rho = \mathcal{L}_0(\mathcal{K}(\rho_s)) + \epsilon\mathcal{L}_1(\mathcal{K}(\rho_s)) = \frac{d}{dt}\left(\mathcal{K}(\rho_s)\right) = \mathcal{K}\left(\frac{d}{dt}\rho_s\right)$$

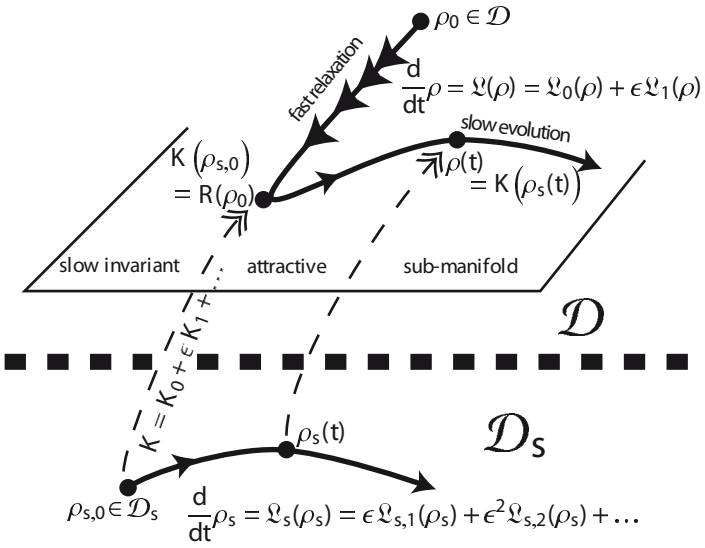

Figure 2.3: Schematic representation of the presented method of adiabatic elimination for open quantum systems based on geometric singular perturbation theory. The dynamics of the complete density operator $\rho \in \mathcal{D}$ is given by $\frac{d}{dt} = \mathcal{L}_0(\rho) + \epsilon\mathcal{L}_1(\rho)$. We use the density operator ρ_s (the "reduced" density operator) to parametrise the slow invariant attractive submanifold via the mapping $\rho = \mathcal{K}(\rho_s) = cK_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) + \ldots$. The slow dynamics of $\rho_s \in \mathcal{D}_s$ ($\dim(\mathcal{D}_s) < \dim(\mathcal{D})$) is given by $\frac{d}{dt}\rho_s = \mathcal{L}_s(\rho_s)) = \mathcal{L}_{s,0}(\rho_s) + \epsilon\mathcal{L}_{s,1}(\rho_s) + \ldots$.

Then, using the asymptotic expansion in power of ϵ for $\mathcal{K}$ and $\mathcal{L}_s$ yields

$$\begin{aligned}
&\mathcal{L}_0\left(\mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) + \epsilon^2\mathcal{K}_2(\rho_s) + \ldots\right) + \epsilon\mathcal{L}_1\left(\mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) + \epsilon^2\mathcal{K}_2(\rho_s) + \ldots\right) \\
= \;& \mathcal{K}_0\left(\mathcal{L}_{s,0}(\rho_s) + \epsilon\mathcal{L}_{s,1}(\rho_s) + \epsilon^2\mathcal{L}_{s,2}(\rho_s) + \ldots\right) + \epsilon\mathcal{K}_1\left(\mathcal{L}_{s,0}(\rho_s) + \epsilon\mathcal{L}_{s,1}(\rho_s) + \epsilon^2\mathcal{L}_{s,2}(\rho_s) + \ldots\right) \\
&+ \epsilon^2\mathcal{K}_2\left(\mathcal{L}_{s,0}(\rho_s) + \epsilon\mathcal{L}_{s,1}(\rho_s) + \epsilon^2\mathcal{L}_{s,2}(\rho_s) + \ldots\right) + \ldots
\end{aligned}$$

Identifying terms of same order versus ϵ yields equations to compute higher order terms from lower order terms:

- Zero order in ϵ:
$$\mathcal{L}_0(\mathcal{K}_0(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,0}(\rho_s)) \tag{2.23}$$
The zero order expansion corresponds to the initialisation of the recurrence. The zero order means we are considering the unperturbed system and therefore we expect a null zero order dynamics: $\mathcal{L}_{s,0} \equiv 0$. From now we will adopt this initialisation, resulting in simpler higher order expansion. The mapping $\mathcal{K}_0$ must therefore verifies $\mathcal{L}_0 \circ \mathcal{K}_0 = 0$. It will be explicitly given in chapter 3 when addressing composite systems.

- First order in ϵ:
$$\mathcal{L}_0(\mathcal{K}_1(\rho_s)) + \mathcal{L}_1(\mathcal{K}_0(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,1}(\rho_s)) \tag{2.24}$$

- Second order in ϵ:
$$\mathcal{L}_0(\mathcal{K}_2(\rho_s)) + \mathcal{L}_1(\mathcal{K}_1(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,2}(\rho_s)) + \mathcal{K}_1(\mathcal{L}_{s,1}(\rho_s)) \,, \tag{2.25}$$

- Third order in ϵ:

$$\mathcal{L}_0(\mathcal{K}_3(\rho_s)) + \mathcal{L}_1(\mathcal{K}_2(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,3}(\rho_s)) + \mathcal{K}_1(\mathcal{L}_{s,2}(\rho_s)) + \mathcal{K}_2(\mathcal{L}_{s,1}(\rho_s)) \tag{2.26}$$
$$\vdots$$

This leads to recurrence relations that have to be solved at any order while ensuring the desired structure for the reduced model. One may see that for each recurrence relation (2.23)- (2.26), we have two unknown super-operator to find: $\mathcal{L}_{s,i}$ and $\mathcal{K}_i$ with respectively $i = \{0, 1, 2, 3, \dots\}$. To solve each equation with two unknown terms, we will begin by applying the mapping $\overline{\mathcal{R}}$ defined in (2.11) to get rid of the $\mathcal{K}_i$ term as illustrated on the first order expansion in ϵ (2.23):

$$\overline{\mathcal{R}}\Big(\mathcal{L}_0(\mathcal{K}_1(\rho_s)) + \mathcal{L}_1(\mathcal{K}_0(\rho_s))\Big) = \overline{\mathcal{R}}\Big(\mathcal{K}_0(\mathcal{L}_{s,1}(\rho_s))\Big)$$

Using the fact that $\overline{\mathcal{R}} = e^{+\infty\mathcal{L}_0}$, $\overline{\mathcal{R}}$ and $\mathcal{L}_0$ commute and by definition, $\mathcal{L}_0 \circ \overline{\mathcal{R}} \equiv 0$, it results:

$$\begin{aligned}\overline{\mathcal{R}}\Big(\mathcal{L}_1(\mathcal{K}_0(\rho_s))\Big) &= \overline{\mathcal{R}}\Big(\mathcal{K}_0(\mathcal{L}_{s,1}(\rho_s))\Big) \\ &= \mathcal{K}_0(\mathcal{L}_{s,1}(\rho_s))\end{aligned} \tag{2.27}$$

where $\mathcal{K}_0$ is derived using the zero order expansion (2.23) thus only $\mathcal{L}_{s,1}$ is unknown and can be computed. We refer to chapter 3 to see how we can ensure a Lindblad form for the reduced dynamics.

Once the first order reduced dynamics $\mathcal{L}_{s,1}$ is derived, then, going back to (2.24), only $\mathcal{K}_1$ remains unknown. It appears through the super-operator $\mathcal{L}_0$. Therefore, we need to be able to define and compute the inverse of the super-operator $\mathcal{L}_0$. This inverse is given by the map $\overline{\mathcal{K}}$ which is linear, trace-preserving and completely positive as shown in the following lemma.

Lemma 1. There exists $\bar{\tau} > 0$ such that the super-operator $\overline{\mathcal{K}}$ sending operator $\boldsymbol{X}$ to

$$\overline{\mathcal{K}}(\boldsymbol{X}) = \frac{1}{\bar{\tau}} \int_0^{+\infty} e^{t\mathcal{L}_0}\Big(\boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})\Big)\, dt + \overline{\mathcal{R}}(\boldsymbol{X}) \tag{2.28}$$

is a linear, trace-preserving and completely positive mapping with

$$-\mathcal{L}_0\Big(\bar{\tau}\overline{\mathcal{K}}(\boldsymbol{X})\Big) = \boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X}).$$

The proof is developed in appendix A.

Remark 1. When for any initial density operator the solution of $\dot{\rho} = \mathcal{L}_0(\rho)$ converges toward a unique density operator $\overline{\rho}$, we have $\overline{\mathcal{R}}(\boldsymbol{X}) = \mathrm{Tr}\,(\boldsymbol{X})\,\overline{\rho}$. In this case, for any given operator $\boldsymbol{W}$ with $\mathrm{Tr}\,(\boldsymbol{W}) = 0$, the general solution of $-\mathcal{L}_0(\boldsymbol{X}) = \boldsymbol{W}$ reads

$$\boldsymbol{X} = \int_0^{+\infty} e^{t\mathcal{L}_0}(\boldsymbol{W})\, dt + \lambda\overline{\rho} = \bar{\tau}\overline{\mathcal{K}}(\boldsymbol{W}) + \lambda\overline{\rho}$$

where λ is an arbitrary complex number. Moreover $\boldsymbol{X} = \bar{\tau}\overline{\mathcal{K}}(\boldsymbol{W})$ is the unique solution with zero trace. The parameter λ corresponds to a gauge degree of freedom. We will show in the following chapter how to fix the gauge in order to guarantee different properties (e.g. trace preservation, complete positivity, or a simple expression) of the expansion.

Remark 2. Note that for any operator $\boldsymbol{X}$ verifying $\overline{\mathcal{R}}(\boldsymbol{X}) = 0$, the term $\overline{\tau}\overline{\mathcal{K}}(\boldsymbol{X})$ is actually independant of $\overline{\tau}$. Such expressions will frequently appear in the following chapters and thus do not require the computation of $\overline{\tau}$.

The same procedure will be applied at any order to compute the reduced model. This method provides a clear path to compute the reduced dynamics up to any desired order in ϵ. The challenging task to ensure a dynamics in a Lindblad form and a completely positive mapping is addressed in chapter 3 for composite systems up to second order. Some preliminaries results on the third order approximation are presented in chapter 5.

Chapter 3

Adiabatic elimination in bipartite quantum systems

In this chapter, we focus on two time-scales bipartite systems. In this type of setting, the system can be physically decomposed in two subsystems. The first one is a stable subsystem converging towards a unique steady state and associated with the fast time-scale while the other subsystem is associated with the slow one. We want to study the dynamics of the slow subsystem of interest in the presence of a small coupling with the fast system. This coupling will be treated as a perturbation of the original, uncoupled system. It may come for example, from inevitable interaction with an environment, or for an interaction specifically designed to shape the reduced dynamics of the system of interest (see section 2.2.2 on reservoir engineering). This type of structure with a physical decomposition between a fast system and a slow one is very common and arises naturally in quantum physics, underlining the need to develop rigorous generic adiabatic elimination techniques. This chapter contains our main results from [ACSR17b] and [ACSR17a] on adiabatic elimination. We solve the recurrence relations presented in 2.4 with specific constraints 2.3.3 up to second order. We derive explicit formulas for the reduced model. On the one hand, they readily give structural results on adiabatic elimination for bipartite systems without any computations. On the other hand, they allow the exact computation of the second order reduced model.

We first present the structure of such bipartite systems in 3.1 and give general results on the first order reduced model. Then, we investigate different types of weak coupling between the two subsystems through this chapter. Section 3.2 is devoted to the general case of Hamiltonian coupling, with special care to the dispersive and resonant interaction. Section 3.3 focuses on cascaded interaction between the subsystems. Some of the considered examples are infinite dimension systems, we postpone the discussion on the application of our method to infinite dimension systems in chapter 4.

3.1 Structure of bipartite systems

3.1.1 Structure

We consider two distinct quantum systems A and B. The state of the system A (respectively B) is described by the density operator ρ_A on $\mathcal{H}_A$ (resp. ρ_B on $\mathcal{H}_B$) of finite dimension and their dynamics are given by :

$$\frac{d}{dt}\rho_A = \mathcal{L}_A(\rho_A)$$
$$\frac{d}{dt}\rho_B = \epsilon\mathcal{L}_B(\rho_B)$$

where

- $0 < \epsilon \ll 1$ is a small parameter describing the different time scales: the system A is on a fast time-scale while the system B is on a slow time-scale.
- $\mathcal{L}_A$ and $\mathcal{L}_B$ are Lindblad-form super-operators on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, describing any possible quantum dynamics and defined by :

$$\mathcal{L}_\xi(\rho) = -i[\boldsymbol{H}_\xi, \rho] + \sum_\mu \mathcal{D}_{\boldsymbol{L}_{\xi,\mu}}(\rho), \quad \xi = A, B \tag{3.1}$$

  with $\boldsymbol{H}_\xi$ a Hermitian operator on $\mathcal{H}_\xi$, $\boldsymbol{L}_{\xi,\mu}$ operators on $\mathcal{H}_\xi$ (not necessarily Hermitian), both time-independent. $\mathcal{D}_{\boldsymbol{X}}$ denotes the dissipation super-operator associated with operator $\boldsymbol{X}$,

$$\mathcal{D}_{\boldsymbol{X}}(\rho) = \boldsymbol{X}\rho\boldsymbol{X}^\dagger - \frac{1}{2}\Big(\boldsymbol{X}^\dagger\boldsymbol{X}\rho + \rho\boldsymbol{X}^\dagger\boldsymbol{X}\Big).$$

- For any initial density operator, the system A asymptotically converges towards a unique equilibrium state denoted by $\overline{\rho}_A = \sum_{n=1}^{\bar{n}} r_n \left|\chi_n\right\rangle\!\left\langle\chi_n\right|$, where the second expression is the spectral decomposition with $r_n > 0$ for all n and $\bar{n}$ is the rank of $\overline{\rho}_A$. Note that asymptotic convergence requires the presence of at least one dissipative (i.e., non Hamiltonian) term in the dynamics. The case of a fast dynamics with a subspace of steady states (corresponding to a decoherence-free space) in investigated in Chapter 5.

In the following, we add a small coupling between both subsystems. The goal is to derive a slow dynamics for the subsystem B only, by adiabatically eliminating the fast variables associated with the system A. In other words, we are interested in the behaviour of the slow system B in the presence of a weak coupling with the subsystem A. The Hilbert space associated with the composite system is the tensor product space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ whose dimension is the product of the individual dimensions. Denoting with $\{|a_j\rangle\}_j$, $\{|b_j\rangle\}_j$ some orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ is given by $\{|a_j\rangle \otimes |b_k\rangle\}_{j,k}$. The state of the composite system is described by a density operator ρ on $\mathcal{H}$ [1] whose dynamics is given by :

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) + \epsilon\mathcal{L}_{int}(\rho) + \epsilon\mathcal{L}_B(\rho) \tag{3.2}$$

[1] Note that a given state ρ of the composite system on $\mathcal{H}$ cannot always be described in terms of one state on $\mathcal{H}_A$ and one on $\mathcal{H}_B$, the dimensions of the Hilbert spaces readily show that the considered composite state can contain more information.

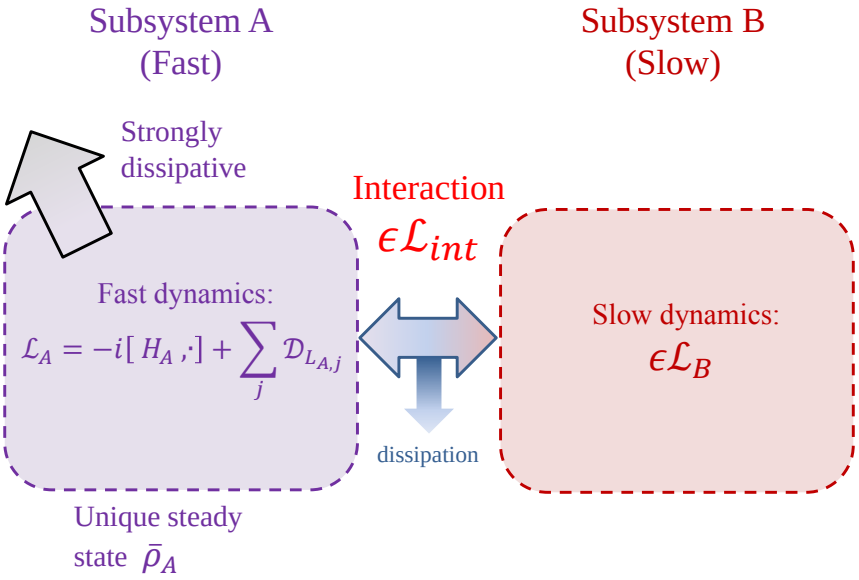

Figure 3.1: We consider a fast system A with stable dynamics $\mathcal{L}_A$ converging towards a unique steady state $\overline{\rho}_A$. It is coupled through the super-operator $\epsilon\mathcal{L}_{int}$ to the slow subsystem B with internal dynamics $\epsilon\mathcal{L}_B$. We investigate the perturbative effect of this coupling on the system of interest B by performing adiabatic elimination of the fast variables associated to the subsystem A. We particularly pay attention to the standard Hamiltonian and cascaded coupling in the next sections.

where $\mathcal{L}_{int}$ is an interaction Lindbladian super-operator acting on both $\mathcal{H}_A$ and $\mathcal{H}_B$. This two time-scales structure for bipartite system will be considered throughout this chapter in order to perform adiabatic elimination. At this point no further assumption is made on the form of the interaction. It will be refined in the forthcoming sections to focus on the most standard coupling : in 3.2 we will address Hamiltonian interactions and in section 3.3 cascaded interaction. Figure 3.1.1 schematically represents the structure of the considered systems.

We make here a slight and common abuse of notation in order to simplify the notations. The super-operator $\mathcal{L}_\xi$ defined in (3.1) acts on operators-on-$\mathcal{H}_\xi$ while the super-operator $\mathcal{L}_\xi$ in (3.2) is an extension acting on operators-on-$\mathcal{H}$. It must be understood like the fact that the super-operator $\mathcal{L}_\xi$ act nontrivially only on $\mathcal{H}_\xi$ i.e. for any operators $\boldsymbol{X}$ on $\mathcal{H}_A$ and $\boldsymbol{Y}$ on $\mathcal{H}_B$, $\mathcal{L}_A(\boldsymbol{X}\otimes\boldsymbol{Y}) := \mathcal{L}_A(\boldsymbol{X})\otimes\boldsymbol{Y}$ and $\mathcal{L}_B(\boldsymbol{X}\otimes\boldsymbol{Y}) := \boldsymbol{X}\otimes\mathcal{L}_B(\boldsymbol{Y})$. In other words, when considering $\mathcal{L}_A$ as a super-operator acting on operators-on-$\mathcal{H}$, the operators $\boldsymbol{H}_A$ and $\boldsymbol{L}_{A,\mu}$ from (3.1) are replaced by $\boldsymbol{H}_A\otimes\mathbf{1}_{\mathcal{H}_B}$ and $\boldsymbol{L}_{A,\mu}\otimes\mathbf{1}_{\mathcal{H}_B}$ (similarly for $\mathcal{L}_B$).

Consider the solution of (3.2) starting from a density operator ρ_0 on $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$. When $\epsilon=0$, the solution of the unperturbed system converges exponentially towards the separable state :

$$\rho_\infty = \overline{\mathcal{R}}(\rho_0) = e^{+\infty\mathcal{L}_A}(\rho_0) = \overline{\rho}_A\otimes\mathrm{Tr}_A(\rho_0) \tag{3.3}$$

where $e^{+\infty\mathcal{L}_A}(\bullet)$ is the propagator (see Section 2.3), $\mathrm{Tr}_A(\rho_0)$ is the partial trace over $\mathcal{H}_A$ of the initial density operator ρ_0. Therefore the unperturbed system features an attractive invariant manifold $\mathcal{D}_0$. It can be parametrised by ρ_s, a density operator on $\mathcal{H}_B$, by

$$\mathcal{D}_0 = \ker(\mathcal{L}_A) = \{\rho = \overline{\rho}_A\otimes\rho_s, \forall\rho_s\in\mathcal{D}_S\}$$

with $\mathcal{D}_S$ the set of density operators on $\mathcal{H}_B$.

Thus, we are in the framework presented in 2.4. Thus, in the presence of a small perturbation corresponding to the weak coupling ($\epsilon \neq 0$), there exists a slow manifold close to $\mathcal{D}_0$. We parametrise the slow manifold, i.e. the perturbation of $\mathcal{D}_0$ by a density operator ρ_s of the same dimension as $\mathcal{D}_0$, and thus of the density operators on $\mathcal{H}_B$. This reduced density operator ρ_s will then be interpreted as approximately the state of the subsystem B (though this is not exact, as in general, both system will be entangled and one cannot properly define the state of one subsystem). Therefore, the slow dynamics $\mathcal{L}_s$ will be interpreted as the effective dynamics governing the slow subsystem of interest B. Our goal is to compute this reduced dynamics $\mathcal{L}_s$ and the application $\mathcal{K}$ mapping the reduced density operator to the complete one, for $\epsilon \neq 0$ but sufficiently small. With this composite structure, the recurrence relations defining the second order reduced model (2.23), (2.24), (2.25) that have to be solved turn into

$$\mathcal{L}_A(\mathcal{K}_0(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,0}(\rho_s)) \tag{3.4}$$

$$\mathcal{L}_A(\mathcal{K}_1(\rho_s)) + \mathcal{L}_{int}(\mathcal{K}_0(\rho_s)) + \mathcal{L}_B(\mathcal{K}_0(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,1}(\rho_s)) \tag{3.5}$$

$$\mathcal{L}_A(\mathcal{K}_2(\rho_s)) + \mathcal{L}_{int}(\mathcal{K}_1(\rho_s)) + \mathcal{L}_B(\mathcal{K}_1(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,2}(\rho_s)) + \mathcal{K}_1(\mathcal{L}_{s,1}(\rho_s)) \tag{3.6}$$

3.1.2 First order adiabatic elimination

We solve in this section the zero order and first order reduced model corresponding respectively to solutions of equations (3.4) and (3.5).

The zero order terms in (3.4) initialises the recurrence. The most natural choice for the zero order reduced dynamics $\mathcal{L}_{s,0}$ is, as presented in 2.4, $\mathcal{L}_{s,0} \equiv 0$ due to the fact that for $\epsilon = 0$ the state of the subsystem B does not evolve. Then the zero order mapping $\mathcal{K}_0$ must verify $\mathcal{L}_A(\mathcal{K}_0(\rho_s)) = 0$. The simplest choice is:

$$\mathcal{K}_0(\rho_s) = \overline{\rho}_A \otimes \rho_s$$

To solve the first order approximation we use the approach presented in 2.4 and apply the mapping $\overline{\mathcal{R}}$ defined in (3.3) on the whole equation (3.5). It yields, after plugging the zero order results in :

$$\overline{\mathcal{R}}\Big(\mathcal{L}_{int}(\overline{\rho}_A \otimes \rho_s) + \overline{\rho}_A \otimes \mathcal{L}_B(\rho_s)\Big) = \overline{\mathcal{R}}\Big(\overline{\rho}_A \otimes \mathcal{L}_{s,1}(\rho_s)\Big) \tag{3.7}$$

And readily gives the first order reduced dynamics :

$$\mathcal{L}_{s,1} = \mathcal{L}_B(\rho_s) + \mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A \otimes \rho_s)\right) \tag{3.8}$$

The following lemma ensures the Lindblad form of the first order reduced dynamics.

Lemma 2. For any super-operator $\mathcal{L}_{int}$ in Lindblad form (2.10) and any density operator ρ_A on $\mathcal{H}_A$, the super-operator given by

$$\rho_B \to \mathrm{Tr}_A\left(\mathcal{L}_{int}(\rho_A \otimes \rho_B)\right)$$

remains a Lindblad super-operator.

Proof. By linearity of the partial trace, it is sufficient to consider the following two cases :

$$\mathcal{L}_{int}(\rho) = -i[\boldsymbol{H}, \rho] \quad \text{and} \quad \mathcal{L}_{int}(\rho) = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \tfrac{1}{2}\Big(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}\Big).$$

- Consider first $\mathcal{L}_{int}(\rho) = -i[\boldsymbol{H}, \rho]$. The Hermitian operator $\mathcal{H}$ can be decomposed without loss of generality as $\boldsymbol{H} = \sum_k \boldsymbol{A}_k \otimes \boldsymbol{B}_k$, for some Hermitian operators $\boldsymbol{A}_k$ on $\mathcal{H}_A$ and $\boldsymbol{B}_k$ on $\mathcal{H}_B$. We then have

$$\mathrm{Tr}_A\left([\boldsymbol{H}, \rho_A \otimes \rho_B]\right) = [\boldsymbol{H}_B, \rho_B]$$

  where $\boldsymbol{H}_B = \sum_k \mathrm{Tr}\left(\boldsymbol{A}_k \rho_A\right) \boldsymbol{B}_k$ is a Hermitian operator on $\mathcal{H}_B$.

- Then consider $\mathcal{L}_{int}(\rho) = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}\Big(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}\Big)$. Decompose $\boldsymbol{L} = \sum_k \boldsymbol{A}_k \otimes \boldsymbol{B}_k$, for some non-necessary Hermitian operators $\boldsymbol{A}_k$ on $\mathcal{H}_A$ and $\boldsymbol{B}_k$ on $\mathcal{H}_B$. We have, using the cyclic property of the trace,

$$\begin{aligned}\mathrm{Tr}_A\Big(\boldsymbol{L}\rho_A \otimes \rho_B\boldsymbol{L}^\dagger - \tfrac{1}{2}\Big(\boldsymbol{L}^\dagger\boldsymbol{L}\rho_A \otimes \rho_B + \rho_A \otimes \rho_B\boldsymbol{L}^\dagger\boldsymbol{L}\Big)\Big)& \\ = \sum_{k,k'} \mathrm{Tr}\left(\boldsymbol{A}_k\rho_A\boldsymbol{A}_{k'}^\dagger\right)\Big(\boldsymbol{B}_k\rho_B\boldsymbol{B}_{k'}^\dagger - \tfrac{1}{2}\Big(\boldsymbol{B}_{k'}^\dagger\boldsymbol{B}_k\rho_B + \rho_B\boldsymbol{B}_{k'}^\dagger\boldsymbol{B}_k\Big)\Big)&.\end{aligned}$$

  The square matrix $(M_{k,k'}) = \Big(\mathrm{Tr}\left(\boldsymbol{A}_k\rho_A\boldsymbol{A}_{k'}^\dagger\right)\Big)$ is Hermitian and non-negative: since $\rho_A \geq 0$, we use the spectral decomposition of $\rho_A = \sum_j |\psi_j\rangle\langle\psi_j|$ and the fact that, for any $|\psi_j\rangle \in \mathcal{H}_A$, the matrix $\Big(\mathrm{Tr}\left(\boldsymbol{A}_k |\psi_j\rangle \langle\psi_j| \boldsymbol{A}_{k'}^\dagger\right)\Big)$ is the Gram matrix associated with the vectors $\boldsymbol{A}_k |\psi_j\rangle$ of $\mathcal{H}_A$. Such Hermitian matrices are always non-negative (see e.g. [Bha07]) ensuring the non-negativity of M. Then, take e.g. the Cholesky decomposition $M = NN^\dagger$. We have

$$\begin{aligned}\sum_{k,k'} M_{k,k'}\Big(\boldsymbol{B}_k\rho_B\boldsymbol{B}_{k'}^\dagger - \tfrac{1}{2}\Big(\boldsymbol{B}_{k'}^\dagger\boldsymbol{B}_k\rho_B + \rho_B\boldsymbol{B}_{k'}^\dagger\boldsymbol{B}_k\Big)\Big)& \\ = \sum_{k,k',k''} N_{k,k"}N^*_{k',k''}\Big(\boldsymbol{B}_k\rho_B\boldsymbol{B}_{k'}^\dagger - \tfrac{1}{2}\Big(\boldsymbol{B}_{k'}^\dagger\boldsymbol{B}_k\rho_B + \rho_B\boldsymbol{B}_{k'}^\dagger\boldsymbol{B}_k\Big)\Big)& \\ = \sum_{k''} \boldsymbol{X}_{k''}\rho_B\boldsymbol{X}_{k''}^\dagger - \tfrac{1}{2}\Big(\boldsymbol{X}_{k''}^\dagger\boldsymbol{X}_{k''}\rho_B + \rho_B\boldsymbol{X}_{k''}^\dagger\boldsymbol{X}_{k''}\Big)&\end{aligned}$$

  with $\boldsymbol{X}_{k''} = \sum_k N_{k,k''}\boldsymbol{B}_k$. Which is in a Lindblad form, concluding the proof.

■

The first order reduced model being computed, we can turn to the calculation of the first order mapping $\mathcal{K}_1$. As stated in section 2.4, it appears through the super-operator $\mathcal{L}_A$ which can be inverted thanks to Lemma 1 ensuring the existence of a linear, completely positive trace preserving (CPTP) map $\overline{\mathcal{K}}_A$ acting on operators-on-$\mathcal{H}_A$, with a positive constant $\overline{\tau}$ verifying $\overline{\mathcal{K}}_A(\overline{\rho}_A) = \overline{\rho}_A$ and corresponding to the inverse of $\mathcal{L}_A$ in the following sense :

For any operator $\boldsymbol{W}$ on $\mathcal{H}_A$ and any $\lambda \in \mathbb{C}$, we have

$$\mathcal{L}_A(\boldsymbol{X}) = \overline{\mathcal{R}}(\boldsymbol{W}) - \boldsymbol{W} \Rightarrow \boldsymbol{X} = \overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{W}) + \lambda\overline{\rho}_A$$

With a small abuse of notation, the map $\overline{\mathcal{K}}_A$ applied to operators-on-$(\mathcal{H}_A \otimes \mathcal{H}_B)$ means $\overline{\mathcal{K}}_A \otimes \mathbf{1}_{\mathcal{H}_B}$.

Using the first order recurrence relation (3.5) together with (3.7) leads to :

$$\begin{aligned}\mathcal{L}_A(\mathcal{K}_1(\rho_s)) &= \overline{\mathcal{R}}\Big(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)+\overline{\rho}_A\otimes\mathcal{L}_B(\rho_s)\Big)-\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)-\overline{\rho}_A\otimes\mathcal{L}_B(\rho_s)\\ &= \overline{\mathcal{R}}\Big(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\Big)-\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\end{aligned}$$

And therefore we readily get:

$$\mathcal{K}_1(\rho_s) = \overline{\tau}\overline{\mathcal{K}}_A\Big(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\Big)+\overline{\rho}_A\otimes\boldsymbol{G}_{1,B}(\rho_s) \tag{3.9}$$

where $\boldsymbol{G}_{1,B}$ is any Hermitian operator on $\mathcal{H}_B$ and corresponds to the gauge degree of freedom resulting from the fact that $\ker\mathcal{L}_A\neq\{0\}$.

At this point, the gauge choice may be arbitrary, but two gauge choices lead to interesting properties and will be considered throughout this manuscript :

- the gauge $\boldsymbol{G}_{1,B}=0$ in (3.9) ensure the CPTP of the parametrisation of the slow manifold $\mathcal{K}$, guaranteeing its expression in a Kraus map form (up to higher order terms) as imposed :

$$\begin{aligned}\mathcal{K}(\rho_s) &= \mathcal{K}_0(\rho_s)+\epsilon\mathcal{K}_1(\rho_s)+O(\epsilon^2)\\ &= \overline{\mathcal{K}}_A\Big(\overline{\rho}_A\otimes\rho_s+\epsilon\overline{\tau}\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\Big)+O(\epsilon^2)\\ &= \overline{\mathcal{K}}_A\left(e^{\epsilon\overline{\tau}\mathcal{L}_{int}}(\mathcal{K}_0(\rho_s))\right)+O(\epsilon^2).\end{aligned}$$

  where we used $\overline{\mathcal{K}}_A(\overline{\rho}_A)=\overline{\rho}_A$ and the Taylor expansion in power of ϵ of the exponential. Since $\overline{\mathcal{K}}_A$, $e^{\epsilon\overline{\tau}\mathcal{L}_{int}}$ and $\mathcal{K}_0$ are linear completely positive trace-preserving maps, the first order approximation $\mathcal{K}_0+\epsilon\mathcal{K}_1$ coincides, up to second order terms, with the linear CPTP map $\overline{\mathcal{K}}_A\circ e^{\epsilon\overline{\tau}\mathcal{L}_{int}}\circ\mathcal{K}_0$.

- the gauge $\boldsymbol{G}_{1,B}=-\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)$ leading to a simpler expression for $\mathcal{K}_1$ from equation (2.28), namely

$$\mathcal{K}_1(\rho_s)=\int_0^\infty e^{t\mathcal{L}_A}\Big(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)-\overline{\mathcal{R}}\Big(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\Big)\Big)dt$$

  but not ensuring a CPTP map in general for $\mathcal{K}$:

$$\begin{aligned}\mathcal{K}(\rho_s) &= \overline{\mathcal{K}}_A\Big(\overline{\rho}_A\otimes\rho_s+\epsilon\overline{\tau}\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)-\epsilon\overline{\tau}\,\overline{\rho}_A\otimes\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)\Big)+O(\epsilon^2)\\ &= \overline{\mathcal{K}}_A\circ e^{\epsilon\overline{\tau}\mathcal{L}_{int}}\circ\mathcal{K}_0\circ e^{-\epsilon\overline{\tau}\mathrm{Tr}_A(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s))}(\rho_s)+O(\epsilon^2).\end{aligned}$$

  This expression is not always completely positive due to the backwards propagation with $\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)$, but it is e.g. when $\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)$ is Hamiltonian (thus in particular when $\mathcal{L}_{int}$ is Hamiltonian).

