Past Quantum States of a Monitored System

Søren Gammelmark, Brian Julsgaard, and Klaus Mølmer*
Department of Physics and Astronomy, Aarhus University, Ny Munkegade 120, DK-8000 Aarhus C, Denmark
(Received 3 May 2013; revised manuscript received 22 August 2013; published 15 October 2013)

A density matrix $\rho(t)$ yields probabilistic information about the outcome of measurements on a quantum system. We introduce here the past quantum state, which, at time T, accounts for the state of a quantum system at earlier times $t < T$. The past quantum state $\Xi(t)$ is composed of two objects, $\rho(t)$ and $E(t)$, conditioned on the dynamics and the probing of the system until t and in the time interval $[t, T]$, respectively. The past quantum state is characterized by its ability to make better predictions for the unknown outcome of any measurement at t than the conventional quantum state at that time. On the one hand, our formalism shows how smoothing procedures for estimation of past classical signals by a quantum probe [M. Tsang, Phys. Rev. Lett. **102**, 250403 (2009)] apply also to describe the past state of the quantum system itself. On the other hand, it generalizes theories of pre- and postselected quantum states [Y. Aharonov and L. Vaidman, J. Phys. A **24**, 2315 (1991)] to systems subject to any quantum measurement scenario, any coherent evolution, and any Markovian dissipation processes.

DOI: 10.1103/PhysRevLett.111.160401 PACS numbers: 03.65.Ta, 03.65.Ca

Quantum systems are described by wave functions or density matrices, which yield probabilistic predictions for the outcome of measurements performed on the systems. Following upon the rules laid out with the foundations of quantum theory in the 1920s, the description of measurements on a so-called open quantum system has in the last few decades evolved into a well-established stochastic theory [1]. According to this theory, the density matrix $\rho(t)$ evolves with time in a manner governed, on the one hand, by the system Hamiltonian and damping terms and, on the other hand, by the backaction associated with the random outcome of measurements performed on the system or its environment. The theory applies to experiments which are commonplace in laboratories today, using superconducting devices [2–5], semiconductor quantum dots [6], nitrogen vacancy centers in diamond [7,8], nuclear spins in silicon [9], trapped ions, atoms and molecules [10–12], photons [13–15], and nanomechanical devices [16]. These systems, indeed, may be used for fundamental tests [2,5,7,10,14], but they also hold the potential for application in precision probing [17] and quantum information science [18].

In this Letter, we introduce a new element in the quantum description of probed quantum systems: the past quantum state. While the density matrix $\rho(t)$ yields predictions about the outcome of the measurement of any observable at time t conditioned on previous measurements, the past quantum state yields better predictions for the same measurement by being conditioned on all measurements carried out until the present time. The past quantum state is the state that we, based on what we know now, assign to a quantum system in the past. It is thus similar to the completely natural assignment of probabilities to past values of classical random quantities, e.g., for a Brownian particle detected at position x to have been at the position y at given earlier times. Here, we provide a generalization of the assignment of probabilities to past classical stochastic processes to the quantum case. Along with the definition and derivation of a past quantum state formalism, we shall answer the pressing questions: What does it mean to make predictions about the past? What are the new results and applications of a theory of past quantum states?

Consider an open quantum system subject to our continuous probing as illustrated in Fig. 1. For convenience, we restrict our analysis to Markovian systems for which well-established methods exist to determine the quantum state evolution conditioned on measurements on the system. We assume the initial quantum state ρ_0, at time $t = 0$, and we probe the system until time T such that, conditioned on the measurement outcomes, the density matrix $\rho(t)$ is given at any time $t \in [0, T]$. Specifically, if a different observer performs a measurement on the system at the intermediate time t, the density matrix $\rho(t)$ provides the probability distribution of the possible measurement outcomes. Probing of the system after time t yields results that further refine our knowledge about the system at t and, indeed, there exists an effect matrix, $E(t)$, assuming the same Hilbert space dimension as $\rho(t)$, which depends on the dynamics and on the information acquired later than t until the present time T such that the pair of matrices,

