Antenna Arrays (Contd.)

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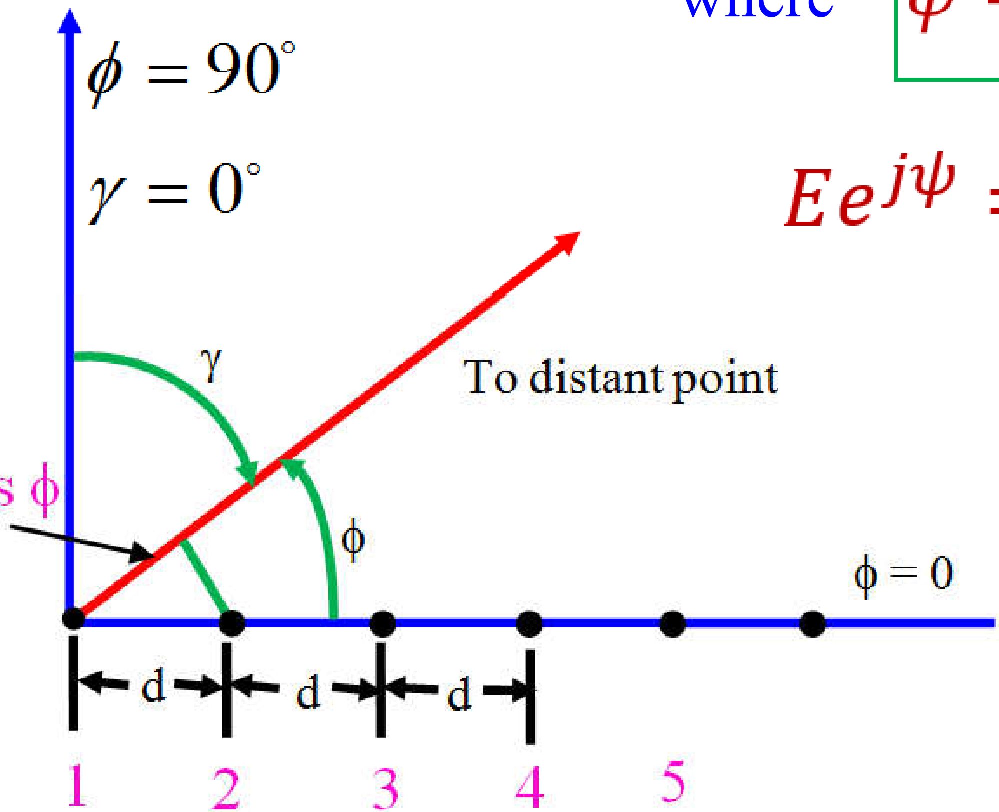
N Isotropic Point Sources of Equal Amplitude and Spacing

$$E = 1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(n-1)\psi}$$

where $\boxed{\psi = \frac{2\pi d}{\lambda}\cos\phi + \delta = d_r\cos\phi + \delta}$



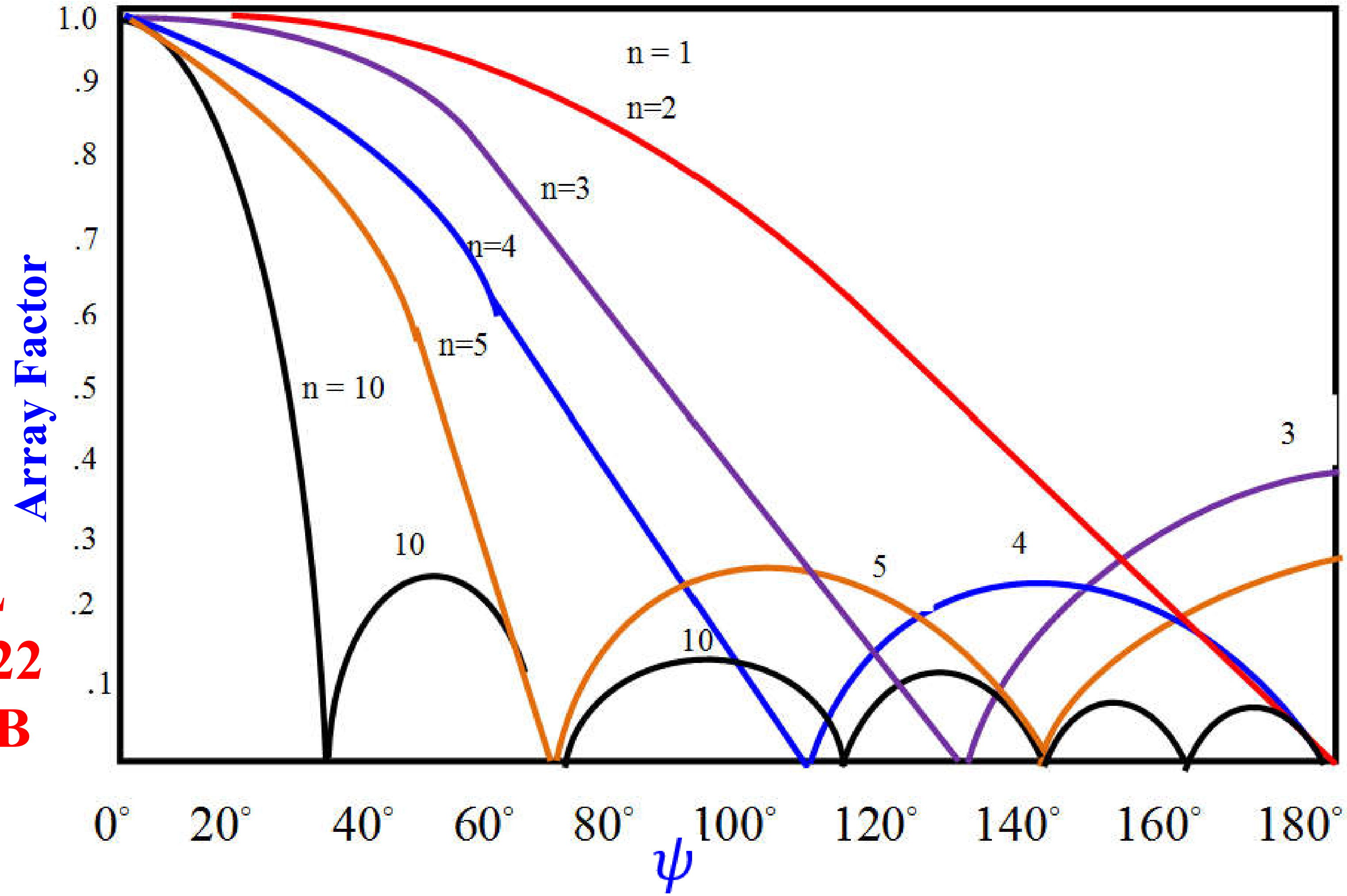
$$Ee^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{jn\psi}$$