Remark 3. In the next sections, we will mainly use the gauge choice $\boldsymbol{G}_{1,B}=-\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)$. In order to get simpler expressions, we will use the following property, including the gauge choice into the operator on which $\mathcal{K}_A$ acts :

$$\begin{aligned}&\overline{\tau}\overline{\mathcal{K}}_A\Big(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\Big)+\overline{\rho}_A\otimes\Big(-\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)\Big)\\ &\quad=\overline{\tau}\overline{\mathcal{K}}_A\Big(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)-\overline{\rho}_A\otimes\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)\Big)\end{aligned}$$

More generally, for any operators $\boldsymbol{X}$ on $\mathcal{H}_A$ and $\boldsymbol{Y}$ on $\mathcal{H}_B$, for the gauge choice $\boldsymbol{G} = -\overline{\tau}\overline{\mathcal{R}}(\boldsymbol{X}\otimes\boldsymbol{Y})$ we have,

$$\overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{X}\otimes\boldsymbol{Y}) - \overline{\tau}\overline{\mathcal{R}}(\boldsymbol{X}\otimes\boldsymbol{Y}) = \overline{\tau}\overline{\mathcal{K}}_A\Big((\boldsymbol{X} - \mathrm{Tr}\,(\boldsymbol{X}\overline{\rho}_A)\,\overline{\rho}_A)\Big)\otimes\boldsymbol{Y}.$$

Note that in this context of bipartite systems, the first order parametrisation of the slow manifold $\rho = \mathcal{K}_0(\rho_s) + \mathcal{K}_1(\rho_s)$ shows that the density operator of the complete system cannot be decomposed as a single tensor product of a density operator on $\mathcal{H}_A$ and on $\mathcal{H}_B$: due to their interactions, both systems get entangled and this first order mapping allows to estimate the residual entanglement

The computations of the second order corrections $\mathcal{K}_2$ and $\mathcal{L}_2$ can be done via (3.6) along the same lines. The obtained expressions will depend on the gauge choice. We conjecture that, at any order n versus ϵ, we can choose $(\mathcal{K}_j, \mathcal{L}_{s,j})_{1\leq j\leq n}$ such that all equations corresponding to orders less that n are satisfied, and also, such that $\sum_{j=0}^{n}\epsilon^j\mathcal{K}_j(\rho_s)$ and $\sum_{j=1}^{n}\epsilon^j\mathcal{L}_{s,j}(\rho_s)$ coincide, up to $n+1$ order terms, with a trace-preserving completely positive map and with a Lindbladian dynamics, respectively. We have seen on several examples that the related expressions can be quite involved and this is the topic of ongoing research. For instance, in general, the autonomous dynamics $\mathcal{L}_B$ undesirably appears in $\mathcal{L}_2$ if we take the gauge choice $\boldsymbol{G}_{1,B} = 0$ which always guarantees the map $(\mathcal{K}_0 + \epsilon\mathcal{K}_1)$ to be completely positive trace-preserving.

We carry out in the upcoming sections further analysis leading to more explicit expressions up to second order, for particular structures that are relevant in typical quantum systems. The associated conclusions, e.g. expressions proving that we retrieve trace-preserving completely positive maps and Lindblad type dynamics at this higher order as well, are presented in the next sections.

3.2 Hamiltonian coupling

3.2.1 Considered Structure

We focus now on the common case of Hamiltonian interaction between the fast and the slow subsystems: weak Hamiltonian coupling of a target system $\mathcal{H}_B$ to environment is a standard model [HR06]. Such models are also frequently encountered in reservoir engineering, where the fast system $\mathcal{H}_A$ is the main dissipator allowing to stabilise the target system $\mathcal{H}_B$ (see example 3.2.3). For this type of coupling, we derive explicit formulas to compute the reduced dynamics and the parametrisation of the slow manifold up to second order. These formulas already allow us to derive general structural results on the reduced dynamics for such systems (see 3.2.2) and to compute the reduced model on a relevant but complex example 3.2.3.

In this case, the dynamics of the complete system, described by a density operator ρ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, is governed by :

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) + \epsilon\Big(-i\,[\boldsymbol{H}_{int}, \rho] + \mathcal{L}_B(\rho)\Big) \tag{3.10}$$

where, without loss of generality, $\boldsymbol{H}_{int} = \sum_{k=1}^{m} \boldsymbol{A}_k \otimes \boldsymbol{B}_k^\dagger$ is the interacting Hamiltonian. Each individual $\boldsymbol{A}_k$ and $\boldsymbol{B}_k$ are not necessarily Hermitian operators on $\mathcal{H}_A$ and $\mathcal{H}_B$, however, $\boldsymbol{H}_{int}$ is a Hermitian operator on $\mathcal{H}$. We choose this particular decomposition for $\boldsymbol{H}_{int}$ as it easily addresses two typical cases of Hamiltonian interactions, namely the *dispersive coupling* where $m = 1$, $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ are Hermitian and diagonal in the energy bases of the respective subsystems; and the *resonant coupling* with $m = 2$ and $\boldsymbol{A}_2 \otimes \boldsymbol{B}_2^\dagger = \boldsymbol{A}_1^\dagger \otimes \boldsymbol{B}_1$ ($\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ are typically annihilation operators). We take a special care for these two standard interactions in section 3.2.2.

Before introducing our theorem for the first order approximation in the case of Hamiltonian coupling, we begin by a technical Lemma on the structure of the inverse of the super-operator $\mathcal{L}_A$:

Lemma 3. Denote by $\rho = \overline{\rho}_A$ the unique density operator solution of $\mathcal{L}_A(\rho) = 0$. For a traceless operator $\boldsymbol{Y}$ such that $\ker(\overline{\rho}_A) \subseteq \ker(\boldsymbol{Y})$, the traceless solution to $\boldsymbol{X} = \mathcal{L}_A^{-1}(\boldsymbol{Y})$ also satisfies $\ker(\overline{\rho}_A) \subseteq \ker(\boldsymbol{X})$.

The proof is presented in appendix A.

This allows to state an explicit solution for the reduced dynamics and the parametrisation of the slow manifold at the first order in ϵ. For the remainder of this section we use the gauge choice $\boldsymbol{G}_{1,B} = -\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A \otimes \rho_s)\right)$ that will lead to simpler explicit expressions.

Theorem 4. **(Hamiltonian coupling, first order)**: The first order reduced dynamics is given in a Lindblad form by :

$$\begin{aligned}\frac{d}{dt}\rho_s &= \epsilon\mathcal{L}_{s,1}(\rho_s)\\ \mathcal{L}_{s,1}(\rho_s) &= -i\sum_{k=1}^{m}\left[\mathrm{Tr}\left(\boldsymbol{A}_k\overline{\rho}_A\right)\boldsymbol{B}_k^\dagger, \rho_s\right] + \mathcal{L}_B(\rho_s)\end{aligned} \tag{3.11}$$

The parametrisation of the slow manifold is given in a Kraus form [2] (up to higher order

[2] The given form does not exactly correspond to a Kraus form as presented in 2.1. The exact Kraus form can be computed by using the spectral decomposition of $\overline{\rho}_A$. However, we choose this representation as it emphasises how the first order parametrisation can be seen as a slight perturbation of the zero order one.

terms) by :

$$\mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) + O(\epsilon^2) = \exp\left(\text{-}i\epsilon\boldsymbol{M}\right)\,(\overline{\rho}_A \otimes \rho_s)\,\exp\left(i\epsilon\boldsymbol{M}^\dagger\right) \;+ O(\epsilon^2) \quad \text{with } \boldsymbol{M} = \sum_{k=1}^{m} \boldsymbol{F}_k \otimes \boldsymbol{B}_k^\dagger \tag{3.12}$$

where

$$\boldsymbol{F}_k\overline{\rho}_A = \overline{\tau}\overline{\mathcal{K}}_A\left(\,\boldsymbol{A}_k\,\overline{\rho}_A\,\right) - \overline{\tau}\mathrm{Tr}\left(\boldsymbol{A}_k\,\overline{\rho}_A\right)\overline{\rho}_A \tag{3.13}$$

satisfies $\mathrm{Tr}\left(\boldsymbol{F}_k\overline{\rho}_A\right) = 0$, for $k = 1, 2, ..., m$.

Proof. The reduced dynamics (3.11) is computed by a direct application of (3.8). From equation (3.9), we get

$$\begin{aligned}\mathcal{K}_1(\rho_s) &= \overline{\tau}\overline{\mathcal{K}}_A\Big(-i\Big[\boldsymbol{H}_{int}, \overline{\rho}_A \otimes \rho_s\Big]\Big) + \overline{\rho}_A \otimes \boldsymbol{G}_{1,B}(\rho_s)\\ &= -i\sum_{k=1}^{m} \overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{A}_k\overline{\rho}_A) \otimes \boldsymbol{B}_k^\dagger\,\rho_s - \overline{\tau}\mathrm{Tr}\left(\boldsymbol{A}_k\,\overline{\rho}_A\right)\overline{\rho}_A \otimes \boldsymbol{B}_k^\dagger\,\rho_s + \text{Herm. conj.}\\ &= -i\boldsymbol{M}(\overline{\rho}_A \otimes \rho_s) + i(\overline{\rho}_A \otimes \rho_s)\boldsymbol{M}^\dagger\end{aligned} \tag{3.14}$$

leading to (3.14) for $\mathcal{K}_0(\rho_s) + \mathcal{K}_1(\rho_s)$ by a Taylor expansion of the exponential. The existence of matrices $\boldsymbol{F}_k$ is ensured by Lemma 3. ■

One may see from (3.8) that the first order reduced dynamics can be directly computed by simply replacing ρ by $\overline{\rho}_A \otimes \rho_s$ in (3.10). This may be interpreted as the fact that at this order, the system A is "frozen" in its steady state $\overline{\rho}_A$. This corresponds to a Zeno effect induced by the strong measurement of the environment. This first order reduced dynamics corresponds to the well known *Zeno dynamics* [MS77, FP02]. This result even holds for non-Hamiltonian coupling, this may be seen from equation (3.8). The first order mapping $\mathcal{K}_0 + \epsilon\mathcal{K}_1$ corresponding to the entanglement between both subsystems can be called in this context a *Zeno entanglement*.

Wee see through e.g. (3.14) that the abstract inversion formula has to be applied only on the given interaction operators, instead of on any possible state. This may lead to simpler computations in particular cases (see chapter 4). Note also that expression (3.14) differs from the general one (3.9) by second order terms and, while the expression without the $O(\epsilon^2)$ terms is completely positive, $\boldsymbol{M}$ is not always Hermitian so corrections of $O(\epsilon^2)$ might be necessary here to exactly preserve the trace.

We now turn to the second order.

3.2.2 Second order reduced dynamics

We state in this section, our main theorem for the computation of the second order approximation for a general Hamiltonian coupling. Then, we present corollaries which are direct applications of the main theorem in the particular common cases of dispersive and resonant interaction.

General case

We start by a technical Lemma needed to ensure the Lindblad form of the second order reduced dynamics. Its proof is done in appendix B.

Lemma 4. Consider the $m \times m$ matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ whose entries are given by

$$X_{k,j} = \mathrm{Tr}\left(\boldsymbol{F}_j \overline{\rho}_A \boldsymbol{A}_k^\dagger + \boldsymbol{A}_j \overline{\rho}_A \boldsymbol{F}_k^\dagger\right)$$
$$Y_{k,j} = \frac{1}{2i}\mathrm{Tr}\left(\boldsymbol{F}_j \overline{\rho}_A \boldsymbol{A}_k^\dagger - \boldsymbol{A}_j \overline{\rho}_A \boldsymbol{F}_k^\dagger\right)$$

where $\boldsymbol{F}_k$ is defined in (3.13). The matrix $\boldsymbol{X}$ is Hermitian and positive semi-definite. Thus there exists a (non-unique) $m \times m$ matrix $\boldsymbol{\Lambda}$ with complex entries such that $\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\dagger$. The matrix $\boldsymbol{Y}$ is Hermitian.

We can then state our main theorem, giving an explicit expression of the second order reduced dynamics.

Theorem 5. **(Hamiltonian coupling, second order)**: Using the $m \times m$ matrices $\boldsymbol{Y}$ and $\boldsymbol{\Lambda}$ provided by Lemma 4, the second order reduced dynamics associated with (3.10), expressed in a Lindblad form, is given by

$$\mathcal{L}_{s,2}(\rho_s) = -i\left[\sum_{k,j=1}^{m} Y_{k,j}\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\,,\,\rho_s\right] + \sum_{p=1}^{m}\mathcal{D}_{\boldsymbol{L}_p}(\rho_s) \tag{3.15}$$

where $\boldsymbol{L}_p = \sum_{j=1}^{m}\Lambda_{j,p}^*\boldsymbol{B}_j^\dagger$. As a result, the reduced dynamics (up to second order) is given by :

$$\frac{d}{dt}\rho_s = -i\epsilon\sum_{k=1}^{m}\left[\mathrm{Tr}\left(\boldsymbol{A}_k\overline{\rho}_A\right)\boldsymbol{B}_k^\dagger\,,\rho_s\right] + \epsilon\mathcal{L}_B(\rho_s) - i\epsilon^2\left[\sum_{k,j=1}^{m} Y_{k,j}\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\,,\,\rho_s\right] + \epsilon^2\sum_{p=1}^{m}\mathcal{D}_{\boldsymbol{L}_p}(\rho_s)$$

Proof. We start from the second order recurrence relation (3.6). We apply the mapping $\overline{\mathcal{R}}$ or similarly the partial trace over A in order to get rid of the first term including the unknown $\mathcal{K}_2$, thanks to $\mathrm{Tr}_A\left(\mathcal{L}_A(\bullet)\right) \equiv 0$. It results

$$\mathrm{Tr}_A\left(\mathcal{L}_{int}\Big(\mathcal{K}_1(\rho_s)\Big)\right) + \mathrm{Tr}_A\left(\mathcal{L}_B\Big(\mathcal{K}_1(\rho_s)\Big)\right) = \mathcal{L}_{s,2}(\rho_s) + \mathrm{Tr}_A\left(\mathcal{K}_1\Big(\mathcal{L}_{s,1}(\rho_s)\Big)\right)$$

From (3.13) and (3.14), we get that $\mathrm{Tr}_A\left(\mathcal{K}_1(\bullet)\right) \equiv 0$. Similar computations, using the fact that $\mathcal{L}_B$ acts non-trivially only on $\mathcal{H}_B$, leads to $\mathrm{Tr}_A\left(\mathcal{L}_B\Big(\mathcal{K}_1(\rho_s)\Big)\right) = 0$. Thus we readily get :

$$\mathcal{L}_{s,2}(\rho_s) = \mathrm{Tr}_A\left(\mathcal{L}_{int}\Big(\mathcal{K}_1(\rho_s)\Big)\right)$$

Note that this expression is analogous to the one describing the first order reduced dynamics (3.11). Next, we plug the expression of $\mathcal{K}_1$ from (3.14) and $\mathcal{L}_{int}$ from (3.10) into the previous equation. A direct expansion leads to

$$\mathcal{L}_{s,2}(\rho_s) = \mathrm{Tr}_A\left(-\sum_{k,j=1}^{m}\left[\boldsymbol{A}_k\otimes\boldsymbol{B}_k^\dagger\,,\,\boldsymbol{F}_j\overline{\rho}_A\otimes\boldsymbol{B}_j^\dagger\rho_s - \overline{\rho}_A\boldsymbol{F}_j^\dagger\otimes\rho_s\boldsymbol{B}_j\right]\right)$$

Using $\sum_k^m \boldsymbol{A}_k\otimes\boldsymbol{B}_k^\dagger = \sum_k^m \boldsymbol{A}_k^\dagger\otimes\boldsymbol{B}_k$, we get

$$\begin{aligned}\mathcal{L}_{s,2}(\rho_s) = \sum_{k,j=1}^{m} &-\mathrm{Tr}\left(\boldsymbol{A}_k^\dagger\boldsymbol{F}_j\overline{\rho}_A\right)\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\rho_s + \mathrm{Tr}\left(\boldsymbol{F}_j\overline{\rho}_A\boldsymbol{A}_k^\dagger\right)\boldsymbol{B}_j^\dagger\rho_s\boldsymbol{B}_k \\ &+\mathrm{Tr}\left(\boldsymbol{A}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger\right)\boldsymbol{B}_k^\dagger\rho_s\boldsymbol{B}_j - \mathrm{Tr}\left(\overline{\rho}_A\boldsymbol{F}_j^\dagger\boldsymbol{A}_k\right)\rho_s\boldsymbol{B}_j\boldsymbol{B}_k^\dagger\end{aligned}$$

Finally, after separating real and imaginary coefficients, and standard algebraic manipulations :

$$\mathcal{L}_{s,2}(\rho_s) = -i\left[\sum_{k,j=1}^{m} Y_{k,j}\,\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\,,\ \rho_s\right] + \sum_{k,j=1}^{m} X_{k,j}\,\Big(\boldsymbol{B}_j^\dagger\rho_s\boldsymbol{B}_k - \tfrac{1}{2}\Big(\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\rho_s + \rho_s\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\Big)\Big)$$

with $X_{k,j}$ and $Y_{k,j}$ defined in Lemma 4. Since $X_{k,j} = \sum_{p=1}^{m}\Lambda_{k,p}\Lambda_{j,p}^*$, by gathering terms with summation indexes (k,j), we get equation (3.15). ■

Theorem 5 shows that the Hamiltonian coupling of a system of interest to a dissipative system adds some Hamiltonian evolution corresponding to second-order correction to the Zeno Hamiltonian, but moreover induces decoherence at the second order on the slow system. It illustrates how the dissipation on the system $\mathcal{H}_A$ "propagates" through the Hamiltonian coupling, leading to dissipation on the subsystem $\mathcal{H}_B$.

Furthermore, from the explicit expression (3.15), and without performing any computation, we get two structural results on the second order adiabatic elimination :

- There is *at most* m decoherence channels, they correspond to the decoherence operators $\boldsymbol{L}_p$ $(p = 1,\dots,m)$ given in (3.15). This parameter m corresponds to the minimal number of tensor-product terms required to express the interaction Hamiltonian in (3.10) and thus is in particular independent of the dimension of $\mathcal{H}_A$ associated to the fast system, of the number of decoherence operators involved in the fast relaxation $\mathcal{L}_A$, etc.
- The decoherence operators of the reduced dynamics are linear combinations of the operators on $\mathcal{H}_B$ involved in the Hamiltonian coupling. The corresponding coefficient given by Lemma 4 can be computed provided we integrate or invert the super-operator $\mathcal{L}_A$ acting only on the fast subsystem.

Note that, with this gauge choice, the initial slow dynamics of the complete system $\mathcal{L}_B$ play no role in the computations of reduced dynamics. It is merely copied in the first order reduced dynamics and have no effect on the second order computations. For these reasons, in considered examples, we will use $\mathcal{L}_B \equiv 0$.

Resonant coupling

We consider the case of resonant coupling where $\boldsymbol{H}_{int} = \boldsymbol{A}\otimes\boldsymbol{B}^\dagger + \boldsymbol{A}^\dagger\otimes\boldsymbol{B}$ (usually $\boldsymbol{A}$ is the annihilation operator on the fast subsystem A). The reduced model can be computed directly by using Theorem 5 as this interaction Hamiltonian is in the desired form . In this case, as $m = 2$, the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ from Lemma 4 are of dimension 2. While in general no further simplifications occur for systems with resonant interaction, in several cases when the fast system is a linear harmonic oscillator, we get a diagonal matrix $\boldsymbol{X}$. The two second order dissipation operators are then respectively proportional to $\boldsymbol{B}$ and $\boldsymbol{B}^\dagger$ and are not a linear combination of both. See Chapter 4 for such examples.

Dispersive coupling

Corollary 1. **(Dispersive coupling, second order)** Consider a system on $\mathcal{H}_A\otimes\mathcal{H}_B$ whose dynamics is governed by the following Lindblad master equation :

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) + \epsilon\Big(-i\,[\boldsymbol{A}\otimes\boldsymbol{B}\,,\,\rho] + \mathcal{L}_B(\rho)\Big)$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are Hermitian operators and $\overline{\rho}_A$ is the unique steady state of the unperturbed system : $\mathcal{L}_A(\overline{\rho}_A) = 0$.

The reduced dynamics, up to second in ϵ is given by :

$$\frac{d}{dt}\rho_s = -i\epsilon\left[\mathrm{Tr}\left(\boldsymbol{A}\overline{\rho}_A\right)\boldsymbol{B}\,,\,\rho_s\right] + \epsilon\mathcal{L}_B(\rho_s) \tag{3.16}$$

$$-i\epsilon^2\left[\frac{\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A} - \boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right)}{2i}\boldsymbol{B}^2\,,\,\rho_s\right] + \epsilon^2\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A} + \boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right)\mathcal{D}_{\boldsymbol{B}}(\rho_s) \tag{3.17}$$

with $\boldsymbol{F}$ given by $\boldsymbol{F}\overline{\rho}_A = -\mathcal{L}_A^{-1}\Big(\boldsymbol{A} - \mathrm{Tr}\left(\boldsymbol{A}\,\overline{\rho}_A\right)\overline{\rho}_A)\,\Big) = \overline{\tau}\overline{\mathcal{K}}_A\Big(\,\boldsymbol{A}\,\overline{\rho}_A - \mathrm{Tr}\left(\boldsymbol{A}\,\overline{\rho}_A\right)\overline{\rho}_A\Big)$.

In this particular case, as the Hamiltonian interaction contains only one term $\boldsymbol{A}\otimes\boldsymbol{B}$, we see that the dissipation operator of the reduced dynamics at second order is readily given by $\boldsymbol{B}$. When we are mainly interested in the type of dissipation (e.g. in quantum error correction in order to know what type of error could occur), our formula readily gives the answer without any computation.

3.2.3 Applications

A tutorial example

We start with a tutorial example of adiabatic elimination on a low quality quantum harmonic oscillator resonantly coupled to another quantum system. For such considered system, adiabatic elimination is standard and the reduced dynamics is well known, using e.g. Langevin equations. Our goal is to show how to apply our method and perform the computations on this simple system. We then apply our method on a more complex example corresponding to a two-photon pumping scheme where adiabatic elimination is not standard.

We consider a strongly dissipative quantum harmonic oscillator resonantly coupled to an unspecified quantum system. The state of the complete system is described by a density operator ρ on $\mathcal{H}_A\otimes\mathcal{H}_B$ whose dynamics is given by :

$$\frac{d}{dt}\rho = \mathcal{D}_{\boldsymbol{a}}(\rho) - i\epsilon\left[\boldsymbol{a}\otimes\boldsymbol{B}^\dagger + \boldsymbol{a}^\dagger\otimes\boldsymbol{B}\,,\,\rho\right]$$

where $\boldsymbol{a}$ is the annihilation operator on $\mathcal{H}_A$ associated with the harmonic oscillator (see chapter 4 for more details) and $\boldsymbol{B}$ is an unspecified operator on $\mathcal{H}_B$. The unique steady state of the fast system is the vacuum state: $\overline{\rho}_A = |0\rangle\langle 0|$. We use Theorem 5 to compute the reduced dynamics up to second order. The first order reduced dynamics is readily given by

$$\mathcal{L}_{s,1} = -i\epsilon\left[\mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\boldsymbol{B}^\dagger + \mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\rho}_A\right)\boldsymbol{B}\,,\,\rho_s\right] = 0$$

The Hamiltonian interaction contains two terms, thus $m = 2$ and the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ from Lemma 4 that have to be computed to get the second order reduced dynamics are 2×2 matrices. The main difficulty is to compute, using (3.13), $\boldsymbol{F}_1\overline{\rho}_A = \overline{\tau}\overline{\mathcal{K}}_A\left(\,\boldsymbol{a}\,\overline{\rho}_A\,\right) - \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}\,\overline{\rho}_A\right)\overline{\rho}_A$ and $\boldsymbol{F}_2\overline{\rho}_A = \overline{\tau}\overline{\mathcal{K}}_A\left(\,\boldsymbol{a}^\dagger\,\overline{\rho}_A\,\right) - \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}^\dagger\,\overline{\rho}_A\right)\overline{\rho}_A$. By definition of $\overline{\mathcal{K}}_A$ in Lemma 1 and using $\boldsymbol{a}\overline{\rho}_A = 0$, we have :

$$\boldsymbol{F}_1\overline{\rho}_A = \int_0^{+\infty} e^{t\mathcal{L}_A}\Big(\boldsymbol{a}\overline{\rho}_A\Big)dt = \int_0^{+\infty} e^{t\mathcal{D}_{\boldsymbol{a}}}\Big(\boldsymbol{a}\overline{\rho}_A\Big)dt = 0$$

$$\boldsymbol{F}_2\overline{\rho}_A = \int_0^{+\infty} e^{t\mathcal{D}_{\boldsymbol{a}}}\Big(\boldsymbol{a}^\dagger\overline{\rho}_A\Big)dt$$

Thus, we have to compute the propagator associated with the fast dynamics for the particular initial condition $\rho_0 = \boldsymbol{a}^\dagger \overline{\rho}_A$. Noticing that $\mathcal{D}_{\boldsymbol{a}}(\boldsymbol{a}^\dagger \overline{\rho}_A) = -\frac{1}{2}\boldsymbol{a}^\dagger \boldsymbol{a}\boldsymbol{a}^\dagger \overline{\rho}_A$, we get

$$e^{t\mathcal{D}_{\boldsymbol{a}}}\left(\boldsymbol{a}^\dagger \overline{\rho}_A\right) = e^{-\frac{t}{2}\boldsymbol{a}^\dagger \boldsymbol{a}}\boldsymbol{a}^\dagger \overline{\rho}_A$$

As a result, $\boldsymbol{F}_2 \overline{\rho}_A = 2(\boldsymbol{a}^\dagger \boldsymbol{a})^{-1} |1\rangle\langle 0| \overline{\rho}_A$, where $\bullet^{-1}$ stands for the Moore-Penrose pseudo-inverse. Using formulae (3.12) we are now able to compute the first order parametrisation:

$$\mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) = \exp\Big(- 2i\epsilon(\boldsymbol{a}^\dagger \boldsymbol{a})^{-1} |1\rangle\langle 0| \Big)(\overline{\rho}_A \otimes \rho_s)\exp\Big(2i\epsilon |0\rangle\langle 1| (\boldsymbol{a}^\dagger \boldsymbol{a})^{-1}\Big)$$

Straightforward computations lead to

$$\boldsymbol{X} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \qquad \boldsymbol{Y} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The last step is to compute $\boldsymbol{\Lambda}$ such that $\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\dagger$. Here we simply choose $\boldsymbol{\Lambda} = \sqrt{\boldsymbol{X}}$ leading to the second order reduced dynamics :

$$\frac{d}{dt}\rho_s = \epsilon^2 \mathcal{D}_{2\boldsymbol{B}}(\rho_s) = 4\epsilon^2 \mathcal{D}_{\boldsymbol{B}}(\rho_s)$$

Two-photon pumping scheme

We illustrate now the application of Theorem 5 on the model of the experiment presented in [Coh17, TGL+]. Such system, following the theoretical proposal [MA+14], is a promising way towards dynamically protected quantum processors. The reservoir is based on a scheme that induces 2-photon loss at order ϵ^2 on the target system, as a dominant effect. We show how our method can calculate a more precise reduced model taking into account *all* the effects at order ϵ^2, and thereby quantify the effects of potential 2-photon excitation and cross-Kerr non-linearity on this reservoir.

The system is composed of two interacting cavities: the first cavity (fast subsystem, $\mathcal{H}_A$) is driven by an electromagnetic field and exchanges energy with the environment, with an energy loss term dominant with respect to both the energy gain term and the electromagnetic field drive. This implies that only the lowest energy level is significantly populated, so we can model the cavity as a two level system, i.e. a qubit with energy levels $|g\rangle$ (ground state),$|e\rangle$ (excited state). The second cavity (slow subsystem, $\mathcal{H}_B$) weakly interacts with this qubit, with auxiliary pump fields matched such that the resonant interaction exchanges 1 energy quantum of $\mathcal{H}_A$ with 2 energy quanta of $\mathcal{H}_B$ [Coh17, TGL+]; an additional residual dispersive interaction ("cross-Kerr") is unavoidable in this setup.

The system model thus writes as follows. The standard Pauli matrices are denoted by $\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}, \boldsymbol{\sigma_z}$. $\boldsymbol{\sigma_-} = \frac{\boldsymbol{\sigma_x} - i\boldsymbol{\sigma_y}}{2} = |g\rangle\langle e|$ and $\boldsymbol{\sigma_+} = \frac{\boldsymbol{\sigma_x} + i\boldsymbol{\sigma_y}}{2} = |e\rangle\langle g|$ denote the energy loss operator and energy gain operator respectively, all on $\mathcal{H}_A$, and $\boldsymbol{b}, \boldsymbol{b}^\dagger$ denote respectively the annihilation and creation operators, on $\mathcal{H}_B$. The dynamics of the fast subsystem is described by the Lindbladian operator

$$\mathcal{L}_A = -iu[\boldsymbol{\sigma_y}, \cdot] + \kappa_- \mathcal{D}_{\boldsymbol{\sigma_-}} + \kappa_+ \mathcal{D}_{\boldsymbol{\sigma_+}}, \tag{3.18}$$

where the coupling constants satisfy

$$\frac{\kappa_+}{\kappa_-} \ll 1 \quad , \quad \frac{|u|}{\kappa_-} \ll 1. \tag{3.19}$$

The interaction of the two systems is described by

$$\begin{aligned} \boldsymbol{H}_{int} &= g\boldsymbol{\sigma}_+ \otimes \boldsymbol{b}^2 + g\boldsymbol{\sigma}_- \otimes \boldsymbol{b}^{\dagger 2} + \chi|e\rangle\langle e| \otimes \boldsymbol{b}^\dagger \boldsymbol{b} \\ &= \boldsymbol{A}_1 \otimes \boldsymbol{B}_1^\dagger + \boldsymbol{A}_2 \otimes \boldsymbol{B}_2^\dagger + \boldsymbol{A}_3 \otimes \boldsymbol{B}_3^\dagger . \end{aligned}$$

The fact that $\boldsymbol{H}_{int}$ is much weaker than $\mathcal{L}_A$ is expressed by $|g|, |\chi| \ll \kappa_+, u, \kappa_-$. Thus, the master equation for the composite system is :

$$\frac{d}{dt}\rho = -iu\left[\boldsymbol{\sigma_y}\,,\,\rho\right] + \kappa_-\mathcal{D}_{\boldsymbol{\sigma_-}}(\rho) + \kappa_+\mathcal{D}_{\boldsymbol{\sigma_+}}(\rho) - i\left[g\boldsymbol{\sigma}_+ \otimes \boldsymbol{b}^2 + g\boldsymbol{\sigma}_- \otimes \boldsymbol{b}^{\dagger 2} + \chi|e\rangle\langle e| \otimes \boldsymbol{b}^\dagger\boldsymbol{b}\,,\,\rho\right]$$

It can be formally expressed in the form (3.10) by applying the change of variable $t \leftrightarrow \tau/\kappa_-$ and denoting $g/\kappa_- = \epsilon\tilde{g}$ and $\chi/\kappa_- = \epsilon\tilde{\kappa}$. Similarly, we can take units such that $\kappa_+, |u|, \kappa_-$ are of order 1 or larger, and g, χ are of order $\epsilon \ll 1$. We also consider $\mathcal{L}_B = 0$. In the absence of perturbation, namely $g = \chi = 0$, the two systems are independent. The fast system converges to the steady state

$$\overline{\rho}_A = \frac{\boldsymbol{I} + x_\infty\boldsymbol{\sigma_x} + z_\infty\boldsymbol{\sigma_z}}{2}$$
$$\text{with } x_\infty = \frac{4u(\kappa_+ - \kappa_-)}{(\kappa_+ + \kappa_-)^2 + 8u^2} \quad \text{and} \quad z_\infty = \frac{\kappa_+^2 - \kappa_-^2}{(\kappa_+ + \kappa_-)^2 + 8u^2},$$

while the slow target system B does not move.

When g, χ are non-zero, the dynamics of the slow dynamics, approximately corresponding to the second cavity $\mathcal{H}_B$, can be written in the form (3.15). By simple computations this yields the first-order dynamics of system B, given by the Zeno Hamiltonian:

$$\epsilon\mathcal{L}_{s,1}(\rho_s) = -i\left[\chi\frac{1+z_\infty}{2}\boldsymbol{b}^\dagger\boldsymbol{b} + gx_\infty\frac{\boldsymbol{b}^{\dagger 2} + \boldsymbol{b}^2}{2},\ \rho_s\right].$$

We now compute the second-order dynamics, choosing in equation (3.9) the gauge $\boldsymbol{G}_{1,B}(\rho_s) = -\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A \otimes \rho_s)\right)$. Algebraic computations, by solving for $\mathcal{L}_A^{-1}$ with the Bloch equations for the qubit, yield $\boldsymbol{X}$ and $\boldsymbol{Y}$ of Lemma 4 in the form:

$$\begin{aligned}
\epsilon^2 X_{1,1} &= \left(\tfrac{z_\infty}{2}(3x_\infty^2 - 2) - \tfrac{x_\infty^2}{2} + z_\infty^2 - (z_\infty - 1)\tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right) g^2/(\kappa_- - \kappa_+) \\
\epsilon^2 X_{2,2} &= \left(\tfrac{z_\infty}{2}(3x_\infty^2 - 2) + \tfrac{x_\infty^2}{2} - z_\infty^2 + (z_\infty + 1)\tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right) g^2/(\kappa_- - \kappa_+) \\
\epsilon^2 X_{3,3} &= \tfrac{z_\infty}{2}\left(z_\infty^2 - x_\infty^2 - 1\right)\chi^2/(\kappa_- - \kappa_+) \\
\epsilon^2 X_{1,2} &= \left(\tfrac{z_\infty}{2}(3x_\infty^2 - 2) - \tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right) g^2/(\kappa_- - \kappa_+) \\
\epsilon^2 X_{1,3} &= x_\infty\left(z_\infty^2 - \tfrac{x_\infty^2}{4} - \tfrac{z_\infty}{2} + \tfrac{1}{2}\tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right)\chi g/(\kappa_- - \kappa_+) \\
\epsilon^2 X_{2,3} &= x_\infty\left(z_\infty^2 - \tfrac{x_\infty^2}{4} + \tfrac{z_\infty}{2} - \tfrac{1}{2}\tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right)\chi g/(\kappa_- - \kappa_+)\ ;
\end{aligned}$$

$$\begin{aligned}
\epsilon^2 Y_{1,2} &= -\left(2z_\infty^2 - x_\infty + 2z_\infty\tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right) g^2/(4i(\kappa_- - \kappa_+)) \\
\epsilon^2 Y_{1,3} &= \left(x_\infty - x_\infty z_\infty - x_\infty^3/2 - z_\infty^2 x_\infty - x_\infty\tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right) g\chi/(4i(\kappa_- - \kappa_+)) \\
\epsilon^2 Y_{2,3} &= \left(x_\infty + x_\infty z_\infty - x_\infty^3/2 - z_\infty^2 x_\infty + x_\infty\tfrac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+}\right) g\chi/(4i(\kappa_- - \kappa_+)) .
\end{aligned}$$

Towards interpreting these expressions, we take into account the relative strengths of the couplings (3.19). More precisely, with $\kappa_- = 1 + n_{\text{thermal}}$, $\kappa_+ = n_{\text{thermal}}$ and $n_{\text{thermal}} \ll 1$, we define $\delta^2 = \kappa_+/\kappa_- \approx n_{\text{thermal}}$ and $\eta = u/\kappa_-$, and we neglect the terms of order three

or higher in δ, η. Note that this approximation is made independently from the one corresponding to the adiabatic elimination.