$$\Xi(t) = (\rho(t), E(t)), \tag{1}$$

together enable better predictions than $\rho(t)$ alone for the outcome of measurements carried out at time t. To discuss in a meaningful way what is meant by predicting a past measurement, we consider the setup shown in Fig. 1. Through an appropriately chosen interaction the physical property of interest is extracted at time t via coupling to another quantum system, an ancillary "meter." This meter is stored "in a safe," or it may be immediately measured

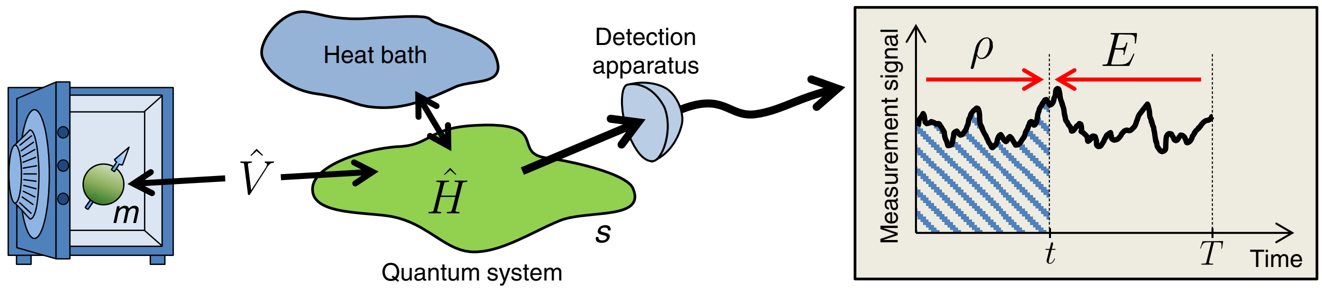

FIG. 1 (color online). Modeling a past measurement: A quantum system "s" is monitored during [0, T] and possibly subject to coherent evolution by the Hamiltonian $\hat{H}$ and coupling to a heat bath. At time t a physical property of s is mapped to a meter "m" using an interaction $\hat{V}$. This meter, or the outcome of a measurement performed on the meter, is stored in a safe immediately after the interaction at time t. The matrices ρ and E, depending on the measurement signal before and after t, respectively, constitute the past quantum state, which predicts better the measurement outcome than ρ alone when the safe is opened at time T.

and the result stored for later inspection. We show that $\Xi(t)$ provides better predictions than $\rho(t)$ of what will eventually be observed when the safe is opened. This qualifies $\Xi(t)$ rather than $\rho(t)$ to be associated to the past quantum state of the system.

The textbook description of projective quantum measurements of a Hermitian operator is a specific case of general measurements associated with the action of different operators $\hat{\Omega}_m$ that fulfill $\Sigma_m \hat{\Omega}_m^\dagger \hat{\Omega}_m = \hat{I}$, where $\hat{I}$ is the identity and m is an index referring to the possible measurement outcomes [18]. For such a generalized measurement a suitable generalization of Born's rule provides the probability at time t for observing the outcome m: $p(m) = \mathrm{Tr}(\hat{\Omega}_m \rho(t) \hat{\Omega}_m^\dagger)/\mathrm{Tr}(\rho(t))$. In the Supplemental Material [19] we prove that after further probing of the system until time $T > t$, the probability that the outcome is m depends on $\Xi(t)$, i.e., on both $\rho(t)$ and $E(t)$:

$$p_p(m) = \frac{\mathrm{Tr}(\hat{\Omega}_m \rho(t) \hat{\Omega}_m^\dagger E(t))}{\sum_m \mathrm{Tr}(\hat{\Omega}_m \rho(t) \hat{\Omega}_m^\dagger E(t))}. \tag{2}$$

This formula is general and covers all possible measurement scenarios and any Markovian dynamical evolution, observed or nonobserved, of our quantum system. As exemplified below and derived formally in the Supplemental Material [19], $E(t)$ can be calculated backward in time following an adjoint equation very similar to the forward evolution of $\rho(t)$. In the absence of probing, E retains its value $E = \hat{I}$ for all times, and Eq. (2) reduces to the conventional expression since only our observations can further the knowledge of the state.

