From Redfield to Lindblad:

$$\frac{d\rho}{dt} = -\int_0^t ds\, \mathrm{tr}_E\left[V_I(t), \left[V_I(s), \overset{\rho(s)}{\rho} \otimes \rho_E\right]\right] \longrightarrow \textcircled{1}$$

$$\mathrm{tr}_E\left[V_I(t), \left[V_I(s), \rho(t)\otimes\rho_E\right]\right]$$

$$= \cancel{\sum_{kh} \mathrm{tr}\left[B_k(t) B_h(s)\rho_E\right] * \to \left[A_k(t), \left[A_h(s), \rho(s)\right]\right]}.$$

$$\mathrm{tr}_E\left[\sum_k A_k\otimes B_k, \left[\sum_h A_h\otimes B_h, \rho(t)\otimes\rho_E\right]\right]$$

$$= \sum_{kh} \mathrm{tr}_E\Big(A_k A_l \rho \otimes B_k B_l \rho_E - A_k \rho A_l \otimes B_k \rho_E B_l - A_l \rho A_k \otimes B_l \rho_E B_k + \rho A_l A_k \otimes \rho_E B_l B_k \Big)$$

$$= \sum_{kh} \langle B_k B_l\rangle \left(A_k A_l \rho - A_l \rho A_k\right) + \langle B_l B_k\rangle \left(\rho A_l A_k - A_k \rho A_l\right)$$

$$= \sum_{kh} \langle B_k B_l\rangle \left[A_k, A_l \rho\right] - \langle B_l B_k\rangle \left[A_k, \rho A_l\right]$$

$$= \sum_{kh} \langle B_k B_l\rangle \left[A_k, A_l \rho\right] - \langle B_l B_k\rangle \left[A_k, \rho A_l\right] \longrightarrow \textcircled{2}$$

Now, note that since $\rho_E = \frac{e^{-\beta H_E}}{Z}$, $\left[\rho_E, e^{\pm i H_E t}\right] = 0$.

$$\langle B_k(t) B_l(s)\rangle = \mathrm{tr}\left[e^{i H_E t} B_k(0) e^{-i H_E (t-s)} B_l(0) e^{-i H_E s} \rho_E\right]$$

$$= \mathrm{tr}\left[e^{i H_E (t-s)} B_k(0) e^{-i H_E (t-s)} B_l(0) \rho_E\right]$$

$$= \langle B_k(t-s) B_l(0)\rangle$$

So, we can write ② as

$$\cancel{\sum_{kh} \langle B_k(t-s) B_l(0)\rangle \left[A_k^I(t), A_h^I(s)\rho(s)\right] - \langle B_l(0) B_k(s-t)\rangle \left[A_k^I(t), \rho(s) A_l(s)\right]}$$

$$\sum_{kh} \langle B_k(t-s) B_l(0)\rangle \left[A_k^I(t), A_l^I(s)\rho(s)\right] - \langle B_l(0) B_k(s-t)\rangle \left[A_k^I(t), \rho(s) A_l^I(s)\right]$$

Also note that

$$\langle B_k(t-s) B_l(0)\rangle^* = \mathrm{tr}\left[B_k(t-s) B_l(0) \rho_E\right]^* = \mathrm{tr}\left[\left(B_k(t-s) B_l(0) \rho_E\right)^\dagger\right]$$

Now,

$$\mathrm{tr}\left[\rho_E B_l(0) B_k(t-s)\right] = \mathrm{tr}\left[\left(B_k(t-s) B_l(0) \rho_E\right)^\dagger\right] = \langle B_k(t-s) B_l(0)\rangle^*$$

$$\langle B_l(0) B_k(t-s)\rangle = \langle B_k(t-s) B_l(0)\rangle^*$$

$$\Rightarrow \boxed{\Gamma_{lk}(s-t) = \Gamma_{kl}(t-s)^*} \longrightarrow \textcircled{3}$$

$$\Rightarrow \boxed{\Gamma^*_{lk}(s-t) = \Gamma_{kl}(t-s)}$$

where, I've defined

$$\Gamma_{kl}(\tau) = \langle B_k(\tau) B_l(0)\rangle = \langle B_k(0) B_l(-\tau)\rangle$$