The second order Hamiltonian operator in (3.15) then reads

$$\epsilon^2 \sum_{k,j} Y_{k,j}\, \boldsymbol{B}_k \boldsymbol{B}_j^\dagger \;=\; \frac{2ig\chi}{\kappa_-}\,\eta\left(\boldsymbol{b}^{\dagger 3}\boldsymbol{b} - \boldsymbol{b}^\dagger \boldsymbol{b}^3\right) + \frac{8ig^2}{\kappa_-}\,\eta^2\left(\boldsymbol{b}^{\dagger 4} - \boldsymbol{b}^4\right)$$

up to terms of order at least three in δ, η. For the dissipative part, up to the same terms,

$$\epsilon^2 X \;=\; \frac{1}{\kappa_-}\begin{pmatrix} (4 - 8\delta^2 - 64\eta^2)\, g^2 & -32\eta^2 g^2 & -8\eta g\chi \\ -32\eta^2 g^2 & 4\delta^2 g^2 & 0 \\ -8\eta g\chi & 0 & (2\delta^2 + 16\eta^2)\,\chi^2 \end{pmatrix}.$$

From this form, it is already clear that the dominant effect of the dissipation involves the two-photon annihilation operator. We next write $\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\dagger$, choosing $\boldsymbol{\Lambda}^\dagger$ as an upper triangular matrix:

$$\epsilon\Lambda^\dagger = \frac{1}{\sqrt{\kappa_-}}\begin{pmatrix} 2(1 - \delta^2 - 8\eta^2)g & -16\eta^2 g & -4\eta\chi \\ 0 & 2\delta g & 0 \\ 0 & 0 & \sqrt{2}\delta\chi \end{pmatrix}$$

so that the three dissipation channels are given by the operators

$$\begin{aligned} \epsilon\boldsymbol{L}_1 &= \tfrac{1}{\sqrt{\kappa_-}}\left(2(1\text{-}\delta^2\text{-}8\eta^2)\, g\boldsymbol{b}^2 - 4\eta\,\chi\boldsymbol{b}^\dagger\boldsymbol{b} - 16\eta^2\, g\boldsymbol{b}^{\dagger 2}\right) \\ \epsilon\boldsymbol{L}_2 &= \tfrac{1}{\sqrt{\kappa_-}}\, 2\delta\, g\boldsymbol{b}^{\dagger 2} \\ \epsilon\boldsymbol{L}_3 &= \tfrac{1}{\sqrt{\kappa_-}}\,\sqrt{2}\delta\,\chi\boldsymbol{b}^\dagger\boldsymbol{b} \end{aligned}$$

With the same approximation, the terms in $\mathcal{L}_1$ involve $(1 + z_\infty)/2 = \delta^2 + 4\eta^2$ and $x_\infty/2 = -2\eta$. Wrapping up, the effects on the slow subsystem are thus, with units such that $\kappa_- = 1$:

- At order $(g\eta)$ and (g^2) respectively: a two-photon pumping Hamiltonian, in $\boldsymbol{b}^{\dagger 2} + \boldsymbol{b}^2$, and a two-photon dissipation, with $\boldsymbol{L}_1 \approx \boldsymbol{b}^2$, precisely as intended in [MA+14];
- At order $(\chi\delta^2, \chi\eta^2)$: a Stark shift Hamiltonian $\boldsymbol{b}^\dagger\boldsymbol{b}$, which just shifts the cavity frequency and can be compensated for;
- At order $(g\chi\eta)$: a Hamiltonian in $(\boldsymbol{b}^\dagger\boldsymbol{b}^3 - \boldsymbol{b}^{\dagger 3}\boldsymbol{b})/i$, whose precise deformation effect would have to be investigated; and a modification of the two-photon dissipation channel $\boldsymbol{L}_1$ by some dephasing effect, leading to terms like $\boldsymbol{b}^2\rho\boldsymbol{b}^\dagger\boldsymbol{b}$.
- At order $(\chi^2\delta^2, g^2\delta^2, g^2\eta^2)$: a Hamiltonian effect in $(\boldsymbol{b}^4 - \boldsymbol{b}^{\dagger 4})/i$, that is essentially a 4-photon drive; two new dissipation channels, namely in $\boldsymbol{b}^{\dagger 2}$ and $\boldsymbol{b}^\dagger\boldsymbol{b}$; and a further modification of $\boldsymbol{L}_1$, now leading to terms like $\boldsymbol{b}^2\rho\boldsymbol{b}^2$.

These are the dominant ones provided $|g|, |\chi| \ll |\eta|, |\delta| \ll 1$; once g, χ become comparable to η, δ, effects of order ϵ^3 might become important as well. It would therefore be necessary to go to the next step of our asymptotic expansion method and compute the third order reduce dynamics.

3.2.4 Second order mapping

We now consider the second order parametrisation $\mathcal{K}_0 + \epsilon\mathcal{K}_1 + \epsilon^2\mathcal{K}_2$ and express it in a Kraus form assuming the slow dynamics $\mathcal{L}_B$ contains only Hamiltonian terms i.e. $\mathcal{L}_B(\bullet) = -i\,[\boldsymbol{H}_B, \bullet]$ for some Hermitian operator $\boldsymbol{H}_B$ on $\mathcal{H}_B$.

Theorem 6. The map $\mathcal{K}_0 + \epsilon\mathcal{K}_1 + \epsilon^2\mathcal{K}_2$ corresponding to the parametrisation of the slow manifold admits the following Kraus-form decomposition (up to higher order terms in ϵ):

$$\begin{aligned}\mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) + \epsilon^2\mathcal{K}_2(\rho_s) = (1 - i\epsilon\boldsymbol{M} + \epsilon^2\boldsymbol{N})(\overline{\rho}_A \otimes \rho_s)(1 + i\epsilon\boldsymbol{M} + \epsilon^2\boldsymbol{N}) \\ +\epsilon^2 \sum_{\theta,\mu} \boldsymbol{P}_{\theta,\mu}(\overline{\rho}_A \otimes \rho_s)\boldsymbol{P}_{\theta,\mu}^\dagger\end{aligned} \tag{3.20}$$

With $\boldsymbol{M} = \sum_{k=1}^m \boldsymbol{F}_k \otimes \boldsymbol{B}_k^\dagger$; $\boldsymbol{N} = \sum_{k,j=1}^{m+1} \boldsymbol{U}_{k,j} \otimes \boldsymbol{B}_k\boldsymbol{B}_j^\dagger - \boldsymbol{V}_j \otimes \mathrm{Tr}\left(\boldsymbol{A}_k^\dagger\overline{\rho}_A\right)\boldsymbol{B}_j^\dagger\boldsymbol{B}_k + \sum_k \boldsymbol{V}_k \otimes \left[\boldsymbol{H}_B\,,\,\boldsymbol{B}_k^\dagger\right]$ and $\boldsymbol{P}_{\theta,\mu} = \sum_{k=1}^{m+1} \boldsymbol{W}_\theta[\boldsymbol{L}_{A,\mu}, \boldsymbol{F}_k] \otimes \boldsymbol{B}_k^\dagger$.

Remark 4. One may notice that the second order parametrisation include a slight shift of the first order parametrisation - described by the operator $\boldsymbol{N}$ - similarly than the passage from the zero to first order parametrisation. Moreover, it adds extra terms $\boldsymbol{P}_{\theta,\mu}$ corresponding to new Kraus operators. This expression highlights that in general we have $\mathrm{Tr}_A\left(\mathcal{K}_0 + \epsilon\mathcal{K}_1 + \epsilon^2\mathcal{K}_2\right) + O(\epsilon^3) \neq \rho_s$. It means that, as both subsystems get entangled, the dynamics of the slow system can be expressed in a Lindblad form (corresponding to a Markovian evolution) at the expense of a direct interpretation of the parameter ρ_s as the density operator (via partial trace) of the subsystem B.

Proof. The strategy of the proof is as follows. We start from the recurrence relation (3.6) defining $\mathcal{K}_2$ through the super-operator $\mathcal{L}_A$. First, we solve this equation assuming $\mathcal{L}_B \equiv 0$ to get $\tilde{\mathcal{K}}_2$. Second, we consider only the terms added in (3.6) when $\mathcal{L}_B(\bullet) = -i\,[\boldsymbol{H}_B, \bullet]$ in order to derive $\hat{\mathcal{K}}_2$. The solution is then given by linearity as $\mathcal{K}_2 = \tilde{\mathcal{K}}_2 + \hat{\mathcal{K}}_2$.

In each case we will expand expression (3.6) and then invert $\mathcal{L}_A$ on each expanded term by linearity. Each time we perform an inversion, we take the gauge choice $\boldsymbol{G} = 0$ ensuring via Lemma 1 a completely positive trace preserving mapping $\mathcal{L}_A^{-1}$. Finally we show how all resulting terms combine in order to get equation (3.20).

Assuming $\mathcal{L}_B \equiv 0$, equation (3.6) reads:

$$\mathcal{L}_A(\tilde{\mathcal{K}}_2(\rho_s)) = \ \mathcal{K}_0(\mathcal{L}_{s,2}(\rho_s)) - \mathcal{L}_{int}(\mathcal{K}_1(\rho_s)) + \mathcal{K}_1(\mathcal{L}_{s,1}(\rho_s))$$

We define the super-operator $\mathcal{S}(\boldsymbol{X}) = \boldsymbol{X} - \mathrm{Tr}\,(\boldsymbol{X})\,\overline{\rho}_A = \boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})$ for any operator $\boldsymbol{X}$ on $\mathcal{H}_A$, simplifying the notations. A direct expansion of the previous expression using (3.14) and the fact that $\boldsymbol{H}_{int} = \sum_{k=1}^m \boldsymbol{A}_k \otimes \boldsymbol{B}_k^\dagger$ is Hermitian leads to :

$$\begin{aligned}\mathcal{L}_A(\tilde{\mathcal{K}}_2(\rho_s)) = \sum_{k,j=1}^m & -\Big(\mathcal{S}(\boldsymbol{A}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger) + \mathcal{S}(\boldsymbol{F}_k\overline{\rho}_A\boldsymbol{A}_j^\dagger)\Big) \otimes \boldsymbol{B}_k^\dagger\rho_s\boldsymbol{B}_j \\ & + \mathcal{S}(\overline{\rho}_A\boldsymbol{F}_j^\dagger\boldsymbol{A}_k) \otimes \rho_s\boldsymbol{B}_j\boldsymbol{B}_k^\dagger + \mathcal{S}(\boldsymbol{A}_k^\dagger\boldsymbol{F}_j\overline{\rho}_A) \otimes \boldsymbol{B}_k\boldsymbol{B}_j^\dagger\rho_s \\ & - \boldsymbol{F}_j\overline{\rho}_A \otimes \boldsymbol{B}_j^\dagger\Big[\mathrm{Tr}\left(\boldsymbol{A}_k^\dagger\overline{\rho}_A\right)\boldsymbol{B}_k\,,\,\rho_s\Big] + \overline{\rho}_A\boldsymbol{F}_j^\dagger \otimes \Big[\mathrm{Tr}\,(\boldsymbol{A}_k\overline{\rho}_A)\,\boldsymbol{B}_k^\dagger\,,\,\rho_s\Big]\boldsymbol{B}_j\end{aligned} \tag{3.21}$$

where $\boldsymbol{F}_k$ is defined in (3.13). By inverting the index notations j and k in the last line, it

can be partially recombined with the first one to get, denoting $\overline{\boldsymbol{A}}_k = \boldsymbol{A}_k - \mathrm{Tr}\left(\boldsymbol{A}_k\overline{\rho}_A\right)$:

$$\begin{aligned}\mathcal{L}_A(\tilde{\mathcal{K}}_2(\rho_s)) = \sum_{k,j=1}^{m} &-\Big(\mathcal{S}(\overline{\boldsymbol{A}}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger) + \mathcal{S}(\boldsymbol{F}_k\overline{\rho}_A\overline{\boldsymbol{A}}_j^\dagger)\Big)\otimes\boldsymbol{B}_k^\dagger\rho_s\boldsymbol{B}_j \\ &+ \mathcal{S}(\overline{\rho}_A\boldsymbol{F}_j^\dagger\boldsymbol{A}_k)\otimes\rho_s\boldsymbol{B}_j\boldsymbol{B}_k^\dagger + \mathcal{S}(\boldsymbol{A}_k^\dagger\boldsymbol{F}_j\overline{\rho}_A)\otimes\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\rho_s \\ &- \boldsymbol{F}_j\overline{\rho}_A\otimes\mathrm{Tr}\left(\boldsymbol{A}_k^\dagger\overline{\rho}_A\right)\boldsymbol{B}_j^\dagger\boldsymbol{B}_k\rho_s - \overline{\rho}_A\boldsymbol{F}_j^\dagger\otimes\mathrm{Tr}\left(\boldsymbol{A}_k\overline{\rho}_A\right)\rho_s\boldsymbol{B}_k^\dagger\boldsymbol{B}_j\end{aligned} \tag{3.22}$$

By virtue of Lemma 3, there exists two set of matrices $\{\boldsymbol{U}_{kj}\}$ and $\{\boldsymbol{V}_j\}$ such that for all $\{j,k\}\in\{1,2,\ldots,m\}^2$,

$$\mathcal{L}_A^{-1}\Big(\mathcal{S}(\boldsymbol{A}_k^\dagger\boldsymbol{F}_j\overline{\rho}_A)\Big) = -\overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{A}_k^\dagger\boldsymbol{F}_j\overline{\rho}_A) \triangleq \boldsymbol{U}_{kj}\overline{\rho}_A \tag{3.23}$$

$$\mathcal{L}_A^{-1}\Big(\boldsymbol{F}_j\overline{\rho}_A\Big) \triangleq \boldsymbol{V}_j\overline{\rho}_A \tag{3.24}$$

We now turn to the operator acting on $\mathcal{H}_A$ in (3.22): $-\Big(\mathcal{S}(\overline{\boldsymbol{A}}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger)+\mathcal{S}(\boldsymbol{F}_k\overline{\rho}_A\overline{\boldsymbol{A}}_j^\dagger)\Big)$. From the definition of $\overline{\mathcal{K}}_A$ and $\boldsymbol{F}_k$, we have $\mathcal{L}_A(\boldsymbol{F}_k\overline{\rho}_A) = \mathcal{L}_A(\overline{\tau}\overline{\mathcal{K}}(\boldsymbol{A}_k\overline{\rho}_A)) = -\boldsymbol{A}_k\overline{\rho}_A + \overline{\mathcal{R}}(\boldsymbol{A}_k\overline{\rho}_A)$. Therefore we get,

$$-\Big(\mathcal{S}(\overline{\boldsymbol{A}}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger) + \mathcal{S}(\boldsymbol{F}_k\overline{\rho}_A\overline{\boldsymbol{A}}_j^\dagger)\Big) = \mathcal{S}\Big(\mathcal{L}_A(\boldsymbol{F}_k\overline{\rho}_A)\boldsymbol{F}_j^\dagger\Big) + \mathcal{S}\Big(\boldsymbol{F}_k\mathcal{L}_A(\overline{\rho}_A\boldsymbol{F}_j^\dagger)\Big)$$

The Lemma 14 presented in appendix B allows to reformulate the previous equation as :

$$\mathcal{S}\Big(\mathcal{L}_A(\boldsymbol{F}_k\overline{\rho}_A)\boldsymbol{F}_j^\dagger\Big) + \mathcal{S}\Big(\boldsymbol{F}_k\mathcal{L}_A(\overline{\rho}_A\boldsymbol{F}_j^\dagger)\Big) = \mathcal{L}_A(\boldsymbol{F}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger) - \mathcal{S}\Big(\sum_\mu[\boldsymbol{L}_{A,\mu},\boldsymbol{F}_k]\overline{\rho}_A[\boldsymbol{L}_{A,\mu},\boldsymbol{F}_j]^\dagger\Big).$$

where $\boldsymbol{L}_{A,\mu}$ are the decoherence operators associated to the fast dynamics $\mathcal{L}_A$ and we used the fact that $\mathrm{Tr}\left(\mathcal{L}_A(\bullet)\right) = 0$. Then we apply the inverse of $\mathcal{L}_A$ which yields

$$\mathcal{L}_A^{-1}\Big(\mathcal{S}\Big(\mathcal{L}_A(\boldsymbol{F}_k\overline{\rho}_A)\boldsymbol{F}_j^\dagger\Big) + \mathcal{S}\Big(\boldsymbol{F}_k\mathcal{L}_A(\overline{\rho}_A\boldsymbol{F}_j^\dagger)\Big)\Big) = \boldsymbol{F}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger + \overline{\tau}\overline{\mathcal{K}}_A\left(\sum_\mu[\boldsymbol{L}_{A,\mu},\boldsymbol{F}_k]\overline{\rho}_A[\boldsymbol{L}_{A,\mu},\boldsymbol{F}_j]^\dagger\right) \tag{3.25}$$

Note that $\overline{\tau}\overline{\mathcal{K}}_A$ is a completely positive map. Therefore there exists a set of matrices $\{\boldsymbol{W}_\theta\}$ such that $\overline{\tau}\overline{\mathcal{K}}_A(\bullet) = \sum_{\theta=1}^m \boldsymbol{W}_\theta\bullet\boldsymbol{W}_\theta^\dagger$. By gathering equations (3.23),(3.24),(3.25) we get the following expression $\tilde{\mathcal{K}}_2$:

$$\begin{aligned}\tilde{\mathcal{K}}_2(\rho_s) = \sum_{k,j,\theta,\mu}&\left(\boldsymbol{F}_k\overline{\rho}_A\boldsymbol{F}_j^\dagger + \boldsymbol{W}_\theta[\boldsymbol{L}_{A,\mu},\boldsymbol{F}_k]\overline{\rho}_A[\boldsymbol{L}_{A,\mu},\boldsymbol{F}_j]^\dagger\boldsymbol{W}_\theta^\dagger\right)\otimes\boldsymbol{B}_k^\dagger\rho_s\boldsymbol{B}_j \\ &+ \overline{\rho}_A\boldsymbol{U}_{k,j}^\dagger\otimes\rho_s\boldsymbol{B}_j\boldsymbol{B}_k^\dagger + \boldsymbol{U}_{k,j}\overline{\rho}_A\otimes\boldsymbol{B}_k\boldsymbol{B}_j^\dagger\rho_s \\ &- \boldsymbol{V}_j\overline{\rho}_A\otimes\mathrm{Tr}\left(\boldsymbol{A}_k^\dagger\overline{\rho}_A\right)\boldsymbol{B}_j^\dagger\boldsymbol{B}_k\rho_s - \overline{\rho}_A\boldsymbol{V}_j^\dagger\otimes\mathrm{Tr}\left(\boldsymbol{A}_k\overline{\rho}_A\right)\rho_s\boldsymbol{B}_k^\dagger\boldsymbol{B}_j\end{aligned}$$

We now consider the existence of a dynamics on the slow subsystem $\mathcal{L}_B$. Without taking into account the previously computed terms, equation (3.6) yields :

$$\mathcal{L}_A(\hat{\mathcal{K}}_2(\rho_s)) + \mathcal{L}_B(\mathcal{K}_1(\rho_s)) = \mathcal{K}_1(\mathcal{L}_B(\rho_s))$$

By using the computation of $\mathcal{K}_1$ in (3.14) and that $\mathcal{L}_B(\bullet) = -i\left[\boldsymbol{H}_B,\bullet\right]$, a direct development yields

$$\begin{aligned}\mathcal{L}_A(\hat{\mathcal{K}}_2(\rho_s)) &= \sum_{k=1}^m \boldsymbol{F}_k\overline{\rho}_A\otimes\Big(\Big[\boldsymbol{H}_B\,,\,\boldsymbol{B}_k^\dagger\,\rho_s\Big] - \boldsymbol{B}_k^\dagger\left[\boldsymbol{H}_B\,,\,\rho_s\right]\Big) + \text{Herm. conj.} \\ &= \sum_{k=1}^m \boldsymbol{F}_k\overline{\rho}_A\otimes\Big[\boldsymbol{H}_B\,,\,\boldsymbol{B}_k^\dagger\Big]\rho_s + \overline{\rho}_A\boldsymbol{F}_k^\dagger\otimes\rho_s\Big[\boldsymbol{H}_B\,,\,\boldsymbol{B}_k^\dagger\Big]^\dagger\end{aligned} \tag{3.26}$$

We inverse then the fast dynamics and use the definition of $\boldsymbol{V}_k$ in equation (3.24) to get :

$$\hat{\mathcal{K}}_2(\rho_s) = \sum_{k=1}^{m} \boldsymbol{V}_k \overline{\rho}_A \otimes \left[\boldsymbol{H}_B\,,\,\boldsymbol{B}_k^\dagger\right] \rho_s + \overline{\rho}_A \boldsymbol{V}_k^\dagger \otimes \rho_s \left[\boldsymbol{H}_B\,,\,\boldsymbol{B}_k^\dagger\right]^\dagger$$

A direct expansion of (3.20) leads to the previous equation for $\mathcal{K}_2 = \tilde{\mathcal{K}}_2 + \hat{\mathcal{K}}_2$ up to order-three-terms in ϵ. We emphasise here that the first order parametrisation $\mathcal{K}_0 + \epsilon\mathcal{K}_1$ given in (3.12) already included (neglected up to now) order two terms. These terms aren't neglected at this order of precision and exactly appears in the expression of $\tilde{\mathcal{K}}_2$. ■

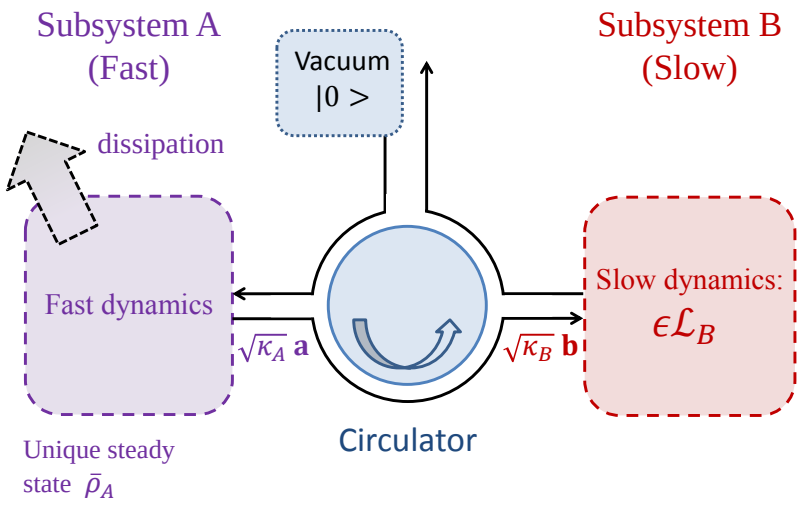

Figure 3.2: Schematic representation of a cascade interconnection between a fast system A and a slow system B. This coupling is obtain by introducing a circulator between an input/output port of the system A associated with the operator $\sqrt{\kappa_A}\boldsymbol{a}$ and an input/output port of the system B associated with the operator $\sqrt{\kappa_B}\boldsymbol{b}$. The dynamics of the fast system A summed to the dynamics due to the cascade interaction is assumed to be stable and to converge towards a unique steady state $\overline{\rho}_A$ in the absence of perturbation.

3.3 Cascaded interaction

We consider in this section a different kind of interaction between the fast and the slow subsystem. Instead of the standard reciprocal Hamiltonian coupling of Section 3.2 we consider a unidirectional coupling which allows the output of the first system to feed the second system while forbidding the reverse process. This is performed by means of a *circulator*, a 3-port device transferring each input to the next one, as illustrated in figure 3.3. It can be used e.g. in quantum networks in order to create entanglement or to transfer quantum states from different nodes [CZK97, SRZ12].

The expression *cascaded system* was first introduced in [Car93b], see also e.g. [GZ10] for more details on this type of interaction. A standard assumption is to consider that the transmissions between the devices are instantaneous. Under this assumption, the SLH formalism developed in [GJ09] is particularly useful to handle this kind of interaction or any network extension and write down the associated dynamics. The study of commutation between the limit of instantaneous transmissions and the adiabatic elimination limit can be found in [GNW10, NG12].

We consider a general system A (respectively B) on a Hilbert space $\mathcal{H}_A$ (resp. $\mathcal{H}_B$), with internal dynamics $\mathcal{L}_A$ (resp. $\mathcal{L}_B$) and an output channel with associated operator $\sqrt{\kappa_A}\boldsymbol{a}$ (resp. $\sqrt{\kappa_B}\boldsymbol{b}$). The coefficients κ_A and κ_B describe the relative strength of the output channels, ρ is the density operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. Using [GJ09], the master equation describing the cascade connection of the system A towards the system B represented in

figure 3.3 is given by :

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) + \mathcal{D}_{\sqrt{\kappa_A}\boldsymbol{a}+\sqrt{\kappa_B}\boldsymbol{b}}(\rho) + \frac{\sqrt{\kappa_A}\sqrt{\kappa_B}}{2}[\boldsymbol{a}^\dagger\boldsymbol{b} - \boldsymbol{a}\boldsymbol{b}^\dagger, \rho] + \tilde{\mathcal{L}}_B(\rho) \tag{3.27}$$

where $\mathcal{L}_A$ and $\tilde{\mathcal{L}}_B$ are Lindbladian super-operators acting only on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and where $\boldsymbol{a}$ and $\boldsymbol{b}$ are operators acting only on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. In order to be able to perform our adiabatic elimination technique we will consider that the system A is fast, while the system B is slow. There is however two different ways to make such a distinction. We will treat both cases with our approach to derive a reduced dynamics in a Lindblad-form up to second order in the small parameter:

- In the first case, considered in section 3.3.1, we take $\kappa_A = \kappa_B = \epsilon$: the fast system A features a weak outcoming field coupled to the subsystem B. The dynamics on the system B is assumed to be slow, thus $\tilde{\mathcal{L}}_B = \epsilon\mathcal{L}_B$. The subsystem A is barely perturbed by the coupling and we can investigate e.g. how a purely quantum state in A propagates to the subsystem B (see chapter 5).
- In the second case, we take $\sqrt{\kappa_A} = 1$ and $\sqrt{\kappa_B} = \epsilon$, with $0 < \epsilon \ll 1$. It corresponds to the fact that the outcoming field of the system A is mostly reflected at the input of the system B and only a small part will get through and interact with the system B. This case is treated in section 3.3.2. It has been shown that it is possible for instance to engineer to coherence times of the subsystem B using a squeezed input field [Gar86].

3.3.1 Cascade with weak outcoming field

We consider equation (3.27) with $\kappa_A = \kappa_B := \epsilon$ and $\tilde{\mathcal{L}}_B := \epsilon\mathcal{L}_B$. The Lindblad master equation of the studied system reads :

$$\begin{aligned}\frac{d}{dt}\rho &= \mathcal{L}_A(\rho) + \frac{\epsilon}{2}\left[\boldsymbol{a}^\dagger\boldsymbol{b} - \boldsymbol{a}\boldsymbol{b}^\dagger\,,\,\rho\right] + \epsilon\mathcal{D}_{\boldsymbol{a}+\boldsymbol{b}}(\rho) + \epsilon\mathcal{L}_B(\rho)\\ &= \mathcal{L}_A(\rho) + \epsilon\Big(\mathcal{D}_{\boldsymbol{a}}(\rho) + \mathcal{D}_{\boldsymbol{b}}(\rho) + \boldsymbol{a}\rho\boldsymbol{b}^\dagger + \boldsymbol{b}\rho\boldsymbol{a}^\dagger - \boldsymbol{b}^\dagger\boldsymbol{a}\rho - \rho\boldsymbol{a}^\dagger\boldsymbol{b}\Big) + \epsilon\mathcal{L}_B(\rho)\end{aligned} \tag{3.28}$$

We assume the existence of a unique steady state $\overline{\rho}_A$ for the unperturbed system i.e. when $\epsilon = 0$ verifying $\mathcal{L}_A(\overline{\rho}_A) = 0$.

Theorem 7. The second order reduced dynamics of two systems with cascaded interactions whose dynamics is given by (3.28) reads :

$$\begin{aligned}\frac{d}{dt}\rho_s = \epsilon\Big(\left[\mathrm{Tr}\left(\overline{\rho}_A\boldsymbol{a}^\dagger\right)\boldsymbol{b} - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\boldsymbol{b}^\dagger\,,\,\rho_s\right] + \mathcal{L}_B(\rho_s)\Big) + \epsilon\mathcal{D}_{\sqrt{1+\epsilon(\alpha+\alpha^*)}\boldsymbol{b}-2\epsilon\beta^*\boldsymbol{b}^\dagger}(\rho_s)\\ +\epsilon^2\Big((\alpha+\alpha^*)\mathcal{D}_{\boldsymbol{b}^\dagger}(\rho_s) + \frac{\alpha^*-\alpha}{2}\left[\boldsymbol{b}^\dagger\boldsymbol{b} - \boldsymbol{b}\boldsymbol{b}^\dagger\,,\,\rho_s\right] + \left[\gamma^*\boldsymbol{b} - \gamma\boldsymbol{b}^\dagger\,,\,\rho_s\right]\Big)\end{aligned} \tag{3.29}$$

where $\alpha = \mathrm{Tr}\left(\boldsymbol{a}\overline{\tau}\overline{\mathcal{K}}_A\left[\overline{\rho}_A\left(\boldsymbol{a}^\dagger - \mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\rho}_A\right)\right)\right]\right)$, $\beta = \mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\tau}\overline{\mathcal{K}}_A\left[\overline{\rho}_A\left(\boldsymbol{a}^\dagger - \mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\rho}_A\right)\right)\right]\right)$ and $\gamma = \mathrm{Tr}\left(\boldsymbol{a}\overline{\tau}\overline{\mathcal{K}}_A(\mathcal{D}_{\boldsymbol{a}}(\overline{\rho}_A))\right)$.

Proof. In order to be consistent with initial notations one would take the coupling operator as :

$$\mathcal{L}_{int}(\rho) = \mathcal{D}_{\boldsymbol{a}}(\rho) + \mathcal{D}_{\boldsymbol{b}}(\rho) + \boldsymbol{a}\rho\boldsymbol{b}^\dagger + \boldsymbol{b}\rho\boldsymbol{a}^\dagger - \boldsymbol{b}^\dagger\boldsymbol{a}\rho - \rho\boldsymbol{a}^\dagger\boldsymbol{b}$$

The zero order terms are straightforwardly given by $\mathcal{L}_{s,0} \equiv 0$ and $\mathcal{K}_0(\rho_s) = \overline{\rho}_A \otimes \rho_s$. Then, by a direct application of equation (3.8) we readily get the first order reduced dynamics, parametrised by the slow density operator ρ_s :

$$\mathcal{L}_{s,1}(\rho_s) = \left[\mathrm{Tr}\left(\overline{\rho}_A \boldsymbol{a}^\dagger\right)\boldsymbol{b} - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\boldsymbol{b}^\dagger\,,\,\rho_s\right] + \mathcal{D}_{\boldsymbol{b}}(\rho_s) + \mathcal{L}_B(\rho_s)$$

As stated in section 3.1, the gauge degree of freedom resulting from the inversion of the super-operator $\mathcal{L}_A$ allows us to select between systems properties. Using the gauge choice $\boldsymbol{G}_{1,B} = 0$ leads in a Kraus form for the first order parametrisation of the slow manifold. Indeed, we have

$$\mathcal{K}_1(\rho_s) = \overline{\tau}\overline{\mathcal{K}}_A\Big(\mathcal{D}_{\boldsymbol{a}}(\overline{\rho}_A)\otimes\rho_s + \boldsymbol{a}\overline{\rho}_A\otimes\rho_s\boldsymbol{b}^\dagger - \boldsymbol{a}\overline{\rho}_A\otimes\boldsymbol{b}^\dagger\rho_s + \overline{\rho}_A\boldsymbol{a}^\dagger\otimes\boldsymbol{b}\rho_s - \overline{\rho}_A\boldsymbol{a}^\dagger\otimes\rho_s\boldsymbol{b} + \overline{\rho}_A\otimes\mathcal{D}_{\boldsymbol{b}}(\rho_s)\Big)$$

And thus $\mathcal{K}_0 + \epsilon\mathcal{K}_1$ can be expressed in the Kraus form

$$\mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) = \overline{\tau}\overline{\mathcal{K}}_A\Big(\Big(\mathbf{1}_{\mathcal{H}} + \epsilon\boldsymbol{M}\Big)\overline{\rho}\otimes\rho_s\Big(\mathbf{1}_{\mathcal{H}} + \epsilon\boldsymbol{M}^\dagger\Big) + \epsilon\boldsymbol{N}(\overline{\rho}_A\otimes\rho_s)\boldsymbol{N}^\dagger\Big)\Big)$$

with $\boldsymbol{M} = -\boldsymbol{a}\otimes\boldsymbol{b}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\otimes\mathbf{1}_{\mathcal{H}_B} - \frac{1}{2}\mathbf{1}_{\mathcal{H}_A}\otimes\boldsymbol{b}^\dagger\boldsymbol{b}$ and $\boldsymbol{N} = \boldsymbol{a}\otimes\mathbf{1}_{\mathcal{H}_B} + \mathbf{1}_{\mathcal{H}_A}\otimes\boldsymbol{b}$.

The gauge choice $\boldsymbol{G}_{1,B} = -\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A\otimes\rho_s)\right)$ leads however to a simpler expression of the second order reduced dynamics and will be used in the remaining of the proof. This choice leads to

$$\mathcal{K}_1(\rho_s) = \boldsymbol{F}\overline{\rho}_A\otimes\rho_s\boldsymbol{b}^\dagger + \overline{\rho}_A\boldsymbol{F}^\dagger\otimes\boldsymbol{b}\rho_s - \boldsymbol{F}\overline{\rho}_A\otimes\boldsymbol{b}^\dagger\rho_s - \overline{\rho}_A\boldsymbol{F}^\dagger\otimes\rho_s\boldsymbol{b} + \boldsymbol{Q}\otimes\rho_s$$

where $\boldsymbol{F}\overline{\rho}_A = \overline{\tau}\overline{\mathcal{K}}_A\left(\boldsymbol{a}\,\overline{\rho}_A\right) - \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}\,\overline{\rho}_A\right)\overline{\rho}_A$ satisfies $\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\right) = 0$ and $\boldsymbol{Q} = \overline{\tau}\overline{\mathcal{K}}_A(\mathcal{D}_{\boldsymbol{a}}(\overline{\rho}_A))$.