$$E - Ee^{j\psi} = 1 - e^{jn\psi}$$

$$\boxed{E = \frac{1 - e^{jn\psi}}{1 - e^{j\psi}} = \frac{\sin(n\psi/2)}{\sin(\psi/2)}}$$

As $\Psi \rightarrow 0$, $E_{max} = n$, $E_{norm} = \dfrac{\sin(n\psi/2)}{n\sin(\psi/2)}$

Radiation Pattern of N Isotropic Elements Array



First SLL
= 20log0.22
= -13.15dB

Radiation Pattern for array of n isotropic radiators of equal amplitude and spacing.

Null Directions for Arrays of N Isotropic Point Sources

$$E_{norm} = \frac{\sin(n\psi/2)}{n \sin(\psi/2)}$$

For Finding Direction of Nulls:

$$\sin\left(\frac{n\psi}{2}\right) = 0 \rightarrow \frac{n\psi}{2} = \pm k\pi \ \textit{where, } k{=}1{,}2{,}3\ldots\ldots$$

$$\psi = \pm\frac{2k\pi}{n}$$

For Broadside Array, δ = 0

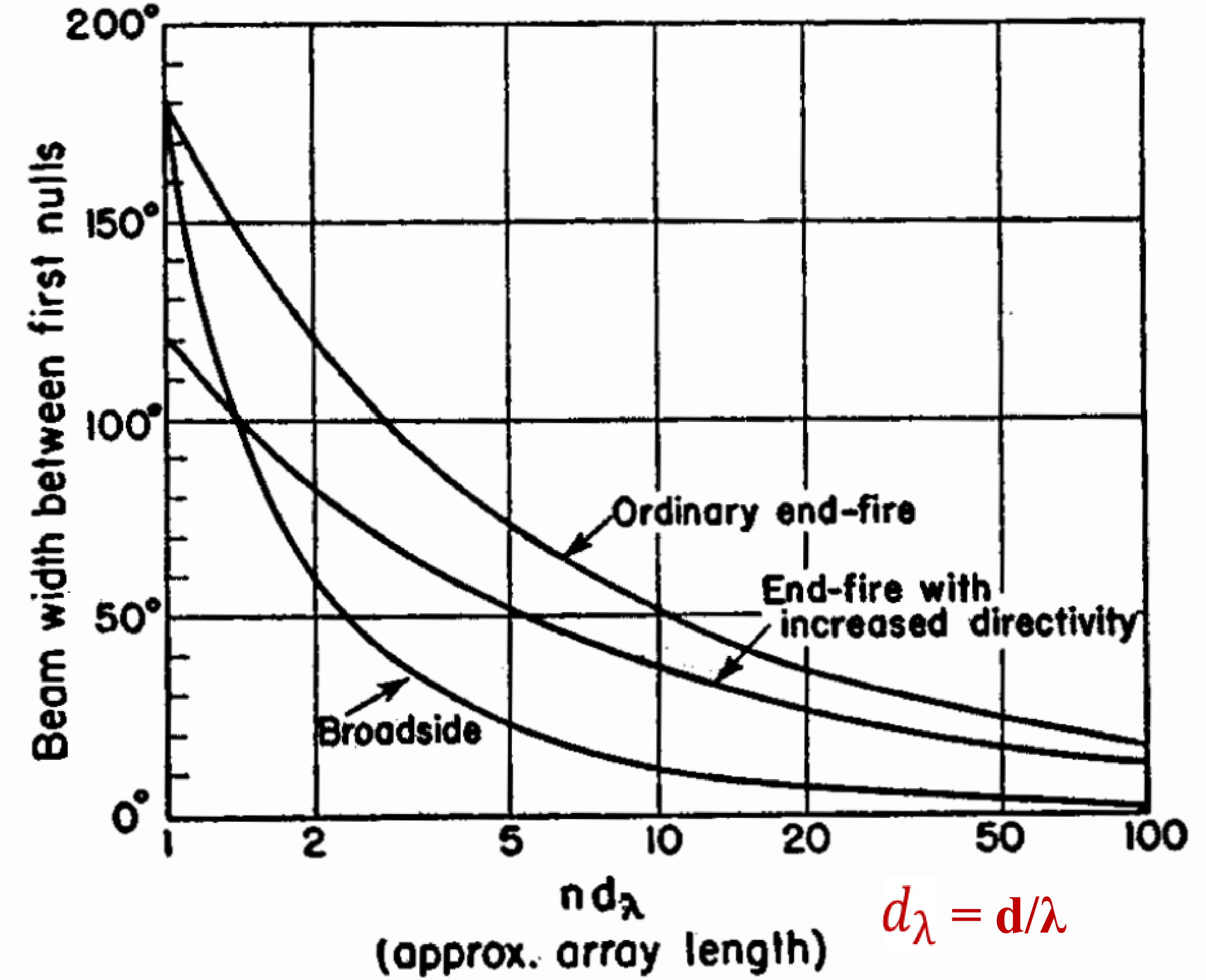
$$\frac{2\pi d}{\lambda} cos\phi_0 = \pm\frac{2k\pi}{n} \rightarrow \phi_0 = \pm\cos^{-1}\left(\frac{k\lambda}{nd}\right)$$

