Partial trace

Definition:

The partial trace tr_B is a mapping from the density matrices ρ_{AB} on a composite space $\mathscr{H}_A \otimes \mathscr{H}_B$ onto density matrices ρ_A on $\mathscr{H}_A$. It is defined as the linear extension of the mapping

$$\mathrm{tr}_B\colon \quad S \otimes T \mapsto \mathrm{tr}(T) S$$

for any matrix S on $\mathscr{H}_A$ and T on $\mathscr{H}_B$.

Let $\{|a_i\rangle\}$ be a basis of $\mathscr{H}_A$, and $\{|b_i\rangle\}$ be a basis of $\mathscr{H}_B$. Any density matrix ρ_{AB} on $\mathscr{H}_A \otimes \mathscr{H}_B$ can then be decomposed as $\rho_{AB} = \sum_{ijkl} c_{ijkl} \big|a_i\big\rangle\big\langle a_j\big| \otimes \big|b_k\big\rangle\big\langle b_l\big|$ and the partial trace reads

$$\mathrm{tr}_B \rho_{AB} = \sum_{ijkl} c_{ijkl} \big|a_i\big\rangle\big\langle a_j\big| \, \langle b_l \,|\, b_k\rangle \,, \tag{1}$$

which is a density matrix ρ_A on $\mathscr{H}_A$. Note, that $\mathrm{tr}\big|b_k\big\rangle\big\langle b_l\big| = \langle b_l \,|\, b_k\rangle$ is a complex number and note the exchange of indices compared to the operator.

Example:

The two-qubit spin singlet $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ corresponds to the density matrix

$$\big|\psi^-\big\rangle\big\langle\psi^-\big| = \frac{1}{2}\big(\big|01\big\rangle\big\langle 01\big| - \big|01\big\rangle\big\langle 10\big| - \big|10\big\rangle\big\langle 01\big| + \big|10\big\rangle\big\langle 10\big|\big),$$

on which we act with the partial trace acording to the definition (1)

$$\begin{aligned} \mathrm{tr}_B\big|\psi^-\big\rangle\big\langle\psi^-\big| &= \frac{1}{2}\big(\big|0\big\rangle\big\langle 0\big| \, \langle 1 \,|\, 1\rangle - \big|0\big\rangle\big\langle 1\big| \, \langle 0 \,|\, 1\rangle - \big|1\big\rangle\big\langle 0\big| \, \langle 1 \,|\, 0\rangle + \big|1\big\rangle\big\langle 1\big| \, \langle 0 \,|\, 0\rangle\big) \\ &= \frac{1}{2}\big(\big|0\big\rangle\big\langle 0\big| + \big|1\big\rangle\big\langle 1\big|\big), \end{aligned}$$

to obtain the reduced state $\rho_A = \frac{1}{2}\mathbb{1}$.

Formula for two qubits:

A generic two-qubit state ρ_{AB} can be expanded with respect to the orthonormal basis $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$ as

$$\rho_{AB} = \rho_{11}\big|00\big\rangle\big\langle 00\big| + \rho_{12}\big|00\big\rangle\big\langle 01\big| + \rho_{13}\big|00\big\rangle\big\langle 10\big| + \ldots + \rho_{44}\big|11\big\rangle\big\langle 11\big|.$$

The partial trace is calculated according to (1)

$$\rho_A = (\rho_{11} + \rho_{22})\big|0\big\rangle\big\langle 0\big| + (\rho_{13} + \rho_{24})\big|0\big\rangle\big\langle 1\big| + (\rho_{31} + \rho_{42})\big|1\big\rangle\big\langle 0\big| + (\rho_{33} + \rho_{44})\big|1\big\rangle\big\langle 1\big|.$$

The factor in front of $\big|1\big\rangle\big\langle 0\big|$, say, can be derived as follows: the preimages of $\big|1\big\rangle\big\langle 0\big|$ under the partial trace are $\big|10\big\rangle\big\langle 00\big|$ and $\big|11\big\rangle\big\langle 01\big|$ and come with factors ρ_{31} and ρ_{42} in ρ_{AB}. In matrix form the formula reads

$$\mathrm{tr}_B \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} = \begin{pmatrix} \mathrm{tr}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} \rho_{13} & \rho_{14} \\ \rho_{23} & \rho_{24} \end{pmatrix} \\ \mathrm{tr}\begin{pmatrix} \rho_{31} & \rho_{32} \\ \rho_{41} & \rho_{42} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix}.$$

Partial trace over A:

Finally, we mention that the partial trace tr_A over the subsystem A is defined in the obvious way and can be calculated similarly to tr_B. For convenience we state the formula for two qubits

$$\mathrm{tr}_A \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} = \begin{pmatrix} \mathrm{tr} \begin{pmatrix} \rho_{11} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix} & \mathrm{tr} \begin{pmatrix} \rho_{12} & \rho_{14} \\ \rho_{32} & \rho_{34} \end{pmatrix} \\ \mathrm{tr} \begin{pmatrix} \rho_{21} & \rho_{23} \\ \rho_{41} & \rho_{43} \end{pmatrix} & \mathrm{tr} \begin{pmatrix} \rho_{22} & \rho_{24} \\ \rho_{42} & \rho_{44} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix}.$$

It is a good exercise to convince oneself that these formulae make sense with respect to the matrix representation of the tensor product

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \otimes \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} a_{00} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} & a_{01} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \\ a_{10} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} & a_{11} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \end{pmatrix}.$$