Similar to the computation of the second order reduced dynamics in the case of Hamiltonian coupling, taking the partial trace over the subsystem A in equation (3.6) yields:

$$\mathcal{L}_{s,2} = \mathrm{Tr}_A\left(\mathcal{L}_{int}\Big(\mathcal{K}_1(\rho_s)\Big) + \mathcal{L}_B(\mathcal{K}_1(\rho_s))\right)$$

Noticing that $\mathcal{L}_B$ and $\mathcal{D}_{\boldsymbol{b}}$ acts non-trivially only on $\mathcal{H}_B$ and that $\mathrm{Tr}\left(\mathcal{K}_1(\bullet)\right) \equiv 0$ together with the fact that $\mathrm{Tr}\left(\mathcal{D}_{\boldsymbol{a}}(\bullet)\right) \equiv 0$ we get,

$$\mathcal{L}_{s,2} = \mathrm{Tr}_A\left(\boldsymbol{a}\mathcal{K}_1(\rho_s)\boldsymbol{b}^\dagger + \boldsymbol{b}\mathcal{K}_1(\rho_s)\boldsymbol{a}^\dagger - \boldsymbol{b}^\dagger\boldsymbol{a}\mathcal{K}_1(\rho_s) - \mathcal{K}_1(\rho_s)\boldsymbol{a}^\dagger\boldsymbol{b}\right)\Big)$$

We denote $\alpha = \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\boldsymbol{F}^\dagger\right)$ and $\beta = \mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\rho}_A\boldsymbol{F}^\dagger\right)$ and $\gamma = \mathrm{Tr}\left(\boldsymbol{a}\boldsymbol{Q}\right)$. A direct expansion yields

$$\begin{aligned}
\mathcal{L}_{s,2} = {} & \beta^*\rho_s\boldsymbol{b}^\dagger\boldsymbol{b}^\dagger + \alpha\boldsymbol{b}\rho_s\boldsymbol{b}^\dagger - \beta^*\boldsymbol{b}^\dagger\rho_s\boldsymbol{b}^\dagger - \alpha\rho_s\boldsymbol{b}\boldsymbol{b}^\dagger + \gamma\rho_s\boldsymbol{b}^\dagger \\
& + \alpha^*\boldsymbol{b}\rho_s\boldsymbol{b}^\dagger + \beta\boldsymbol{b}\boldsymbol{b}\rho_s - \alpha^*\boldsymbol{b}\boldsymbol{b}^\dagger\rho_s - \beta\boldsymbol{b}\rho_s\boldsymbol{b} + \gamma^*\boldsymbol{b}\rho_s \\
& - \beta^*\boldsymbol{b}^\dagger\rho_s\boldsymbol{b}^\dagger - \alpha\boldsymbol{b}^\dagger\boldsymbol{b}\rho_s + \beta^*\boldsymbol{b}^\dagger\boldsymbol{b}^\dagger\rho_s + \alpha\boldsymbol{b}^\dagger\rho_s\boldsymbol{b} - \gamma\boldsymbol{b}^\dagger\rho_s \\
& - \alpha^*\rho_s\boldsymbol{b}^\dagger\boldsymbol{b} - \beta\boldsymbol{b}\rho_s\boldsymbol{b} + \alpha^*\boldsymbol{b}^\dagger\rho_s\boldsymbol{b} + \beta\rho_s\boldsymbol{b}\boldsymbol{b} - \gamma^*\rho_s\boldsymbol{b}
\end{aligned}$$

Contrary to Hamiltonian coupling, it is possible to show on particular examples that the super-operator $\mathcal{L}_{s,2}$ can't be cast in a Lindblad-form. However we show that $\epsilon\mathcal{L}_{s,1} + \epsilon^2\mathcal{L}_{s,2}$ can be expressed in a Lindblad-form (up to order three terms), as imposed in Section 2.3.3. The intuitive reason is that the super-operator $\mathcal{D}_{\boldsymbol{b}}$ appearing in $\mathcal{L}_{s,1}$ is entirely related to the cascaded structure (in opposition, the term $\mathcal{L}_B$ is independent of it and can be

chosen arbitrarily) and as so, has to be taken into account while performing adiabatic elimination. We are thus looking for a Lindblad-form expression for the super-operator $\mathcal{D}_{\boldsymbol{b}}(\rho_s) + \epsilon\mathcal{L}_{s,2}(\rho_s)$. The expression of $\mathcal{L}_{s,2}$ contains only two different operators on $\mathcal{H}_B$: $\boldsymbol{b}$ and $\boldsymbol{b}^\dagger$. Therefore, we search a second order reduced dynamics in the following form :

$$\mathcal{D}_{\boldsymbol{b}}(\rho_s) + \epsilon\mathcal{L}_{s,2}(\rho_s) = \sum_{j=1}^{j_{max}=2} \mathcal{D}_{x_j\boldsymbol{b}+y_j\boldsymbol{b}^\dagger}(\rho_s) - i\left[\boldsymbol{H}_{s,2}\,,\,\rho_s\right] + o(\epsilon) \tag{3.30}$$

where $\boldsymbol{H}_{s,2} = -i\left[\omega_1\boldsymbol{b}^\dagger\boldsymbol{b} + \omega_2\boldsymbol{b}\boldsymbol{b}^\dagger\,,\,\rho_s\right] + \left[\theta\boldsymbol{b} - \theta^*\boldsymbol{b}^\dagger\,,\,\rho_s\right]$ with $\{x_j\}, \{y_j\}, \theta \in \mathbb{C}$ and $\omega_1, \omega_2 \in \mathbb{R}$.

In principle j_{max} can be arbitrary large. However from the computations of the reduced dynamics for the Hamiltonian case, we saw that there were at most as much decoherence channel as the number of operators appearing in the coupling. We may assume that it is true also for the case of cascaded coupling, which turn out to be true as presented below.

We expand the right and expression and using a term-by-terms identification we have to solve the following system of equations :

$$\begin{cases} 1 + \epsilon(\alpha + \alpha^*) = \sum_j |x_j|^2 + o(\epsilon) \\ -\frac{1}{2} - \epsilon\alpha = \sum_j -\frac{1}{2}|x_j|^2 - i\omega_1 + o(\epsilon) \\ -2\epsilon\beta = \sum_j x_j y_j^* + o(\epsilon) \\ \epsilon(\alpha + \alpha^*) = \sum_j |y_j|^2 + o(\epsilon) \\ -\epsilon\alpha^* = \sum_j -\frac{1}{2}|y_j|^2 - i\omega_2 + o(\epsilon) \\ \epsilon\gamma^* = \theta + o(\epsilon) \end{cases}$$

Using a slight adaptation of Lemma 15 to this case we get that $\alpha + \alpha^* \geq 0$. Therefore this set of equation admits a solution. There may be several solutions as the Lindblad decomposition (3.30) is not unique. A simple solution is given e.g. by :

$$x_1 = \sqrt{1 + \epsilon(\alpha + \alpha^*)} \qquad x_2 = 0$$
$$y_1 = -2\epsilon\beta^* \qquad y_2 = \sqrt{\epsilon(\alpha + \alpha^*)}$$
$$\omega_1 = \frac{i\epsilon}{2}(\alpha^* - \alpha) \qquad \omega_2 = \frac{i\epsilon}{2}(\alpha - \alpha^*)$$

Inserting these expressions into (3.30), one gets equation (3.29), concluding the proof. ■

3.3.2 Cascade with strongly reflected field

Second order adiabatic elimination

We consider now equation (3.27) with $\kappa_A = 1$ and $\sqrt{\kappa_B} = \epsilon$. corresponding to a composite system with cascaded interaction where the outcoming field of the fast system is strongly reflected at the input of the slow system. The corresponding master equation governing

the dynamics of the density operator ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is given by:

$$\begin{aligned}\frac{d}{dt}\rho &= \mathcal{L}_A(\rho) + \mathcal{D}_{\boldsymbol{a}+\epsilon\boldsymbol{b}}(\rho) + \frac{\epsilon}{2}[\boldsymbol{a}^\dagger\boldsymbol{b} - \boldsymbol{a}\boldsymbol{b}^\dagger, \rho] + \epsilon^2\mathcal{L}_B(\rho) \\ &= \mathcal{L}_A(\rho) + \mathcal{D}_{\boldsymbol{a}}(\rho) + \epsilon\Big(\boldsymbol{a}[\rho, \boldsymbol{b}^\dagger] + [\boldsymbol{b}, \rho]\boldsymbol{a}^\dagger\Big) + \epsilon^2\Big(\mathcal{D}_{\boldsymbol{b}}(\rho) + \mathcal{L}_B(\rho)\Big)\end{aligned} \tag{3.31}$$

We assume that there exists a unique steady state density operator $\overline{\rho}_A$ solution of $\mathcal{L}_A(\rho) + \mathcal{D}_{\boldsymbol{a}}(\rho) = 0$. Note that unlike the Hamiltonian interaction case, the fast dynamics is given by the super-operator $\mathcal{L}_A(\rho)$ but also by $\mathcal{D}_{\boldsymbol{a}}(\rho)$ due to the unidirectional coupling. The completely positive map $\overline{\mathcal{K}}_A$ of Lemma 1 is defined accordingly. Thus for any operator $\boldsymbol{X}$ on $\mathcal{H}_A$ we have,

$$-\mathcal{L}_A\left(\overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{X})\right) - \mathcal{D}_{\boldsymbol{a}}\left(\overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{X})\right) = \boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})$$

Before presenting the second order reduced dynamics associated with system (3.31), we introduce an assumption needed to ensure the Lindblad form of the reduced model.

Assumption 1. For $\overline{\boldsymbol{a}} = \boldsymbol{a} - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\mathbf{1}_{\mathcal{H}_A}$, we define $\alpha = \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}\overline{\mathcal{K}}_A\left(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right)\right)$ and $\beta = \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\mathcal{K}}_A(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger)\right)$. We assume that they verify :

$$(\alpha + \alpha^* + 1)(\alpha + \alpha) \geq 4|\beta|^2$$

We conjecture that this assumption is true for any quantum system whose dynamics is governed by (3.31). However, even if α and β seems closely related, the complexity of the mapping $\overline{\mathcal{K}}_A$ make the analysis difficult for a general setting.

Theorem 8. The reduced dynamics of the cascaded system whose dynamics is given by (3.31) can be expressed in a Lindblad-form if assumption 1 holds. In this case, it is given by the following expression :

$$\begin{aligned}\frac{d}{dt}\rho_s =& \epsilon\left[\mathrm{Tr}\left(\overline{\rho}_A\boldsymbol{a}^\dagger\right)\boldsymbol{b} - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\boldsymbol{b}^\dagger, \rho_s\right] + \epsilon^2\Big(\mathcal{L}_B(\rho_s) + \mathcal{D}_{x_1\boldsymbol{b}+y_1\boldsymbol{b}^\dagger}(\rho_s) + \mathcal{D}_{y_2\boldsymbol{b}^\dagger}(\rho_s)\Big) \\ &- i(\alpha - \alpha^*)\left[\boldsymbol{b}^\dagger\boldsymbol{b} - \boldsymbol{b}\boldsymbol{b}^\dagger\,,\, \rho_s\right]\end{aligned} \tag{3.32}$$

where $\{x_1, y_1, y_2\} \in \mathbb{C}^3$ are given by

$$\begin{cases} x_1 = \sqrt{1 + \alpha + \alpha^*} \\ y_1 = \dfrac{-2\beta^*}{\sqrt{1 + \alpha + \alpha^*}} \\ y_2 = \sqrt{\dfrac{(1 + \alpha + \alpha^*)(\alpha + \alpha^*) - 4|\beta|^2}{1 + \alpha + \alpha^*}} \end{cases} \tag{3.33}$$

with $\alpha = \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}\overline{\mathcal{K}}_A\left(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right)\right)$ and $\beta = \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\mathcal{K}}_A(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger)\right)$.

Proof. As usually, from (3.4), the zero order approximation is readily given by $\mathcal{L}_{s,0}(\rho_s) = 0$ and $\mathcal{K}_0(\rho_s) = \overline{\rho}_A \otimes \rho_s$.

The first order reduced dynamics can be directly computed using equation (3.8). A straightforward calculation yields:

$$\mathcal{L}_{s,1}(\rho_s) = \left[\mathrm{Tr}\left(\overline{\rho}_A\boldsymbol{a}^\dagger\right)\boldsymbol{b} - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\boldsymbol{b}^\dagger, \rho_s\right] \tag{3.34}$$

The super-operator $\mathcal{K}_1$ corresponding to the first order parametrisation of the slow manifold and that can be interpreted as the entanglement between the two subsystems is given by (3.9). From this expression, with the gauge choice $\boldsymbol{G}_{1,B}(\rho_s) = -\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A \otimes \rho_s)\right)$ we get

$$\mathcal{K}_1(\rho_s) = \overline{\tau}\overline{\mathcal{K}}_A\left(\overline{\boldsymbol{a}}\,\overline{\rho}_A\right) \otimes \left(\rho_s \boldsymbol{b}^\dagger - \boldsymbol{b}^\dagger \rho_s\right) + \overline{\tau}\overline{\mathcal{K}}_A\left(\overline{\rho}_A \overline{\boldsymbol{a}}^\dagger\right) \otimes \left(\boldsymbol{b}\rho_s - \rho_s \boldsymbol{b}\right)$$

where $\overline{\boldsymbol{a}} = \boldsymbol{a} - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\boldsymbol{1}_{\mathcal{H}_A}$. We made this particular gauge choice in order to ensure $\mathrm{Tr}_A\left(\mathcal{K}(\rho_s)\right) = \rho_s$ and therefore, we may in this sense associate ρ_s with the density operator of the subsystem B. As presented in section 3.1, this gauge choice does not ensure in general a Kraus map for the first order parametrisation of the slow manifold. However, we show that with this cascaded structure, the map $\mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s)$ with this gauge choice is also a completely positive trace-preserving map up-to second order terms. To exhibit such property, we express this map in an explicit Kraus operator form.

Using Lemma 3 and Corollary 2 we get that $\overline{\mathcal{K}}_A\left(\overline{\boldsymbol{a}}\,\overline{\rho}_A\right) = \overline{\mathcal{K}}_A\left(\overline{\boldsymbol{a}}\,\overline{\rho}_A\right)\overline{\rho}_A^{-1}\overline{\rho}_A$ and also $\overline{\mathcal{K}}_A\left(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right) = \overline{\mathcal{K}}_A\left(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right)\overline{\rho}_A^{-1}\overline{\rho}_A$ where $\overline{\rho}_A^{-1}$ stands for the Moore-Penrose pseudo-inverse of $\overline{\rho}_A$. Therefore we derive the following Kraus operator form for the first order entanglement:

$$\begin{aligned}
\rho &= \mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) + O(\epsilon^2) = \boldsymbol{M}(\overline{\rho}_A \otimes \rho_s)\boldsymbol{M}^\dagger \\
\boldsymbol{M} &= \left(I + \epsilon\overline{\tau}\left(\overline{\mathcal{K}}_A\left(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right)\overline{\rho}_A^{-1} \otimes b - \overline{\mathcal{K}}_A\left(\overline{\boldsymbol{a}}\,\overline{\rho}_A\right)\overline{\rho}_A^{-1} \otimes \boldsymbol{b}^\dagger\right)\right)
\end{aligned}$$

In order to compute the second order dynamics we proceed in the same manner as the Hamiltonian interaction case: first, we choose the gauge $\boldsymbol{G}_{1,B}(\rho_s) = -\overline{\tau}\mathrm{Tr}_A\left(\mathcal{L}_{int}(\overline{\rho}_A \otimes \rho_s)\right)$ in order to facilitate the expression of $\mathcal{L}_{s,2}$. Then, take the partial trace Tr_A on equation (3.6) leading to the cancellation of the unknown term $\mathcal{L}_A(\mathcal{K}_2(\rho_s))$. Using $\mathrm{Tr}_A\left(\mathcal{L}_A(\bullet)\right) \equiv 0$ with $\mathrm{Tr}_A\left(\mathcal{K}_0(\bullet)\right) \equiv \bullet$ and the fact that $\mathrm{Tr}_A\left(\mathcal{K}_1(\bullet)\right) \equiv 0$ due to our choice of the gauge degree of freedom we get:

$$\mathcal{L}_{s,2}(\rho_s) = \mathrm{Tr}_A\left(\boldsymbol{a}[\boldsymbol{b}^\dagger, \mathcal{K}_1(\rho_s)] + [\mathcal{K}_1(\rho_s), \boldsymbol{b}]\boldsymbol{a}^\dagger\right) + \left(\mathcal{D}_{\boldsymbol{b}}(\rho_s) + \mathcal{L}_B(\rho_s)\right)$$

Note that the term $\mathcal{D}_{\boldsymbol{b}}$ is intrinsically related to the cascaded structure while $\mathcal{L}_B$ represents the slow dynamics on the subsystem B and is independent of it (in particular, it could satisfy $\mathcal{L}_B \equiv 0$). Therefore the term $\mathcal{L}_B$ must be treated independently from the other terms. For this reason we impose $\mathcal{L}_{s,2} = \widetilde{\mathcal{L}}_{s,2} + \mathcal{L}_B$ with

$$\widetilde{\mathcal{L}}_{s,2} = \mathrm{Tr}_A\left(\boldsymbol{a}[\boldsymbol{b}^\dagger, \mathcal{K}_1(\rho_s)] + [\mathcal{K}_1(\rho_s), \boldsymbol{b}]\boldsymbol{a}^\dagger\right)) + \mathcal{D}_{\boldsymbol{b}}$$

A direct expansion leads to the following expression:

$$\widetilde{\mathcal{L}}_{s,2}(\rho_s) = \mathcal{D}_{x_1\boldsymbol{b}+y_1\boldsymbol{b}^\dagger}(\rho_s) + \mathcal{D}_{x_2\boldsymbol{b}+y_2\boldsymbol{b}^\dagger}(\rho_s) + \frac{\alpha^* - \alpha}{2}\left[\boldsymbol{b}^\dagger\boldsymbol{b} - \boldsymbol{b}\boldsymbol{b}^\dagger\,,\,\rho_s\right] \tag{3.35}$$

where $\{x_1, x_2, y_1, y_2\} \in \mathbb{C}^4$ are solutions of the set of equations:

$$\left\{\begin{array}{rcl}
|x_1|^2 + |x_2|^2 &=& \alpha + \alpha^* + 1 \\
|y_1|^2 + |y_2|^2 &=& \alpha + \alpha^* \\
x_1 y_1^* + x_2 y_2^* &=& -2\beta
\end{array}\right. \tag{3.36}$$

with $\alpha = \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}\overline{\mathcal{K}}_A\left(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right)\right)$ and $\beta = \overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\mathcal{K}}_A(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger)\right)$.

This set of equation can be viewed as the norm and the scalar product between the two vectors $\{x_1, x_2\}^\intercal \in \mathbb{C}^2$ and $\{y_1, y_2\}^\intercal \in \mathbb{C}^2$. Therefore, there exists a (non-unique) solution to (3.36) if and only if holds:

$$\alpha + \alpha^* \geq 0 \tag{3.37}$$
$$(\alpha + \alpha^* + 1)(\alpha + \alpha^*) \geq 4|\beta|^2 \tag{3.38}$$

Using a slight adaptation of Lemma 15, we get that condition (3.37) holds for any system described by the master equation (3.31). Condition (3.38) corresponds to the assumption 1. The non-unique solution of the system of equations (3.36) is expected due to the fact that the Lindblad decomposition is also not unique. A solution is given by

$$\begin{cases} x_1 = \sqrt{1 + \alpha + \alpha^*} \\ x_2 = 0 \\ y_1 = \dfrac{-2\beta^*}{\sqrt{1 + \alpha + \alpha^*}} \\ y_2 = \sqrt{\dfrac{(1 + \alpha + \alpha^*)(\alpha + \alpha^*) - 4|\beta|^2}{1 + \alpha + \alpha^*}} \end{cases}$$

Therefore, gathering equations (3.34) and (3.35) we get the second order reduced dynamics (3.32), concluding the proof.

■

Application to an arbitrary system with a squeezed drive

To illustrate these results, we consider a driven linear cavity producing a squeezed output field, unidirectionally "feeding" the slow subsystem. Such kind system was first studied in [Gar86] considering the slow subsystem as a qubit and shows how one can engineer the coherence times of this qubit using a squeezed input field. It has recently been realized experimentally in [MWB+13] validating the theoretical results. Our aim is to show through this example, how to apply our method, emphasizing that it doesn't need the specification of the slow subsystem and readily retrieve the known results of the adiabatic elimination of the linear cavity when the slow subsystem is a qubit.

Using e.g. [GJ09] , we get the following dynamics for the system:

$$\frac{d}{dt}\rho = g[\boldsymbol{a}^2 - \boldsymbol{a}^{\dagger 2}, \rho] + \mathcal{D}_{\sqrt{\kappa}\boldsymbol{a} + \epsilon\boldsymbol{b}}(\rho) + \frac{\epsilon}{2}[\boldsymbol{a}^\dagger \boldsymbol{b} - \boldsymbol{a}\boldsymbol{b}^\dagger, \rho] \tag{3.39}$$

where $\boldsymbol{a}$ is the annihilation operator on the cavity, $\boldsymbol{b}$ is an operator on the unspecified slow subsystem (usually the annihilation operator for a cavity and the energy loss operator for a qubit). The term $g[\boldsymbol{a}^2 - \boldsymbol{a}^{\dagger 2}, \rho]$ corresponds to the standard squeezing Hamiltonian by an appropriate phase choice for $\boldsymbol{a}$ (g is real here). The parameter κ is the dissipation rate of the cavity. The time-scale separation is given by $\kappa \gg \epsilon^2$. This model is valid under the assumption $\kappa > 4g$, otherwise the cavity subsystem is unstable and its energy grows to infinity (and therefore additional phenomenon such as Kerr effect have to be taken into account). As any dynamics $\mathcal{L}_B$ on the slow subsystem plays no role for second order computations, we assume for simplicity $\mathcal{L}_B \equiv 0$.

To derive the reduced dynamics up to second order on this example, we use formula (3.32) where the fast system A is the cavity and the slow system B is unspecified. Therefore

we have only to compute the three coefficients $\mathrm{Tr}\left(\sqrt{\kappa}\boldsymbol{a}\overline{\rho}_A\right)$, $\alpha = \kappa\overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}\overline{\mathcal{K}}_A\left(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right)\right)$ and $\beta = \kappa\overline{\tau}\mathrm{Tr}\left(\boldsymbol{a}^\dagger\overline{\mathcal{K}}_A(\overline{\rho}_A\overline{\boldsymbol{a}}^\dagger\right)$. As the fast dynamics of (3.39) corresponds to the evolution of a linear quantum harmonic oscillator, these coefficients are easier to compute by taking their Heisenberg representation counterpart i.e:

$$\begin{aligned}\mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right) &= \mathrm{Tr}\left(\boldsymbol{a}\ e^{\infty\mathcal{L}_A}(\rho_0)\right) = \mathrm{Tr}\left(e^{\infty\mathcal{L}_A^*}(\boldsymbol{a})\ \rho_0\right)\\ \alpha &= \kappa\mathrm{Tr}\left(e^{\infty\mathcal{L}_A^*}\left(\overline{\boldsymbol{a}}^\dagger\int_0^\infty e^{t\mathcal{L}_A^*}(\boldsymbol{a})\ dt\right)\rho_0\right)\\ \beta &= \kappa\mathrm{Tr}\left(e^{\infty\mathcal{L}_A^*}\left(\overline{\boldsymbol{a}}^\dagger\int_0^\infty e^{t\mathcal{L}_A^*}(\boldsymbol{a}^\dagger)dt\right)\rho_0\right)\end{aligned}$$

where ρ_0 is the initial state, ${\mathcal{L}_A}^*$ is the dual of $\mathcal{L}_A$. With a small abuse of notation, $e^{\infty{\mathcal{L}_A}^*}$ means $\lim_{t\to\infty} e^{t{\mathcal{L}_A}^*}$.

Then, some usual computations lead to (see chapter 4 for the key elements of the computation):

$$\begin{aligned}\mathrm{Tr}\left(e^{\infty\mathcal{L}_A^*}(\boldsymbol{a})\ \rho_0\right) &= 0\\ \alpha &= \frac{32\kappa^2 g^2}{((\kappa+4g)(\kappa-4g))^2}\\ \beta &= -\frac{64g^3\kappa+4g\kappa^3}{((\kappa+4g)(\kappa-4g))^2}\end{aligned}$$

In this case one can check that $(\alpha+\alpha^*+1)(\alpha+\alpha^*) = 4|\beta|^2$ verifies condition (3.38). The coefficients to be commputed are therefore given by (3.33) and we get, $y_2 = 0, x_1 = \sqrt{2\alpha+1}, y_1 = -\sqrt{2\alpha}$. Using the commutation property of the annihilation operator $\left[\boldsymbol{b},\boldsymbol{b}^\dagger\right] = \mathbf{1}$, the Hamiltonian term in (3.32) vanishes. As a result we get the second order reduced dynamics:

$$\frac{d}{dt}\rho_s = \epsilon^2\mathcal{D}_{\sqrt{2\alpha+1}\boldsymbol{b}-\sqrt{2\alpha}\boldsymbol{b}^\dagger}(\rho_s) \tag{3.40}$$

Recovering the result of [Gar86] when the slow subsystem is a qubit and $\boldsymbol{b} = \boldsymbol{\sigma}_-$ the energy loss operator.

3.4 Summary

In this chapter, we have considered finite dimension bipartite quantum systems. Using the different time-scales, we used the method presented in chapter 2 in order to compute the reduced model while ensuring a physical meaning of the reduced system. This corresponds to find a solution for the recurrence relations (3.4), (3.5) and (3.6). We showed how our method ensure a Lindblad master equation for the reduced dynamics and a completely positive trace preserving map (Kraus map) for the parametrisation of the slow manifold at the first order in the small parameter describing the different time-scales. Then, we focused on different usual coupling between the fast stable subsystem and the slow one namely any Hamiltonian interactions or cascaded interactions. We were able to derive explicit formulas allowing to compute the second order reduced dynamics by solving the second order recurrence relation. These formulas are given by Theorem 5 for Hamiltonian interactions and by Theorems 7 and 8 for cascaded interactions. They already allows to give general structural results on the second order adiabatic elimination especially on the number and the type of decoherence operators we may expect for the reduced dynamics. Moreover, they permit to precisely compute the reduced dynamics on relevant examples with non-trivial interactions. The key element of these computations is to be able to calculate the inverse of the fast dynamics $\mathcal{L}_A$. This is a very difficult problem for a general setting. We show in the next chapter 4 how to perform such computations for several class of quantum systems.

Our method opens a clear path to compute higher order models using the recurrence relations and we conjecture that it ensures a reduced dynamics in a Lindblad form at any order. The key element to do so would be to properly choose the gauge degree of freedom associated with the inversion of the fast dynamics. Some preliminary results on the third order are presented in chapter 5.

Chapter 4

Application to quantum oscillators

The aim of this chapter is to show the practical appliance of our formulas in order to compute the second order reduced model for various dynamics. In a wide class of quantum experiments, the fast subsystem is a low-quality (strongly damped) quantum harmonic oscillator. As such system plays an important role in quantum physics, we devote this chapter to the study of systems where the fast system is a quantum harmonic oscillator. We begin this chapter by notions and general textbook material on the quantum harmonic oscillator in section 4.1. We proceed then in section 4.2 to the computation of the reduced dynamics for *linear*[1] harmonic oscillators. The goal of this section is twofold. First, we illustrate more precisely how to practically compute the second order reduced dynamics on relevant systems. The main difficulty to get the reduced model by using our results from chapter 3 is being able the computation of the inverse of the fast dynamics corresponding to the super-operator $\overline{\mathcal{K}}_A$. We present several techniques to perform such computations namely by going into the Heisenberg picture when dealing with linear quantum harmonic oscillators and using an explicit expression of the propagator when it is possible. Such techniques were used to get the results of the examples presented in 3. Second, we emphasise that harmonic oscillators are infinite dimension systems. While our adiabatic elimination technique is rigorous only for finite dimension systems, we show how it can readily be applied on infinite dimension systems and how we are able to perform the computations by using the different properties of the infinite dimension operators. Although the extension of Fenichel's 2 and Carr's 3 theorems to infinite dimension have to be carefully studied (which is not the aim of this manuscript) we retrieve the expected results on known examples. In the last section 4.3 we consider non-linear harmonic oscillators. We show on a class of non-linear oscillator how our method allows the computation of the reduced dynamics. We first consider a system where the non-linearity lies within the interaction between the fast and the slow subsystem. Then we consider a system where the fast dynamics includes non-linear terms. We emphasise that for such system, the computation of the reduced dynamics using the Heisenberg picture of section 4.2 or Langevin equations as it is usually performed in physics is impossible.

[1] A quantum harmonic oscillator is said linear if its dynamics can be describe with an Hamiltonian quadratic and some decoherence operators linear with respect to the system coordinates

4.1 The quantum harmonic oscillator

Quantum harmonic oscillators play a central role in quantum systems. It may be seen as one of the main "building block", together with two-level systems, in quantum physics to describe a large variety of setup. By linearisation, all kinds of systems close to equilibrium can be approximate by harmonic oscillator. Complex systems such as the environment of a quantum system undergoing some relaxation process can be modelled using oscillators. A simple description of an environment is made of an infinite set of quantum oscillators weakly coupled with the system of interest and spanning a large frequency range. The quantum harmonic oscillator is one of the few quantum systems with explicit solutions, highlighting the interest of such modelling of various systems with harmonic oscillator. We refer to [HR06] for an excellent presentation of harmonic oscillators.

4.1.1 From classical to quantum harmonic oscillator

We consider a one dimension classical harmonic oscillator consisting a particle of mass m coupled to a spring with associated spring constant k. The system can be described by the dynamical variables $\{x, p\}$ where x represents the position of the particle and p its momentum related by $p = m\frac{dx}{dt} = mv$ with v the velocity. The dynamics of such system is described by the familiar Newtonian equations :

$$p = m\frac{dx}{dt} \qquad \frac{dp}{dt} = -kx$$

The mechanical energy of the system is given by

- The quadratic potential associated to the spring $V(x) = \frac{1}{2}kx^2$.
- The kinetic energy $T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$.

Leading to the well known Hamiltonian associated with an harmonic oscillator :

$$H_{classical} = V + T = \frac{1}{2}kx^2 + \frac{p^2}{2m}$$

From Hooke's law, we know that the oscillation frequency is $\omega = \sqrt{\frac{k}{m}}$. The Hamiltonian can thus be expressed as

$$H_{classical} = \frac{m\omega^2}{2}x^2 + \frac{p^2}{2m} \tag{4.1}$$

We are now able to go from the description of a classical harmonic oscillator to a quantum one. Such quantisation procedure consists in this case by replacing classical variables by their quantum counterpart. The goal is that the dynamics of the quantum harmonic oscillator corresponds, in the classical limit, as the one presented above.

In quantum physics, the momentum p and the position x become the Hermitian operators $\boldsymbol{P}$ and $\boldsymbol{X}$. Similarly, the Hamiltonian becomes

$$H_{quant} = \frac{m\omega^2}{2}\boldsymbol{X}^2 + \frac{\boldsymbol{P}^2}{2m}.$$

The operators $\boldsymbol{X}$ and $\boldsymbol{P}$ verify the commutation relation

$$[\boldsymbol{X}\,,\,\boldsymbol{P}] = \boldsymbol{X}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{X} = i\hbar$$

where $\hbar$ is Planck constant. It is convenient from now to define two non-Hermitian operators $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ by :

$$\boldsymbol{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{X} + \frac{i\boldsymbol{P}}{m\omega}\right) \qquad \boldsymbol{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{X} - \frac{i\boldsymbol{P}}{m\omega}\right)$$

Using the commutation relation between $\boldsymbol{P}$ and $\boldsymbol{X}$ we readily get

$$\left[\boldsymbol{a}\,,\,\boldsymbol{a}^\dagger\right] = \mathbf{1}$$

These operators allow to advantageously reformulate the Hamiltonian (4.1) :

$$H_{quant} = \hbar\omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{1}{2}\right)$$

The constant term in the previous expression represents the vacuum fluctuation energy. It can be cancelled by an appropriate choice of the origin energy. The Hamiltonian of the quantum harmonic oscillator is thus described by the operator $\boldsymbol{a}^\dagger\boldsymbol{a}$:

$$\tilde{H}_{quant} = \hbar\omega\left(\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$

The operator $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$ is called the *number operator*. The eigenstates and eigenvalues of this operator corresponds to the one of the Hamiltonian and are therefore of interest. The spectrum of $\boldsymbol{N}$ is non-degenerate and contains all the non-negative integers. We denote its eigenstates by $|n\rangle$ such that $\boldsymbol{N}\,|n\rangle = n\,|n\rangle$. They are called the *Fock states* and verifies $\langle n|\,m\rangle = \delta_{n,m}$, $\sum_n |n\rangle\langle n| = \mathbf{1}$ thus they form an orthogonal basis (the Fock basis) of the Hilbert space associated with the harmonic oscillator. We define the energy E_n of the Fock state $|n\rangle$ as the eigenvalue of the Hamiltonian $\tilde{H}_{quant}$ associated to this state. We readily get $E_n = n\hbar\omega$, corresponding to the energy of n quanta $\hbar\omega$. This explain the name *number operator* for $\boldsymbol{N}$ as it counts the number of energy quanta, in unit of $\hbar\omega$, associated with the system. An energy quanta is also called *photon*. For any function f, we have the following commutation property :

$$\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+\mathbf{1})\boldsymbol{a} \qquad \boldsymbol{a}^\dagger f(\boldsymbol{N}) = f(\boldsymbol{N}-\mathbf{1})\boldsymbol{a}^\dagger$$

The use of this Fock basis is suitable for the study of the quantum harmonic oscillator and the operator $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ have a simple effect on the Fock states :

$$\begin{aligned}\boldsymbol{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle\\ \boldsymbol{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle\end{aligned}$$

One see from these expression that the effect of the operator $\boldsymbol{a}$ (respectively $\boldsymbol{a}^\dagger$) is to destroy (create) an energy quanta. For this reason they are respectively called the *annihilation operator* and *creation operator*.