Null Direction and First Null Beamwidth

Null directions and beam width between first nulls for linear arrays of n isotropic point sources of equal amplitude and spacing

Type of array	Null directions (array any length)	Null directions (long array)	Beam width between first nulls(long array)
General case	$\phi_0 = \arccos\left[\left(\pm\frac{2K\pi}{n} - \delta\right)\frac{1}{d_r}\right]$		
Broadside	$\gamma_0 \simeq \arcsin\left(\pm\frac{K\lambda}{nd}\right)$	$\gamma_0 \simeq \pm\frac{K\lambda}{nd}$	$2\gamma_{01} \simeq \frac{2\lambda}{nd}$
Ordinary end-fire	$\phi_0 = 2\arcsin\left(\pm\sqrt{\frac{K\lambda}{2nd}}\right)$	$\phi_0 \simeq \pm\sqrt{\frac{2K\lambda}{nd}}$	$2\phi_{01} \simeq 2\sqrt{\frac{2\lambda}{nd}}$
End-fire with increased directivity	$\phi_0 = 2\arcsin\left[\pm\sqrt{\frac{\lambda}{4nd}(2K-1)}\right]$	$\phi_0 \simeq \pm\sqrt{\frac{\lambda}{nd}(2K-1)}$	$2\phi_{01} \simeq 2\sqrt{\frac{\lambda}{nd}}$

First Null Beamwidth (FNBW)



d_λ = **d/λ**

For long array, *(n-1)d* is equal to array length *L*

Directions of Max SLL for Arrays of N Isotropic Point Sources

$$\sin\frac{n\psi}{2} = \pm 1 \rightarrow \frac{n\psi}{2} = \pm\frac{(2k+1)\pi}{2} \quad \textit{where } k=1,2,3\ldots\ldots$$

$$\psi = \pm\frac{(2k+1)\pi}{n}$$

Magnitude of SLL: $AF = \left|\frac{\sin\frac{n\psi}{2}}{n\sin\frac{\psi}{2}}\right| = \left|\frac{1}{n\sin\left(\frac{(2k+1)\pi}{2n}\right)}\right|$

For very large n:

$$AF = \left|\frac{1}{n\times\left(\frac{(2k+1)\pi}{2n}\right)}\right| = \frac{2}{(2k+1)\pi} = 0.212 \quad \textbf{for k =1 (First SLL)}$$

SLL in dB = 20Log 0.212 = -13.5dB

Direction of Minor Lobe Maxima

Type of array	Direction of minor lobe maxima
General case	$\phi_m = \arccos\left[\left(\pm\frac{(2K+1)\pi}{n} - \delta\right)\frac{1}{d_r}\right]$
Broadside	$\phi_m \simeq \arccos\left(\pm\frac{(2K+1)\lambda}{2nd}\right)$
Ordinary end-fire	$\phi_m \simeq \arccos\left(\pm\frac{(2K+1)\lambda}{2nd} + 1\right)$
End-fire with increased directivity	$\phi_m \simeq \arccos\left[\frac{\lambda}{2nd}[1 \pm (2K+1)] + 1\right]$

Half-Power Beamwidth (HPBW) of Array

For calculating HPBW, find *Ψ*, where radiated power is reduced to half of its maximum value

$$AF = \left|\frac{\sin\frac{n\psi}{2}}{n\sin\frac{\psi}{2}}\right| = 1/\sqrt{2}$$

For large n, HPBW is small : $AF \simeq \left|\frac{\sin\frac{n\psi}{2}}{n\,\frac{\psi}{2}}\right| = 1/\sqrt{2}$

Solution:
***nΨ/2* = 1.3915**

For Broadside: $\psi = \frac{2\pi d}{\lambda}\cos\phi$ **= 2.783/*n***

$$\cos\phi = \sin(90 - \phi) = 1.3915/(\pi n d/\lambda) = 0.443/L_\lambda \text{ (radian)}$$

$$\text{HPBW} \simeq 2 \times (90 - \phi) = 50.8^0 / L_\lambda$$

Aperture, Directivity and Beamwidth

Array (or aperture)‡	Directivity formula	Directivity for L_λ or d_λ equal to 1	10	100	1000	Half-power beam widths
Linear broadside array of length L_λ	$2L_\lambda$	2	20	200	2000	$\frac{50.8^\circ}{L_\lambda} \times 360^\circ$
Ordinary end-fire array of length L_λ	$2\pi L_\lambda$	6.3	63	630	6300	$\frac{108^\circ}{\sqrt{L_\lambda}}$
Increased-directivity end-fire array of length L_λ	$4\pi L_\lambda$	12.6	126	1 260	12 600	$\frac{52^\circ}{\sqrt{L_\lambda}}$
Square broadside aperture with side length L_λ	$4\pi L_\lambda^2$	12.6	1260	126 000	1.26×10^7	$\frac{50.8^\circ}{L_\lambda} \times \frac{50.8^\circ}{L_\lambda}$
Circular broadside aperture with diameter d_λ	$\pi^2 d_\lambda^2$	9.9	990	99 000	9.9×10^6	$\frac{58^\circ}{d_\lambda}$