In the special case of a projective measurement of an observable $\hat{A}$, Eq. (2) applies with $\hat{\Omega}_m = \hat{\Pi}_m$ denoting orthogonal projection operators on the eigenstates of $\hat{A}$. Past mean values, variances, and higher moments of $\hat{A}$ then follow in the usual manner from $p_p(m)$.

We note that if $\rho(t)$ predicted a measurement outcome with certainty at time t, our later probing will never lead to disagreement with this prediction: If the system was in the mth eigenstate $|a_m\rangle$ of an observable $\hat{A}$, i.e., $\hat{A}|a_m\rangle = a_m|a_m\rangle$, at time t, then $\rho(t) = |a_m\rangle\langle a_m| = \hat{\Pi}_m$ and the past probability for measuring the nth eigenstate then becomes: $p_p(n) = \delta_{n,m}$ according to Eq. (2) and the orthogonality relation $\hat{\Pi}_m \hat{\Pi}_n = \hat{\Pi}_m \delta_{n,m}$.

It also follows that variances of past measurement outcomes of noncommuting operators $\hat{A}$ and $\hat{B}$ are not limited by Heisenberg's uncertainty relation. Consider for example the particularly simple situation where the detection until time t has led to the preparation of the system in an eigenstate $|a_m\rangle$ of $\hat{A}$ while a projective measurement on eigenstate $|b_n\rangle$ of $\hat{B}$ is applied right after t. Clearly, under this circumstance, the unrevealed outcome of a measurement of $\hat{A}$ ($\hat{B}$) at time t is known to be a_m (b_n) with certainty. In our formalism, this applies because $\rho = |a_m\rangle\langle a_m|$ and $E \propto |b_n\rangle\langle b_n|$ in Eq. (2), and the outcome of a projective measurement $\hat{\Omega}_l = \hat{\Pi}_l$, in either of the eigenbases is predicted with unit probability.

Our general formalism also permits an analysis of so-called weak value measurements [20]. In this case the strength of the interaction between a system observable $\hat{A}$ and the meter can be parametrized by a small number $\epsilon \ll 1$, such that the disturbance resulting from the measurement is proportional to ϵ^2 and thus may be neglected. Nonetheless, averaged over sufficiently many experimental realizations the meter readout will reveal the mean value of the system observable $\hat{A}$ by the formula: $\langle \hat{A} \rangle_w = \mathrm{Tr}(\hat{A}\rho_p)$ where we have defined the past density matrix $\rho_p = \rho E/\mathrm{Tr}(\rho E)$ [19]. Here ρ and E are exactly the constituents of the past quantum state (1) and we thus provide a generalization of existing weak value expressions: $\langle \hat{A} \rangle_w = \mathrm{Re}\{\langle \varphi|\hat{A}|\psi\rangle / \langle \varphi|\psi\rangle\}$ or $\langle \hat{A} \rangle_w = \mathrm{Re}\{\mathrm{Tr}(\hat{A}\rho_i E_m)/\mathrm{Tr}(\rho_i E_m)\}$, which apply, respectively, for a system initially prepared in a pure state $|\psi\rangle$ or mixed state ρ_i and subsequently projected into the final state $|\varphi\rangle$ or detected by a generalized measurement operator E_m [20,21]. In these two examples ρ_p takes the form $|\psi\rangle\langle\varphi|/\langle\varphi|\psi\rangle$ or $\rho_i E_m/\mathrm{Tr}(\rho_i E_m)$, which has recently led to the recognition of these expressions as pre- and postselected "connection states" [22]. In our general theory, the quantum state may at any time in the past be viewed as a preselection ρ by the earlier measurement and a postselection E by the later ones, with unitary evolution, dissipation processes, and direct measurement on the system also accounted for. We note that ρ_p is not described by a single evolution equation, but ρ and E must be calculated separately.