Conversely, the Fock states can be defined only by use of the creation operator $\boldsymbol{a}^\dagger$ and the ground state $|0\rangle$ corresponding to the lower energy of the system by :

$$|n\rangle = \frac{(\boldsymbol{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle \tag{4.2}$$

4.1.2 Coherent states

Another type of states playing an important role in the description of quantum harmonic oscillator are the *coherent states* (or Glauber states). The state of a classical harmonic oscillator can be described by using the variables x and p and is represented in the phase space diagram by the complex variable $x+ip$. The coherent state $|\alpha\rangle$ with $\alpha = \alpha' + i\alpha" \in \mathbb{C}$ is its quantum counterpart. It is defined as the eigenstate of the annihilation operator $\boldsymbol{a}$ associated with the eigenvalue α :

$$\boldsymbol{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \qquad \alpha \in \mathbb{C}$$

These coherent states minimise the Heisenberg uncertainty principle, verifying $\Delta\boldsymbol{X}\Delta\boldsymbol{P} = \hbar/2$. Their decomposition in the Fock basis is given by :

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}\,|k\rangle \tag{4.3}$$

We can see from this expression that the distribution along the Fock states is a Poissonian distribution. The coherent state is also normalised as $\langle\alpha|\ \alpha\rangle = 1$. By taking advantage of equation (4.2) we can define a coherent state only by means of the annihilation operator and the ground state :

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\boldsymbol{a}^\dagger)\,|0\rangle \tag{4.4}$$

We emphasise here that a coherent state $|\alpha\rangle$ with $\alpha = n \in \mathbb{N}$ is different from the Fock state $|n\rangle$. From equation (4.4) we see that they are equal if and only if $\alpha = 0$. We will use Greek letters to denote coherent states and Latin letters to denote Fock states. When the context may be not clear, we will use a subscript c (for coherent) or f (for Fock) in order to distinguish both of them (e.g. $|2\rangle_c$). The scalar product between two coherent states, representing their overlap is given by

$$\langle\alpha|\ \beta\rangle = e^{\frac{-|\alpha|^2}{2}}e^{\frac{-|\beta|^2}{2}}e^{\alpha^*\beta} \tag{4.5}$$

Now we introduce the *displacement operator* $\boldsymbol{D}_\alpha$ which is widely use when considering coherent state and quantum harmonic oscillator. Its name come from the fact that it displaces the vacuum state (ground state $|0\rangle$) towards a coherent state of amplitude α :

$$\boldsymbol{D}_\alpha\,|0\rangle = |\alpha\rangle \qquad \alpha \in \mathbb{C}$$

The displacement operator is unitary and defined by

$$\boldsymbol{D}_\alpha = \exp\left[\alpha\boldsymbol{a}^\dagger - \alpha^*\boldsymbol{a}\right]$$

As $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ do not commute, we have to be careful when manipulating this expression. It can be expended by using the Glauber identities [BR03]. Consider two operators $\boldsymbol{A}$ and $\boldsymbol{B}$ which commutes with their commutator: $[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] = [\boldsymbol{B},[\boldsymbol{A},\boldsymbol{B}]] = 0$, then we have :

$$\exp[\boldsymbol{A}+\boldsymbol{B}] = e^{\boldsymbol{A}}e^{\boldsymbol{B}}e^{-[\boldsymbol{A},\boldsymbol{B}]/2}$$

Noting that $[\boldsymbol{a},\boldsymbol{a}^\dagger] = \boldsymbol{1}$ we can use this formula to get, for the displacement operator :

$$\boldsymbol{D}_\alpha = e^{\frac{-|\alpha|^2}{2}}e^{\alpha\boldsymbol{a}^\dagger}e^{\alpha^*\boldsymbol{a}}$$

Note that we have $\boldsymbol{D}_\alpha^\dagger = \boldsymbol{D}_{-\alpha}$. The displacement operator is closely related to the annihilation and creation operators and has a simple action on them. To derive this action, we compute $\boldsymbol{D}_\alpha \boldsymbol{a} \boldsymbol{D}_\alpha^\dagger$. To do so, we introduce the Baker-Campbell-Hausdorff formula very useful to perform computations with non-commuting operators. It states that for any operator $\boldsymbol{A}$ and $\boldsymbol{B}$ we have :

$$e^{\boldsymbol{A}} \boldsymbol{B} e^{-\boldsymbol{A}} = \boldsymbol{B} + [\boldsymbol{A}, \boldsymbol{B}] + \frac{1}{2}[\boldsymbol{A}, [\boldsymbol{A}, \boldsymbol{B}]] + \dots$$

Using this formula with the commutation relations between $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ we readily get that

$$\boldsymbol{D}_\alpha \boldsymbol{a} \boldsymbol{D}_\alpha^\dagger = \boldsymbol{a} - \alpha \mathbf{1}$$

The displacement operator will appear frequently when dealing with linear quantum harmonic oscillators and we summary here several properties which will be widely used throughout the computations :

$$\begin{aligned} \boldsymbol{D}_\alpha^\dagger &= \boldsymbol{D}_\alpha^{-1} = \boldsymbol{D}_{-\alpha} \\ \boldsymbol{D}_\alpha \boldsymbol{a}^\dagger \boldsymbol{D}_{-\alpha} &= \boldsymbol{a}^\dagger - \alpha^* \mathbf{1} \\ \boldsymbol{D}_{\alpha+\beta} &= \boldsymbol{D}_\alpha \boldsymbol{D}_\beta e^{-i\Im(\alpha^*\beta)} \end{aligned}$$

where $\Im$ denotes the imaginary part.

4.2 Linear harmonic oscillator

We consider in this Section a linear quantum harmonic oscillator acting as the fast subsystem interacting via linear coupling to a unspecified slow subsystem. In this type of setting, thanks to the linearity of the fast oscillator, it is possible to perform the computations of the second order reduced dynamics by going into the Heisenberg picture in order to calculate the required coefficients depicted in Theorem 5. The aim of this section is to present, through a concrete example how to perform these computations. This example is chosen in order to cover several standard linear harmonic oscillators and therefore readily gives the computational results for systems with such fast dynamics. The method can however be applied to more general systems provided that the fast subsystem is a linear harmonic oscillator and that the coupling is also a linear one. It was used in chapter 3 to perform the computations of the second order reduced dynamics on the illustrative examples. We present in this section the detail of the calculations.

4.2.1 Second order reduced dynamics

We consider a fast system A, which is a strongly damped linear quantum harmonic oscillator. The associated Hilbert space is $\mathcal{H}_A = \{\sum_{n\geq 0} \psi_n |n\rangle \,|(\psi_n) \in \ell^2\}$ (where $\{|n\rangle\}$ is the Fock basis, ℓ^2 the space of square summable series). This subsystem is weakly coupled to a system B on a Hilbert space $\mathcal{H}_B$ with no internal dynamics (this assumption is only for simplicity as any dynamics on the slow subsystem plays no role in the computations of the reduced model). For the sake of clearness we consider that the subsystem B is a qubit, thought it doesn't matter for the computations.

As usually, we assume that the typical dissipation rate of the cavity κ is much greater than the coupling rate g. This gives rise to two time scales : a fast one for the quantum harmonic oscillator A and a slower one for the subsystem B, corresponding to the structure presented in 3.1. We will consider the standard resonant Hamiltonian coupling between the cavity and the qubit. If we are mainly interest in the evolution of the qubit, we can apply our adiabatic elimination technique to eliminate the cavity and study the perturbation effect of the coupling on the qubit.

The Lindblad master equation governing the dynamics of the considered example is given by :

$$\begin{aligned}\frac{d\rho}{dt} = u[\boldsymbol{a}^\dagger - \boldsymbol{a}, \rho] + \kappa(1+n_{th})\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\right)\\ +\kappa n_{th}\left(\boldsymbol{a}^\dagger\rho\boldsymbol{a} - \frac{1}{2}(\boldsymbol{a}\boldsymbol{a}^\dagger\rho + \rho\boldsymbol{a}\boldsymbol{a}^\dagger)\right) - ig[\boldsymbol{a}\boldsymbol{\sigma}_+ + \boldsymbol{a}^\dagger\boldsymbol{\sigma}_-, \rho]\end{aligned}$$

where $\boldsymbol{a}$ is the annihilation operator of the subsystem A. κ is the dissipation rate corresponding to the strong damping of the harmonic oscillator. It is driven by a coherent field of amplitude $u \in \mathbb{C}$ and subject to thermal noise, with n_{th} the number of thermal photons. $\boldsymbol{\sigma}_+$ and $\boldsymbol{\sigma}_-$ are respectively the energy gain and loss operators. The coefficient g represents the coupling strength between the two subsystems, with the relation $\kappa \gg g$.

The unique steady state of the harmonic oscillator is $\overline{\rho}_A = \mathfrak{D}_\alpha \rho_{th} \mathfrak{D}_{-\alpha}$. Where $\mathfrak{D}_\alpha$ is the displacement operator of amplitude $\alpha = 2u/\kappa \in \mathbb{C}$ and ρ_{th} is the thermal state associated to n_{th}:

$$\rho_{th} = \frac{1}{1+n_{th}} \sum_{n=0}^{\infty} \left(\frac{n_{th}}{1+n_{th}}\right)^n |n\rangle\langle n|$$

We are therefore able to use Theorem 5 in order to compute the second order reduced dynamics. We show how to perform the computations by going into the Heisenberg picture.

A straightforward computation yields $\mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right) = \mathrm{Tr}\left(\boldsymbol{a}\rho_{th}\right) + \alpha = \alpha$. Theorem (5) immediately gives the first order approximation :

$$\mathfrak{L}_{s,1}(\rho_S) = -i\Big[\alpha\boldsymbol{\sigma}_+ + \alpha^*\boldsymbol{\sigma}_-, \rho_S\Big]$$

In order to get the second order approximation we have to compute the 4 coefficients defined in Lemma (4). We will detail the computations of the first coefficient (corresponding to $\boldsymbol{A}_1 = \boldsymbol{a}$) : $\mathrm{Tr}\left(\overline{\tau\mathcal{K}}_A(\boldsymbol{a}\overline{\rho}_A - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\overline{\rho}_A)\boldsymbol{a}^\dagger\right)$. The other coefficients are computed in a similar way. By virtue of definition (2.28) we have :

$$\begin{aligned}\mathrm{Tr}\left(\overline{\tau\mathcal{K}}_A(\boldsymbol{a}\overline{\rho}_A - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\overline{\rho}_A)\boldsymbol{a}^\dagger\right) &= \mathrm{Tr}\left(\boldsymbol{a}^\dagger\int_0^\infty e^{t\mathcal{L}_A}\Big((\boldsymbol{a}-\alpha\mathbf{1})\overline{\rho}_A\Big)dt\right)\\ &= \mathrm{Tr}\left((\boldsymbol{a}^\dagger-\alpha^*\mathbf{1})\int_0^\infty e^{t\mathcal{L}_A}\Big((\boldsymbol{a}-\alpha\mathbf{1})\overline{\rho}_A\Big)dt\right)\end{aligned}$$

Where we used the fact that $\mathrm{Tr}\left(e^{t\mathcal{L}_A}\Big((\boldsymbol{a}-\alpha\mathbf{1})\overline{\rho}_A\Big)\right) = 0$. Using the cyclic property of the trace, the trick is to go from the Schrödinger evolution of the operator $(\boldsymbol{a}-\alpha\mathbf{1})\overline{\rho}_A$ to the Heisenberg evolution of $\boldsymbol{a}^\dagger$, as briefly mentioned in section 3.3.1 :

$$\mathrm{Tr}\left(\overline{\tau\mathcal{K}}_A(\boldsymbol{a}\overline{\rho}_A - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\overline{\rho}_A)\boldsymbol{a}^\dagger\right) = \mathrm{Tr}\left((\boldsymbol{a}-\alpha\mathbf{1})\overline{\rho}_A\int_0^\infty e^{t\mathcal{L}_A^*}(\boldsymbol{a}^\dagger-\alpha^*\mathbf{1})dt\right)$$

where $\mathcal{L}_A^*$ is the adjoint of $\mathcal{L}_A$ (see section 2.1.4). Thanks to the linearity of the fast harmonic oscillator, it possible to explicitly compute $e^{t\mathcal{L}_A^*}$. The way to perform such computation is illustrated in section 4.2.2. From equation (4.6), we get that

$$\begin{aligned}\mathrm{Tr}\left(\overline{\tau\mathcal{K}}_A(\boldsymbol{a}\overline{\rho}_A - \mathrm{Tr}\left(\boldsymbol{a}\overline{\rho}_A\right)\overline{\rho}_A)\boldsymbol{a}^\dagger\right) &= \mathrm{Tr}\left((\boldsymbol{a}-\alpha\mathbf{1})\overline{\rho}_A\int_0^\infty e^{-\kappa t/2}(\boldsymbol{a}^\dagger-\alpha^*)dt\right)\\ &= n_{th}\end{aligned}$$

After completely similar calculations, we get the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ from Lemma 4:

$$\boldsymbol{X} = \begin{pmatrix} 2n_{th} & 1 \\ 0 & 2(1+n_{th})\end{pmatrix} \qquad \boldsymbol{Y} = 0$$

Choosing for simplicity $\boldsymbol{\Lambda} = \sqrt{\boldsymbol{X}}$, we get the following second order reduced dynamics :

$$\frac{d}{dt}\rho_s = -ig\Big[\alpha\boldsymbol{\sigma}_+ + \alpha^*\boldsymbol{\sigma}_-, \rho_S\Big] + \frac{4g^2}{\kappa}\Big(n_{th}\mathcal{D}_{\boldsymbol{\sigma}_+}(\rho_s) + (1+n_{th})\mathcal{D}_{\boldsymbol{\sigma}_-}(\rho_s)\Big)$$

Note that here, contrary to the general case, the second order dissipation operators correspond exactly to the interaction operators and are not a mixture of them.

4.2.2 Computations in the Heisenberg picture

The aim of this section is to illustrate how to compute the Heisenberg evolution of an operator for a linear harmonic oscillator. We show the method on this example (which includes a drive and thermal noise), this can however be extended to any linear dynamics. These computations are standard, they are given here as they are helpful for anyone

interest in the computation of the reduced model of a linear harmonic oscillator. We perform the computations for the operators $\boldsymbol{N}$ and $\boldsymbol{a}$ as they frequently appears in the interaction Hamiltonian (corresponding to dispersive or resonant coupling).

For any operator $\boldsymbol{X}$ on $\mathcal{H}_A$, we denote $\boldsymbol{X}_t \triangleq e^{t\mathfrak{L}_A^*}(\boldsymbol{X})$. Therefore $\boldsymbol{X}_t$ stand for the evolving operator in the Heisenberg picture while $\boldsymbol{X}$ is a standard (non evolving) operator. They are related by $\boldsymbol{X}_0 = \boldsymbol{X}$.

We start by solving the adjoint master equation for the annihilation operator $\boldsymbol{a}_t$:

$$\frac{d\boldsymbol{a}_t}{dt} = -[u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{a}_t] + \kappa(1+n_{th})\left(\boldsymbol{a}^\dagger\boldsymbol{a}_t\boldsymbol{a} - \frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}_t + \boldsymbol{a}_t\boldsymbol{a}^\dagger\boldsymbol{a})\right) + \kappa n_{th}\left(\boldsymbol{a}\boldsymbol{a}_t\boldsymbol{a}^\dagger - \frac{1}{2}(\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a}_t + \boldsymbol{a}_t\boldsymbol{a}\boldsymbol{a}^\dagger)\right)$$

Then assume that $\boldsymbol{a}_t = f(t)\boldsymbol{a} + g(t)$ with $f(0) = 1$ and $g(0) = 0$. Using the commutation relations of section 4.1 we get :

$$\frac{df(t)}{dt}\boldsymbol{a} + \frac{dg(t)}{dt} = uf(t) - f(t)\frac{\kappa}{2}\boldsymbol{a}$$

Denoting $\alpha = 2u/\kappa$, we straightforwardly get

$$\boldsymbol{a}_t = e^{t\mathcal{L}_A^*}(\boldsymbol{a}) = e^{-\kappa t/2}(\boldsymbol{a} - \alpha) + \alpha \tag{4.6}$$

If we are interested in the evolution of the number operator $\boldsymbol{N}_t$ in the Heisenberg picture, we have :

$$\frac{d\boldsymbol{N}_t}{dt} = \mathfrak{L}_A^*(\boldsymbol{N}_t) = -[u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{N}_t] + \kappa(1+n_{th})\left(\boldsymbol{a}^\dagger\boldsymbol{N}_t\boldsymbol{a} - \frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{N}_t + \boldsymbol{N}_t\boldsymbol{a}^\dagger\boldsymbol{a})\right) + \kappa n_{th}\left(\boldsymbol{a}\boldsymbol{N}_t\boldsymbol{a}^\dagger - \frac{1}{2}(\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{N}_t + \boldsymbol{N}_t\boldsymbol{a}\boldsymbol{a}^\dagger)\right)$$

Now assume $\boldsymbol{N}_t = f(t)\boldsymbol{N} + g(t)\boldsymbol{a} + g^*(t)\boldsymbol{a}^\dagger + h(t)$ with $f(0) = 1$ and $g(0) = h(0) = 0$. Once again using commutation relations we get :

$$\frac{d\boldsymbol{N}_t}{dt} = f(t)\Big(u\boldsymbol{a}^\dagger + u^*\boldsymbol{a} - \kappa\boldsymbol{N} + \kappa n_{th}\Big) + g(t)\Big(u - \frac{\kappa}{2}\boldsymbol{a}\Big) + g^*(t)\Big(u^* - \frac{\kappa}{2}\boldsymbol{a}^\dagger\Big)$$

Therefore we have to solve :

$$\begin{aligned}
\frac{df(t)}{dt} &= -\kappa f(t) \\
\frac{dg(t)}{dt} &= f(t)u^* - \frac{\kappa}{2}g(t) \\
\frac{dh(t)}{dt} &= f(t)\kappa n_{th} + g(t)u + g^*(t)u^*
\end{aligned}$$

A direct computation leads to the following solution :

$$e^{t\mathcal{L}_A^*}(\boldsymbol{N}) = e^{-\kappa t}\boldsymbol{N} + \alpha^*(e^{-\kappa t/2} - e^{-\kappa t})\boldsymbol{a} + \alpha(e^{-\kappa t/2} - e^{-\kappa t})\boldsymbol{a}^\dagger - e^{-\kappa t}n_{th} + |\alpha|^2(e^{-\kappa t} - 2e^{-\kappa t/2}) + n_{th} + |\alpha|^2$$

4.3 Non-linear quantum oscillator

4.3.1 A class of non-linear oscillators

We consider in this section the class of bipartite composite system considered in 3.1 : a subsystem A associated with a Hilbert space $\mathcal{H}_A = \{\sum_{n\geq 0} \psi_n |n\rangle \,|(\psi_n) \in \ell^2\}$ (where $\{|n\rangle\}$ is the Fock basis) evolves on a fast time-scale and converges towards a unique steady state while the subsystem B on a Hilbert space $\mathcal{H}_B$ is on a slow time-scale. Moreover we assume that the dynamics of the fast subsystem can be described by using only one dissipation operator. In other word, we consider the following fast dynamics :

$$\mathcal{L}_A(\bullet) = \kappa \left(\boldsymbol{L} \bullet \boldsymbol{L}^\dagger - \frac{1}{2} \left(\boldsymbol{L}^\dagger \boldsymbol{L} \bullet + \bullet \boldsymbol{L}^\dagger \boldsymbol{L} \right) \right) \tag{4.7}$$

for some operator $\boldsymbol{L}$ on $\mathcal{H}_A$ verifying $e^{\boldsymbol{L}^\dagger \boldsymbol{L}} \boldsymbol{L} = \boldsymbol{L} e^{\boldsymbol{L}^\dagger \boldsymbol{L} + \mathbf{1}} e^{i\theta}$, for any $\theta \in \mathbb{R}$ (note that in particular the annihilation operator $\boldsymbol{a}$ verifies this property). The dynamics of the complete composite system is given accordingly to section 3.1 by :

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) + \epsilon \mathcal{L}_{int}(\rho) + \epsilon \mathcal{L}_{slow}(\rho)$$

where ρ is the density operator on $\mathcal{H}_A \otimes \mathcal{H}_B$, $\mathcal{L}_{slow}$ is a super-operator acting non-trivially only on $\mathcal{H}_B$ and $\mathcal{L}_{int}$ is the interaction between the subsystem A and B. The small parameter $0 < \epsilon \ll 1$ emphasize the different time-scales in the composite system.

It is possible in this case to get an explicit expression of the propagator associated with the subsystem A [Ued89, MSB+16] as stated by the following lemma.

Lemma 5. For a density operator ρ_A on $\mathcal{H}_A$ and any initial state $\rho_A(0)$, the solution of the Lindblad master equation

$$\frac{d}{dt}\rho_A = \mathcal{L}_A(\rho_A)$$

where $\mathcal{L}_A$ is in the form (4.7), is given by the following series formulation of the propagator :

$$\rho_A(t) = e^{t\mathcal{L}_A}(\rho_A(0)) = \sum_{p=0}^{\infty} \frac{(1-e^{-\kappa t})^p}{p!} e^{-\frac{\kappa t}{2}\boldsymbol{L}^\dagger \boldsymbol{L}} \boldsymbol{L}^p \rho_A(0) \boldsymbol{L}^{\dagger p} e^{-\frac{\kappa t}{2}\boldsymbol{L}^\dagger \boldsymbol{L}} \tag{4.8}$$

Proof. The proof is immediate by computing the derivative with respect to t and using the properties of the operator $\boldsymbol{L}$. ■

This explicit formulae for the propagator allows to readily compute the inverse of the super-operator $\mathcal{L}_A$ needed in order to calculate the reduced dynamics. According to equation (2.28), for any operator $\boldsymbol{W}$ verifying $\mathrm{Tr}\,(\boldsymbol{W}) = 0$, the zero trace solution of $\mathcal{L}_A(\boldsymbol{X}) = -\boldsymbol{W}$ is given by :

$$\begin{aligned} \boldsymbol{X} &\triangleq -\mathcal{L}_A^{-1}(\boldsymbol{W}) \\ &= \sum_{p=0}^{\infty} \int_0^{\infty} \frac{(1-e^{-\kappa t})^p}{p!} e^{-\frac{\kappa t}{2}\boldsymbol{L}^\dagger \boldsymbol{L}} \boldsymbol{L}^p \boldsymbol{W} \boldsymbol{L}^{\dagger p} e^{-\frac{\kappa t}{2}\boldsymbol{L}^\dagger \boldsymbol{L}} dt \end{aligned} \tag{4.9}$$

We illustrate how these formulas readily allow to compute the second order reduced model, using results from chapter 3 on composite systems with some non-linearities. We begin by an example where the non-linearity lies within the coupling term $\mathcal{L}_{int}$

corresponding to a small anharmonicity of the oscillator. Then, we study a case with a cascaded coupling and a non-linear fast dynamics. For the last system, it is not possible to use the approach presented in 4.2 as we can't get a closed set of equations to solve. To our knowledge, using this formulae is currently the only way to compute a reduced model on quantum harmonic oscillator with this type of non-linear elements.

4.3.2 Application to non-linear coupling

We consider the two-photon pumping scheme from [MA+14] whose goal is to induce 2-photon loss at order ϵ^2 on the target slow subsystem. This system was used in section 3.2 to illustrate the application of theorem 5 (we refer to this section for a more detailed presentation of this system). We assumed there that the fast subsystem had thermal noise but was sufficiently low-quality in order to consider that only the lowest energy levels were populated and therefore can be treated as a two-level system. Here we consider that the thermal noise is negligible but we will treat the fast subsystem as an harmonic oscillator thus relax the two-level system assumption. Moreover, we take into account a small detuning Δ between the frequencies of the two cavities. As the driving field on the fast subsystem needs to be strong in order to guarantee satisfying error protection, we have to take into account the (small) anharmonicity of the fast oscillator. This non-linearity results in a Kerr-effect corresponding to an Hamiltonian term $(\boldsymbol{a}^\dagger\boldsymbol{a})^2$ acting on the fast subsystem. The non-linearity of this term makes the model reduction model more complicated. We illustrate here how to perform the computations by using the explicit formulas of the propagator.

We denote $\boldsymbol{a}$ the annihilation operator on $\mathcal{H}_A$ (similarly, $\boldsymbol{b}$ on $\mathcal{H}_B$), $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$ the number operator, u is the complex drive amplitude, $\kappa \in \mathbb{R}$ the dissipation rate of the fast subsystem and $\Delta \in \mathbb{R}$ is the detuning.

The dynamics of the fast subsystem are described by the Lindblad super-operator :

$$\mathcal{L}_A(\rho) = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] - i\Delta[\boldsymbol{N}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\left(\rho\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}^\dagger\boldsymbol{a}\rho\right)\right)$$

It is well known that for any initial density operator, the solution of $\dot{\rho} = \mathcal{L}_A(\rho)$ converges towards the unique steady state $\overline{\rho}_A = |\alpha\rangle\langle\alpha|$, the coherent state of complex amplitude $\alpha = u/(\kappa/2 + i\Delta)$.

Before turning to the computation of the reduced dynamics, we show how to apply the explicit formulas for the propagator on this system. Note that the fast dynamics $\mathcal{L}_A$ is not in the form (4.7). In order to use (4.8) we apply the change of variable $\rho = \mathcal{D}_\alpha e^{-i\Delta\boldsymbol{N}t}\xi e^{i\Delta\boldsymbol{N}t}\mathcal{D}_\alpha^\dagger$, where $\mathcal{D}_\alpha$ is the displacement operator of amplitude $\alpha = u/(\kappa/2 + i\Delta)$. From $\dot{\rho} = \mathcal{L}_A(\rho)$ it results after standard computations to :

$$\frac{d}{dt}\xi = \kappa\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger - \tfrac{1}{2}\left(\xi\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}^\dagger\boldsymbol{a}\xi\right)\right) \tag{4.10}$$

We are now being able to use equation (4.8) to compute the propagator associated with (4.10). By using then the backward change of variables we get, for any initial density operator ρ :

$$e^{t\mathcal{L}_A}(\rho) = \sum_{n=0}^{+\infty}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right)\boldsymbol{D}_\alpha\left(e^{-\left(\frac{\kappa}{2}+i\Delta\right)t\boldsymbol{N}}\boldsymbol{a}^n\right)\boldsymbol{D}_\alpha^\dagger\,\rho\,\boldsymbol{D}_\alpha\left(\left(\boldsymbol{a}^\dagger\right)^n e^{-\left(\frac{\kappa}{2}-i\Delta\right)t\boldsymbol{N}}\right)\boldsymbol{D}_\alpha^\dagger \tag{4.11}$$

where $\mathcal{D}_\alpha$ is the displacement operator of amplitude α. As a result, the inversion formula (4.9) yields for any traceless operator $\boldsymbol{W}$,

$$\begin{aligned}&\boldsymbol{X} \triangleq -\mathcal{L}_A^{-1}(\boldsymbol{W})\\ &= \sum_{n=0}^{+\infty} \int_0^{+\infty} \left(\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) \boldsymbol{D}_\alpha \left(e^{-\left(\frac{\kappa}{2}+i\Delta\right)t\boldsymbol{N}} \boldsymbol{a}^n \right) \boldsymbol{D}_\alpha^\dagger \, \boldsymbol{W} \, \boldsymbol{D}_\alpha \left(\left(\boldsymbol{a}^\dagger\right)^n e^{-\left(\frac{\kappa}{2}-i\Delta\right)t\boldsymbol{N}} \right) \boldsymbol{D}_\alpha^\dagger \right) dt\end{aligned}$$

We now turn to the computation of the second order dynamics and use these formulas to get the reduced model.

The dynamics of the composite system is given by :

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) + \Big(-i[\boldsymbol{H}_{\text{int}}, \rho] + \mathcal{L}_B(\rho) \Big)$$

As the slow dynamics $\mathcal{L}_B$ plays no role in the computations of the reduced model (see chapter 3) we take $\mathcal{L}_B \equiv 0$ for simplicity. The Hamiltonian coupling is given by :

$$\boldsymbol{H}_{\text{int}} = \sum_{k=1}^{4} \boldsymbol{A}_k \otimes \boldsymbol{B}_k^\dagger = g[\boldsymbol{a}(\boldsymbol{b}^\dagger)^2 + \boldsymbol{a}^\dagger \boldsymbol{b}^2, \rho] + \chi(\boldsymbol{a}^\dagger \boldsymbol{a})(\boldsymbol{b}^\dagger \boldsymbol{b}) + \tfrac{\chi_a}{2}(\boldsymbol{a}^\dagger \boldsymbol{a})^2$$

where g is the resonant coupling rate, χ is the cross-Kerr rate and χ_a the self-Kerr rate. We assume the following ordering of the parameters : $\kappa \gg \max(|g|, |\chi|, |\chi_a|)$ corresponding to the standard assumption that the subsystem A is strongly dissipative i.e. on a fast time-scale. As the interaction is Hamiltonian, we can therefore apply theorem 5 to derive the second order reduced dynamics.

Using the notations of Theorem 5, we take

$$\begin{aligned}&(\boldsymbol{A}_1, \boldsymbol{B}_1) = (\boldsymbol{a}, g\boldsymbol{b}^2), && (\boldsymbol{A}_2, \boldsymbol{B}_2) = (\boldsymbol{a}^\dagger, g(\boldsymbol{b}^\dagger)^2),\\ &(\boldsymbol{A}_3, \boldsymbol{B}_3) = (\boldsymbol{a}^\dagger \boldsymbol{a}, \chi \boldsymbol{b}^\dagger \boldsymbol{b}), && (\boldsymbol{A}_4, \boldsymbol{B}_4) = ((\boldsymbol{a}^\dagger \boldsymbol{a})^2, \tfrac{\chi_a}{2}\boldsymbol{I}).\end{aligned}$$

The first order reduced dynamics is readily given by :

$$\mathcal{L}_{s,1}(\rho_s) = -i\alpha g[\boldsymbol{b}^2 + (\boldsymbol{b}^\dagger)^2, \rho_s] - i\left[\alpha^2 \chi \boldsymbol{b}^\dagger \boldsymbol{b}, \rho_s\right]$$

In order to get the second order reduced dynamics we compute the operators $\boldsymbol{F}_k \overline{\rho}_A$ $(k = 1, \ldots, 4)$ defined in equation (3.13):

$$\boldsymbol{F}_k \overline{\rho}_A = \int_0^\infty e^{t\mathcal{L}_A} \Big(\boldsymbol{A}_k \overline{\rho}_A - \operatorname{Tr}\left(\boldsymbol{A}_k \overline{\rho}_A\right) \overline{\rho}_A \Big) dt$$

Using the fact that for any $n \geq 1$, we have $\langle\alpha| \, \mathcal{D}_\alpha (\boldsymbol{a}^\dagger)^n = 0$ and the explicit expression of the propagator (4.11) we get :

$$\boldsymbol{F}_k \overline{\rho}_a = \int_0^{+\infty} \left(\boldsymbol{D}_\alpha \left(e^{-\left(\frac{\kappa}{2}+i\Delta\right)t\boldsymbol{N}} \right) \boldsymbol{D}_\alpha^\dagger \boldsymbol{A}_k \, |\alpha\rangle\langle\alpha| - \langle\alpha| \, \boldsymbol{A}_k \, |\alpha\rangle \, |\alpha\rangle\langle\alpha| \right) dt.$$

Direct computations then lead to

$$\begin{aligned}&\boldsymbol{F}_1 = 0, \quad \boldsymbol{F}_2 = \tfrac{1}{\frac{\kappa}{2}+i\Delta}(\boldsymbol{a}^\dagger - \alpha^*), \quad \boldsymbol{F}_3 = \alpha \boldsymbol{F}_2,\\ &\boldsymbol{F}_4 = \tfrac{\alpha(1+2|\alpha|^2)}{\frac{\kappa}{2}+i\Delta}(\boldsymbol{a}^\dagger - \alpha^*) + \tfrac{1}{2}\tfrac{\alpha^2}{\frac{\kappa}{2}+i\Delta}(\boldsymbol{a}^\dagger - \alpha^*)^2\end{aligned}$$

We are now being able to compute the 4×4 matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined in Lemma 4 and needed to compute the second order reduced dynamics through Theorem 5.

After standard calculations, we get the following matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$:

$$\boldsymbol{X} = \frac{4\kappa}{\kappa^2 + 4\Delta^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & \alpha(1+2|\alpha|^2) \\ 0 & \alpha^* & |\alpha|^2 & |\alpha|^2(1+2|\alpha|^2) \\ 0 & \alpha^*(1+2|\alpha|^2) & |\alpha|^2(1+2|\alpha|^2) & |\alpha|^2(1+2|\alpha|^2)^2 + |\alpha|^4 \end{pmatrix}$$

$$\boldsymbol{Y} = \frac{-\Delta}{\kappa^2 + 4\Delta^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & \alpha(1+2|\alpha|^2) \\ 0 & \alpha^* & |\alpha|^2 & |\alpha|^2(1+2|\alpha|^2) \\ 0 & \alpha^*(1+2|\alpha|^2) & |\alpha|^2(1+2|\alpha|^2) & |\alpha|^2(1+2|\alpha|^2)^2 + |\alpha|^4 \end{pmatrix}$$

The last step to get the second order reduced dynamics is to compute the matrix $\boldsymbol{\Lambda}$ verifying $\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\dagger$. This decomposition is not unique. Using the fact that $\boldsymbol{X}$ is positive semi-definite, we use the Cholesky decomposition leading to a triangular matrix $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda} = \sqrt{\frac{4\kappa}{\kappa^2 + 4\Delta^2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha^* & 0 & 0 \\ 0 & \alpha^*(1+2|\alpha|^2) & 0 & |\alpha|^2 \end{pmatrix}.$$

From the structural results presented after Theorem 5, we know that there is at most 4 slow-dissipation operators. In this case, we see that there is only one as two vanishes and one is co-linear to identity. Gathering first and second order results, we get the following slow second-order evolution:

$$\frac{d}{dt}\rho_s = -i\alpha g[\boldsymbol{b}^2 + (\boldsymbol{b}^\dagger)^2, \rho_s] - i\left[\alpha^2\chi\boldsymbol{b}^\dagger\boldsymbol{b}, \rho_s\right] - i[\boldsymbol{H}_{s,2}, \rho_s] + \left(\tfrac{4\kappa}{\kappa^2+4\Delta^2}\right)\mathcal{D}_{\boldsymbol{L}_s}(\rho)$$

with

$$\boldsymbol{L}_s = g\boldsymbol{b}^2 + \alpha^*\left(\chi\boldsymbol{b}^\dagger\boldsymbol{b} + \tfrac{\chi_a(1+2|\alpha|^2}{2}\boldsymbol{I}\right) + O(|\alpha|^2)$$

$$\boldsymbol{H}_{s,2} = \frac{-\Delta g^2}{\kappa^2 + 4\Delta^2}\left((\boldsymbol{b}^\dagger)^2\boldsymbol{b}^2 + \alpha(\boldsymbol{b}^\dagger)^2\left(\tfrac{\chi}{g}\boldsymbol{b}^\dagger\boldsymbol{b} + \tfrac{\chi_a}{2g}\boldsymbol{I}\right) + \alpha^*\left(\tfrac{\chi}{g}\boldsymbol{b}^\dagger\boldsymbol{b} + \tfrac{\chi_a}{2g}\boldsymbol{I}\right)(\boldsymbol{b}^\dagger)^2\right) + O(|\alpha|^2)$$

The terms of order $|\alpha|^2$ have not been included in the Hamiltonian for convenience as the complete expression involves nine different terms . It is however possible to retrieve the exact expression using and $\boldsymbol{Y}$.

By using equation (3.12) we get the first-order Kraus map relating the parametrisation via ρ_s of the physical ρ. reads

$$\rho = \Big(\boldsymbol{I} - i\boldsymbol{M}\Big)\Big(\,|\alpha\rangle\langle\alpha| \otimes \rho_s\Big)\Big(\boldsymbol{I} + i\boldsymbol{M}^\dagger\Big)$$

where

$$\boldsymbol{M} = (\boldsymbol{a}^\dagger - \alpha^*) \otimes \left(\tfrac{g}{\frac{\kappa}{2}+i\Delta}\boldsymbol{b}^2 + \tfrac{\alpha\chi}{\frac{\kappa}{2}+i\Delta}\boldsymbol{b}^\dagger\boldsymbol{b} + \tfrac{\alpha(1+2|\alpha|^2)\chi_a}{\frac{\kappa}{2}+i\Delta}\boldsymbol{I}\right) + \tfrac{\alpha^2\chi_a}{\frac{\kappa}{2}+i\Delta}(\boldsymbol{a}^\dagger - \alpha^*)^2 \otimes \boldsymbol{I}.$$

By taking advantage of the fact that $\boldsymbol{M}^\dagger |\alpha\rangle\langle\alpha| \otimes \rho_s \equiv 0$, such parametrisation can be reformulated, up to second order terms as,

$$\rho = e^{-i(\boldsymbol{M}+\boldsymbol{M}^\dagger)}\Big(|\alpha\rangle\langle\alpha| \otimes \rho_s \Big) e^{i(\boldsymbol{M}+\boldsymbol{M}^\dagger)}$$

where the unitary transformation $e^{-i(\boldsymbol{M}+\boldsymbol{M}^\dagger)}$ induces first-order entanglement between the fast and the slow subsystems.