Grating Lobes for Arrays of N Isotropic Point Sources

To Avoid Grating Lobes:

$$\psi = \frac{2\pi d}{\lambda}(\cos\phi - \cos\phi_m) < 2\pi$$

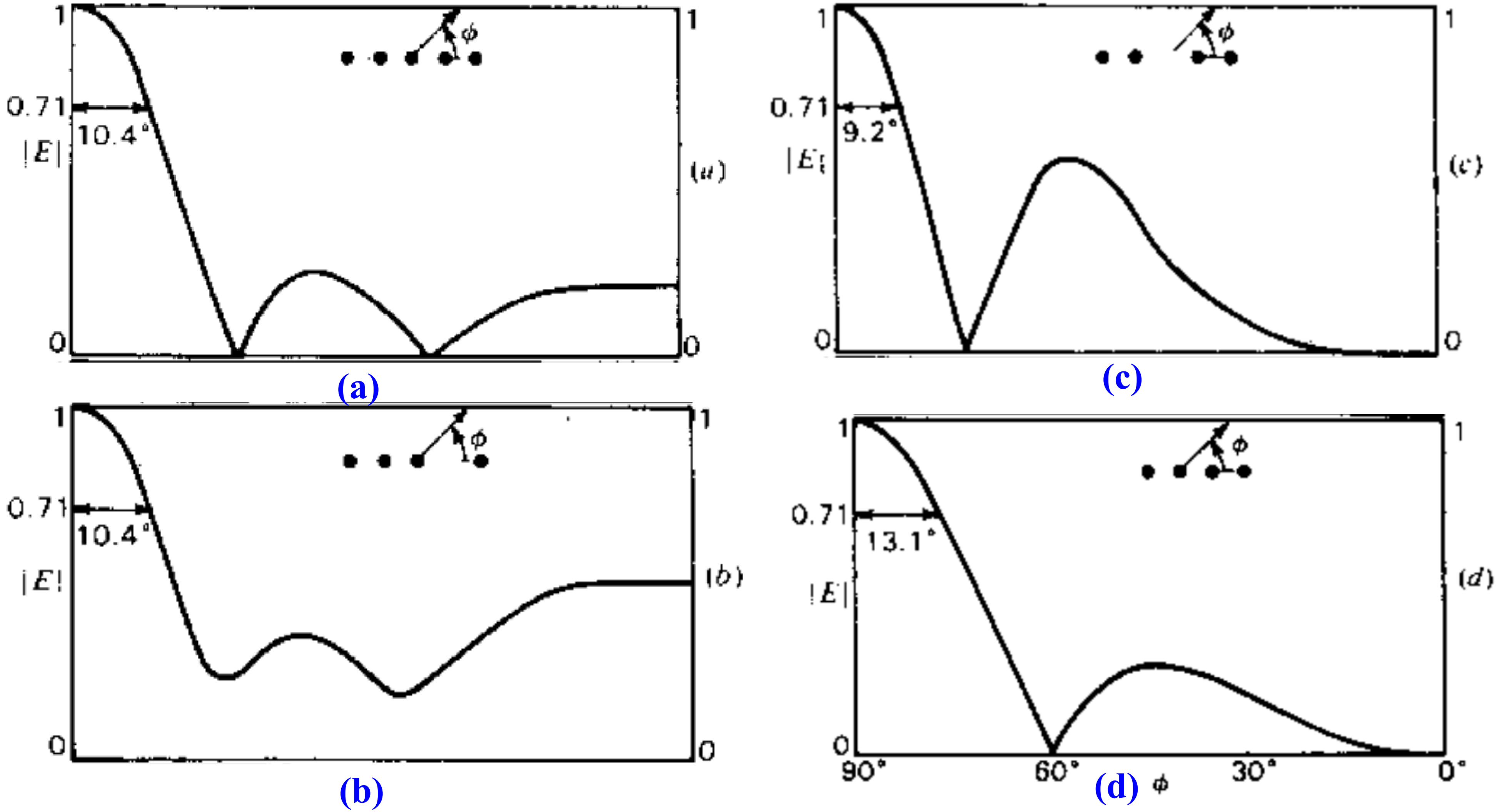
where ϕ_m is direction of max. radiation

$$\frac{d}{\lambda} < \frac{1}{\cos\phi - \cos\phi_m} \rightarrow \boxed{\frac{d}{\lambda} < \frac{1}{1 + |\cos\phi_m|}}$$

