Decoding Quantum Codes

Ben Criger

June 3, 2016

Introduction: What is Decoding?

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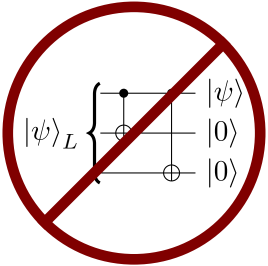

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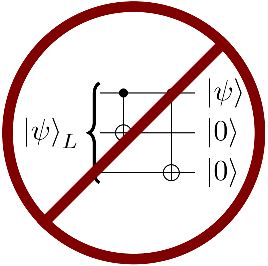

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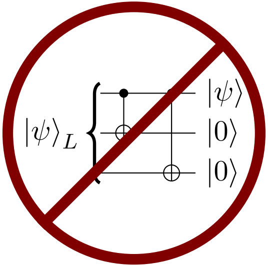

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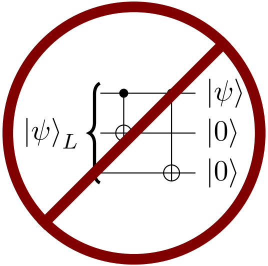

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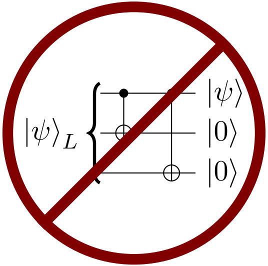

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To get a feel for decoding, let's focus on classical codes for the moment.

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Repetition Code　　　　Hamming Code

A message is multiplied by G to encode, and transmitted over a noisy channel, where it can be randomly flipped.

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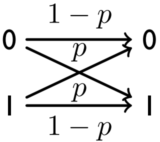

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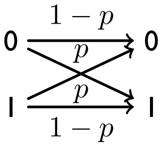

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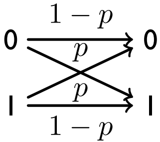

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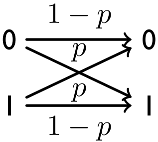

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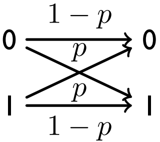

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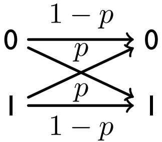

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  - *Maximum Likelihood* (ML): $e' = \underset{v|Hv=s}{\operatorname{argmax}} [p(v)]$
  - *Minimum Weight* (MW): $e' = \underset{v|Hv=s}{\operatorname{argmin}} [|v|]$, equivalent to ML when error rates are small and equal.

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This lets us use H as a look-up table for MW decoding:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}}_{\text{parity-check matrix}} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\substack{\text{noisy} \\ \text{message}}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{syndrome}} \therefore e' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\substack{\text{original} \\ \text{message}}}$$

Codes like this are called *perfect* or *non-degenerate*.

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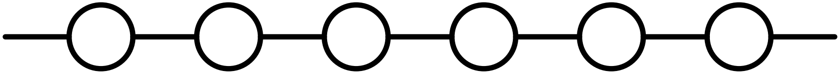

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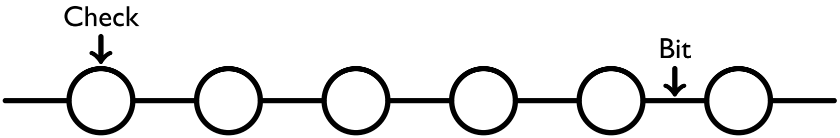

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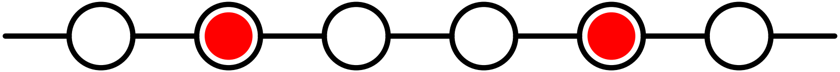

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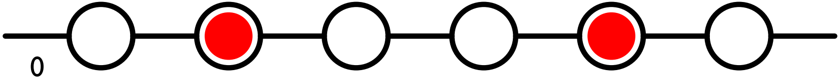

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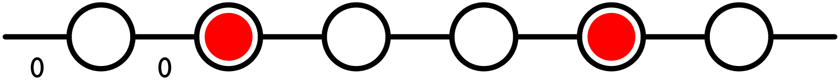

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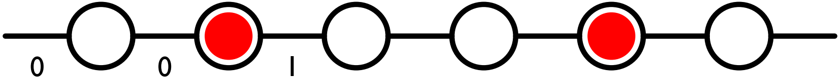

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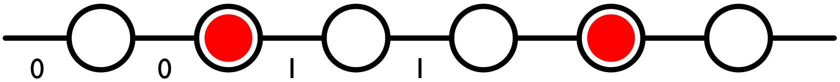

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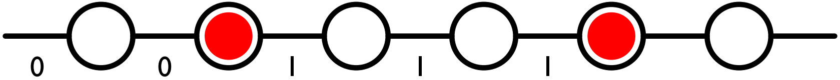

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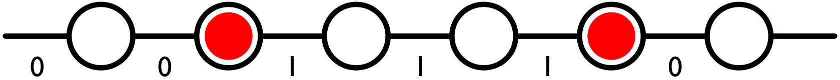

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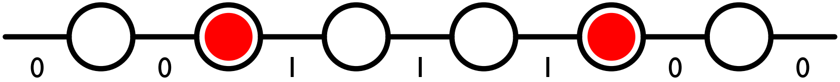

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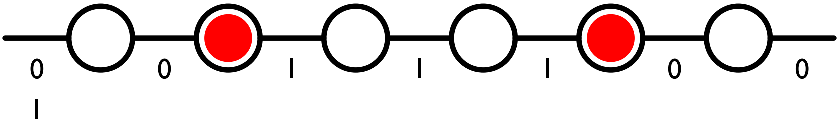