To illustrate the past state formalism and some of its results for a physical problem, let us turn to an example with a quantum system subject to coherent evolution, dissipation, and continuous homodynelike monitoring. In such problems, the usual quantum state $\rho(t)$ is conditioned on the measurement outcomes prior to the time t and it formally obeys the corresponding stochastic master equation [1,23]:

$$d\rho_t = -i[\hat{H}, \rho_t]dt + \sqrt{\eta}(\hat{c}\rho_t + \rho_t\hat{c}^\dagger)dY_t + \sum_m\left(\hat{L}_m\rho_t\hat{L}_m^\dagger - \frac{1}{2}\{\hat{L}_m^\dagger\hat{L}_m, \rho_t\}\right)dt, \quad (3)$$

where $d\rho_t = \rho_{t+dt} - \rho_t$, $\hat{H}$ is the interaction Hamiltonian, $\hat{c}$ is the observable measured with the detector efficiency η, and dY_t/dt is the observed homodyne photocurrent with mean value $\sqrt{\eta}\mathrm{Tr}((\hat{c}+\hat{c}^\dagger)\rho_t)$ and unit variance. $\hat{L}_m$ are Lindblad operators describing the dissipative coupling to the environment (including $\hat{c}$ as one of the $\hat{L}_m$ terms). The effect matrix E solves a corresponding adjoint equation with final condition $E = \hat{I}$ at T [19]:

$$dE_t = i[\hat{H}, E_t]dt + \sqrt{\eta}(\hat{c}^\dagger E_t + E_t\hat{c})dY_{t-dt} + \sum_m(\hat{L}_m^\dagger E_t\hat{L}_m - \frac{1}{2}\{\hat{L}_m^\dagger\hat{L}_m, E_t\})dt, \quad (4)$$

where dt is positive and $dE_t \equiv E_{t-dt} - E_t$, propagating backward from T to t using the same measurement record dY_t as in Eq. (3). We note that these equations are not trace preserving but can easily be adapted as such if required, e.g., for numerical evaluation.

For concreteness, we consider now a quantum two-level system subject to coherent driving with Rabi frequency χ according to the Hamiltonian $\hat{H} = (1/2)(\chi\hat{\sigma}_+ + \chi^*\hat{\sigma}_-)$ and to continuous probing of $\hat{\sigma}_z$ through the measurement operator $\hat{c} = \sqrt{k}\hat{\sigma}_z$. Here $\hat{\sigma}_j$ are Pauli spin operators and k is the measurement strength. Such a measurement could be implemented by, e.g., polarization rotation of a radiation field coupled to the spin-1/2 particle [24]. We have performed simulations with $\eta = 1$, a pure initial state $\rho_0 = |\uparrow\rangle\langle\uparrow|$, and an imaginary χ such that the coherent driving rotates the spin around the y axis of the Bloch sphere. In Fig. 2, the expectation values of the Pauli operators, $\hat{\sigma}_z$ and $\hat{\sigma}_x$, for the quantum system are compared using the usual forward state $\rho(t)$ and the past quantum state $\Xi(t)$.