For Broadside Array: $\frac{d}{\lambda} < 1 \rightarrow d < \lambda$

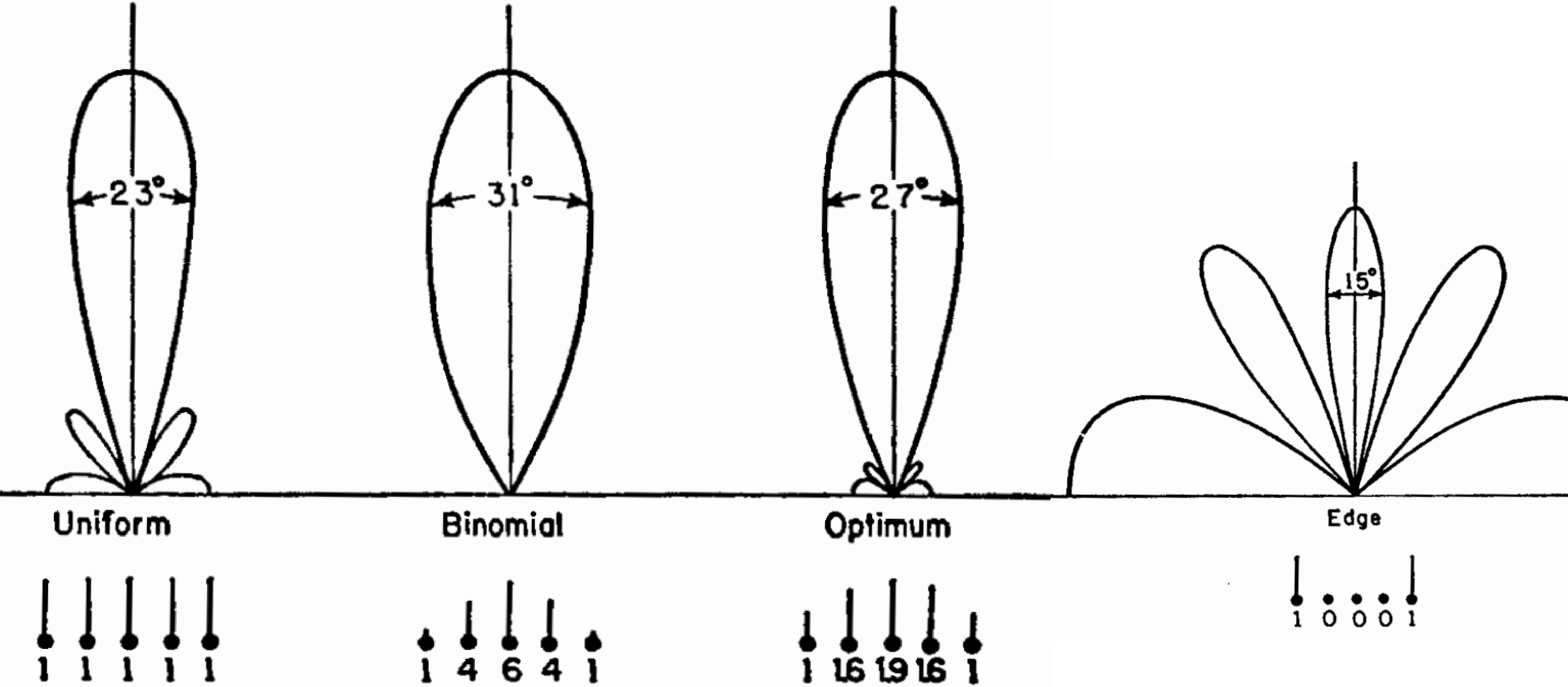
For Endfire Array: $d < \frac{\lambda}{2}$

Arrays with Missing Source



adiation Pattern of linear array of 5 isotropic point sources of equal amplitude and λ/2 spacing (a) all 5 source
(b) one source (next to the edge) OFF (c) one source (at the centre) OFF, and (d) one source (at the edge) OF

Radiation Pattern of Broadside Arrays with Non-Uniform Amplitude
(5 elements with spacing = $\lambda/2$, Total Length = 2λ)