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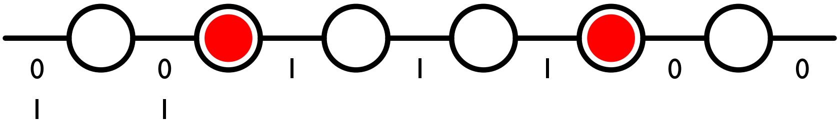

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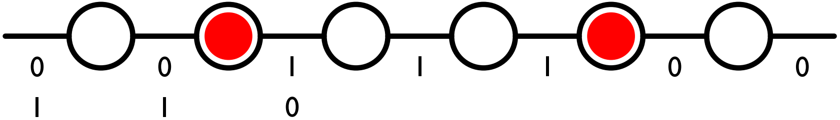

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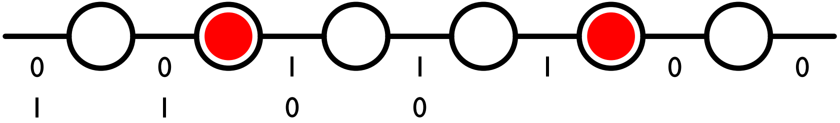

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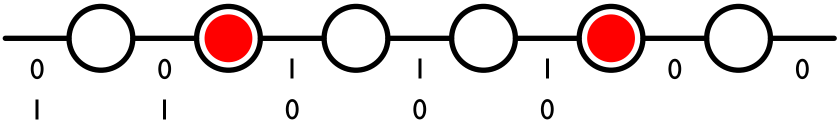

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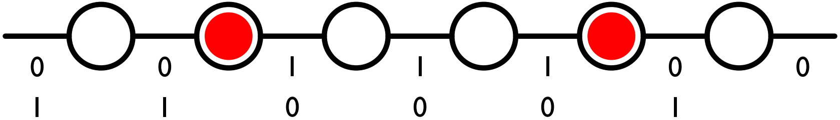

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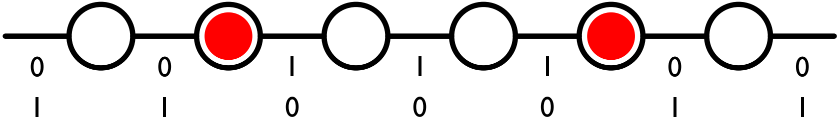

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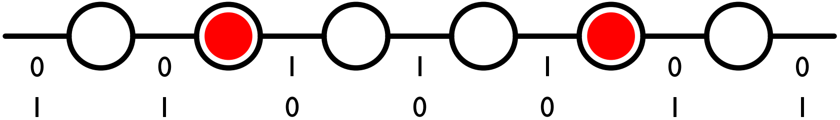

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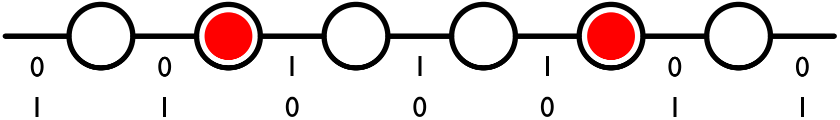

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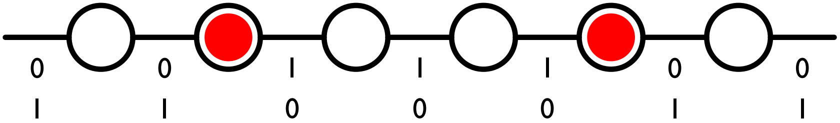

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Place bits in the cells of a lattice (squares of a square tiling, cubes of a cubic tiling) and checks between neighbouring cells. How many errors are consistent with a given syndrome, and what structure do the syndromes have?

The CSS Construction

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These stabilisers protect a quantum state against Z or X errors with the same support as correctable errors in the original classical code.

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Prove that a Z operator commutes with X stabilisers (and vice versa) derived from a parity-check matrix H iff it is mapped from a codeword of the code defined by H.

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We will focus on codes which are inherently planar, so that operations can be performed locally 'on a chip'.

The Toric/Surface/Rotated Surface Code

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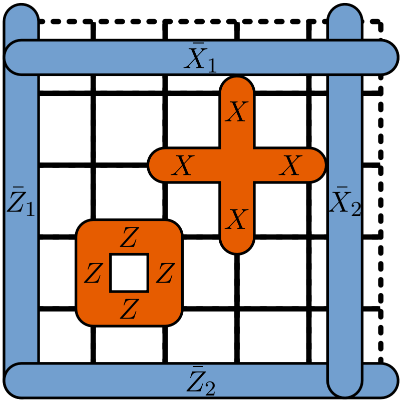

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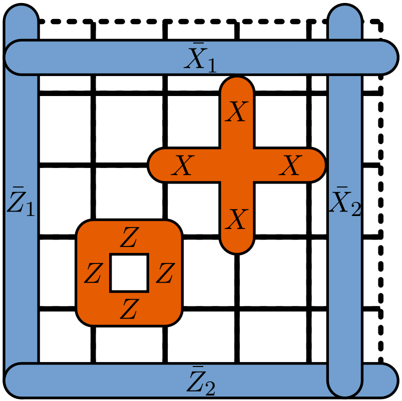

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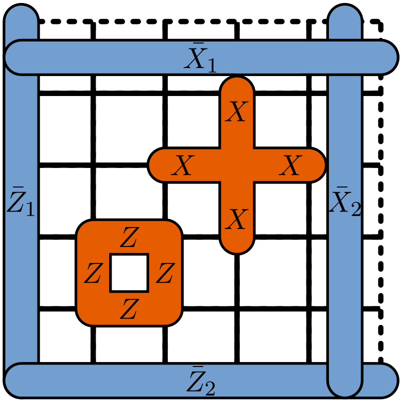