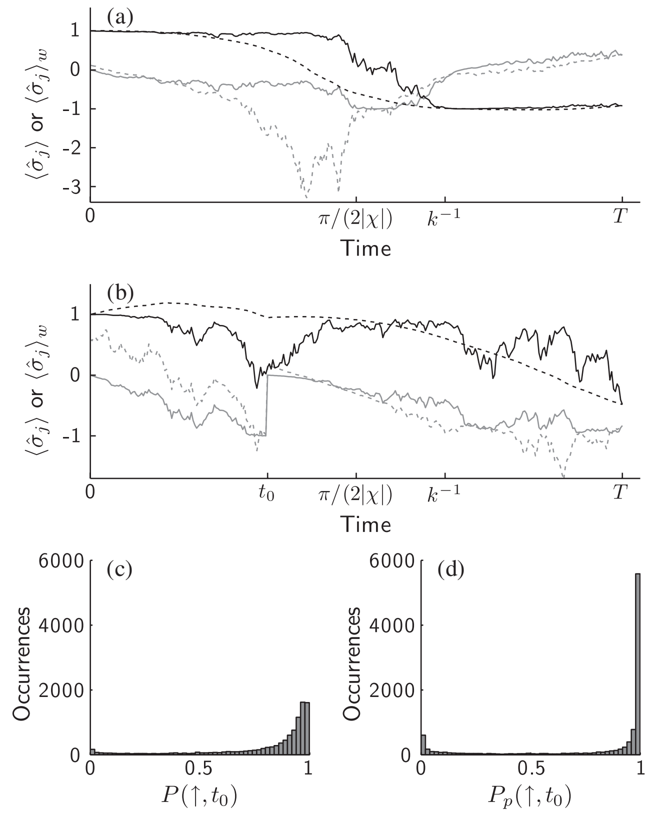

FIG. 2. Forward and past expectation values of a continuously monitored spin-1/2 particle: In panels (a) and (b) the curves show: $\langle\hat{\sigma}_z\rangle$ (solid black line), $\langle\hat{\sigma}_x\rangle$ (solid gray line), $\mathrm{Re}\{\langle\hat{\sigma}_z\rangle_w\}$ (dashed black line), and $\mathrm{Re}\{\langle\hat{\sigma}_x\rangle_w\}$ (dashed gray line). In panel (b) an observer has performed a projective measurement of $\hat{\sigma}_z$ on the system at time t_0 without revealing the result. This leads to $\langle\hat{\sigma}_x\rangle$ resetting to zero and the spectral boundary $-1 \le \mathrm{Re}\{\langle\hat{\sigma}_z\rangle_w\} \le 1$ automatically enforced. Panels (c) and (d) show histograms based on 10 000 simulations, of the distribution of probabilities for a projective measurement revealing the spin-up based on $\rho(t_0)$ and $\Xi(t_0)$, respectively.

The analysis in Fig. 2(a) applies to the case where the system has not been disturbed by further measurements by other observers. This implies that the past density matrix $\rho_p(t)$ may be used to yield predictions for the outcome of weak value measurements of any system observable. $\rho_p(t) = \rho E/\mathrm{Tr}(\rho E)$ is not Hermitian in general, and $\langle\hat{A}\rangle_w = \mathrm{Tr}(\hat{A}\rho_p)$ may be complex and acquire values beyond the spectrum of $\hat{A}$. Nonetheless, despite its possible values beyond the interval $[-1, 1]$, $\langle\hat{\sigma}_j(t)\rangle_w = \mathrm{Tr}(\hat{\sigma}_j\rho_p(t))$, rather than $\langle\hat{\sigma}_j(t)\rangle = \mathrm{Tr}(\hat{\sigma}_j\rho(t))$, represents the correct estimate of the disturbance of the meter system. For a spin-1/2 meter system, the real and imaginary parts of $\langle\hat{\sigma}_j(t)\rangle_w$ correspond to mean rotation angles of the meter spin around different axes [25].