SLL ≤ -13 dB | **No SLL** | **SLL ≤ -20 dB** | **Grating Lobes**

All 5 sources are in same phase but relative amplitudes are different

Binomial Amplitude Distribution Arrays

Binomial Amplitude Coefficients are defined by

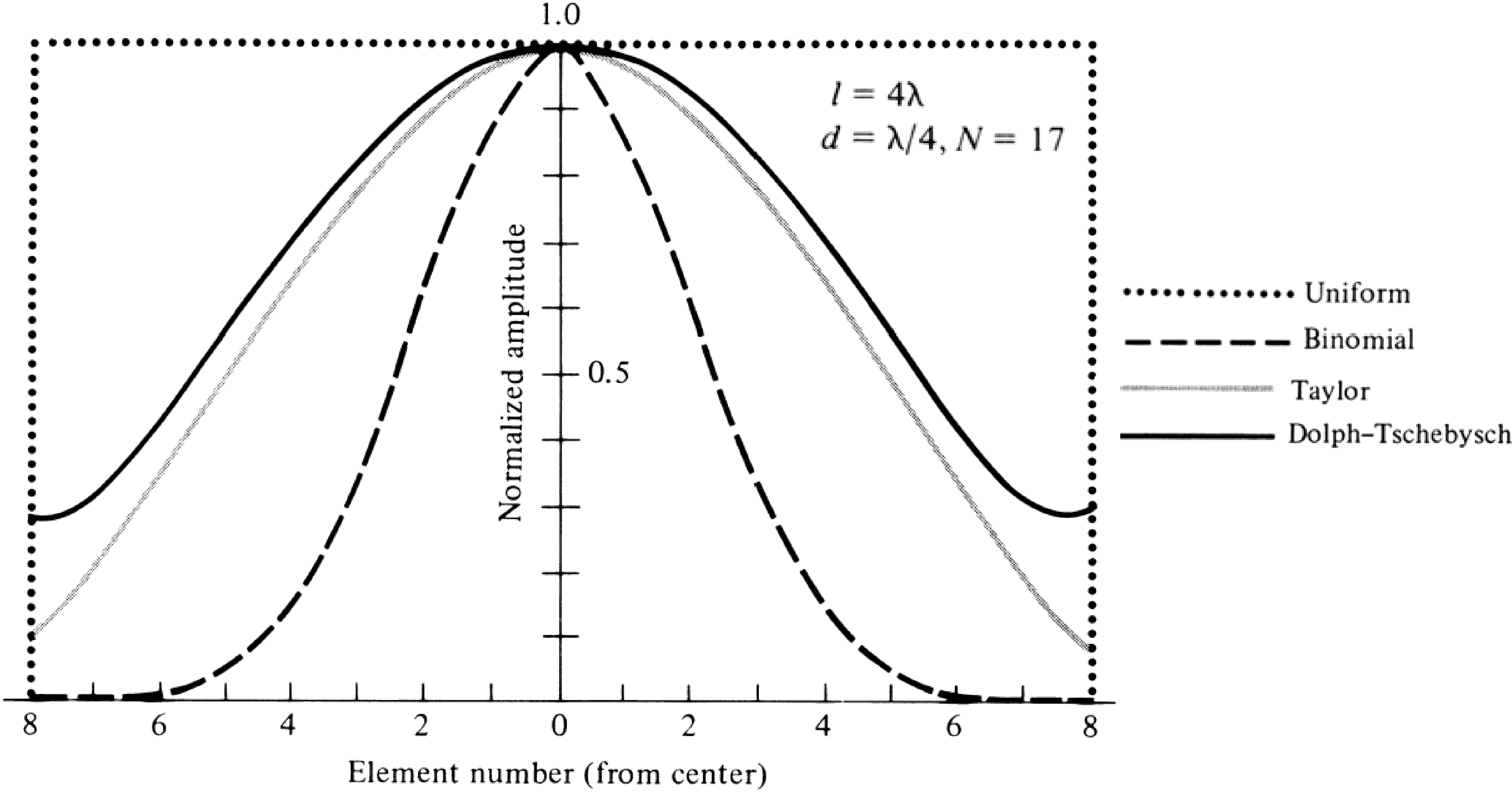
$$(1+x)^{m-1} = 1 + \frac{(m-1)x}{1!} + \frac{(m-1)(m-2)x^2}{2!} + \ldots$$

$m = 1$	1
$m = 2$	1 1
$m = 3$	1 2 1
$m = 4$	1 3 3 1
$m = 5$	1 4 6 4 1
$m = 6$	1 5 10 10 5 1