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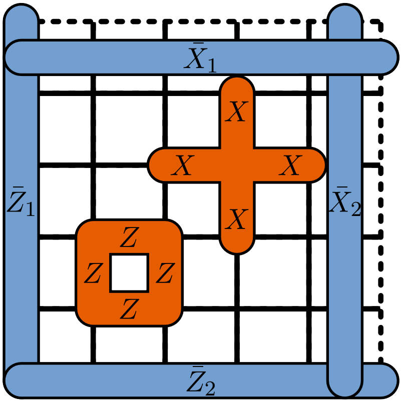

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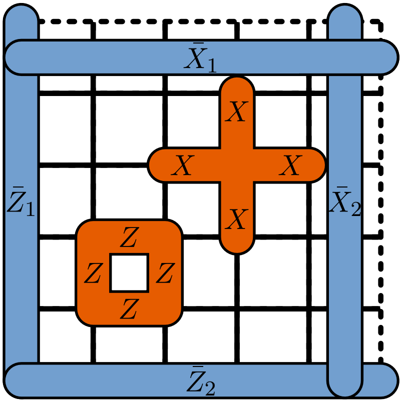

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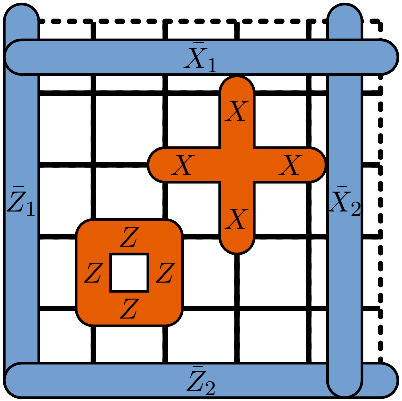

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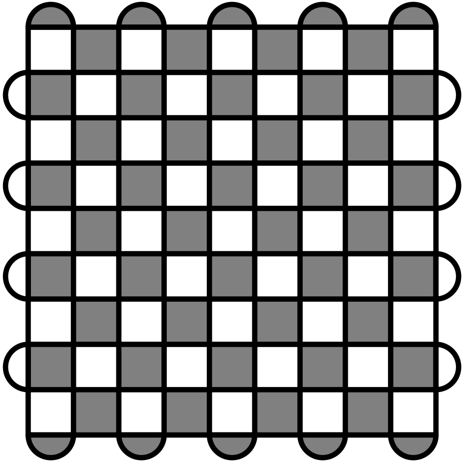

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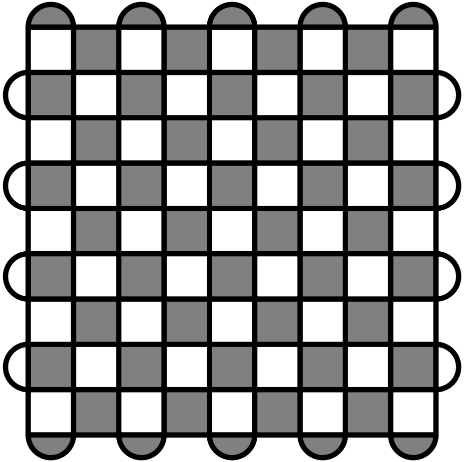

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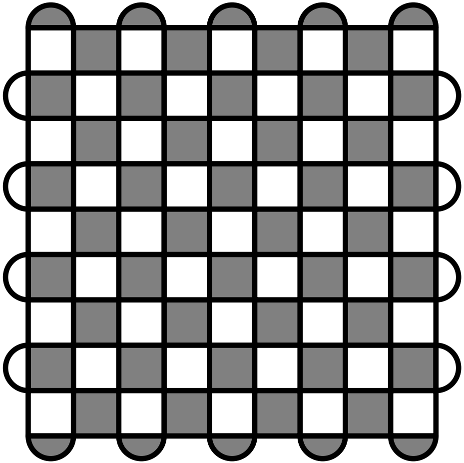

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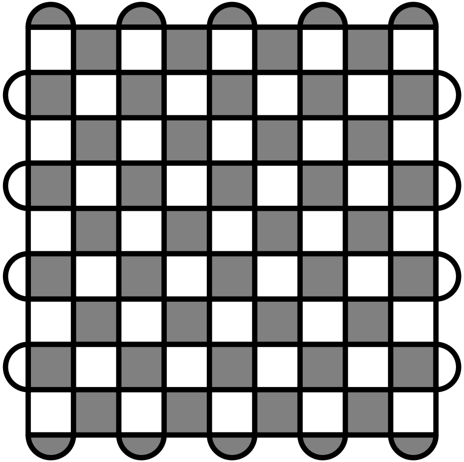

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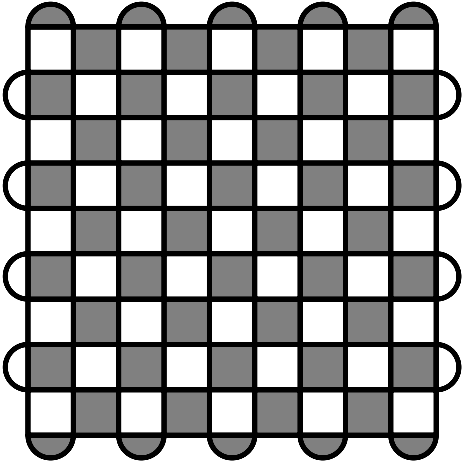

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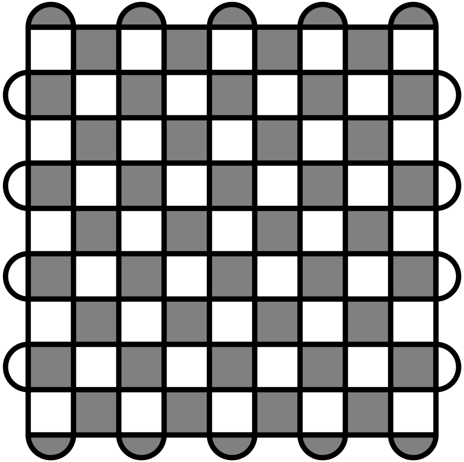