Ok, now, lets write

$$A^I_k(t) = \int d\omega\, e^{i\omega t}\, \tilde{A}_k(\omega)$$

~~for eqn (2)~~

$$= \sum_{kh}\int d\omega \int d\Omega\, \Gamma_{kl}(t-s)\left[\tilde{A}_k(\omega), \tilde{A}_l(\Omega)\rho\right] \to e^{i(\omega+\Omega)t} -$$

$$\to \Gamma_{lk}(s-t)\, e^{i(\omega+\Omega)t}\left[\tilde{A}_k(\omega), \rho\,\tilde{A}_l(\Omega)\right].$$

$$A^I_l(s) = \int d\Omega\, e^{i\Omega s}\, \tilde{A}_l(\Omega).$$

So, we go to eqn 1, to write

$$\dot{\rho} = -\int_0^t ds\, \Gamma_{kl}(t-s)\left[A^I_k(t), A_l(s)\rho(s)\right] - \Gamma_{lk}(s-t)\left[A^I_k(t), \rho(s)A_l(s)\right]$$

Now, extend the upper limit to ∞, and fourier transform A^I_k, A^I_l to get

$$\dot{\rho} = -\sum_{kl}\int_0^\infty ds\, d\omega\, d\Omega \Big\{ \Gamma_{kl}(t-s)\, e^{i(\omega t+\Omega s)}\left[\tilde{A}_k(\omega), \tilde{A}_l(\Omega)\rho(s)\right] - \Gamma_{lk}(s-t)\, e^{i(\omega t+\Omega s)}\left[\tilde{A}_k(\omega), \rho(s)\tilde{A}_l(\Omega)\right]\Big\}$$

Now, appeal to Born & set $\rho(s) \to \rho(t)$ So that you can do the "s" integral.

$$\dot{\rho} = -\sum_{kl}\int_0^\infty ds \int_{-\infty}^{\infty} d\omega\, d\Omega \Big\{ e^{i(\omega+\Omega)t}\, e^{i\Omega(s-t)} \to$$

$$\Gamma_{kl}(t-s)\left[\tilde{A}^\dagger_k(\omega), \tilde{A}_l(\Omega)\rho(t)\right] - \Gamma_{lk}(s-t)\left[\tilde{A}_k(\omega), \rho\tilde{A}_l(\Omega)\right]\Big\}$$

Change variables from $s \to s-t = \tau$

$ds = d\tau$ & limits are from

$s \to (0 \to \infty) \Rightarrow \tau \to (0, +\infty)$.

Hence, we can write

~~$\dot{\rho} = -\int_{-\infty} d\omega\, d\Omega \int_0^\infty d\tau$~~

$$\dot{\rho} = -\sum_{kl}\int_{-\infty}^{\infty} d\omega\, d\Omega\, e^{i(\omega+\Omega)t}\int_0^\infty d\tau\, e^{i\Omega\tau} \to$$

$$\to \Big\{ \Gamma_{kl}(-\tau)\left[\tilde{A}_k(\omega), \tilde{A}_l(\Omega)\rho(t)\right] - \Gamma_{lk}(\tau)\left[\tilde{A}_k(\omega), \rho(t)\tilde{A}_l(\Omega)\right]\Big\}.$$

write $\int_0^\infty d\tau\, \Gamma_{kl}(\pm\tau)\, e^{i\Omega\tau} = \tilde{\Gamma}_{kl}(\pm\Omega)$

$$\dot{\rho} = -\sum_{kl}\int_{-\infty}^{\infty} d\omega\, d\Omega\, e^{i(\omega+\Omega)t} \to$$

$$\to \Big\{ \tilde{\Gamma}_{kl}(\Omega)\left[\tilde{A}_k(\omega), \tilde{A}_l(\Omega)\rho(t)\right] - \tilde{\Gamma}_{lk}(-\Omega)\left[\tilde{A}_k(\omega), \rho(t)\tilde{A}_l(\Omega)\right]\Big\}.$$