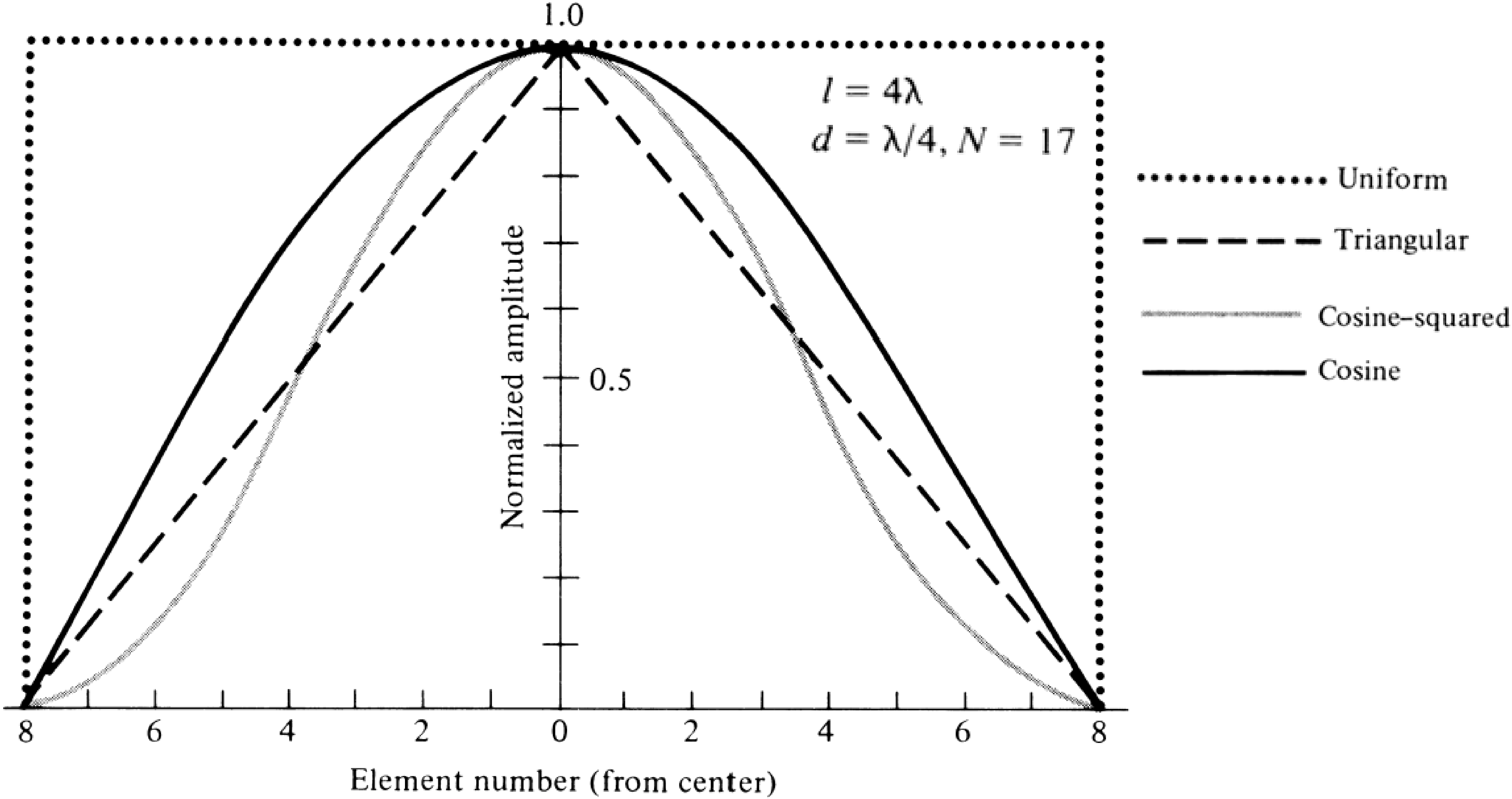
No side lobe level but broad beamwidth

→ Gain decreases (practically not used)

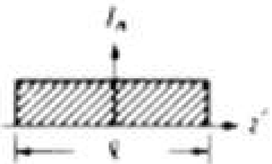
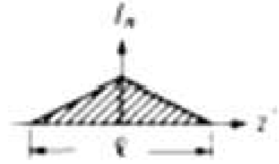
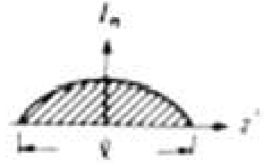
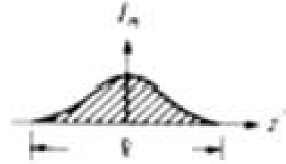
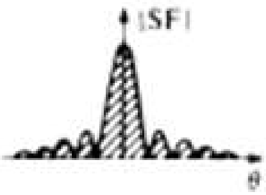
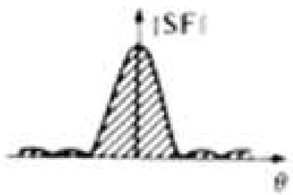
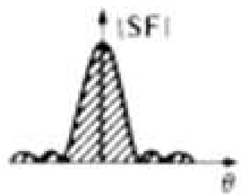
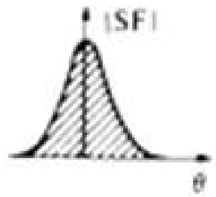
Non-Uniform Amplitude Distribution



Non-Uniform Amplitude Distribution (Contd.)



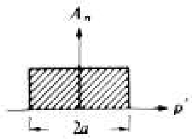
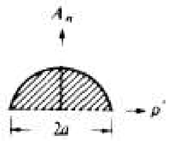
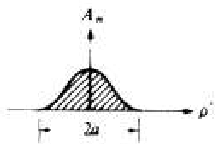
Current Distribution for Line-Sources and Linear Array

Distribution	**Uniform**	**Triangular**	**Cosine**	**Cosine-Squared**
Distribution I_n (analytical)	I_0	$I_1\left(1-\frac{2}{l}\lvert z'\rvert\right)$	$I_2\cos\left(\frac{\pi}{l}z'\right)$	$I_3\cos^2\left(\frac{\pi}{l}z'\right)$
Distribution (graphical)				
Space factor (SF) $u=\left(\frac{\pi l}{\lambda}\right)\cos\theta$	$I_0 l\frac{\sin(u)}{u}$	$I_1\frac{l}{2}\left[\frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}}\right]^2$	$I_2 l\frac{\pi}{2}\frac{\cos(u)}{(\pi/2)^2-u^2}$	$I_3\frac{l}{2}\frac{\sin(u)}{u}\left[\frac{\pi^2}{\pi^2-u^2}\right]$
Space factor $\lvert SF\rvert$				

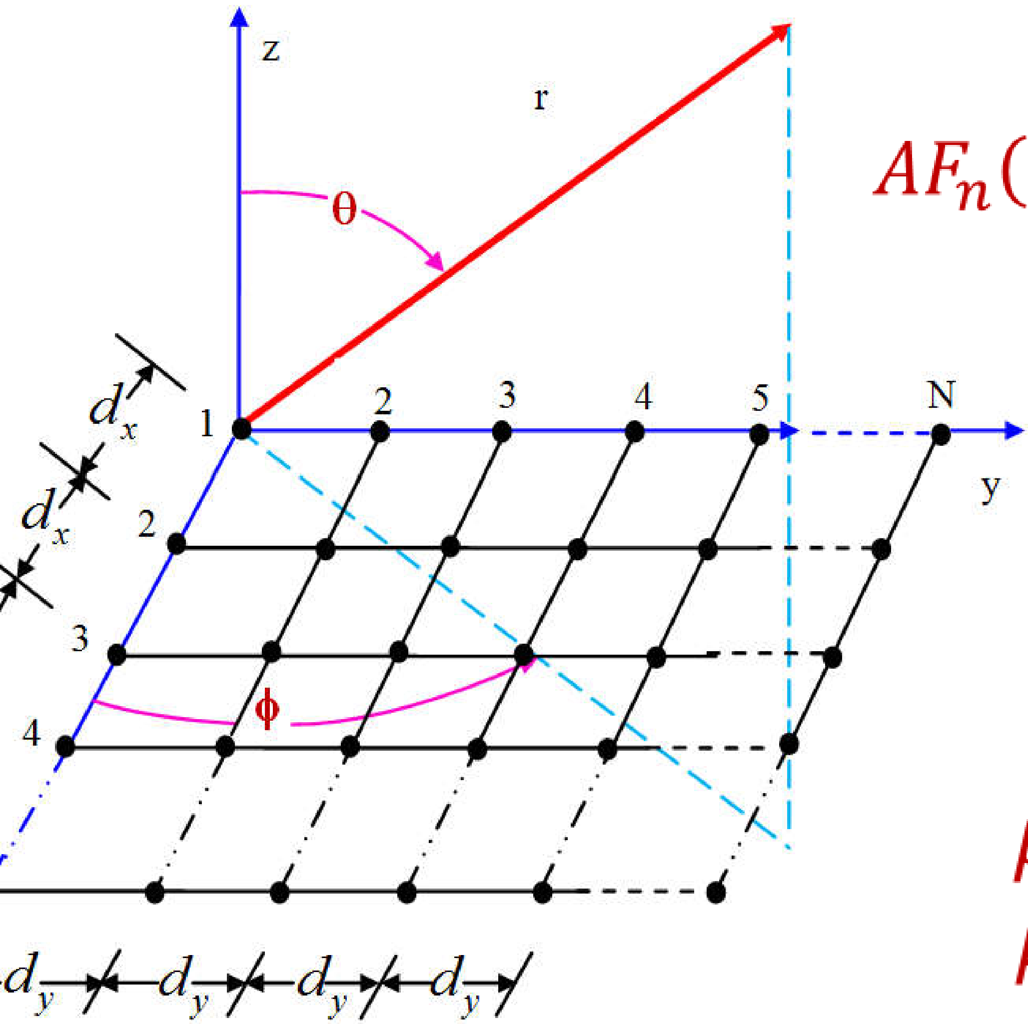
Radiation Characteristics for Line-Sources and Linear Arrays