Figure 2(b) exemplifies the case where an observer has performed a projective measurement of $\hat{\sigma}_z$ at time t_0 without revealing the result, and Eq. (2) enables a past prediction $\langle\hat{\sigma}_z(t_0)\rangle_p$ of this outcome using $\Xi(t_0)$. For all other times $t \in [0, T]$ the projective measurement must be taken into account in the evolution of ρ and E. To account for the decoherence by the measurement at t_0, we evolve the density matrix, $\rho(t_{0+}) = \hat{P}_{\rm up}\rho(t_{0-})\hat{P}_{\rm up} + \hat{P}_{\rm down}\rho(t_{0-})\hat{P}_{\rm down}$ by projection operators, $\hat{P}_{\rm up} = |\uparrow\rangle\langle\uparrow|$ and $\hat{P}_{\rm down} = |\downarrow\rangle\langle\downarrow|$. Similarly, to obtain the value of the effect matrix E prior to t_0, we have to apply the operation $E(t_{0-}) = \hat{P}_{\rm up}E(t_{0+})\hat{P}_{\rm up} + \hat{P}_{\rm down}E(t_{0+})\hat{P}_{\rm down}$. It is particularly interesting to compare the predictions of the unrevealed measurement outcome using the conventional and the past quantum state formalism. For past predictions associated with projective measurements, $\langle\hat{\sigma}_z(t_0)\rangle_p$ is real and it remains within its spectral boundaries as it should to yield agreement with experiments. The result, however, differs from the prediction by the conventional density matrix, and we quantify this difference by the distribution of probabilities for the two-level spin direction to be registered as up or down. In each of 10 000 independent simulations we have extracted at time t_0 the probability of observing the spin-up eigenstate according to the conventional and the past quantum states $\rho(t)$ and $\Xi(t)$. Histograms for their values, $P(\uparrow, t_0)$, $P_p(\uparrow, t_0)$, are shown in Figs. 2(c) and 2(d), and they show that the past quantum state more often provides probabilities close to zero and unity. By picking the obvious strategy of predicting the spin-up (spin-down) measurement result, whenever $P(\uparrow, t_0)$ or $P_p(\uparrow, t_0)$ is larger (smaller) than one half, the outcome is correct in 88% of the simulation occurrences for the conventional state $\rho(t_0)$ and in 94% of the occurrences in the past quantum state $\Xi(t_0)$.

While the past quantum state enables a sharper prediction for the projective measurement shown in Figs. 2(c) and 2(d), we note that the weak value $\langle\hat{\sigma}_z\rangle_w$ is actually smoother than the forward estimate $\langle\hat{\sigma}_z\rangle$ in Figs. 2(a) and 2(b). This is because the information dY_t becoming available from the measurement at time t, and governing the stochastic evolution of $\rho(t)$, is already included in the past quantum state via $E(t)$, and there is no additional noise associated with a measurement that commutes with the operator in question (σ_z). In the Supplemental Material [19] we show that while $\langle\hat{\sigma}_z\rangle$ varies with noisy increments $\propto dY_t \propto \sqrt{dt}$ due to Eq. (3), the changes in $\langle\hat{\sigma}_z\rangle_w$ at time t are formally independent of dY_t and vary smoothly according to the integrated noise via $\rho(t)$ and $E(t)$.

To exemplify an estimation process with mixed coherent and incoherent degrees of freedom, we consider in Fig. 3 a coherently driven, spontaneously decaying two-level atom which can jump incoherently between two sites, a and b. In Fig. 3(a) the position of the atom (as used in the simulations but in experimental realizations hidden from us) is shown along with the instants of detection of photons emitted from the two-level atom. Owing to different site environments, the coherent atom dynamics is determined by different parameter values at each site. We can use the