Simplify $\tilde{\Gamma}_{lk}(-\Omega) = \tilde{\Gamma}^{*}_{kl}(-\Omega)$, to write

proof ★

$$\dot{\rho} = -\sum_{kl}\int_{-\infty}^{\infty} d\omega\, d\Omega\, e^{i(\omega+\Omega)t} \longrightarrow$$

$$\Big\{ \tilde{\Gamma}_{kl}(\Omega)\left[\tilde{A}_k(\omega), \tilde{A}_l(\Omega)\rho(t)\right] - \tilde{\Gamma}^{*}_{kl}(-\Omega)\left[\tilde{A}_k(\omega), \rho(t)\tilde{A}_l(\Omega)\right] \Big\}.$$

Now, we use $\omega+\Omega=0 \Rightarrow$ {exponential $e^{i(\Omega+\omega)t}$ oscillates rapidly for $\Omega+\omega \neq 0$. So we make the approx $\omega = -\Omega$}, to get

$$\dot{\rho} \approx -\sum_{kl}\int_{-\infty}^{\infty} d\omega \Big\{ \tilde{\Gamma}_{kl}(-\omega)\left[\tilde{A}_k(\omega), \tilde{A}_l(-\omega)\rho(t)\right] - \tilde{\Gamma}^{*}_{kl}(+\omega)\left[\tilde{A}_k(\omega), \rho(t)\tilde{A}_l(-\omega)\right] \Big\}.$$

Now, if $\Gamma_{kl}(\tau) = \Gamma^{*}_{lk}(-\tau)$, then

$$\tilde{\Gamma}_{kl}(\omega) = \cancel{\int_{0}^{\infty} d\tau\, e^{i\omega\tau}\, \Gamma_{kl}(\tau)}$$

$$= \cancel{\int_{0}^{-\infty} d(-\tau)\, e^{i\omega(-\tau)}\, \Gamma_{kl}(-\tau)}$$

$$= \cancel{\int_{0}^{-\infty} dz\, e^{i\omega z}\, \Gamma^{*}_{lk}(\tau)}$$

$$\tilde{\Gamma}_{kl}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, \Gamma_{kl}(\tau)$$

$$\tilde{\Gamma}^{*}_{kl}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\, \Gamma^{*}_{kl}(\tau)$$

$$= -\int_{+\infty}^{-\infty} dz\, e^{i\omega z}\, \Gamma^{*}_{kl}(-z)$$

$$= \int_{-\infty}^{\infty} dz\, e^{i\omega z}\, \Gamma_{lk}(z)$$

$$\boxed{\tilde{\Gamma}^{*}_{kl}(\omega) = \tilde{\Gamma}_{lk}(\omega)}.$$

Now, proof of ★.

$$\tilde{\Gamma}_{kl}(-\Omega) = \int_{-\infty}^{\infty} d\tau\, e^{-i\Omega\tau}\, \Gamma_{kl}(\tau)$$

$$\tilde{\Gamma}^{*}_{kl}(-\Omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\Omega\tau}\, \Gamma^{*}_{kl}(\tau)$$

$$= -\int_{+\infty}^{-\infty} dz\, e^{-i\Omega z}\, \Gamma^{*}_{kl}(z)$$

$$= \int_{-\infty}^{\infty} dz\, e^{-i\Omega z}\, \Gamma^{*}_{lk}(z)$$

$$\boxed{\tilde{\Gamma}^{*}_{kl}(-\Omega) = \tilde{\Gamma}_{lk}(-\Omega)}.$$

duh! Same as proof above!