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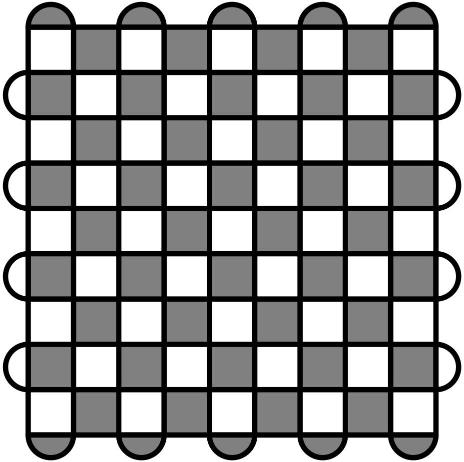

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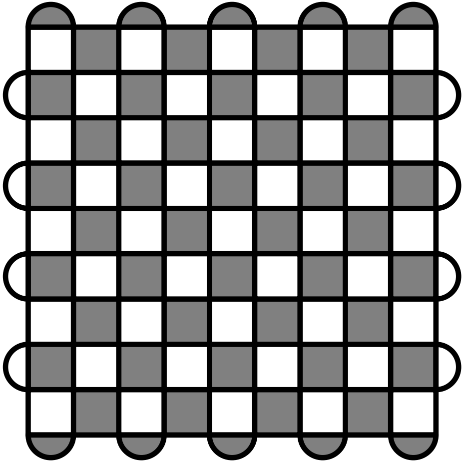

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Decoding by Minimum-Weight Perfect Matching

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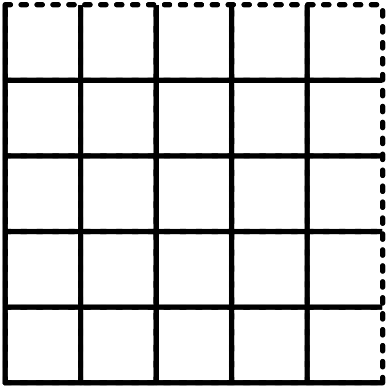

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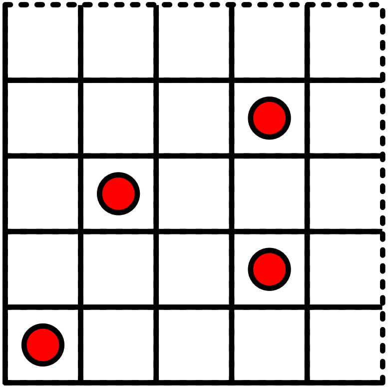

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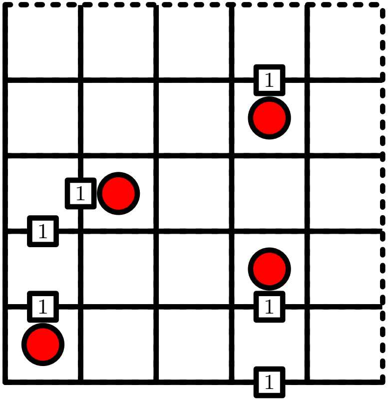

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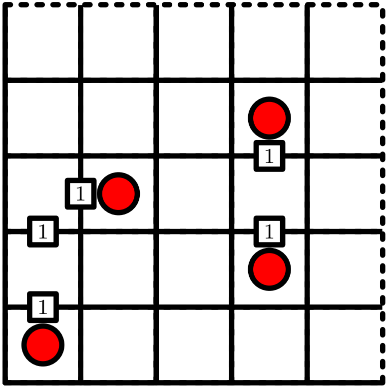

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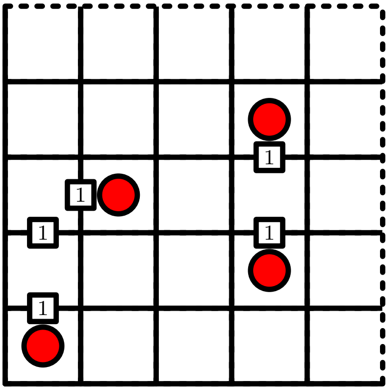

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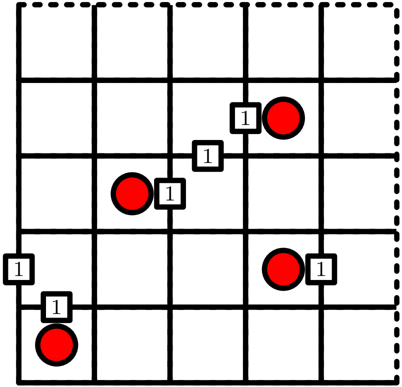

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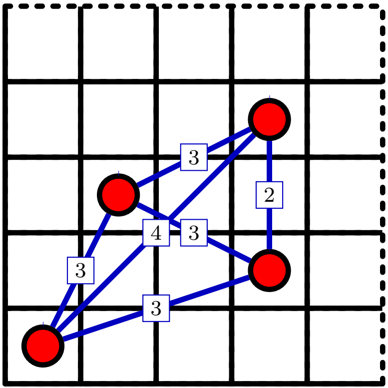

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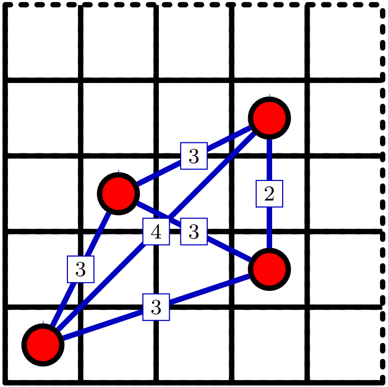

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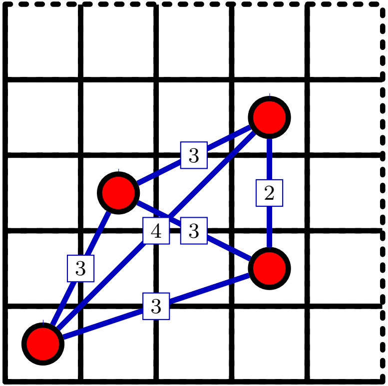