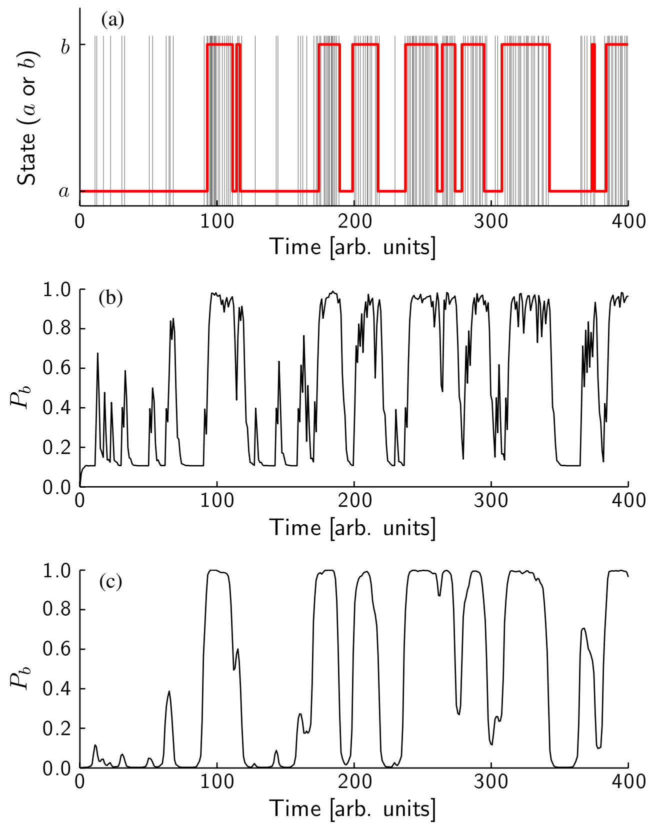

FIG. 3 (color online). Forward and past estimate of discrete position jumps of a driven two-level atom: In panel (a), the actual random position jumps of the atom are shown (red line) along with the photon detection times (vertical black lines). At position a (b), the atoms have a low (high) average emission rate. Panel (b) shows the probability for finding the atom at site b calculated using the forward density matrix, while panel (c) employs the past quantum state using the same data. Panel (c) is clearly more decisive and in much better agreement with the true state [red line in panel (a)].

forward and past quantum states, based on the photodetection record only, to estimate the location of the atom as shown in Figs. 3(b) and 3(c), respectively. As demonstrated in Refs. [26–29], a similar formalism enables the estimation of unknown classical perturbations, applied to a probed quantum system. These works generalize the so-called smoothing procedure applied in the classical probability theory of hidden Markov models (HMMs) [30] to hybrid quantum classical systems. Such hybrid systems may, indeed, be embedded in a full quantum model, and we comment further on the formal similarities between our past quantum state and the smoothing procedure in classical HMMs in the Supplemental Material [19]. The past quantum state estimate is both much less noisy and much more decisive, enabling a more accurate state estimation and, e.g., a better estimate of the a-b jump rate. In addition to estimating such classical properties, our formalism also provides a better estimation of the quantum system itself and we are currently investigating the application of our theory to the state of a photon field in a cavity [13] and to the state of a superconducting qubit [3–5].

In conclusion, we have introduced the past quantum state $\Xi(t)$ of a monitored quantum system which provides more correct predictions for measurements on the system than the conventional density matrix. $\Xi(t)$ thus provides an important contribution to the general theoretical description of quantum systems subject to measurement. The past quantum state $\Xi(t)$ depends on events occurring in the future beyond the time t. While "spooky action from the future" via postselection has stimulated fascinating scientific debate [31], the predictions we make can be interpreted as correlations between system observables at the past time t and a probing signal acquired in the, also past, interval $[0, T]$. This distinction has also been made clear in the formal work on pre- and postselected states [32,33].

This work was supported by the Villum Foundation.