$$\dot{\rho} = -\sum_{kl}' \int_{-\infty} d\omega \Big\{ \tilde{\Pi}_{kl}(-\omega) \left[\tilde{A}_k(\omega) , \tilde{A}_l(-\omega) \rho(t) \right]$$

$$- \tilde{\Pi}_{kl}(\omega) \left[\tilde{A}_k(\omega) , \rho(t) \tilde{A}_l(-\omega) \right] \Big\}$$

$$\dot{\rho} = -\sum_{kl}' \Big\{ \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(-\omega) \left[\tilde{A}_k(\omega) , \tilde{A}_l(-\omega) \rho(t) \right]$$

$$- \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(-\omega) \left[\tilde{A}_k(-\omega) , \rho(t) \tilde{A}_l(\omega) \right]$$

went $\omega \to -\omega$ & flipped the $\int$ again.

$$\cancel{\dot{\rho} = -\sum_{kl}' \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(-\omega) \Big\{ \left[\tilde{A}_k(\omega) , \tilde{A}_l(-\omega) \rho(t) \right] - \left[\tilde{A}_l(-\omega) , \rho(t) \tilde{A}_l(\omega) \right] \Big\}}$$

$$\dot{\rho} = -\sum_{kl} \Big\{ \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(-\omega) \left[\tilde{A}_k(\omega) , \tilde{A}_l(-\omega) \rho(t) \right]$$

$$- \sum_{kl}' \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(+\omega) \left[\tilde{A}_k(-\omega) , \tilde{A}_l(\omega) \rho(t) \right]$$

$$\dot{\rho} = -\sum_{kl} \Big\{ \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(\omega) \Big\{ \left[\tilde{A}_k(\omega) , \tilde{A}_l(-\omega) \rho(t) \right]$$

$$- \left[\tilde{A}_l(-\omega) , \rho(t) \tilde{A}_k(\omega) \right] \Big\}$$

expanding

$$\dot{\rho} = -\sum_{kl} \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(\omega) \Big\{ A_k(\omega) A_l(-\omega) \rho - A_l(-\omega) \rho A_k(\omega) - A_l(-\omega) \rho A_k(\omega) + \rho(t) A_k(\omega) A_l(-\omega)$$

$$\dot{\rho} = -\sum_{kl}' \int_{-\infty}^{\infty} d\omega \; \tilde{\Pi}_{kl}(\omega) \left[2 A_l(-\omega) \rho A_k(\omega) - \{ A_k(\omega) A_l(-\omega) , \rho \} \right]$$

Now, since $\tilde{\Pi}_{kl}(\omega)$ is Hermitian, write

$$\tilde{\Pi}_{kl}(\omega) = \sum_m' C_{km}(\omega) \lambda_m(\omega) C^*_{lm}(\omega)$$

just the spectral decomp.

$$\dot{\rho} = -\sum_m' \int_{-\infty}^{\infty} d\omega \; \lambda_m(\omega) \left[2 L_m \rho L_m^\dagger - \{ L_m^\dagger L_m , \rho \} \right]$$

this is Lindblad form.

where

$$L_m = \sum_l C^*_{lm}(\omega) \tilde{A}_l(-\omega)$$

$$L_m^\dagger = \sum_l C_{lm}(\omega) \tilde{A}_l^\dagger(-\omega)$$

$$L_m^\dagger = \sum_l C_{km}(\omega) \tilde{A}_k(\omega)$$

why? If $A_t = A_t^\dagger$ then

$$A(\omega) = A^\dagger(-\omega)$$

Proof: $A(\omega) = \int dt \, e^{i\omega t} A(t)$

$$A^\dagger(\omega) = \int dt \, e^{-i\omega t} A^\dagger(t)$$

$$A^\dagger(\omega) = \int d\tau \, e^{i\omega t} A^\dagger(-t)$$

$$= \int d\tau \, e^{i\omega t} A(-t)$$

$$= A(-\omega) \text{ QED.}$$