We emphasise that due to the fact that the fast dynamics is linear and that the non-linearity is simply caused by the operator $\boldsymbol{N}^2$, it would have been possible to compute the second order reduced dynamics using the approach of section 4.2 by solving the Heisenberg equations. However, with such approach, the calculation of the parametrisation via ρ_s of the physical complete density operator ρ would not be possible. We now turn to an example where even the second order reduced dynamics cannot be computed using the linear approach of section 4.2.

4.3.3 Application to non-linear fast dynamics

We consider in this section a fast quantum harmonic oscillator A on $\mathcal{H}_A$ with a non-linear dynamics and weakly coupled in a cascade way to a slow subsystem B corresponding to the case considered in section 3.3.1. The unique steady state of the fast subsystem is a purely quantum state, a *cat state* (see below). Due to the cascade coupling, the slow subsystem is "driven" by this quantum state. This system thus allows to study the effect of a purely quantum drive on a system of interest. The state of the system is described by the density operator ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ whose dynamics is given by :

$$\frac{d}{dt}\rho = [u\boldsymbol{a}_k^\dagger - u^*\boldsymbol{a}_k, \rho] + \frac{\sqrt{\kappa_A\kappa_B}}{2}[\boldsymbol{a}\boldsymbol{b}^\dagger - \boldsymbol{a}^\dagger\boldsymbol{b}, \rho] + \kappa\mathcal{D}_{\boldsymbol{a}_k}(\rho) + \mathcal{D}_{\sqrt{\kappa_A}\boldsymbol{a}+\sqrt{\kappa_B}\boldsymbol{b}}(\rho) \tag{4.12}$$

where $\boldsymbol{a}$ is the annihilation operator on the fast subsystem. The operator $\boldsymbol{a}_k = e^{i\pi\boldsymbol{a}^\dagger\boldsymbol{a}}\boldsymbol{a}$ corresponds to the subsystem being in a non-linear medium. Due to this non-linearity, obtaining the reduced dynamics through Langevin equations or using the Heinsenberg picture is impossible. u is the complex drive amplitude, $\kappa, \kappa_A, \kappa_B$ are the dissipation rates. The operator $\boldsymbol{b}$ is an unspecified operator on $\mathcal{H}_B$ (typically the annihilation operator when the subsystem B is a harmonic oscillator). The two time-scales are given by the parameter ordering $\kappa \gg \max\{\kappa_A, \kappa_B\}$. In the absence of perturbation ($\epsilon = 0$) the fast subsystem converges towards the unique steady state, called *cat-phase state* :

$$\overline{\rho}_A = \frac{1}{\sqrt{2}}\Big(|i\alpha\rangle + i\,|-i\alpha\rangle \Big)\Big(\langle i\alpha| - i\,\langle -i\alpha| \Big) \qquad \text{with } \alpha = \tfrac{2u}{\kappa}. \tag{4.13}$$

To show this, use the change of variables $\rho = \mathcal{U}\xi\mathcal{U}^\dagger$, where $\mathcal{U}$ is the Kerr transformation defined by $\mathcal{U} = \exp\Big[-i\frac{\pi}{2}(\boldsymbol{N}^2 - \boldsymbol{N})\Big]$, on the unperturbed system

$$\frac{d}{dt}\rho = [u\boldsymbol{a}_k^\dagger - u^*\boldsymbol{a}_k, \rho] + \kappa\mathcal{D}_{\boldsymbol{a}_k}(\rho).$$

Using the properties of the Kerr transform

$$\begin{aligned} \boldsymbol{a}_k^\dagger\mathcal{U} &= \mathcal{U}\boldsymbol{a}^\dagger \\ \boldsymbol{a}_k\mathcal{U} &= \mathcal{U}\boldsymbol{a} \end{aligned} \tag{4.14}$$

we derive the following master equation for the density operator ξ :

$$\frac{d}{dt}\xi = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\mathcal{D}_{\boldsymbol{a}}(\rho).$$

Therefore, we get that ξ converges towards the unique steady state $|\alpha\rangle\langle\alpha|$ with $\alpha = \frac{2u}{\kappa}$. The unique steady state for ρ is thus given by $\mathcal{U}|\alpha\rangle\langle\alpha|\mathcal{U}^\dagger = \overline{\rho}_A$. Direct computations leads to equation (4.13).

We now turn to the computation of the reduced dynamics. By using Theorem 7, to get the first order we only have to compute $\mathrm{Tr}\,(\boldsymbol{a}\overline{\rho}_A)$:

$$\mathrm{Tr}\,(\boldsymbol{a}\overline{\rho}_A) = \alpha\,\langle\alpha|\,e^{i\pi\boldsymbol{N}}\,|\alpha\rangle = \langle\alpha|\,-\alpha\rangle = \alpha e^{-2|\alpha|^2}$$

where we used the properties of the Kerr transform (4.14) and of the coherent states presented in section 4.1.2.

The second order reduced dynamics is derived by means of the coefficients $\widetilde{\alpha}$, β and γ defined in Theorem 7[2]. While it's possible to directly perform the computations by using the explicit formulas for the propagator (4.8), (4.9), the simplest way is first to make the change of variables associated with the Kerr transformation to the composite system. The resulting Lindblad master equation is given by :

$$\begin{aligned}\frac{d}{dt}\xi &= [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \xi] + \frac{\sqrt{\kappa_A\kappa_B}}{2}[\boldsymbol{a}_k\boldsymbol{b}^\dagger - \boldsymbol{a}_k^\dagger\boldsymbol{b}, \xi] + \kappa\mathcal{D}_{\boldsymbol{a}}(\xi) + \mathcal{D}_{\sqrt{\kappa_A}\boldsymbol{a}_k+\sqrt{\kappa_B}\boldsymbol{b}}(\xi)\\ &\triangleq \mathcal{L}_A(\xi) + \frac{\sqrt{\kappa_A\kappa_B}}{2}[\boldsymbol{a}_k\boldsymbol{b}^\dagger - \boldsymbol{a}_k^\dagger\boldsymbol{b}, \xi] + \mathcal{D}_{\sqrt{\kappa_A}\boldsymbol{a}_k+\sqrt{\kappa_B}\boldsymbol{b}}(\xi)\end{aligned}$$

Where the unique steady state is $\overline{\xi}_A = |\alpha\rangle\langle\alpha|$ and $\kappa \gg \max\{\kappa_A, \kappa_B\} > 0$. By an appropriate choice of unit and in order to match exactly the framework of Theorem 7, we denote $\epsilon = \kappa_A$. We assume for simplicity that we also have $\kappa_B = \epsilon$. The general case can be readily retrieved by considering the change $\boldsymbol{b} \leftrightarrow \kappa_B/\epsilon\boldsymbol{b}$. It results,

$$\frac{d}{dt}\xi = \mathcal{L}_A(\xi) + \frac{\epsilon}{2}[\boldsymbol{a}_k\boldsymbol{b}^\dagger - \boldsymbol{a}_k^\dagger\boldsymbol{b}, \xi] + \epsilon\mathcal{D}_{\boldsymbol{a}_k+\boldsymbol{b}}(\xi)$$

The propagator associated with the fast dynamics is given by :

$$e^{t\mathcal{L}_A}(\xi) = \sum_{n=0}^{+\infty}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right)\boldsymbol{D}_\alpha\left(e^{-\frac{\kappa}{2}t\boldsymbol{N}}\boldsymbol{a}^n\right)\boldsymbol{D}_\alpha^\dagger\;\xi\;\boldsymbol{D}_\alpha\left(\left(\boldsymbol{a}^\dagger\right)^n e^{-\frac{\kappa}{2}t\boldsymbol{N}}\right)\boldsymbol{D}_\alpha^\dagger$$

We detail now the computation of the coefficient $\widetilde{\alpha} = \mathrm{Tr}\left(\boldsymbol{a}_k^\dagger\overline{\tau}\overline{\mathcal{K}}_A((\boldsymbol{a}_k - \mathrm{Tr}\left(\boldsymbol{a}_k\overline{\xi}_A\right))\overline{\xi}_A\right)$ from Theorem 7 needed to get the second order reduced dynamics . The coefficient β is computed in a similar way. From equation (2.28) in appendix, we get :

$$\widetilde{\alpha} = \mathrm{Tr}\left(\boldsymbol{a}_k^\dagger\int_0^\infty e^{t\mathcal{L}_A}\left((\boldsymbol{a}_k - \mathrm{Tr}\left(\boldsymbol{a}_k\overline{\xi}_A\right))\overline{\xi}_A\right)dt\right)$$

[2]The coefficient $\widetilde{\alpha}$ here is simply denoted α in Theorem 7, however, to avoid any confusion with the parameter $\alpha = 2u/\kappa$ we changed the notation.

Using the fact that for any $n > 0$, we have $\langle\alpha|\,\mathcal{D}_\alpha(\boldsymbol{a}^\dagger)^n = 0$, we get

$$\begin{aligned}
\mathrm{Tr}\left(\boldsymbol{a}_k^\dagger e^{t\mathcal{L}_A}\left(\boldsymbol{a}_k\overline{\xi}_A\right)\right) &= \mathrm{Tr}\left(\boldsymbol{a}_k^\dagger \boldsymbol{D}_\alpha e^{-\frac{\kappa}{2}t\boldsymbol{N}}\boldsymbol{D}_\alpha^\dagger\ \boldsymbol{a}_k\,|\alpha\rangle\langle\alpha|\ \ \boldsymbol{D}_\alpha e^{-\frac{\kappa}{2}t\boldsymbol{N}}\boldsymbol{D}_\alpha^\dagger\right)\\
&= \alpha\mathrm{Tr}\left(\boldsymbol{a}_k^\dagger \boldsymbol{D}_\alpha e^{-\frac{\kappa}{2}t\boldsymbol{N}}\boldsymbol{D}_\alpha^\dagger\ \ |-\alpha\rangle\langle 0|\ \ e^{-\frac{\kappa}{2}t\boldsymbol{N}}\boldsymbol{D}_\alpha^\dagger\right)\\
&= \alpha\mathrm{Tr}\left(e^{-\frac{\kappa}{2}t\boldsymbol{N}}\ \ |-2\alpha\rangle\langle\alpha|\,\boldsymbol{a}_k^\dagger\boldsymbol{D}_\alpha\right)\\
&= |\alpha|^2\,\langle -2\alpha|\,e^{-\frac{\kappa}{2}t\boldsymbol{N}}\,|-2\alpha\rangle\\
&= |\alpha|^2 e^{-2|\alpha|^2}e^{2|\alpha|^2e^{-\kappa t}}\,\left\langle -2\alpha|\ -2\alpha e^{-\kappa t/2}\right\rangle
\end{aligned}$$

where we used between line 1 and 2 that for any coherent state of amplitude α, we have $e^{-i\pi\boldsymbol{N}}\,|\alpha\rangle = |-\alpha\rangle$. To get the last line, we decompose the coherent state in the Fock basis using (4.3) to perform the computations. By taking advantage of equation (4.5) giving the overlap between two different coherent states, we get :

$$\widetilde{\alpha} = |\alpha|^2 e^{-4|\alpha|^2}\int_0^\infty\left(e^{4|\alpha|^2e^{-\kappa t/2}} - 1\right)dt$$

Finally, using equation 5.1.40 from [AS65] it can be expressed using the special function $Ei(x)$ and the Euler's constant $\bar{\gamma}$ by :

$$\widetilde{\alpha} = \frac{2|\alpha|^2e^{-4|\alpha|^2}}{\kappa}\left(\mathrm{Ei}(4|\alpha|^2) - \ln(4|\alpha|^2) - \bar{\gamma}\right)$$

Similar computations leads to

$$\begin{aligned}
\beta &= \alpha^2e^{-4|\alpha|^2}\int_0^\infty\left((1-2e^{-\kappa t/2})e^{4|\alpha|^2e^{-\kappa t/2}} - 1\right)dt\\
&= \frac{\alpha^2}{|\alpha|^2}\left(\widetilde{\alpha} - \frac{e^{-4|\alpha|^2}}{\kappa}\left(e^{4|\alpha^2|} - 1\right)\right) \qquad \text{assuming } \alpha \neq 0
\end{aligned}$$

Last, we have to calculate the coefficient $\gamma = \mathrm{Tr}\left(\boldsymbol{a}_k\overline{\tau}\overline{\mathcal{K}}(\mathcal{D}_{\boldsymbol{a}_k}(\xi_A))\right)$. After simple but tedious computations we get :

$$\begin{aligned}
\gamma &= \int_0^\infty \alpha e^{-2|\alpha|^2}e^{-\kappa t/2}\frac{4|\alpha|^2-1}{2} + \alpha|\alpha|^2e^{-2|\alpha|^2}(-1+(1-2e^{-\kappa t/2})e^{8|\alpha|^2e^{-\kappa t/2}}e^{-8|\alpha|^2e^{-\kappa t}})\\
&= \frac{\alpha e^{-2|\alpha|^2}(4|\alpha|^2-1)}{\kappa} + \frac{2\alpha}{\kappa}|\alpha|^2e^{-2|\alpha|^2}\int_0^1\frac{e^{8|\alpha|^2(u-u^2)}-1}{u}du\\
&- \frac{\sqrt{2}\alpha}{\kappa}|\alpha|\sqrt{\pi}\,\mathrm{erf}(\sqrt{2}|\alpha|)
\end{aligned}$$

where erf is the error function. We can now apply Theorem 7 to get the second order reduced dynamics associated with the dynamics (4.12) :

$$\begin{aligned}
\frac{d}{dt}\rho_s = \epsilon e^{-2|\alpha^2}\Big(\left[\alpha^*\boldsymbol{b} - \alpha\boldsymbol{b}^\dagger\,,\,\rho_s\right]\Big) + \epsilon\mathcal{D}_{\sqrt{1+2\epsilon\widetilde{\alpha}}\boldsymbol{b}-2\epsilon\beta\boldsymbol{b}^\dagger}(\rho_s)\\
+2\epsilon^2\Big(\widetilde{\alpha}\mathcal{D}_{\boldsymbol{b}^\dagger}(\rho_s) + \left[\gamma^*\boldsymbol{b} - \gamma\boldsymbol{b}^\dagger\,,\,\rho_s\right]\Big)
\end{aligned}$$

Remark 5. We have considered in this example the parameter ordering $\kappa \gg \max\{\kappa_A, \kappa_B\}$ corresponding to a cascade with weak outcoming field presented in section 3.3.1. In a difference case, with the other ordering permitting adiabatic elimination, namely $\{\kappa, \kappa_A\} \gg \kappa_B$ corresponding to the cascade with strongly reflected field of section 3.3.2, we see from the definition within Theorem 8 that the coefficient to be computed in order to get the reduced dynamics are equal to $\tilde{\alpha}$ and β^*. As they verify Assumption 1, we can apply Theorem 8 and directly derive the second order reduced dynamics.

Chapter 5

Extensions

We present in this chapter various extensions of our main results on adiabatic elimination from chapter 3. In the first section, we completely remain in the framework of chapter 3. The goal is to extend our adiabatic elimination technique up to third order terms. We succeed in this objective provided that the Hamiltonian interaction includes only one tensor-product term. This already allows to make structural results on the third order approximation and we conjecture with a strong symmetry argument, that it holds for any type of Hamiltonian coupling. This is still the topic of ongoing research.

The second section, deals with a totally different setting than the bipartite systems considered in chapter 3. We take interest in systems with decoherence-free spaces [LW03]. Such systems are taking more and more importance in quantum information as they are a promising way to build a quantum computer [LCW98, MA+14]. The need to develop generic adiabatic elimination methods for such systems is therefore becoming important. A first order method has been presented in [ZV14]. We go further and use our approach to develop the second order approximation and have a parametrisation of the slow manifold in a Kraus map form. This underlines that our geometric approach presented in chapter 2 seems applicable for very different structure of two time-scales quantum systems.

5.1 Preliminary results on the third order approximation

We present in this section preliminary results on the third order approximation for composite systems with an Hamiltonian interaction of the form $\boldsymbol{H}_{int} = \boldsymbol{A} \otimes \boldsymbol{B}$ (such as in dispersive coupling). We rely on a technical assumption needed to express the third order reduced dynamics in a Lindblad form. We believe this assumption to be generically true. This is the topic of ongoing research with the generalisation to any Hamiltonian coupling.

We consider the structure of a bipartite quantum system with Hamiltonian interaction presented in Section 3.2 ($\mathcal{L}_{int} = -i[\boldsymbol{H}_{int}, \cdot]$). The dynamics of the complete system, composed of a subsytem A on a Hilbert space $\mathcal{H}_A$ and a subsystem B on $\mathcal{H}_B$ and described by the density operator ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$, is, according to (3.10):

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) + \epsilon\Big(-i\left[\boldsymbol{H}_{int}, \rho\right] + \mathcal{L}_B(\rho)\Big)$$

where the interaction Hamiltonian H_{int} is assumed to be dispersive i.e. $\boldsymbol{H}_{int} = \boldsymbol{A} \otimes \boldsymbol{B}$. The unique steady state of the fast dynamics is denote $\overline{\rho}_A$, verifying $\mathcal{L}_A(\overline{\rho}_A) = 0$. Moreover, we assume that the slow dynamics $\mathcal{L}_B$ includes only Hamiltonian terms i.e $\mathcal{L}_B(\bullet) = -i[\boldsymbol{H}_B, \bullet]$. This case was considered (for general Hamiltonian coupling) in Section 3.2.4 and we showed in Theorem 6 how the second order parametrisation can be expressed in a Kraus form.

For convenience, before turning to the computation of the third order, we briefly remind notations and results for the second order reduced model in this particular case. We refer to Chapter 3 for more details.

Corollary 1 shows that the second order reduced dynamics described by the super-operator $\mathcal{L}_{s,2}$ can be expressed in the Lindblad form (3.16). Thus the reduced dynamics, up to second in ϵ is given by :

$$\begin{aligned}\frac{d}{dt}\rho_s =& -i\epsilon\left[\mathrm{Tr}\left(\boldsymbol{A}\overline{\rho}_A\right)\boldsymbol{B}\,,\,\rho_s\right] - i\epsilon[\boldsymbol{H}_B, \rho_s] \\ & - i\epsilon^2\left[\frac{\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A} - \boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right)}{2i}\boldsymbol{B}^2\,,\,\rho_s\right] + \epsilon^2\eta^2\mathcal{D}_{\boldsymbol{B}}(\rho_s)\end{aligned} \tag{5.1}$$

with $\eta^2 = \mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A} + \boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right)$ and $\boldsymbol{F}$ is given by $\boldsymbol{F}\overline{\rho}_A = -\mathcal{L}_A^{-1}\Big(\boldsymbol{A} - \mathrm{Tr}\left(\boldsymbol{A}\,\overline{\rho}_A\right)\overline{\rho}_A)\Big) = \overline{\tau}\overline{\mathcal{K}}_A\Big(\boldsymbol{A}\,\overline{\rho}_A - \mathrm{Tr}\left(\boldsymbol{A}\,\overline{\rho}_A\right)\overline{\rho}_A\Big)$.

The second order parametrisation $\mathcal{K}_2$ is computed in the proof of Theorem 6. Its expression is given by equations (3.21) and (3.26) which turn to be, for this dispersive coupling case :

$$\begin{aligned}\mathcal{L}_A(\mathcal{K}_2(\rho_s)) =& -\Big(\mathcal{S}(\boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger) + \mathcal{S}(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A})\Big) \otimes \boldsymbol{B}\rho_s\boldsymbol{B} \\ & + \mathcal{S}(\overline{\rho}_A\boldsymbol{F}^\dagger\boldsymbol{A}) \otimes \rho_s\boldsymbol{B}^2 + \mathcal{S}(\boldsymbol{A}\boldsymbol{F}\overline{\rho}_A) \otimes \boldsymbol{B}^2\rho_s \\ & - \boldsymbol{F}\overline{\rho}_A \otimes \boldsymbol{B}\Big[\mathrm{Tr}\left(\boldsymbol{A}\overline{\rho}_A\right)\boldsymbol{B}\,,\rho_s\Big] + \overline{\rho}_A\boldsymbol{F}^\dagger \otimes \Big[\mathrm{Tr}\left(\boldsymbol{A}\overline{\rho}_A\right)\boldsymbol{B}\,,\rho_s\Big]\boldsymbol{B} \\ & + \boldsymbol{F}\overline{\rho}_A \otimes \left[\boldsymbol{H}_B\,,\,\boldsymbol{B}\right]\rho_s - \overline{\rho}_A\boldsymbol{F}^\dagger \otimes \rho_s\left[\boldsymbol{H}_B\,,\,\boldsymbol{B}\right]\end{aligned}$$

where $\mathcal{S}(\boldsymbol{X}) = \boldsymbol{X} - \mathrm{Tr}\left(\boldsymbol{X}\right)\overline{\rho}_A = \boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})$ for any operator $\boldsymbol{X}$ on $\mathcal{H}_A$. We introduce now the notation $\overline{\boldsymbol{A}} = \boldsymbol{A} - \mathrm{Tr}\left(\boldsymbol{A}\overline{\rho}_A\right)\boldsymbol{1}$ and the following definition of the operators $\boldsymbol{T}$, $\boldsymbol{U}$

and $\boldsymbol{V}$:

$$\begin{aligned}\mathcal{L}_A^{-1}\Big(\mathcal{S}(\overline{\boldsymbol{A}}\overline{\rho}_A\boldsymbol{F}^\dagger)\Big) &= -\overline{\tau}\overline{\mathcal{K}}_A(\overline{\boldsymbol{A}}\overline{\rho}_A\boldsymbol{F}^\dagger) \triangleq \boldsymbol{T}\\ \mathcal{L}_A^{-1}\Big(\mathcal{S}(\overline{\boldsymbol{A}}\boldsymbol{F}\overline{\rho}_A)\Big) &= -\overline{\tau}\overline{\mathcal{K}}_A(\overline{\boldsymbol{A}}\boldsymbol{F}\overline{\rho}_A) \triangleq \boldsymbol{U}\\ \mathcal{L}_A^{-1}\Big(\boldsymbol{F}\overline{\rho}_A\Big) &= -\overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{F}\overline{\rho}_A) \triangleq \boldsymbol{V}\end{aligned}$$

They lead to a simple expression of the second order parametrisation $\mathcal{K}_2$, namely :

$$\begin{aligned}\mathcal{K}_2(\rho_s) = &-\Big(\boldsymbol{T}+\boldsymbol{T}^\dagger\Big)\otimes\boldsymbol{B}\rho_s\boldsymbol{B}\\ &+\boldsymbol{U}^\dagger\otimes\rho_s\boldsymbol{B}^2+\boldsymbol{U}\otimes\boldsymbol{B}^2\rho_s\\ &+\boldsymbol{V}\otimes[\boldsymbol{H}_B\,,\,\boldsymbol{B}]\,\rho_s-\boldsymbol{V}^\dagger\otimes\rho_s\,[\boldsymbol{H}_B\,,\,\boldsymbol{B}]\,.\end{aligned}\tag{5.2}$$

Note that this expression is different from the one ensuring a Kraus map for $\mathcal{K}_0+\epsilon\mathcal{K}_1+\epsilon^2\mathcal{K}_2$. It has however the advantage of verifying $\mathrm{Tr}_A\left(\mathcal{K}_0+\epsilon\mathcal{K}_1+\epsilon^2\mathcal{K}_2\right)=\rho_s$, facilitating the interpretation of ρ_s as the density operator of the subsystem B.

Theorem 9. **(Third order, one tensor-product term coupling)**

Assume that the dissipative term in the second order reduced dynamics is non-zero i.e. in (5.1) we have $\eta\neq 0$. Then the third order reduced dynamics is given by the following Lindblad master equation :

$$\begin{aligned}\frac{d}{dt}\rho_s = &-i\epsilon\left[\mathrm{Tr}\left(\boldsymbol{A}\overline{\rho}_A\right)\boldsymbol{B}\,,\,\rho_s\right]-i\epsilon[\boldsymbol{H}_B,\rho_s]-i\epsilon^2\left[\frac{\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A}-\boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right)}{2i}\boldsymbol{B}^2\,,\,\rho_s\right]\\ &-i\epsilon^3\frac{u+u^*}{2}[\boldsymbol{B}^3,\rho_s]-i\epsilon^3\left[\frac{v}{2}\boldsymbol{B}[\boldsymbol{H}_B,\boldsymbol{B}]-\frac{v^*}{2}[\boldsymbol{H}_B,\boldsymbol{B}]\boldsymbol{B},\rho_s\right]\\ &+\epsilon^2\mathcal{D}_{\boldsymbol{L}}+O(\epsilon^2)\end{aligned}$$

with : $t=-\mathrm{Tr}\left(\overline{\tau}\overline{\mathcal{K}}_A(\overline{\boldsymbol{A}}\overline{\rho}_A\boldsymbol{F}^\dagger)\boldsymbol{A}\right)$, $u=-\mathrm{Tr}\left(\overline{\tau}\overline{\mathcal{K}}_A(\overline{\boldsymbol{A}}\boldsymbol{F}\overline{\rho}_A)\boldsymbol{A}\right)$ and $v=-\mathrm{Tr}\left(\overline{\tau}\overline{\mathcal{K}}_A(\boldsymbol{F}\overline{\rho}_A)\boldsymbol{A}\right)$. The dissipation operator $\boldsymbol{L}$ is defined as $\boldsymbol{L}=\eta\boldsymbol{B}+\frac{i\epsilon}{\eta}\Big((t+t^*+u)\boldsymbol{B}^2+v[\boldsymbol{H}_B,\boldsymbol{B}]\Big)$.

Proof. In order to get the third order reduced dynamics, we have to solve the third order recurrence relation (2.26). In this case of composite structure this equation yields :

$$\mathcal{L}_A(\mathcal{K}_3(\rho_s))+\mathcal{L}_{int}(\mathcal{K}_2(\rho_s))=\mathcal{K}_0(\mathcal{L}_{s,3}(\rho_s))+\mathcal{K}_1(\mathcal{L}_{s,2}(\rho_s))+\mathcal{K}_2(\mathcal{L}_{s,1}(\rho_s))\tag{5.3}$$

We use the usual trick of taking the mapping $\overline{\mathcal{R}}$ (or similarly the partial trace over A) in equation (5.3) leading to the cancellation of the $\mathcal{L}_A$ term. Using that $\mathrm{Tr}_A\left(\mathcal{K}_1(\bullet)\right)\equiv 0$ and from (3.21) that $\mathrm{Tr}_A\left(\mathcal{K}_2(\bullet)\right)\equiv 0$ we get :

$$\mathcal{L}_{s,3}(\rho_s)=\mathrm{Tr}_A\Big(\mathcal{L}_{int}\Big(\mathcal{K}_2(\rho_s)\Big)\Big)$$

Using the expression 5.2 for $\mathcal{K}_2$ and that $\mathcal{L}_{int}(\bullet)=-i[\boldsymbol{A}\otimes\boldsymbol{B},\bullet]$ we get :

$$\begin{aligned}\mathcal{L}_{s,3}(\rho_s) &= -i\mathrm{Tr}_A\Big([\boldsymbol{A}\otimes\boldsymbol{B},-(\boldsymbol{T}+\boldsymbol{T}^\dagger)\otimes\boldsymbol{B}\rho_s\boldsymbol{B}+\boldsymbol{U}^\dagger\otimes\rho_s\boldsymbol{B}^2+\boldsymbol{U}\otimes\boldsymbol{B}^2\rho_s\Big)\\ &\quad -i\mathrm{Tr}_A\Big([\boldsymbol{A}\otimes\boldsymbol{B},\boldsymbol{V}\otimes[\boldsymbol{H}_B\,,\,\boldsymbol{B}]\,\rho_s-\boldsymbol{V}^\dagger\otimes\rho_s\,[\boldsymbol{H}_B\,,\,\boldsymbol{B}]\Big)\\ &= i\Big((t+t^*+u)\boldsymbol{B}^2\rho_s\boldsymbol{B}-(t+t^*+u^*)\boldsymbol{B}\rho_s\boldsymbol{B}^2-u\boldsymbol{B}^3\rho_s+u^*\rho_s\boldsymbol{B}^3\\ &\quad -v\boldsymbol{B}\,[\boldsymbol{H}_B\,,\,\boldsymbol{B}]\,\rho_s+v^*\boldsymbol{B}\rho_s\,[\boldsymbol{H}_B\,,\,\boldsymbol{B}]+v\,[\boldsymbol{H}_B\,,\,\boldsymbol{B}]\,\rho_s\boldsymbol{B}-v^*\rho_s\,[\boldsymbol{H}_B\,,\,\boldsymbol{B}]\,\boldsymbol{B}\Big)\end{aligned}$$

with $t = \mathrm{Tr}\left(\boldsymbol{AT}\right), u = \mathrm{Tr}\left(\boldsymbol{AU}\right), v = \mathrm{Tr}\left(\boldsymbol{AV}\right)$.

One can see that this expression of $\mathcal{L}_{s,3}$ is Hermitian. However, the asymmetry of this equation due to the term $\boldsymbol{B}\rho_s\boldsymbol{B}^2$ and its Hermitian conjugate prevent any Lindblad-form expression for $\mathcal{L}_{s,3}$. In order to get a Lindblad dynamics for the reduced system, we will proceed similarly than the case of cascade coupling 3.3 and combine the second and third order in order to get a Lindblad expression. Indeed, $\mathcal{L}_{s,2} + \epsilon\mathcal{L}_{s,3}$ admits the following Lindblad form (up to higher order terms) :

$$\begin{aligned}\mathcal{L}_{s,2} + \epsilon\mathcal{L}_{s,3} = -i\left[\frac{\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A} - \boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right)}{2i}\boldsymbol{B}^2\,,\,\rho_s\right] - i\epsilon\frac{u+u^*}{2}[\boldsymbol{B}^3,\rho_s] \\ -i\epsilon\left[\frac{v}{2}\boldsymbol{B}[\boldsymbol{H}_B,\boldsymbol{B}] - \frac{v^*}{2}[\boldsymbol{H}_B,\boldsymbol{B}]\boldsymbol{B},\rho_s\right] + \mathcal{D}_{\boldsymbol{L}} + O(\epsilon^4)\end{aligned}$$

where $\boldsymbol{L} = \eta\boldsymbol{B} + \frac{i\epsilon}{\eta}\Big((t+t^*+u)\boldsymbol{B}^2 + v[\boldsymbol{H}_B,\boldsymbol{B}]\Big)$, using that by assumption we have $\eta \neq 0$. This can be proved by a direct expansion, concluding the proof. ■

Remark 6. The Theorem 9 allows to make several structural results on the third order reduced dynamics :

- The third order does not add new decoherence channel. It merely modifies the existing one. This come from the asymmetry in the direct computation of $\mathcal{L}_{s,3}$ and we don't see any reason why this would be different for the most general type of Hamiltonian coupling $\boldsymbol{H}_{int} = \sum_k \boldsymbol{A}^k \otimes \boldsymbol{B}_k^\dagger$. This is currently investigated.
- The initial slow dynamics $\mathcal{L}_B$, which played no role in the computation of the reduced dynamics up to second order have a role in the third order. It appears both in the third order Hamiltonian and dissipation operator, each time with the particular form of a commutator with the interaction operator.

5.2 Systems with decoherence-free space

The realisation of a universal quantum computer is currently an exiting goal as such computer is intrinsically more powerful than its classical counterpart. Indeed, it has be shown that quantum information processing allows speedup compared to classical one [Sho94]. In order to obtain this speedup, one must be able to prepare, manipulate and measure superposition of quantum states. This is however very difficult as these superposition of states are usually not robust to decoherence caused by unwanted coupling to the environment. How to cope with the decoherence is currently an important issue. A first approach is to encode a logical qubit in several redundant physical qubits, similarly than for classical bit. See e.g. [Ste96] for more details on the theory of quantum *error correction* codes. It relies on the assumption that most occurring errors are not correlated between the different redundant qubits encoding a single logical qubit. Such assumption may be questionable in cases e.g. where the different qubits are spatially closed. Another approach, is to encode a logical qubit in a so-called decoherence-free space. In such space, due to symmetry reasons, the qubit is protected against decoherence [LW03]. It was shown in [BKLW00] that universal quantum computation is possible. This promising *passive error-preventing* scheme using decoherence-free space has been recently performed experimentally [MA+14, LTP+15]. It is therefore of great importance to be able to describe and characterise such systems. In this objective, we present now a generic adiabatic elimination method for such system. Our approach is similar to [ZV14], we go further and derive results for the second order reduced dynamics and the parametrisation of the slow manifold.

5.2.1 A class of perturbed master equuations

We consider in this section, a Hilbert space $\mathcal{H}$ of finite dimension. We denote by $\mathcal{D}$ the compact convex set of density operators ρ on $\mathcal{H}$. The dynamics on $\mathcal{D}$ is described by a Lindblad master equation. Assuming the existence of two time-scales, we have :

$$\frac{d}{dt}\rho = \mathcal{L}_0(\rho) + \epsilon\mathcal{L}_1(\rho) \tag{5.4}$$

where $0 < \epsilon$ is a small positive parameter and the linear super-operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are of Lindbladian forms (2.10) characterised by two finite families of operators on $\mathcal{H}$, denoted by $(\boldsymbol{L}_{0,\nu})$ and $(\boldsymbol{L}_{1,\nu})$, and two Hamiltonian terms, described by two Hermitian operators $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$.

We assume as usual that, for $\epsilon = 0$, the unperturbed master equation $\dot{\rho} = \mathcal{L}_0(\rho)$ converges to a stationary regime, but, contrary to the previous sections, we do not impose a unique steady state. That is to say, we assume that the unperturbed master equation admits a sub-manifold of stationary operators and any solution of the unperturbed system with initial condition in $\mathcal{D}$ converges towards this sub-manifold. Denote by

$$\mathcal{D}_0 = \Big\{\rho \in \mathcal{D} \;\Big|\; \mathcal{L}_0(\rho) = 0\Big\}$$

this stationary manifold: it is compact and convex. We assume additionally that $\mathcal{D}_0$ coincides with the set of density operators with support in $\mathcal{H}_0$, a subspace of $\mathcal{H}$. In other words the unperturbed master equation features a decoherence-free space $\mathcal{H}_0$. Accordingly to previous sections we denote $\overline{\mathcal{R}}(\rho_0) = \lim_{t\to\infty} e^{t\mathcal{L}_0}(\rho_0)$ corresponding to the mapping of

any initial state to the final state and verifying, for some operator $(\boldsymbol{M}_\mu)$ on $\mathcal{H}$, the Kraus map form (2.11):

$$\overline{\mathcal{R}}(\rho_0) = \sum_\mu \boldsymbol{M}_\mu \rho_0 \boldsymbol{M}_\mu^\dagger \tag{5.5}$$

with $\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \mathbf{1}$, the identity operator on $\mathcal{H}$.