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- Required: An algorithm to find the minimum weight set of 1D strings, given syndrome minimum lengths between syndrome pairs

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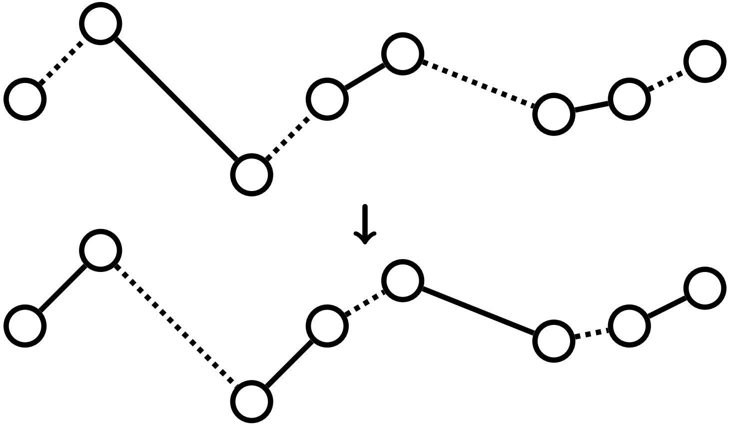

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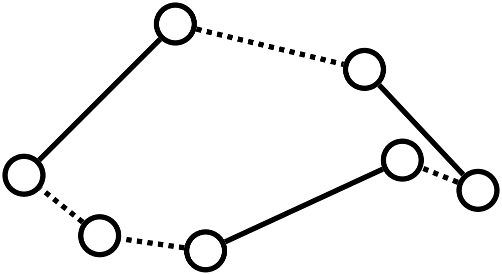

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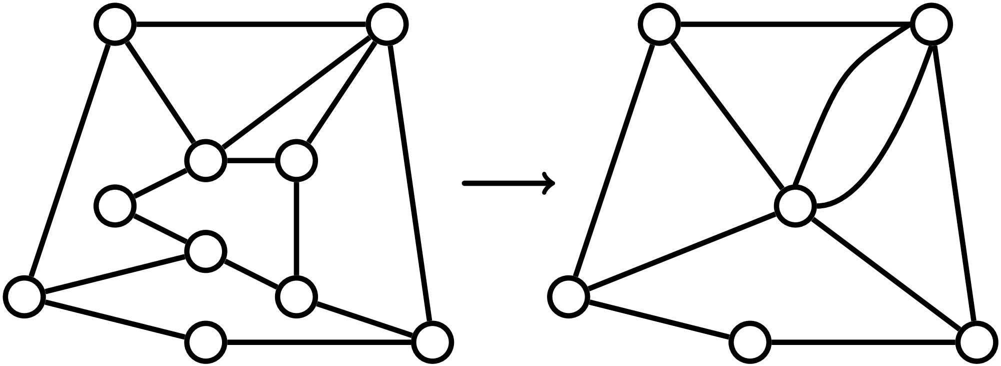

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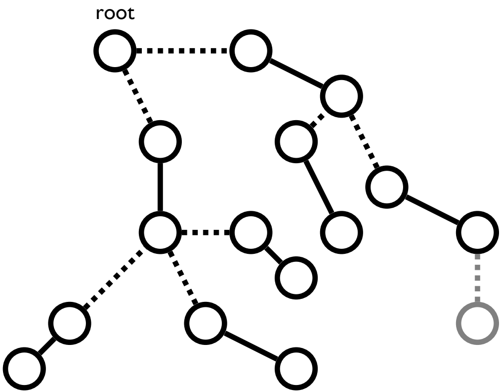

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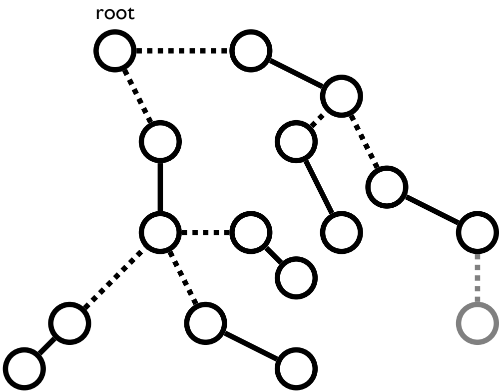

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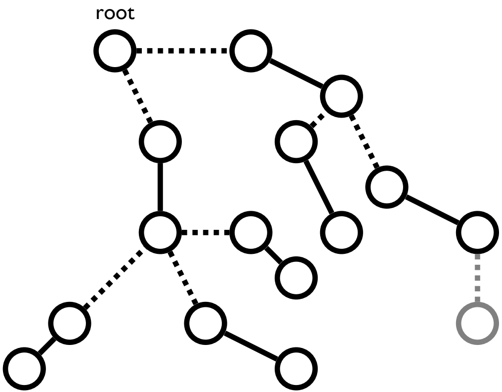

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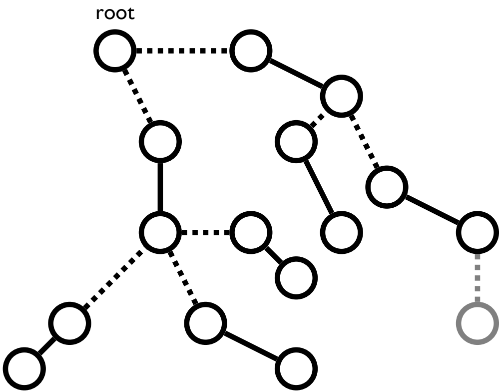

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- if no such neighbour exists, the tree is *frustrated*, and no perfect matching exists.
- For un-weighted perfect matchings: repeatedly grow these trees until a perfect matching or frustrated tree is found