*moelmer@phys.au.dk

[1] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, New York, 2010).
[2] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nat. Phys. **6**, 442 (2010).
[3] R. Vijay, D. H. Slichter, and I. Siddiqi, Phys. Rev. Lett. **106**, 110502 (2011).
[4] M. Hatridge, S. Shankar, M. Mirrahimi, F. Schackert, K. Geerlings, T. Brecht, K. M. Sliwa, B. Abdo, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Science **339**, 178 (2013).
[5] J. P. Groen, D. Ristè, L. Tornberg, J. Cramer, P. C. de Groot, T. Picot, G. Johansson, and L. DiCarlo, Phys. Rev. Lett. **111**, 090506 (2013).
[6] I. T. Vink, K. C. Nowack, F. H. L. Koppens, J. Danon, Y. V. Nazarov, and L. M. K. Vandersypen, Nat. Phys. **5**, 764 (2009).
[7] G. Waldherr, P. Neumann, S. F. Huelga, F. Jelezko, and J. Wrachtrup, Phys. Rev. Lett. **107**, 090401 (2011).
[8] J. Cai, F. Jelezko, M. B. Plenio, and A. Retzker, New J. Phys. **15**, 013020 (2013).
[9] J. J. Pla, K. Y. Tan, J. P. Dehollain, W. H. Lim, J. J. L. Morton, F. A. Zwanenburg, D. N. Jamieson, A. S. Dzurak, and A. Morello, Nature (London) **496**, 334 (2013).
[10] G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt, and C. F. Roos, Nature (London) **460**, 494 (2009).
[11] A. Kubanek, M. Koch, C. Sames, A. Ourjoumtsev, P. W. H. Pinkse, K. Murr, and G. Rempe, Nature (London) **462**, 898 (2009).
[12] Th. Basché, S. Kummer, and C. Bräuchle, Nature (London) **373**, 132 (1995).
[13] S. Gleyzes, S. Kuhr, C. Guerlin, J. Bernu, S. Deléglise, U. B. Hoff, M. Brune, J.-M. Raimond, and S. Haroche, Nature (London) **446**, 297 (2007).
[14] M. E. Goggin, M. P. Almeida, M. Barbieri, B. P. Lanyon, J. L. O'Brien, A. G. White, and G. J. Pryde, Proc. Natl. Acad. Sci. U.S.A. **108**, 1256 (2011).
[15] S. Kocsis, B. Braverman, S. Ravets, M. J. Stevens, R. P. Mirin, L. K. Shalm, and A. M. Steinberg, Science **332**, 1170 (2011).
[16] A. Naik, O. Buu, M. D. LaHaye, A. D. Armour, A. A. Clerk, M. P. Blencowe, and K. C. Schwab, Nature (London) **443**, 193 (2006).
[17] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[18] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.111.160401 for a derivation of our central Eq. (2) and equations to obtain the operator $E(t)$, for application of our theory to projective and weak measurements, and for a discussion of its relation to classical hidden Markov models.
[20] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[21] L. M. Johansen and A. Luis, Phys. Rev. A **70**, 052115 (2004).
[22] A. G. Kofman, S. K. Özdemir, and F. Nori, arXiv:1303.6031.
[23] K. Jacobs and D. A. Steck, Contemp. Phys. **47**, 279 (2006).
[24] Y. Takahashi, K. Honda, N. Tanaka, K. Toyoda, K. Ishikawa, and T. Yabuzaki, Phys. Rev. A **60**, 4974 (1999).
[25] S. Wu and K. Mølmer, Phys. Lett. A **374**, 34 (2009).
[26] M. Tsang, Phys. Rev. Lett. **102**, 250403 (2009).
[27] M. Tsang, Phys. Rev. A **80**, 033840 (2009).
[28] M. Tsang, Phys. Rev. A **81**, 013824 (2010).
[29] M. Tsang, H. M. Wiseman, and C. M. Caves, Phys. Rev. Lett. **106**, 090401 (2011).
[30] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, New York, 2007), 3rd ed..
[31] Y. Aharonov, S. Popescu, and J. Tollaksen, Phys. Today **63**, 27 (2010).
[32] Y. Aharonov and L. Vaidman, J. Phys. A **24**, 2315 (1991).
[33] Y. Aharonov, S. Popescu, J. Tollaksen, and L. Vaidman, Phys. Rev. A **79**, 052110 (2009).