An *invariant operator* attached to the dynamics $\dot{\rho} = \mathcal{L}_0(\rho)$ is a Hermitian operator $\boldsymbol{J}$ such that for any time $t \geq 0$ and any initial state $\rho_0 = \rho(0)$, we have $\mathrm{Tr}\,(\boldsymbol{J}\rho(t)) = \mathrm{Tr}\,(\boldsymbol{J}\rho_0)$. Such invariant operators $\boldsymbol{J}$ are characterized by the fact that $\mathcal{L}_0^*(\boldsymbol{J}) = 0$ where the adjoint map to $\mathcal{L}_0$ is given by

$$\mathcal{L}_0^*(\boldsymbol{A}) = i[\boldsymbol{H}_0, \boldsymbol{A}] + \sum_\nu \boldsymbol{L}_{0,\nu}^\dagger \boldsymbol{A} \boldsymbol{L}_{0,\nu} - \tfrac{1}{2}\boldsymbol{L}_{0,\nu}^\dagger \boldsymbol{L}_{0,\nu} \boldsymbol{A} - \tfrac{1}{2}\boldsymbol{A}\boldsymbol{L}_{0,\nu}^\dagger \boldsymbol{L}_{0,\nu}$$

for any Hermitian operator $\boldsymbol{A}$. In other words, $\boldsymbol{J}$ is an invariant of the dynamics, corresponding to the fact that its dynamics in the Heisenberg picture is zero.

Thus by taking the limit for t tending to $+\infty$ in $\mathrm{Tr}\,(\boldsymbol{J}\rho(t)) = \mathrm{Tr}\,(\boldsymbol{J}\rho_0)$, we have, for all Hermitian operators ρ_0, $\mathrm{Tr}\left(\boldsymbol{J}\overline{\mathcal{R}}(\rho_0)\right) = \mathrm{Tr}\,(\boldsymbol{J}\rho_0)$. Denote by $\overline{\mathcal{R}}^*$ the adjoint map associated to $\overline{\mathcal{R}}$:

$$\overline{\mathcal{R}}^*(\boldsymbol{A}) = \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{A} \boldsymbol{M}_\mu \tag{5.6}$$

for any Hermitian operator $\boldsymbol{A}$ on $\mathcal{H}$. Then, $\mathrm{Tr}\left(\overline{\mathcal{R}}^*(\boldsymbol{J})\rho_0\right) = \mathrm{Tr}\,(\boldsymbol{J}\rho_0)$ for all ρ_0, implying $\overline{\mathcal{R}}^*(\boldsymbol{J}) = \boldsymbol{J}$. I.e. invariant operators $\boldsymbol{J}$ are characterized by $\mathcal{L}_0^*(\boldsymbol{J}) = 0$ and satisfy $\overline{\mathcal{R}}^*(\boldsymbol{J}) = \boldsymbol{J}$.

Denote by $\boldsymbol{P}_0$ the operator on $\mathcal{H}$ corresponding to orthogonal projection onto $\mathcal{H}_0$. Consequently, for any Hermitian operator ρ, we have $\boldsymbol{P}_0\overline{\mathcal{R}}(\rho) = \overline{\mathcal{R}}(\rho)\boldsymbol{P}_0 = \overline{\mathcal{R}}(\rho)$. Thus $\mathrm{Tr}\left(\overline{\mathcal{R}}^*(\boldsymbol{P}_0)\rho\right) = \mathrm{Tr}\left(\overline{\mathcal{R}}(\rho)\right) = \mathrm{Tr}\,(\rho)$ for all ρ which implies:

$$\overline{\mathcal{R}}^*(\boldsymbol{P}_0) = \mathbf{1}. \tag{5.7}$$

Moreover, for any vector $|c\rangle$ in $\mathcal{H}_0$, $\overline{\mathcal{R}}(|c\rangle\langle c|) = |c\rangle\langle c|$. This implies that, for the Kraus map (5.5), there exists a family of complex numbers λ_μ such that $\sum_\mu |\lambda_\mu|^2 = 1$ and

$$\forall\, |c\rangle \in \mathcal{H}_0,\ \ \boldsymbol{M}_\mu\, |c\rangle = \lambda_\mu\, |c\rangle\,. \tag{5.8}$$

5.2.2 First order expansion for arbitrary perturbations

We consider here the perturbed master equation with two time-scales (5.4) whose unperturbed part $\dot{\rho} = \mathcal{L}_0(\rho)$ satisfies the assumptions of Section 5.2.1: any trajectory converges to a steady-state; the set of steady-states $\mathcal{D}_0$ coincides with the set of density operators with support on a subspace $\mathcal{H}_0$ of $\mathcal{H}$. This section develops a first-order expansion versus ϵ of (5.4) around $\mathcal{D}_0$.

Denote by $\mathcal{H}_s$ (subscript s for slow), an abstract Hilbert space with the same dimension as $\mathcal{H}_0$. Denote by $\mathcal{D}_s$ the set of density operators on $\mathcal{H}_s$. Denote by $\{|\nu_s\rangle\}$ (resp. $\{|\nu_0\rangle\}$) a Hilbert basis of $\mathcal{H}_s$ (resp. $\mathcal{H}_0$). Consider the Kraus map $\mathcal{K}_0$ defined by

$$\forall \rho_s \in \mathcal{D}_s,\ \ \mathcal{K}_0(\rho_s) = \boldsymbol{S}_0 \rho_s \boldsymbol{S}_0^\dagger \in \mathcal{D} \tag{5.9}$$

where $\boldsymbol{S}_0 = \sum_\nu |\nu_0\rangle\,\langle\nu_s|$. We have $\boldsymbol{S}_0\boldsymbol{S}_0^\dagger = \boldsymbol{P}_0$, the orthogonal projector onto $\mathcal{H}_0$ and $\boldsymbol{S}_0^\dagger\boldsymbol{S}_0 = \mathbf{1}_s$, the identity operator on $\mathcal{H}_s$.

As presented in Chapter 2, we are looking for an expansion based on linear super-operators $\{\mathcal{K}_m\}_{m\geq 0}$ between $\mathcal{D}_s$ and $\mathcal{D}$ and on Lindblad dynamics $\{\mathcal{L}_{s,m}\}_{m\geq 0}$ on $\mathcal{D}_s$ such that any solution $t \mapsto \rho_s(t) \in \mathcal{D}_s$ of the reduced Lindblad master equation

$$\frac{d}{dt}\rho_s = \mathcal{L}_s(\rho_s) = \sum_{m\geq 0} \epsilon^m \mathcal{L}_{s,m}(\rho_s) \tag{5.10}$$

yields, via the map

$$\rho(t) = \mathcal{K}(\rho_s(t)) = \sum_{m\geq 0} \epsilon^m \mathcal{K}_m(\rho_s(t)) \,, \tag{5.11}$$

a trajectory of the perturbed system (5.4).

We get from Chapter 2 that the recurrence relations corresponding to the invariance condition of the slow manifold that have to be solved are given by equations (2.24) and (2.25) :

The first order terms in epsilon define $\mathcal{K}_1$ and $\mathcal{L}_{s,1}$ by :

$$\mathcal{L}_0(\mathcal{K}_1(\rho_s)) + \mathcal{L}_1(\mathcal{K}_0(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,1}(\rho_s)) \tag{5.12}$$

The second order terms in epsilon define $\mathcal{K}_2$ and $\mathcal{L}_{s,2}$ by in :

$$\mathcal{L}_0(\mathcal{K}_2(\rho_s)) + \mathcal{L}_1(\mathcal{K}_1(\rho_s)) = \ \mathcal{K}_0(\mathcal{L}_{s,2}(\rho_s)) + \mathcal{K}_1(\mathcal{L}_{s,1}(\rho_s)) \,, \tag{5.13}$$

Where we used the solution of the zeroth order terms (2.23). They are given by $\mathcal{K}_0$ defined in (5.9), and consequently, as we have $\mathcal{L}_0\Big(\mathcal{K}_0(\rho_s)\Big) \equiv 0$, the zero order reduced dynamics is $\mathcal{L}_{s,0}(\rho_s) = 0$.

The following lemma proves that the super-operator $\mathcal{L}_{s,1}(\rho_s)$ defined by equation (5.12) is always of Lindblad form.

Lemma 6. **(First order reduced dynamics, any perturbation)**

The first order reduced dynamics is given by

$$\frac{d}{dt}\rho_s = \epsilon \mathcal{L}_{s,1}(\rho_s)$$

The super-operator $\mathcal{L}_{s,1}$ is in a Lindblad form, whose expression can be computed using equation (5.14) or (5.15) depending on the form of the super-operator $\mathcal{L}_1$.

Assume that $\mathcal{L}_1(\rho) = -i[\boldsymbol{H}_1, \rho]$ for some Hermitian operator $\boldsymbol{H}_1$ on $\mathcal{H}$. Then, if $\mathcal{L}_{s,1}$ satisfies (5.12), we have

$$\mathcal{L}_{s,1}(\rho_s) = -i[\boldsymbol{H}_{s,1}, \rho_s] \tag{5.14}$$

where $\boldsymbol{H}_{s,1} = \boldsymbol{S}_0^\dagger \boldsymbol{H}_1 \boldsymbol{S}_0$ is a Hermitian operator on $\mathcal{H}_s$.

Assume that $\mathcal{L}_1(\rho) = \boldsymbol{L}_1 \rho \boldsymbol{L}_1^\dagger - \frac{1}{2}\Big(\boldsymbol{L}_1^\dagger \boldsymbol{L}_1 \rho + \rho \boldsymbol{L}_1^\dagger \boldsymbol{L}_1\Big)$ for some operator $\boldsymbol{L}_1$ on $\mathcal{H}$. Then, if $\mathcal{L}_{s,1}$ satisfies (5.12), we have

$$\mathcal{L}_{s,1}(\rho_s) = \sum_\mu \boldsymbol{A}_\mu \rho_s \boldsymbol{A}_\mu^\dagger - \tfrac{1}{2}\Big(\boldsymbol{A}_\mu^\dagger \boldsymbol{A}_\mu \rho_s + \rho_s \boldsymbol{A}_\mu^\dagger \boldsymbol{A}_\mu\Big) \tag{5.15}$$

with $\boldsymbol{A}_\mu = \boldsymbol{S}_0^\dagger \boldsymbol{M}_\mu \boldsymbol{L}_1 \boldsymbol{S}_0$ and the Kraus operators $\boldsymbol{M}_\mu$ defined by (5.5).

The result for a general Lindbladian dynamics (2.10) for $r = 1$ follows by linearity.

Proof. By definition we have $\overline{\mathcal{R}} \circ \mathcal{K}_0 = \mathcal{K}_0$ and $\overline{\mathcal{R}} \circ \mathcal{L}_0 = \mathcal{L}_0 \circ \overline{\mathcal{R}} = 0$. Therefore we have, $\overline{\mathcal{R}}\Big(\mathcal{L}_1\Big(\mathcal{K}_0(\rho_s)\Big)\Big) = \mathcal{K}_0\Big(\mathcal{L}_{s,1}(\rho_s)\Big)$. Left multiplication by $\boldsymbol{S}_0^\dagger$ and right multiplication by $\boldsymbol{S}_0$ yields

$$\mathcal{L}_{s,1}(\rho_s) = \boldsymbol{S}_0^\dagger \overline{\mathcal{R}}\left(\mathcal{L}_1\Big(\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\Big)\right)\boldsymbol{S}_0$$

since $\boldsymbol{S}_0^\dagger\boldsymbol{S}_0 = \mathbf{1}_s$ is the identity operator on $\mathcal{H}_s$.

For $\mathcal{L}_1(\rho) = -i[\boldsymbol{H}_1, \rho]$ we have, exploiting the fact that $\boldsymbol{M}_\mu\boldsymbol{S}_0 = \lambda_\mu\boldsymbol{S}_0$ and $\boldsymbol{S}_0^\dagger\boldsymbol{S}_0 = \mathbf{1}_s$:

$$\begin{aligned}
\boldsymbol{S}_0^\dagger\overline{\mathcal{R}}\left(\mathcal{L}_1\Big(\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\Big)\right)\boldsymbol{S}_0 &= -i\sum_\mu \boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu\Big(\boldsymbol{H}_1\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger - \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\boldsymbol{H}_1\Big)\boldsymbol{M}_\mu^\dagger\boldsymbol{S}_0 \\
&= -i\sum_\mu \Big(\lambda_\mu^*\boldsymbol{S}_0^\dagger\Big)\,\boldsymbol{M}_\mu\boldsymbol{H}_1\boldsymbol{S}_0\rho_s + i\sum_\mu \rho_s\boldsymbol{S}_0^\dagger\boldsymbol{H}_1\boldsymbol{M}_\mu^\dagger\Big(\lambda_\mu\boldsymbol{S}_0\Big) \\
&= -i\sum_\mu \boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu^\dagger\boldsymbol{M}_\mu\boldsymbol{H}_1\boldsymbol{S}_0\rho_s + i\sum_\mu \rho_s\boldsymbol{S}_0^\dagger\boldsymbol{H}_1\boldsymbol{M}_\mu^\dagger\boldsymbol{M}_\mu\boldsymbol{S}_0 \\
&= -i\left[\boldsymbol{S}_0^\dagger\boldsymbol{H}_1\boldsymbol{S}_0\ ,\ \rho_s\right] \qquad (5.16)
\end{aligned}$$

since $\sum_\mu \boldsymbol{M}_\mu^\dagger\boldsymbol{M}_\mu = \mathbf{1}$. We get the Zeno Hamiltonian $\boldsymbol{H}_{s,1} = \boldsymbol{S}_0^\dagger\boldsymbol{H}_1\boldsymbol{S}_0$.

For $\mathcal{L}_1(\rho) = \boldsymbol{L}_1\rho\boldsymbol{L}_1^\dagger - \frac{1}{2}\Big(\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\rho + \rho\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\Big)$, similar computations yield

$$\begin{aligned}
\boldsymbol{S}_0^\dagger\overline{\mathcal{R}}\left(\mathcal{L}_1\Big(\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\Big)\right)\boldsymbol{S}_0 =& \sum_\mu \boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu\boldsymbol{L}_1\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\boldsymbol{M}_\mu^\dagger\boldsymbol{S}_0 \\
&- \tfrac{1}{2}\sum_\mu \boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu\Big(\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger - \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\Big)\boldsymbol{M}_\mu^\dagger\boldsymbol{S}_0 \\
=& \left(\sum_\mu \boldsymbol{A}_\mu\rho_s\boldsymbol{A}_\mu^\dagger\right) - \tfrac{1}{2}\boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\boldsymbol{S}_0\rho_s - \rho_s\boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\boldsymbol{S}_0
\end{aligned}$$

with $\boldsymbol{A}_\mu = \boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu\boldsymbol{L}_1\boldsymbol{S}_0$. It remains to prove that $\sum_\mu \boldsymbol{A}_\mu^\dagger\boldsymbol{A}_\mu = \boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\boldsymbol{S}_0$ for showing that we indeed have a Lindblad formulation. This results from the following computations:

$$\begin{aligned}
\sum_\mu \boldsymbol{A}_\mu^\dagger\boldsymbol{A}_\mu &= \sum_\mu \boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\boldsymbol{M}_\mu^\dagger\boldsymbol{S}_0\boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu\boldsymbol{L}_1\boldsymbol{S}_0 \\
&= \boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\overline{\mathcal{R}}^*(\boldsymbol{S}_0\boldsymbol{S}_0^\dagger)\boldsymbol{L}_1\boldsymbol{S}_0 \\
&= \boldsymbol{S}_0^\dagger\boldsymbol{L}_1^\dagger\boldsymbol{L}_1\boldsymbol{S}_0,
\end{aligned}$$

where we use that $\boldsymbol{S}_0\boldsymbol{S}_0^\dagger = \boldsymbol{P}_0$ and $\overline{\mathcal{R}}^*(\boldsymbol{P}_0) = \mathbf{1}$. ∎

5.2.3 Second order expansion for Hamiltonian perturbations

We assume here that $\mathcal{L}_0$ is defined by a single operator $\boldsymbol{L}_0$:

$$\mathcal{L}_0(\rho) = \boldsymbol{L}_0\rho\boldsymbol{L}_0^\dagger - \tfrac{1}{2}\Big(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\rho + \rho\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\Big),$$

and that the perturbation $\mathcal{L}_1$ is Hamiltonian, $\mathcal{L}_1(\rho) = -i[\boldsymbol{H}_1, \rho]$, where $\boldsymbol{H}_1$ is a Hermitian operator. Before turning to the computation of the second order reduced dynamics, we show how to express the first order parametrisation of the slow manifold in a Kraus map form under the considered assumptions.

We begin by a technical lemma, need in the different proofs of this section. It states that attractivity and invariance of the steady states, belonging to $\mathcal{D}_0$ the density operators with support on $\mathcal{H}_0$, implies not only $\mathcal{L}_0(\mathcal{K}_0(\rho_s)) = 0$ but even $\boldsymbol{L}_0\boldsymbol{S}_0 = 0$ and $\boldsymbol{S}_0^\dagger\boldsymbol{L}_0^\dagger = 0$.

Lemma 7. Denote $\{|\nu_0\rangle\}$ a Hilbert basis of the decoherence-free space $\mathcal{H}_0$. The vectors $|\nu_0\rangle$ are the eigenvectors of the operator $\boldsymbol{L}_0$ associated with the eigenvalue 0.

Proof. We construct a Hilbert basis of $\mathcal{H}$ containing the Hilbert basis of $\mathcal{H}_0$:

$$\{|0_h\rangle, |1_h\rangle, \ldots, |m'_h\rangle, |0_0\rangle, |1_0\rangle, \ldots, |\nu_0\rangle, \ldots |m_0\rangle\} \triangleq \{|n\rangle\}.$$

with $m'_h, m_0 \in \mathbb{N} < \infty$. For any density operator $|\nu_0\rangle\langle\nu_0| \in \mathcal{D}_0$, the decoherence-free condition impose that $\mathcal{L}_0(|\nu_0\rangle\langle\nu_0|) = 0$. Therefore for all n, ν we have ,

$$\langle n| \mathcal{L}_0(|\nu_0\rangle\langle\nu_0|) |n\rangle = 0$$

Using that $\forall n \neq \nu_0$, $\langle n| \nu_0\rangle = 0$ we get that

$$\langle n| \boldsymbol{L}_0 |\nu_0\rangle\langle\nu_0| \boldsymbol{L}_0^\dagger |n\rangle = 0$$

Therefore, for all ν, we have $\boldsymbol{L}_0 |\nu_0\rangle = \lambda |\nu_0\rangle$ for some $\lambda \in \mathbb{C}$. Note that λ is independent of ν (this results from the same computation on $|\nu_0\rangle + |\nu'_0\rangle \in \mathcal{H}_0$). We now show that necessarily $\lambda = 0$. Straightforward computation yields that for all ν,

$$\mathcal{L}_0(|\nu_0\rangle\langle\nu_0|) |\nu_0\rangle = 0 = \frac{|\lambda|^2}{2} |\nu_0\rangle - \frac{\lambda}{2} \boldsymbol{L}_0^\dagger |\nu_0\rangle .$$

If $\lambda \neq 0$ then it results $\boldsymbol{L}_0^\dagger |\nu_0\rangle = \lambda^* |\nu_0\rangle$. This is impossible, as it implies that for any density matrix $\rho \in \mathcal{H}$ and for all ν we have, $\langle \nu_0| \mathcal{L}_0(\rho) |\nu_0\rangle = 0$ which contradict the attractivity property of the decoherence-free space. ■

The following lemma gives a simple expression for $\mathcal{K}_1(\rho_s)$ solution of (5.12) as well as the Kraus expression for $\mathcal{K}_0 + \epsilon\mathcal{K}_1$ (up to higher order terms).

Lemma 8. **(First order parametrisation)**
Assume that $\mathcal{L}_0(\rho) = \boldsymbol{L}_0\rho\boldsymbol{L}_0^\dagger - \frac{1}{2}\left(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\rho + \rho\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\right)$ and $\mathcal{L}_1(\rho) = -i[\boldsymbol{H}_1, \rho]$. Then $\mathcal{L}_{s,1}(\rho_s) = -i\left[\boldsymbol{S}_0^\dagger\boldsymbol{H}_1\boldsymbol{S}_0\ ,\ \rho_s\right]$ and $\mathcal{K}_1(\rho_s) = -i\left[\boldsymbol{C}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right]$ satisfy (5.12) where $\boldsymbol{C}_1$ is the Hermitian operator

$$\boldsymbol{C}_1 = 2(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{P}_0 + 2\boldsymbol{P}_0\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}$$

with $\boldsymbol{P}_0$ the orthogonal projector onto $\mathcal{H}$, and $(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}$ standing for the Moore-Penrose pseudo-inverse of the Hermitian operator $\boldsymbol{L}_0^\dagger\boldsymbol{L}_0$.

The associated first order ρ_s-parametrisation of the slow invariant attractive manifold,

$$\mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) = \left(\mathbf{1} - i\epsilon(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\right)\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\left(\mathbf{1} + i\epsilon(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\right) + O(\epsilon^2),$$

corresponds, up to second-order terms, to a trace-preserving completely positive map as it is expressed in a Kraus form.

Proof. With $\boldsymbol{S}_0\mathcal{L}_{s,1}(\rho_s)\boldsymbol{S}_0^\dagger = -i\left[\boldsymbol{P}_0\boldsymbol{H}_1P_0, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right]$, equation (5.12) reads

$$\begin{aligned}\mathcal{L}_0(\mathcal{K}_1(\rho_s)) &= -i\left[\boldsymbol{P}_0\boldsymbol{H}_1\boldsymbol{P}_0, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right] + i\left[\boldsymbol{H}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right] \\ &= -i\left[\boldsymbol{P}_0\boldsymbol{H}_1\boldsymbol{P}_0 - \boldsymbol{H}_1\ ,\ \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right].\end{aligned}$$

With $\mathcal{K}_1(\rho_s) = -i\Big[\boldsymbol{C}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\Big]$ we have also

$$\begin{aligned}\mathcal{L}_0(\mathcal{K}_1(\rho_s)) = &- i\boldsymbol{L}_0\left[\boldsymbol{C}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right]\boldsymbol{L}_0^\dagger \\ &+ \tfrac{i}{2}\left(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\left[\boldsymbol{C}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right] + \left[\boldsymbol{C}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right]\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\right).\end{aligned}$$

It results from Lemma 7 that $\boldsymbol{L}_0\boldsymbol{S}_0 = 0$ and $\boldsymbol{S}_0^\dagger\boldsymbol{L}_0^\dagger = 0$. We thus have

$$\boldsymbol{L}_0\left[\boldsymbol{C}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right]\boldsymbol{L}_0^\dagger = 0.$$

Since additionally, $\boldsymbol{P}_0\boldsymbol{S}_0 = \boldsymbol{S}_0$, $\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\boldsymbol{P}_0 = 0$ and $\boldsymbol{L}_0^\dagger\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1} = \mathbf{1} - \boldsymbol{P}_0$, we have

$$\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\left[\boldsymbol{C}_1, \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right] = 2(\mathbf{1} - \boldsymbol{P}_0)\boldsymbol{H}_1\boldsymbol{P}_0\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger.$$

Thus

$$\begin{aligned}\mathcal{L}_0(\mathcal{K}_1(\rho_s)) &= i(\mathbf{1} - \boldsymbol{P}_0)\boldsymbol{H}_1\boldsymbol{P}_0\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger - i\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\boldsymbol{P}_0\boldsymbol{H}_1(\mathbf{1} - \boldsymbol{P}_0) \\ &= -i\left[\boldsymbol{P}_0\boldsymbol{H}_1\boldsymbol{P}_0 - \boldsymbol{H}_1\ ,\ \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger\right].\end{aligned}$$

■

The second order term $\mathcal{L}_{s,2}(\rho_s)$ is solution of (5.13). Using, once again, $\overline{\mathcal{R}} \circ \mathcal{L}_0 \equiv 0$ and $\overline{\mathcal{R}} \circ \mathcal{K}_0 = \mathcal{K}_0$, we get

$$\mathcal{L}_{s,2}(\rho_s) = \boldsymbol{S}_0^\dagger\overline{\mathcal{R}}\Big(\mathcal{L}_1\left(\mathcal{K}_1(\rho_s)\right) - \mathcal{K}_1\left(\mathcal{L}_{s,1}(\rho_s)\right)\Big)\boldsymbol{S}_0. \tag{5.17}$$

The following lemma shows that $\mathcal{L}_{s,2}(\rho_s)$ admits a Lindbladian form.

Lemma 9. **(Second order, Hamiltonian perturbation)**

The super-operator $\mathcal{L}_{s,2}$ corresponding to the second order reduced dynamics defined by (5.17) admits the following Lindbladian formulation

$$\mathcal{L}_{s,2}(\rho_s) = \sum_\mu \boldsymbol{B}_\mu\rho_s\boldsymbol{B}_\mu^\dagger - \tfrac{1}{2}\Big(\boldsymbol{B}_\mu^\dagger\boldsymbol{B}_\mu\rho_s + \rho_s\boldsymbol{B}_\mu^\dagger\boldsymbol{B}_\mu\Big)$$

with $\boldsymbol{B}_\mu = 2\boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0$, $\boldsymbol{M}_\mu$ defined by (5.5) and $(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}$ standing for the Moore-Penrose pseudo-inverse of $\boldsymbol{L}_0^\dagger\boldsymbol{L}_0$.

Proof. We have $\overline{\mathcal{R}}\Big(\mathcal{K}_1\left(\mathcal{L}_{s,1}(\rho_s)\right)\Big) = 0$. This results from ($\mathcal{K}_0^s$ stands for $\boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger = \mathcal{K}_0(\rho_s)$)

$$\begin{aligned}\mathcal{K}_1\left(\mathcal{L}_{s,1}(\rho_s)\right) &= -i[\boldsymbol{C}_1,\ -i\,\boldsymbol{S}_0[\boldsymbol{S}_0^\dagger\boldsymbol{H}_1\boldsymbol{S}_0,\ \rho_s]\boldsymbol{S}_0^\dagger] \\ &= -[\boldsymbol{C}_1, \boldsymbol{P}_0\boldsymbol{H}_1\mathcal{K}_0^s - \mathcal{K}_0^s\boldsymbol{H}_1\boldsymbol{P}_0] \\ &= -2(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1(\boldsymbol{P}_0\boldsymbol{H}_1\mathcal{K}_0^s - \mathcal{K}_0^s\boldsymbol{H}_1\boldsymbol{P}_0) \\ &\quad + 2(\boldsymbol{P}_0\boldsymbol{H}_1\mathcal{K}_0^s - \mathcal{K}_0^s\boldsymbol{H}_1\boldsymbol{P}_0)\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\end{aligned} \tag{5.18}$$

where we have used Lemma 8 and $\boldsymbol{P}_0\mathcal{K}_0 = \mathcal{K}_0$.

Repeating computations similar to (5.16), we see that for any operator $\boldsymbol{A}$ on $\mathcal{H}$, $\overline{\mathcal{R}}(\boldsymbol{A}\boldsymbol{P}_0) = \overline{\mathcal{R}}(\boldsymbol{P}_0\boldsymbol{A}) = \boldsymbol{P}_0\boldsymbol{A}\boldsymbol{P}_0$. Since $\boldsymbol{P}_0\mathcal{K}_0^s = \mathcal{K}_0^s\boldsymbol{P}_0 = \mathcal{K}_0^s$ we moreover have $\overline{\mathcal{R}}(\boldsymbol{A}\mathcal{K}_0^s) = \boldsymbol{P}_0\boldsymbol{A}\mathcal{K}_0^s$ and $\overline{\mathcal{R}}(\mathcal{K}_0^s\boldsymbol{A}) = \mathcal{K}_0^s\boldsymbol{A}\boldsymbol{P}_0$. This gives the result of applying $\overline{\mathcal{R}}$ on all the terms in (5.18), and since $\boldsymbol{P}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1} = (\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{P}_0 = 0$, we conclude that $\overline{\mathcal{R}}(\mathcal{K}_1(\mathcal{L}_{s,1})) = 0$.

Thus $\mathcal{L}_{s,2}(\rho_s) = \boldsymbol{S}_0^\dagger \overline{\mathcal{R}}\Big(\mathcal{L}_1\left(\mathcal{K}_1(\rho_s)\right)\Big)\boldsymbol{S}_0$. Exploiting similar simplifications, we have

$$\begin{aligned}\mathcal{L}_1\left(\mathcal{K}_1(\rho_s)\right) &= -\boldsymbol{H}_1(\boldsymbol{C}_1\mathcal{K}_0^s - \mathcal{K}_0^s\boldsymbol{C}_1) + (\boldsymbol{C}_1\mathcal{K}_0^s - \mathcal{K}_0^s\boldsymbol{C}_1)\boldsymbol{H}_1\\ &= \boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{C}_1 + \boldsymbol{C}_1\mathcal{K}_0^s\boldsymbol{H}_1 - (\boldsymbol{H}_1\boldsymbol{C}_1\mathcal{K}_0^s + \mathcal{K}_0^s\boldsymbol{C}_1\boldsymbol{H}_1)\\ &= 2\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1} + 2(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1\\ &\quad - 2\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s - 2\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\end{aligned}$$

and, using $\boldsymbol{S}_0^\dagger\overline{\mathcal{R}}(\boldsymbol{A}\mathcal{K}_0^s) = \boldsymbol{S}_0^\dagger\boldsymbol{P}_0\boldsymbol{A}\mathcal{K}_0^s = \boldsymbol{S}_0^\dagger\boldsymbol{A}\mathcal{K}_0^s$ and the definition $\mathcal{K}_0^s = \boldsymbol{S}_0\rho_s\boldsymbol{S}_0^\dagger$, we get

$$\begin{aligned}\mathcal{L}_{s,2}(\rho_s) &= 2\boldsymbol{S}_0^\dagger\overline{\mathcal{R}}\Big(\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1} + (\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1\Big)\boldsymbol{S}_0\\ &\quad - 2\boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0\rho_s - 2\rho_s\boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0.\end{aligned}$$

Since for all $\boldsymbol{A}$, $\overline{\mathcal{R}}(\mathcal{L}_0(\boldsymbol{A})) = 0$, we have the identity

$$\overline{\mathcal{R}}(\boldsymbol{L}_0\boldsymbol{A}\boldsymbol{L}_0^\dagger) = \overline{\mathcal{R}}\Big(\tfrac{1}{2}\Big(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\boldsymbol{A} + \boldsymbol{A}\boldsymbol{L}_0^\dagger\boldsymbol{L}_0\Big)\Big).$$

With $\boldsymbol{A} = (\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}$ we get

$$\begin{aligned}&2\overline{\mathcal{R}}\Big(\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{L}_0^\dagger\Big) =\\ &\overline{\mathcal{R}}\Big((\mathbf{1}-\boldsymbol{P}_0)\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1} + (\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\mathbf{1}-\boldsymbol{P}_0)\Big)\\ &= \overline{\mathcal{R}}\Big(\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1} + (\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1\Big)\end{aligned}$$

since $\overline{\mathcal{R}}\Big(\boldsymbol{P}_0\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\Big) = \boldsymbol{P}_0\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{P}_0$ and $(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{P}_0 = 0$. Thus

$$\begin{aligned}\mathcal{L}_{s,2}(\rho_s) &= 4\boldsymbol{S}_0^\dagger\overline{\mathcal{R}}\Big(\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{L}_0^\dagger\Big)\boldsymbol{S}_0\\ &\quad - 2\boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0\rho_s - 2\rho_s\boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0.\end{aligned}$$

Using the decomposition (5.5) of $\overline{\mathcal{R}}$ we have

$$4\boldsymbol{S}_0^\dagger\overline{\mathcal{R}}\Big(\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\mathcal{K}_0^s\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{L}_0^\dagger\Big)\boldsymbol{S}_0 = \sum_\mu \boldsymbol{B}_\mu\rho_s\boldsymbol{B}_\mu^\dagger\,.$$

We conclude by the following computations:

$$\begin{aligned}\tfrac{1}{2}\sum_\mu \boldsymbol{B}_\mu^\dagger\boldsymbol{B}_\mu &= 2\sum_\mu \boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{L}_0^\dagger\boldsymbol{M}_\mu^\dagger\boldsymbol{S}_0\boldsymbol{S}_0^\dagger\boldsymbol{M}_\mu\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0\\ &= 2\boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{L}_0^\dagger\overline{\mathcal{R}}^*(\boldsymbol{P}_0)\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0\\ &= 2\boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{L}_0^\dagger\boldsymbol{L}_0(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0\\ &= 2\boldsymbol{S}_0^\dagger\boldsymbol{H}_1(\boldsymbol{L}_0^\dagger\boldsymbol{L}_0)^{-1}\boldsymbol{H}_1\boldsymbol{S}_0.\end{aligned}$$

■

Chapter 6

Conclusions and perspectives

In this thesis we have studied adiabatic elimination for open quantum systems using a geometric approach.

In chapter 2, after presenting general notions on open quantum systems and singular perturbation theory, we introduced the model of considered quantum system throughout this manuscript. The dynamics is assumed to be on different time-scales, a fast and a slow one, characterised by a small parameter. One of the key issues of adiabatic elimination is to ensure a physical meaning for the reduced model. Therefore, we imposed two constraints, on the reduced model, to tackle this objective. First, we want to ensure that the dynamics of the reduced system is in a Lindblad master equation form as it is the only form of dynamics carrying a physical significance. Secondly, the parametrisation of the slow manifold, that can be seen as a mapping from the reduced density operator to the density operator of the entire system, is a Kraus map form i.e. completely positive trace-preserving. To answer this problem, we presented in the last section of chapter 2 our method. We treat the slow dynamics as a perturbation and, use geometric singular perturbation theory and center manifold theory. we have derived an asymptotic expansion for the perturbation dynamics and for the slow manifold, in terms of powers of the time-scale separation leading to recurrence relations linking the reduced and the complete model. This asymptotic expansion naturally leads to the characterisation of the order of precision of the approximation.

The chapter 3 is devoted to the resolution of the recurrence relations allowing to compute the reduce dynamics and the parametrisation of the slow manifold. It contains our main results stated in different Theorems. In this chapter, we considered the case of composite open quantum systems. We consider bipartite systems, where a subsystem is assumed to be on a fast time-scale and to converge towards a unique steady state. It is weakly coupled to another subsystem, on a slow time-scale. We derived general formulas for the computation of the first order reduced model. Our method ensure a reduced dynamics in a Lindblad form as well as a parametrisation of the slow manifold in a Kraus map form. Then, focusing on cases with Hamiltonian interaction between the fast and the slow subsystem, as it is common, we derived explicit formula for the computation of the second order approximation ensuring the desired constraints. This formula allowed us to make general structural results on the second order reduced dynamics. Then, we considered another type of standard coupling between the two subsystems, namely a cascade interaction by means of a circulator. We got the second order reduced dynamics in a Lindblad form as well, provided a small assumption.

We took great care in the practical application of our method of adiabatic elimination

and chapter 4 proposes different techniques to compute the reduced model up to second order, provided that the fast subsystem is an harmonic oscillator. These techniques are illustrated with several examples chosen for their commonness and/or relevance.