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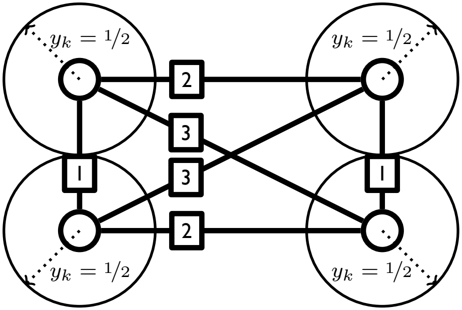
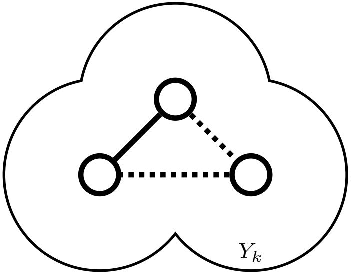
There exists an edge set on a weighted graph such that a perfect matching on that set is also minimum-weight, called the *tight edges*.





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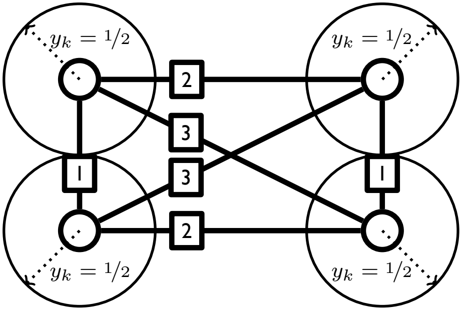

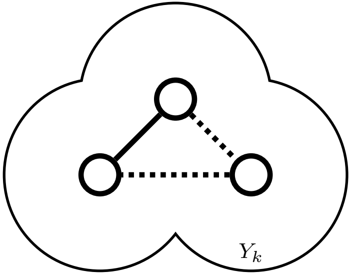

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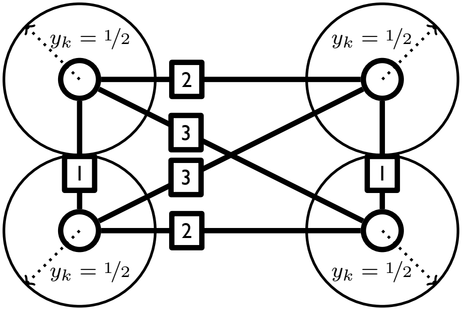

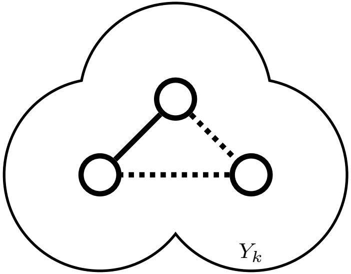

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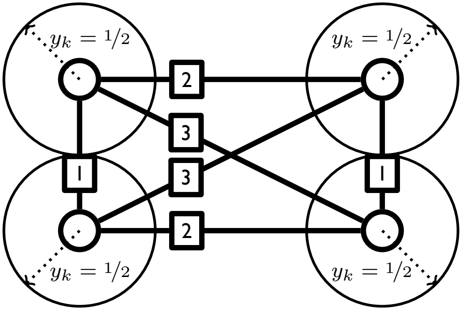

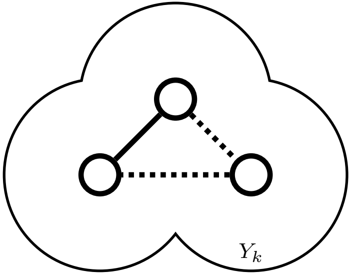

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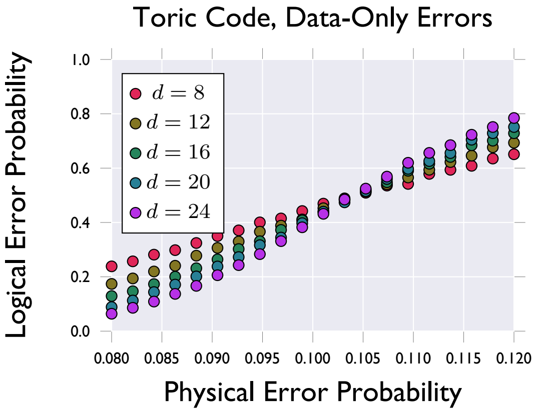
Why do we tolerate this complexity?

Performance

When subjected to independent bit- and phase-flip errors, a threshold of $\sim 10.3\%$ can be obtained with the toric code:



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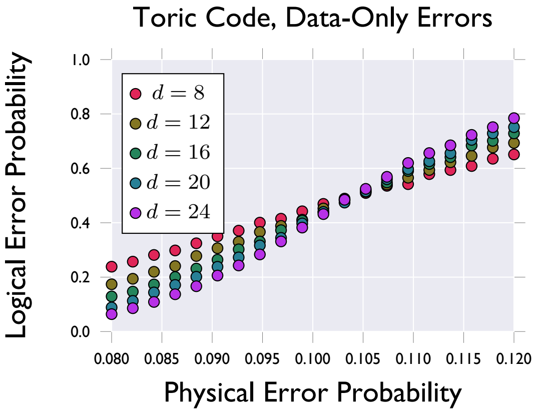

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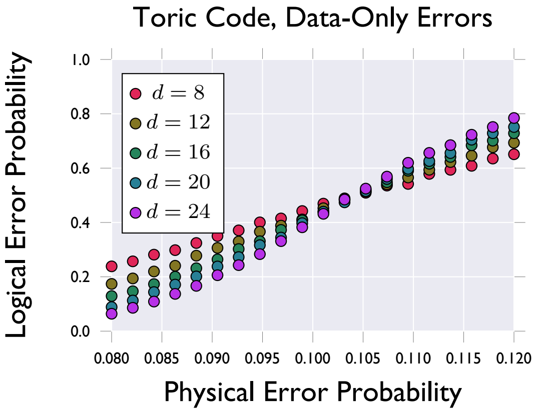