In the last chapter 5, we presented different results which have to be completed in some sense. First, we derived an explicit expression of the third order reduced dynamics for composite systems with interaction between fast and slow subsystems described by a one tensor-product term Hamiltonian. Secondly, we considered a completely different type of systems, namely systems with decoherence-free space. For such systems, we were able to compute the first order reduced model. We also got the second order reduced dynamics provided that the fast dynamics is described by only one decoherence operator. This illustrates that our method is suitable in different cases : composite systems with Hamiltonian and cascaded interactions and systems with decoherence-free space, representing a large variety of systems usually considered.

Now, we wish to draw some potential development paths on the presented work. These paths can be divided into two lines. First, extend the second order results for more general type of dynamics thus allowing to treat more quantum systems. Second, derive explicit formulas allowing to compute higher order approximations. This would allow to get accurate results for less significant time-scale separations and/or to bound more precisely the effects of the coupling, for systems where higher order perturbations become important to characterise. We conjecture that this geometric adiabatic elimination technique will lead to formulas conveying a physical interpretation of the reduced model (by ensuring the Lindblad form of the reduced dynamics, and by providing a completely positive trace-preserving mapping from reduced model to actual system states) at higher orders as well. The key element to do so, would be to properly choose a gauge degree of freedom appearing in the computations.

Conclusions et perspectives

Dans cette thèse, nous avons étudié le problème de l'élimination adiabatique dans les systèmes quantiques ouverts en proposant une nouvelle approche géométrique.

Dans le chapitre 2, après avoir présenté des notions générales sur les systèmes quantiques ouverts et la théorie des perturbations singulières, nous introduisons le modèle des systèmes quantiques considérés tout au long de ce manuscrit. Nous considérons les systèmes quantiques dont la dynamique possède deux échelles de temps, une lente et une rapide, caractérisée par un petit paramètre. Une des difficultés principales de l'élimination adiabatique pour les systèmes quantiques est d'assurer une signification physique du modèle réduit. Ainsi, nous imposons deux contraintes, sur le modèle réduit, pour satisfaire cet objectif. Premièrement, nous devons assurer que la dynamique du système réduit soit sous forme de Lindblad, ce qui est la seule forme ayant un sens physique. Deuxièmement, la paramétrisation de la variété lente, qui peut être vue comme une application de l'opérateur densité réduit vers l'opérateur densité du système complet, doit être une application de Kraus c'est-à-dire préservant la trace et complètement positive. Pour résoudre ce problème, nous présentons dans la dernière section du chapitre 2 notre méthode. Nous traitons la dynamique lente comme une perturbation et utilisons la théorie géométrique des perturbations singulières avec la théorie des variétés centres. Après un développement asymptotique de la dynamique perturbatrice et de la variété lente, en fonction des puissances du paramètre représentant la séparation des échelles de temps, nous obtenons des relations de récurrence liant le modèle réduit et complet. Ce développement asymptotique amène naturellement à la caractérisation de la précision de l'approximation.

Le chapitre 3 est consacré à la résolution des relations de récurrence permettant de calculer la dynamique réduite et la paramétrisation de la variété lente. Il contient nos principaux résultats présentés dans différents théorèmes. Nous considérons des systèmes composites dans lesquels un sous-système est supposé sur une échelle de temps rapide et convergeant vers un état d'équilibre unique. Ce sous-système est faiblement couplé à un autre, qui est sur une échelle de temps lente. Nous obtenons des formules générales afin de calculer le modèle réduit au premier ordre. Notre méthode assure une dynamique réduite sous forme de Lindblad ainsi qu'une paramétrisation de la variété lente sous forme de Kraus. Ensuite, en étudiant le cas standard d'interaction Hamiltonienne entre les deux sous-systèmes, nous obtenons une formule explicite pour calculer l'approximation au second tout en respectant les contraintes imposées. Cette formule nous permet de dresser des résultats généraux sur la structure de la dynamique réduite au second ordre. Ensuite, nous considérons un autre type de couplage standard entre les deux sous-systèmes, à savoir un couplage en cascade effectué au moyen d'un circulateur. Nous obtenons également la dynamique réduite au second ordre, dans une forme de Lindblad, au moyen d'une légère hypothèse.

Un soin particulier est accordé à l'application pratique de notre méthode d'élimination adiabatique et le chapitre 4 propose différentes techniques afin de calculer le modèle réduit

jusqu'au second ordre dans le cas où le système rapide est un oscillateur harmonique. Ces techniques sont illustrées par plusieurs exemples choisis pour leurs apparitions fréquentes et/ou leur pertinence.

Dans le dernier chapitre 5, nous avons présenté différents résultats qui restent à compléter en un certain sens. Premièrement, nous avons obtenu une expression explicite de la dynamique réduite au troisième ordre pour les systèmes composites dans lesquels l'interaction entre le sous-système lent et rapide est décrite par un Hamiltonien composé d'un unique terme sous forme de produit tensoriel. Deuxièmement, nous avons considéré un type de système complètement différent, à savoir les systèmes possédant un sous-espace ne subissant pas de décohérence (decoherence-free space). Pour de tels systèmes, nous avons pu calculer le modèle réduit au premier ordre. Nous avons également des formules explicites pour calculer la dynamique réduite au second ordre lorsque la dynamique rapide est décrite par un unique opérateur de décohérence. Cela illustre notamment que notre méthode est applicable dans des cas très différents: les systèmes composites avec des couplages Hamiltoniens ou en cascade et les systèmes avec espace sans décohérence, représentant une large gamme de systèmes usuellement considérés.

Enfin, nous souhaitons présenter quelques pistes potentielles de développement du travail présenté. Nous voyons deux principales pistes. Premièrement, étendre nos résultats au second ordre pour des dynamiques plus générales, permettant de traiter des systèmes quantiques généraux. Deuxièmement, obtenir des formules explicites pour calculer les approximations aux ordres supérieurs. Cela permettrait d'obtenir des résultats précis pour des séparations d'échelles de temps moins significatives et/ou de caractériser plus précisément les effets du couplage, pour les systèmes dans lesquels les perturbations d'ordre deviennent importantes à analyser. Nous conjecturons que cette méthode géométrique d'élimination adiabatique permettra d'obtenir des formules préservant l'interprétation physique du modèle réduit (en assurant une dynamique réduite sous forme de Lindblad et une application du modèle réduit au système entier complètement positive et préservant la trace) pour les ordres plus élevés également. L'élément clef pour y arriver, serait de choisir judicieusement une liberté de jauge apparaissant dans les calculs.

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Appendices

Appendix A

Inverse of Lindblad super-operators via Kraus maps

Consider the Lindblad master equation $\dot{\rho} = \mathcal{L}(\rho)$ where ρ is a density operator on a finite dimensional Hilbert space $\mathcal{H}$. Assume that for any operator $\boldsymbol{X}$ on $\mathcal{H}$, $e^{t\mathcal{L}}(\boldsymbol{X})$ converges exponentially towards a fixed point depending on $\boldsymbol{X}$. This means that there exists a complete-positive and trace-preserving map $\overline{\mathcal{R}}$ such that $\lim_{t\mapsto+\infty} e^{t\mathcal{L}}(\boldsymbol{X}) = \overline{\mathcal{R}}(\boldsymbol{X})$. Thus we have $\mathcal{L}(\overline{\mathcal{R}}(\boldsymbol{X})) \equiv 0 \equiv \overline{\mathcal{R}}(\mathcal{L}(\boldsymbol{X})) \equiv 0$ since $e^{t\mathcal{L}}(\overline{\mathcal{R}}(\boldsymbol{X})) \equiv \overline{\mathcal{R}}(\boldsymbol{X}) \equiv \overline{\mathcal{R}}(e^{t\mathcal{L}}(\boldsymbol{X}))$.

Lemma 10. There exists $\bar{\tau} > 0$ such that the super-operator $\overline{\mathcal{K}}$ sending operator $\boldsymbol{X}$ to

$$\overline{\mathcal{K}}(\boldsymbol{X}) = \frac{1}{\bar{\tau}} \int_0^{+\infty} e^{t\mathcal{L}}\Big(\boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})\Big)\, dt + \overline{\mathcal{R}}(\boldsymbol{X}) \tag{A.1}$$

is a linear, trace-preserving and completely positive mapping with

$$-\mathcal{L}\Big(\bar{\tau}\overline{\mathcal{K}}(\boldsymbol{X})\Big) = \boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X}).$$

Proof. Due to exponential convergence of $e^{t\mathcal{L}}(\boldsymbol{X})$ towards $\overline{\mathcal{R}}(\boldsymbol{X})$, the indefinite integral

$$\overline{\mathcal{M}}(\boldsymbol{X}) \triangleq \int_0^{+\infty} e^{t\mathcal{L}}\Big(\boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})\Big)\, dt$$

is absolutely convergent. Since $\frac{d}{dt}e^{t\mathcal{L}}\Big(\boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})\Big) = \mathcal{L}\left(e^{t\mathcal{L}}\Big(\boldsymbol{X} - \overline{\mathcal{R}}(\boldsymbol{X})\Big)\right)$, we have $\mathcal{L}(\overline{\mathcal{K}}(\boldsymbol{X})) = -\frac{\boldsymbol{X}-\overline{\mathcal{R}}(\boldsymbol{X})}{\bar{\tau}}$. Since, for each $t \geq 0$ the propagator $e^{t\mathcal{L}}$ is trace preserving and $\mathrm{Tr}\,(\boldsymbol{X}) = \mathrm{Tr}\,\Big(\overline{\mathcal{R}}\boldsymbol{X}\Big)$, simple computations yield $\mathrm{Tr}\,\Big(\overline{\mathcal{M}}(\boldsymbol{X})\Big) = 0$ and thus $\mathrm{Tr}\,\Big(\overline{\mathcal{K}}(\boldsymbol{X})\Big) = \mathrm{Tr}\,(\boldsymbol{X})$.

To prove complete-positivity, consider the extension of $\mathcal{L}$, $\overline{\mathcal{K}}$, $\overline{\mathcal{R}}$ on the tensor product $\mathcal{H} \otimes \widetilde{\mathcal{H}}$ where $\widetilde{\mathcal{H}}$ is any Hilbert space of finite dimension. Let us prove that for $\bar{\tau}$ large enough, such extension of $\overline{\mathcal{K}}$ is non-negative, i. e., that for any $|\Phi\rangle, |\Psi\rangle \in \mathcal{H} \otimes \widetilde{\mathcal{H}}$, we have $\left\langle \Psi \middle| \overline{\mathcal{K}}\left(|\Phi\rangle\langle\Phi|\right) \middle| \Psi \right\rangle \geq 0$. Consider an Hilbert basis $(|n\rangle)_{1\leq n\leq d}$ of $\mathcal{H}$ those dimension is denoted by d. Take

$$|\Phi\rangle = \sum_{n=1}^{d} |n\rangle \otimes |\phi_n\rangle\,, \quad |\Psi\rangle = \sum_{\nu=1}^{d} |\nu\rangle \otimes |\psi_\nu\rangle$$

where, for each n and ν in $\{1, \ldots, d\}$, $|\phi_n\rangle, |\psi_\nu\rangle \in \widetilde{\mathcal{H}}$. Then standard computations give

$$\left\langle \Psi \middle| \overline{\mathcal{M}}\left(|\Phi\rangle\langle\Phi|\right) \middle| \Psi \right\rangle = \sum_{n',\nu',n,\nu=1}^{d} z^*_{n',\nu'}\ M_{n',\nu',n,\nu}\ z_{n,\nu}$$

with $z_{n,\nu} = \langle \phi_n | \, \psi_\nu \rangle$ and

$$M_{n',\nu',n,\nu} = \int_0^{+\infty} \left\langle \nu' \middle| e^{t\mathcal{L}}(|n'\rangle\langle n|) - \overline{\mathcal{R}}(|n'\rangle\langle n|) \middle| \nu \right\rangle \, dt.$$

Similarly

$$\left\langle \Psi \middle| \overline{\mathcal{R}} \left(|\Phi\rangle\langle\Phi| \right) \middle| \Psi \right\rangle = \sum_{n',\nu',n,\nu=1}^{d} z^*_{n',\nu'} \; R_{n',\nu',n,\nu} \; z_{n,\nu}$$

with $R_{n',\nu',n,\nu} = \left\langle \nu' \middle| \overline{\mathcal{R}}(|n'\rangle\langle n|) \middle| \nu \right\rangle$. This means that, with the vector $z = (z_{n,\nu})_{n,\nu\in\{1,\ldots,d\}}$ of dimension d^2 and $M = \left(M_{n',\nu',\, n,\nu} \right)$ with $R = \left(R_{n',\nu',\, n,\nu} \right)$ considered as $d^2 \times d^2$ Hermitian matrices, we have the following quadratic forms

$$\left\langle \Psi \middle| \overline{\mathcal{M}} \left(|\Phi\rangle\langle\Phi| \right) \middle| \Psi \right\rangle = z^\dagger M z, \quad \left\langle \Psi \middle| \overline{\mathcal{R}} \left(|\Phi\rangle\langle\Phi| \right) \middle| \Psi \right\rangle = z^\dagger R z$$

where z depends on $|\Phi\rangle$ and $|\Psi\rangle$, where M and R depend only on $\overline{\mathcal{M}}$ and $\overline{\mathcal{R}}$. We have thus to prove that exists $\bar{\tau} > 0$ such that $M + \bar{\tau} R \geq 0$. Since $\overline{\mathcal{R}}$ is a complete-positive map, the $d^2 \times d^2$ Hermitian matrix R is non-negative.

Take z, such that $Rz = 0$. Take $T > 0$. We have

$$\left\langle \Psi \middle| \int_0^T e^{t\mathcal{L}} \Big(\, |\Phi\rangle\langle\Phi| - \overline{\mathcal{R}}(|\Phi\rangle\langle\Phi|) \Big) \, dt \middle| \Psi \right\rangle = \left\langle \Psi \middle| \int_0^T e^{t\mathcal{L}} \Big(\, |\Phi\rangle\langle\Phi| \, \Big) \, dt \middle| \Psi \right\rangle$$

since $e^{t\mathcal{L}} \Big(\overline{\mathcal{R}}(|\Phi\rangle\langle\Phi|) \Big) = \overline{\mathcal{R}}(|\Phi\rangle\langle\Phi|)$ and $\left\langle \Psi \middle| \overline{\mathcal{R}}(|\Phi\rangle\langle\Phi|) \middle| \Psi \right\rangle = z^\dagger R z = 0$. We have

$$\left\langle \Psi \middle| \int_0^T e^{t\mathcal{L}} \Big(\, |\Phi\rangle\langle\Phi| \, \Big) \, dt \middle| \Psi \right\rangle = \int_0^T \sum_{n,\nu,n',\nu'} z^*_{n',\nu'} \left\langle \nu' \middle| e^{t\mathcal{L}}(|n'\rangle\langle n|) \middle| \nu \right\rangle z_{n,\nu} \, dt.$$

Since for each $t \geq 0$, $e^{t\mathcal{L}}$ is completely positive, then there exists a Kraus decomposition

$$e^{t\mathcal{L}}(\boldsymbol{X}) = \sum_\mu \boldsymbol{W}_{\mu,t} \boldsymbol{X} \boldsymbol{W}^\dagger_{\mu,t}$$

with operators $\boldsymbol{W}_{\mu,t}$ on $\mathcal{H}$ such that $\sum_\mu \boldsymbol{W}^\dagger_{\mu,t} \boldsymbol{W}_{\mu,t} = \boldsymbol{I}$. We have

$$\left\langle \nu' \middle| e^{t\mathcal{L}}(|n'\rangle\langle n|) \middle| \nu \right\rangle = \sum_\mu \left\langle \nu' \middle| \boldsymbol{W}_{\mu,t} \middle| n' \right\rangle \left\langle n \middle| \boldsymbol{W}^\dagger_{\mu,t} \middle| \nu \right\rangle.$$

Consequently

$$\sum_{n,\nu,n',\nu'} z^*_{n',\nu'} \left\langle \nu' \middle| e^{t\mathcal{L}}(|n'\rangle\langle n|) \middle| \nu \right\rangle z_{n,\nu} = \sum_\mu \left| \sum_{n,\nu=1}^{d} \left\langle n \middle| W^\dagger_{\mu,t} \middle| \nu \right\rangle z_{n,\nu} \right|^2.$$

Since $z^\dagger M z$ is the limit when T tends to $+\infty$ of

$$\left\langle \Psi \middle| \int_0^T e^{t\mathcal{L}} \Big(\, |\Phi\rangle\langle\Phi| \, \Big) \, dt \middle| \Psi \right\rangle,$$

we have for any $T > 0$

$$z^\dagger M z \geq \sum_\mu \int_0^T \left| \sum_{n,\nu=1}^{d} \left\langle n \middle| W^\dagger_{\mu,t} \middle| \nu \right\rangle z_{n,\nu} \right|^2 \, dt \geq 0.$$

Assume now that $z^\dagger M z = 0$. The above inequality implies that for any $t > 0$

$$\sum_{n,\nu=1}^{d} \left\langle n \middle| W_{\mu,t}^\dagger \middle| \nu \right\rangle z_{n,\nu} = 0.$$

Recall that, by assumption, $\sum_{n,\nu} \left\langle \nu' \middle| \overline{\mathcal{R}}(|n'\rangle\langle n|) \middle| \nu \right\rangle \; z_{n,\nu} = 0$ for any $n', \nu' \in \{1, \ldots, d\}$. Consequently

$$\begin{aligned}
&\sum_{n,\nu} \left(\int_0^T \left\langle \nu' \middle| e^{t\mathcal{L}}(|n'\rangle\langle n|) - \overline{\mathcal{R}}(|n'\rangle\langle n|) \middle| \nu \right\rangle \; dt \right) z_{n,\nu} \\
&\qquad = \sum_{n,\nu} \left(\int_0^T \left\langle \nu' \middle| e^{t\mathcal{L}}(|n'\rangle\langle n|) \middle| \nu \right\rangle \; dt \right) z_{n,\nu} \\
&= \int_0^T \sum_{\mu} \left\langle \nu' \middle| \boldsymbol{W}_{\mu,t} \middle| n' \right\rangle \left(\sum_{n,\nu} \left\langle n \middle| W_{\mu,t}^\dagger \middle| \nu \right\rangle z_{n,\nu} \right) = 0.
\end{aligned}$$

Thus for any z such that $Rz = 0$ and $z^\dagger M z = 0$, we have necessarily $Mz = 0$ by taking the limit for T tending to $+\infty$.

To summarize we have shown that

1. $R \geq 0$;
2. if $z^\dagger R z = 0$ then $z^\dagger M z \geq 0$;
3. if $z^\dagger R z = z^\dagger M z = 0$ then $Mz = 0$.

According to Lemma 11 proved below, exists $\bar{\tau} > 0$ such that $M + \bar{\tau} R \geq 0$.

■

Lemma 11. Consider two Hermitian matrices of same dimension R and M such that R is non negative, such that $z^\dagger R z = 0$ implies that $z^\dagger M z \geq 0$, and such $z^\dagger R z = z^\dagger M z = 0$ implies $Mz = 0$. Then for $\tau \geq 0$ large enough, $M + \tau R \geq 0$.

Proof. Up to a unitary transformation, we have the block decomposition associated with $\ker R$ and $\ker R^\perp$:

$$M = \begin{pmatrix} A & C^\dagger \\ C & B \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix},$$

with A, B and D Hermitian matrices with $D > 0$. For any z such that $z^\dagger R z = 0$ we have $z^\dagger M z \geq 0$, this means that $A \geq 0$. Up to some unitary transformation on A only, we can always assume the following sub-block decomposition for A, C and x

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \bar{A} \end{pmatrix}, \quad C = \begin{pmatrix} \tilde{C} & \bar{C} \end{pmatrix}, \quad x = \begin{pmatrix} \tilde{x} \\ \bar{x} \end{pmatrix},$$

with $\bar{A} > 0$. According to these block decompositions, $z^\dagger R z = z^\dagger M z = 0$ means that $y = 0$ and $\bar{x} = 0$ with $\tilde{x}$ arbitrary. But $Mz = 0$ means that

$$\begin{pmatrix} \tilde{C} & \bar{C} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} = \tilde{C}\tilde{x} = 0$$

for all $\tilde{x}$. Thus $\tilde{C} = 0$. To summarize, up to a unitary transformation, we have the following decomposition for M and R

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{A} & \bar{C}^\dagger \\ 0 & \bar{C} & B \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D \end{pmatrix}$$

with $\bar{A} > 0$ and $D > 0$. For τ large enough $\tau\bar{A} \geq \bar{C}^\dagger D^{-1}\bar{C}$ and thus $M + \tau R \geq 0$ (see, e.g, [Bha07, Theorem 1.3.3, page 14]). ■

Several times throughout the manuscript, we claim a factorization property with $\overline{\rho}_A$ for the result of $\mathcal{L}_A^{-1}$. This property is a direct consequence of the following two short lemmas, first introduced in [ACSR17a]. Then, we use these lemmas to derive Corollary 2 in the particular case of composite systems with a cascaded structure, as presented in Section 3.3.

Lemma 12. Let $\mathcal{L}_A(\overline{\rho}_A) = 0$, with $\overline{\rho}_A$ a density operator and $\mathcal{L}_A$ is defined by an hermitian operator $\boldsymbol{H}_A$ and some operators $\boldsymbol{L}_{A,\mu}$, all acting on $\mathcal{H}_A$ by :

$$\mathcal{L}_A(\rho) = -i\left[\boldsymbol{H}_A\,,\,\rho\right] + \sum_\mu \boldsymbol{L}_{A,\mu}\rho\boldsymbol{L}_{A,\mu}^\dagger - \frac{1}{2}\left(\boldsymbol{L}_{A,\mu}^\dagger\boldsymbol{L}_{A,\mu}\rho + \rho\boldsymbol{L}_{A,\mu}^\dagger\boldsymbol{L}_{A,\mu}\right). \tag{A.2}$$

Then for any $|\nu\rangle \in \ker(\overline{\rho}_A)$ we have $\sqrt{\overline{\rho}_A}\boldsymbol{L}_{A,\mu}^\dagger|\nu\rangle = 0$, for all μ.

Proof. For $|\nu\rangle \in \ker(\overline{\rho}_A)$ we have $\langle\nu|\,\mathcal{L}_A(\overline{\rho}_A)\,|\nu\rangle = \sum_k \langle\nu|\,\boldsymbol{L}_{A,\mu}\overline{\rho}_A\boldsymbol{L}_{A,\mu}^\dagger\,|\nu\rangle$. Since $\mathcal{L}_A(\overline{\rho}_A) = 0$ each term of this positive sum must vanish. ■

Lemma 13. Denote by $\rho = \overline{\rho}_A$ the unique density operator solution of $\mathcal{L}_A(\rho) = 0$. For a traceless operator $\boldsymbol{Y}$ such that $\ker(\overline{\rho}_A) \subseteq \ker(\boldsymbol{Y})$, the traceless solution to $\boldsymbol{X} = \mathcal{L}_A^{-1}(\boldsymbol{Y})$ also satisfies $\ker(\overline{\rho}_A) \subseteq \ker(\boldsymbol{X})$.

Proof. Note that the operators have such kernels if and only if they can be written $\boldsymbol{X} = \tilde{\boldsymbol{X}}\overline{\rho}_A$, $\boldsymbol{Y} = \tilde{\boldsymbol{Y}}\overline{\rho}_A$. Since $\mathcal{L}_A$ is a bijection on the space of traceless operators, the property is equivalent to show that $\boldsymbol{Y}\,|\nu\rangle = \mathcal{L}_A(\tilde{\boldsymbol{X}}\overline{\rho}_A)\,|\nu\rangle = 0$ for all $|\nu\rangle \in \ker(\overline{\rho}_A)$. By using $\overline{\rho}_A\,|\nu\rangle = 0$ and Lemma 12, we directly get

$$\mathcal{L}_A(\tilde{\boldsymbol{X}}\overline{\rho}_A)\,|\nu\rangle = \boldsymbol{X}\left(i\overline{\rho}_A\boldsymbol{H}_A - \tfrac{1}{2}\overline{\rho}_A\sum_\mu \boldsymbol{L}_{A_\mu}^\dagger\boldsymbol{L}_{A_\mu}\right)|\nu\rangle\ .$$

Subtracting $0 = \mathcal{L}_A(\overline{\rho}_A)$ inside the bracket, applying $\overline{\rho}_A\,|\nu\rangle = 0$ and Lemma 12 once again, we do get 0. ■

Corollary 2. Let the density operator $\overline{\rho}_A$ be solution of $\mathcal{L}_A(\overline{\rho}_A) = \widetilde{\mathcal{L}}_A(\overline{\rho}_A) + \mathcal{D}_{\boldsymbol{a}}(\overline{\rho}_A) = 0$ with $\widetilde{\mathcal{L}}_A$ of the form (A.2) . For the traceless operator $\boldsymbol{Y} = \overline{\rho}_A\boldsymbol{a}^\dagger - \mathrm{Tr}\left(\overline{\rho}_A\boldsymbol{a}^\dagger\right)\overline{\rho}_A$, the traceless solution to $\boldsymbol{X} = \mathcal{L}_A^{-1}(\boldsymbol{Y})$ satisfies $\ker(\overline{\rho}_A) \subseteq \ker(\boldsymbol{X})$.

Proof. From Lemma 12 we get that for any $|\nu\rangle \in \ker(\overline{\rho}_A)$ we have $\sqrt{\overline{\rho}_A}\boldsymbol{a}^\dagger\,|\nu\rangle = 0$. Therefore $\ker(\overline{\rho}_A) \subseteq \ker(\boldsymbol{Y})$, then apply Lemma 3. ■

Appendix B

Computation of $\mathcal{L}_{s,2}$: positivity of quadratic form on $\{\boldsymbol{B}_k, \boldsymbol{B}_k^\dagger\}$

We define the super-operator $\mathcal{L}_A$ by an Hermitian operator $\boldsymbol{H}_A$ and some operators $\boldsymbol{L}_{A,\mu}$, all acting on $\mathcal{H}_A$ such that :

$$\mathcal{L}_A(\rho) = -i\,[\boldsymbol{H}_A\,,\,\rho] + \sum_\mu \boldsymbol{L}_{A,\mu}\rho\boldsymbol{L}_{A,\mu}^\dagger - \frac{1}{2}\left(\boldsymbol{L}_{A,\mu}^\dagger\boldsymbol{L}_{A,\mu}\rho + \rho\boldsymbol{L}_{A,\mu}^\dagger\boldsymbol{L}_{A,\mu}\right).$$

The density operator $\overline{\rho}_A$ on $\mathcal{H}_A$ verifies $\mathcal{L}_A(\overline{\rho}_A) = 0$.

Lemma 14. : For any operators $\boldsymbol{X}$ and $\boldsymbol{Y}$ on $\mathcal{H}_A$, it holds that

$$\mathcal{L}_A(\boldsymbol{X}\overline{\rho}_A)\boldsymbol{Y}^\dagger + \boldsymbol{X}\mathcal{L}_A(\overline{\rho}_A\boldsymbol{Y}^\dagger) = \mathcal{L}_A(\boldsymbol{X}\overline{\rho}_A\boldsymbol{Y}^\dagger) - \sum_\mu [\boldsymbol{L}_{A,\mu}, \boldsymbol{X}]\overline{\rho}_A[\boldsymbol{L}_{A,\mu}, \boldsymbol{Y}]^\dagger\,.$$

Proof. To check this, expand both sides using the definition of $\mathcal{L}_A$. Using on the left hand side the facts that

$$[\boldsymbol{H}_A\,,\,\boldsymbol{X}\overline{\rho}_A]\,\boldsymbol{Y}^\dagger + \boldsymbol{X}\left[\boldsymbol{H}_A\,,\,\overline{\rho}_A\boldsymbol{Y}^\dagger\right] = \left[\boldsymbol{H}_A\,,\,\boldsymbol{X}\overline{\rho}_A\boldsymbol{Y}^\dagger\right] + \boldsymbol{X}\,[\boldsymbol{H}_A, \overline{\rho}_A]\,\boldsymbol{Y}^\dagger$$

$$\text{with } -\sum_\mu \frac{\overline{\rho}_A\boldsymbol{L}_{A,\mu}^\dagger\boldsymbol{L}_{A,\mu} + \boldsymbol{L}_{A,\mu}^\dagger\boldsymbol{L}_{A,\mu}\overline{\rho}_A}{2} = i\,[\boldsymbol{H}_A\,,\,\overline{\rho}_A] - \sum_\mu \boldsymbol{L}_{A,\mu}\overline{\rho}_A\boldsymbol{L}_{A,\mu}^\dagger$$

one gets the proof. ■

Lemma 15. The matrix $\boldsymbol{X}$ defined in Lemma 4 by

$$X_{k,j} = \mathrm{Tr}\left(\boldsymbol{F}_j\overline{\rho}_A\boldsymbol{A}_k^\dagger + \boldsymbol{A}_j\overline{\rho}_A\boldsymbol{F}_k^\dagger\right) \qquad \text{with } \boldsymbol{F}_j\overline{\rho}_A = \overline{\tau}\overline{\mathcal{K}}_A\left(\boldsymbol{A}_j\,\overline{\rho}_A\right) - \overline{\tau}\mathrm{Tr}\left(\boldsymbol{A}_j\,\overline{\rho}_A\right)\overline{\rho}_A$$

is always positive semidefinite. In particular, for $\boldsymbol{H}_{int} = \boldsymbol{A}\otimes\boldsymbol{B}$ with $\boldsymbol{A}$ and $\boldsymbol{B}$ Hermitian, the coefficient $\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A} + \boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right)$ is always non-negative.

Proof. The proof for the particular case is simple and introduces the main idea. Apply Lemma 14 with $\boldsymbol{X} = \boldsymbol{Y} = \boldsymbol{F}$; use that $-\mathcal{L}_A(\boldsymbol{F}\overline{\rho}_A) = \boldsymbol{A}\overline{\rho}_A - \mathrm{Tr}\left(\boldsymbol{A}\overline{\rho}_A\right)\overline{\rho}_A$; take the trace over $\mathcal{H}_A$ and use that $\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\right) = 0 = \mathrm{Tr}\left(\mathcal{L}_A\right)$. This yields:

$$\mathrm{Tr}\left(\boldsymbol{F}\overline{\rho}_A\boldsymbol{A} + \boldsymbol{A}\overline{\rho}_A\boldsymbol{F}^\dagger\right) = \sum_\mu \mathrm{Tr}\left([\boldsymbol{L}_{A,\mu}, \boldsymbol{F}]\overline{\rho}_A[\boldsymbol{L}_{A,\mu}, \boldsymbol{F}]^\dagger\right)$$

where the right hand side is obviously nonnegative.

Applying the same idea with $(\boldsymbol{X}, \boldsymbol{Y}) = (\boldsymbol{F}_j, \boldsymbol{F}_k)$, shows that the components of X can be re-expressed as

$$X_{j,k} \;=\; \sum_{\mu} \mathrm{Tr}\left([\boldsymbol{L}_{A,\mu}, \boldsymbol{F}_k]\, \overline{\rho}_A\, [\boldsymbol{L}_{A,\mu}, \boldsymbol{F}_j]^\dagger \right) =: \sum_{\mu} X_{j,k}^{(\mu,\overline{\rho}_A)}.$$

For each μ and replacing $\overline{\rho}_A$ by any pure state $\left|\overline{\psi}\right\rangle \left\langle\overline{\psi}\right|$, we would thus have

$$X_{j,k}^{(\mu,|\overline{\psi}\rangle\langle\overline{\psi}|)} = \left\langle v_j^{(\mu,\overline{\psi})} \middle|\, v_k^{(\mu,\overline{\psi})} \right\rangle \quad \text{with} \quad \left| v_k^{(\mu,\overline{\psi})} \right\rangle = [\boldsymbol{L}_{A,\mu}, \boldsymbol{F}_k] \left|\overline{\psi}\right\rangle .$$

The corresponding matrix $X^{(\mu,|\overline{\psi}\rangle\langle\overline{\psi}|)}$ is a Gram matrix, which is always positive semi-definite, see e.g. [Bha07, exercise 1.1.1, page 3]. The same then obviously holds for X, which is obtained by taking the sum over μ and a convex combination over different $\left|\overline{\psi}\right\rangle \left\langle\overline{\psi}\right|$. ■

Résumé

Cette thèse traite du problème de la réduction de modèle pour les systèmes quantiques ouverts possédant différentes échelles de temps, également connu sous le nom d'élimination adiabatique. L'objectif est d'obtenir une méthode générale d'élimination adiabatique assurant la structure quantique du modèle réduit.

On considère un système quantique ouvert, décrit par une équation maîtresse de Lindblad possédant deux échelles de temps, la dynamique rapide faisant converger le système vers un état d'équilibre. Les systèmes associés à un état d'équilibre unique ou une variété d'états d'équilibre ("decoherence-free space") sont considérés. La dynamique lente est traitée comme une perturbation. En utilisant la séparation des échelles de temps, on développe une nouvelle technique d'élimination adiabatique pour obtenir, à n'importe quel ordre, le modèle réduit décrivant les variables lentes. Cette méthode, basée sur un développement asymptotique et la théorie géométrique des perturbations singulières, assure une bonne interprétation physique du modèle réduit au second ordre en exprimant la dynamique réduite sous une forme de Lindblad et la paramétrisation définissant la variété lente dans une forme de Kraus (préservant la trace et complètement positif). On obtient ainsi des formules explicites, pour calculer le modèle réduit jusqu'au second ordre, dans le cas des systèmes composites faiblement couplés, de façon Hamiltonienne ou en cascade; des premiers résultats au troisième ordre sont présentés. Pour les systèmes possédant une variété d'états d'équilibre, des formules explicites pour calculer le modèle réduit jusqu'au second ordre sont également obtenues.

Mots Clés

Elimination adiabatique ; Perturbations singulières ; Systèmes quantiques ouverts ; Systèmes quantiques composites ; Systèmes multi-échelles ; Réduction de modèle ; Equation maîtresse de Lindblad

Abstract

This thesis addresses the model reduction problem for open quantum systems with different time-scales, also called adiabatic elimination. The objective is to derive a generic adiabatic elimination technique preserving the quantum structure for the reduced model.

We consider an open quantum system, described by a Lindblad master equation with two time-scales, where the fast time-scale drives the system towards an equilibrium state. The cases of a unique steady state and a manifold of steady states (decoherence-free space) are considered. The slow dynamics is treated as a perturbation. Using the time-scale separation, we developed a new adiabatic elimination technique to derive at any order the reduced model describing the slow variables. The method, based on an asymptotic expansion and geometric singular perturbation theory, ensures the physical interpretation of the reduced second-order model by giving the reduced dynamics in a Lindblad form and the mapping defining the slow manifold as a completely positive trace-preserving map (Kraus map) form. We give explicit second-order formulas, to compute the reduced model, for composite systems with weak - Hamiltonian or cascade - coupling between the two subsystems and preliminary results on the third order. For systems with decoherence-free space, explicit second order formulas are as well derived.

Keywords

Adiabatic elimination ; Singular perturbations ; Open quantum systems ; Quantum composite systems ; Different time-scales ; Model reduction ; Lindblad master equation