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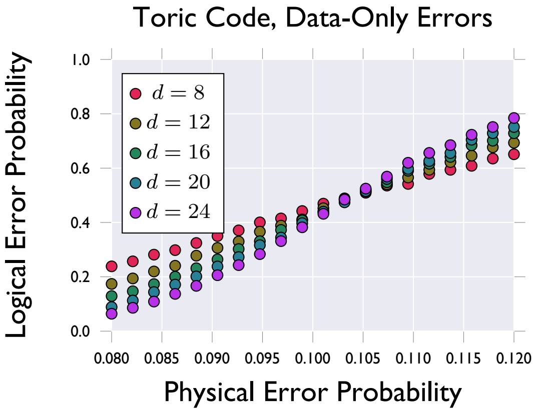

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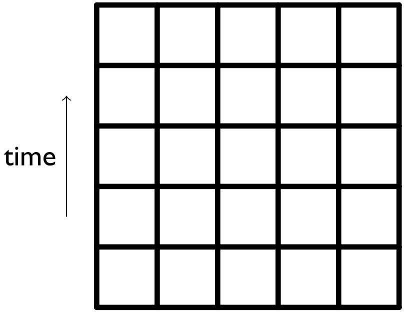

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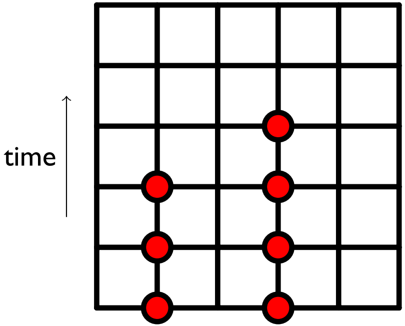

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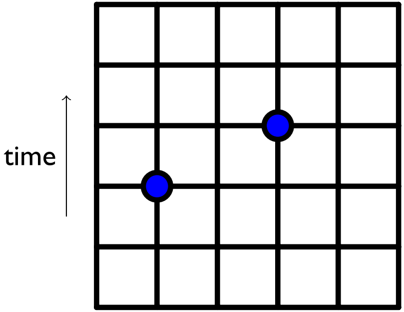

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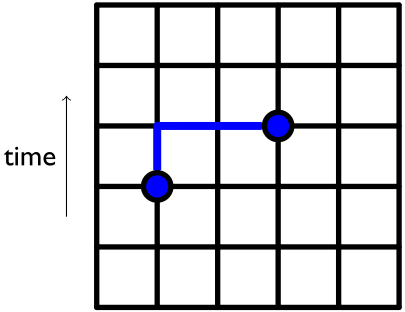

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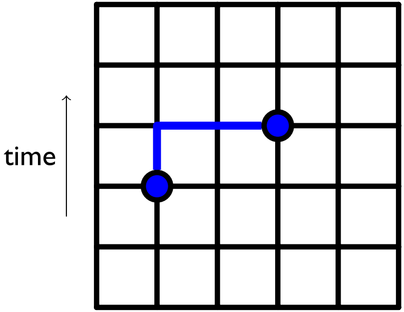

$$p_{\text{chain}} = p^d (1-p)^{n-d} \times q^m (1-q)^{n_s - m}$$

$$\sim \left(\frac{p}{1-p}\right)^d \left(\frac{q}{1-q}\right)^m$$

$$\log(p_{\text{chain}}) = d \log\left(\frac{p}{1-p}\right) + m \log\left(\frac{q}{1-q}\right)$$

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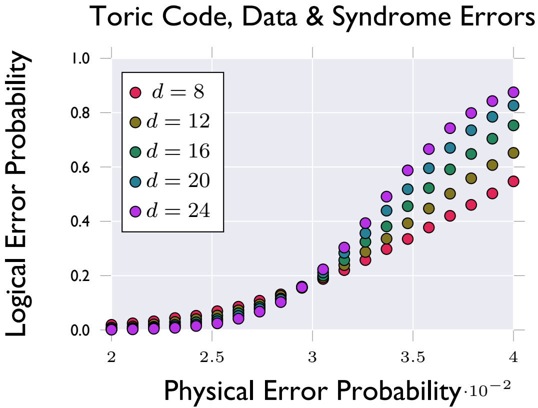
This toy model results in a reduced threshold of $\sim 2.9\%$:



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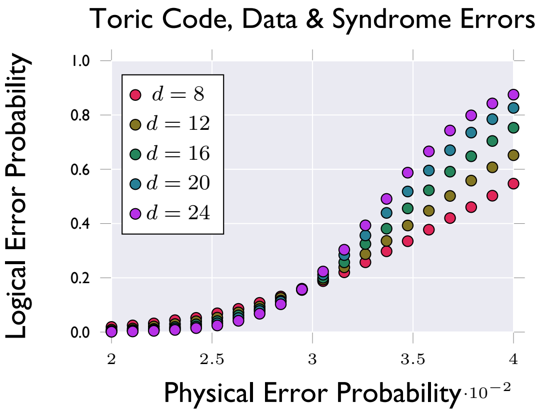

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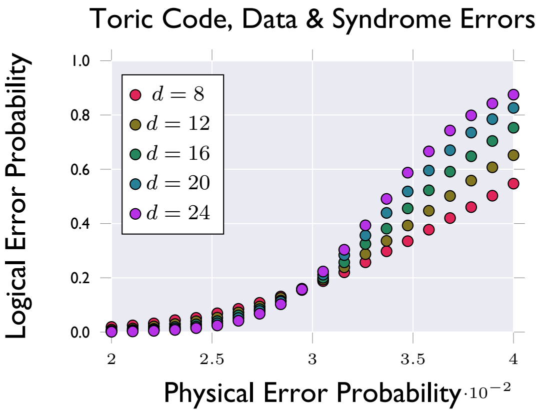

- More advanced likelihood models based on circuit noise can result in thresholds $0.6\% - 1.4\%$.
- This result implies that real fault tolerance is possible if classical computation is "free".

Alternate Codes and Decoders

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If errors take time $\mathcal{O}(n^k)$ to correct, and errors are time-dependent, then *error strength increases with* n.

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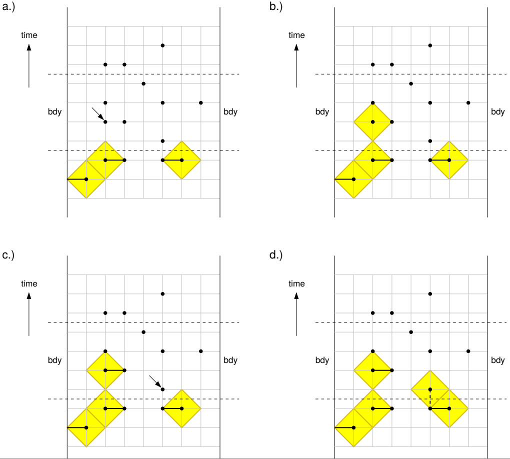
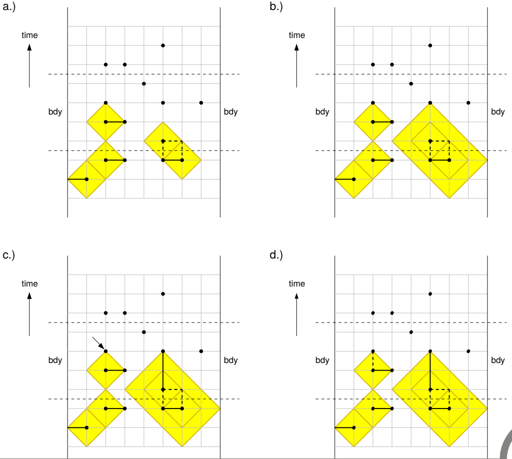
We must interleave decoding with faulty measurement.





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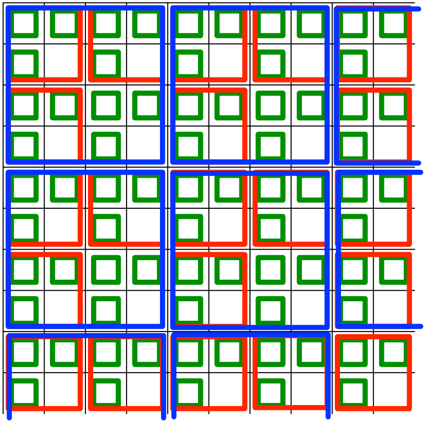

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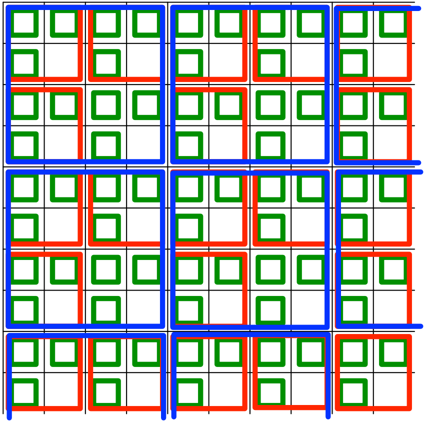

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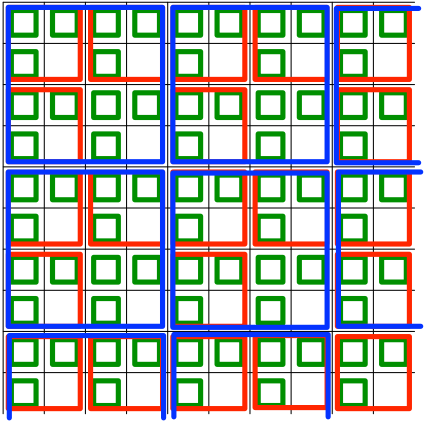

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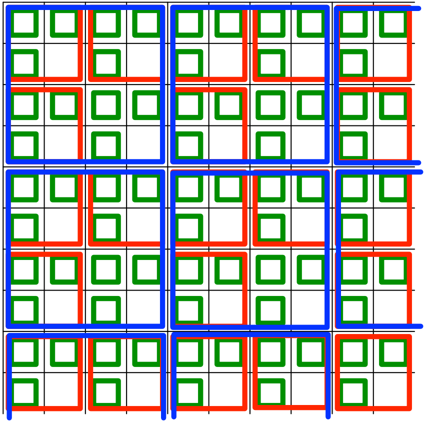

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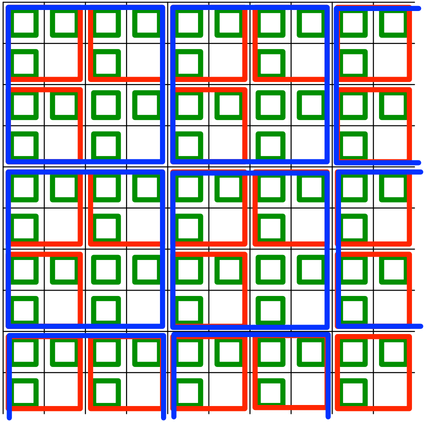

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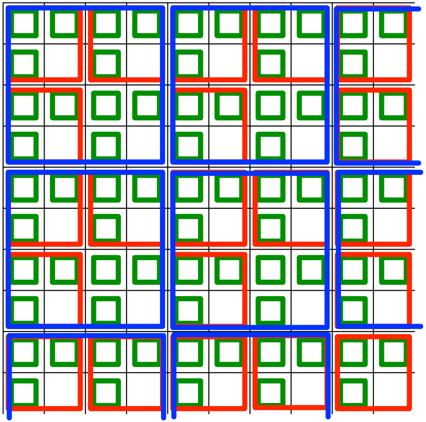

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Questions?