Distribution	Uniform	Triangular	Cosine	Cosine-Squared
Half-power beamwidth (degrees) $l \gg \lambda$	$\frac{50.6}{(l/\lambda)}$	$\frac{73.4}{(l/\lambda)}$	$\frac{68.8}{(l/\lambda)}$	$\frac{83.2}{(l/\lambda)}$
First-null beamwidth (degrees) $l \gg \lambda$	$\frac{114.6}{(l/\lambda)}$	$\frac{229.2}{(l/\lambda)}$	$\frac{171.9}{(l/\lambda)}$	$\frac{229.2}{(l/\lambda)}$
First sidelobe max. (to main max.) (dB)	-13.2	-26.4	-23.2	-31.5
Directivity factor (l large)	$2\left(\frac{l}{\lambda}\right)$	$0.75\left[2\left(\frac{l}{\lambda}\right)\right]$	$0.810\left[2\left(\frac{l}{\lambda}\right)\right]$	$0.667\left[2\left(\frac{l}{\lambda}\right)\right]$

adiation Characteristics for Circular Aperture and Circular Arra

Distribution	**Uniform**	**Radial Taper**	**Radial Taper Squared**
Distribution (analytical)	$I_0\left[1-\left(\frac{\rho'}{a}\right)^2\right]^0$	$I_1\left[1-\left(\frac{\rho'}{a}\right)^2\right]^1$	$I_2\left[1-\left(\frac{\rho'}{a}\right)^2\right]^2$
Distribution (graphical)			
Space factor (SF) $u=\left(2\pi\frac{a}{\lambda}\right)\sin\theta$	$I_0 2\pi a^2\frac{J_1(u)}{u}$	$I_1 4\pi a^2\frac{J_2(u)}{u}$	$I_2 16\pi a^2\frac{J_3(u)}{u}$
Half-power beamwidth (degrees) $a \gg \lambda$	$\frac{29.2}{(a/\lambda)}$	$\frac{36.4}{(a/\lambda)}$	$\frac{42.1}{(a/\lambda)}$
First-null beamwidth (degrees) $a \gg \lambda$	$\frac{69.9}{(a/\lambda)}$	$\frac{93.4}{(a/\lambda)}$	$\frac{116.3}{(a/\lambda)}$
First sidelobe max. (to main max.) (dB)	−17.6	−24.6	−30.6
Directivity factor	$\left(\frac{2\pi a}{\lambda}\right)^2$	$0.75\left(\frac{2\pi a}{\lambda}\right)^2$	$0.56\left(\frac{2\pi a}{\lambda}\right)^2$

Rectangular Planar Array



$$AF_n(\theta,\phi) = \left\{\frac{1}{M}\frac{\sin\left(\frac{M}{2}\psi_x\right)}{\sin\left(\frac{\psi_x}{2}\right)}\right\}\left\{\frac{1}{N}\frac{\sin\left(\frac{N}{2}\psi_y\right)}{\sin\left(\frac{\psi_y}{2}\right)}\right\}$$

where, $\psi_x = kd_x\sin\theta\cos\phi + \beta_x$

$\psi_y = kd_y\sin\theta\sin\phi + \beta_y$

$\beta_x = -kd_x\sin\theta_0\cos\phi_0$ for $\psi_x = 0$

$\beta_y = -kd_y\sin\theta_0\sin\phi_0$ for $\psi_y = 0$

Rectangular Planar Array

$$\tan\phi_0 = \frac{\beta_y d_x}{\beta_x d_y}$$

and $\sin^2\theta_0 = \left(\frac{\beta_x}{kd_x}\right)^2 + \left(\frac{\beta_y}{kd_y}\right)^2$ **where k = 2π/λ**

The principal maximum(m = n = 0) and grating lobes can be located by:

$kd_x(\sin\theta\cos\phi - \sin\theta_0\cos\phi_0) = \pm 2m\pi$ **m = 0, 1, 2,….**

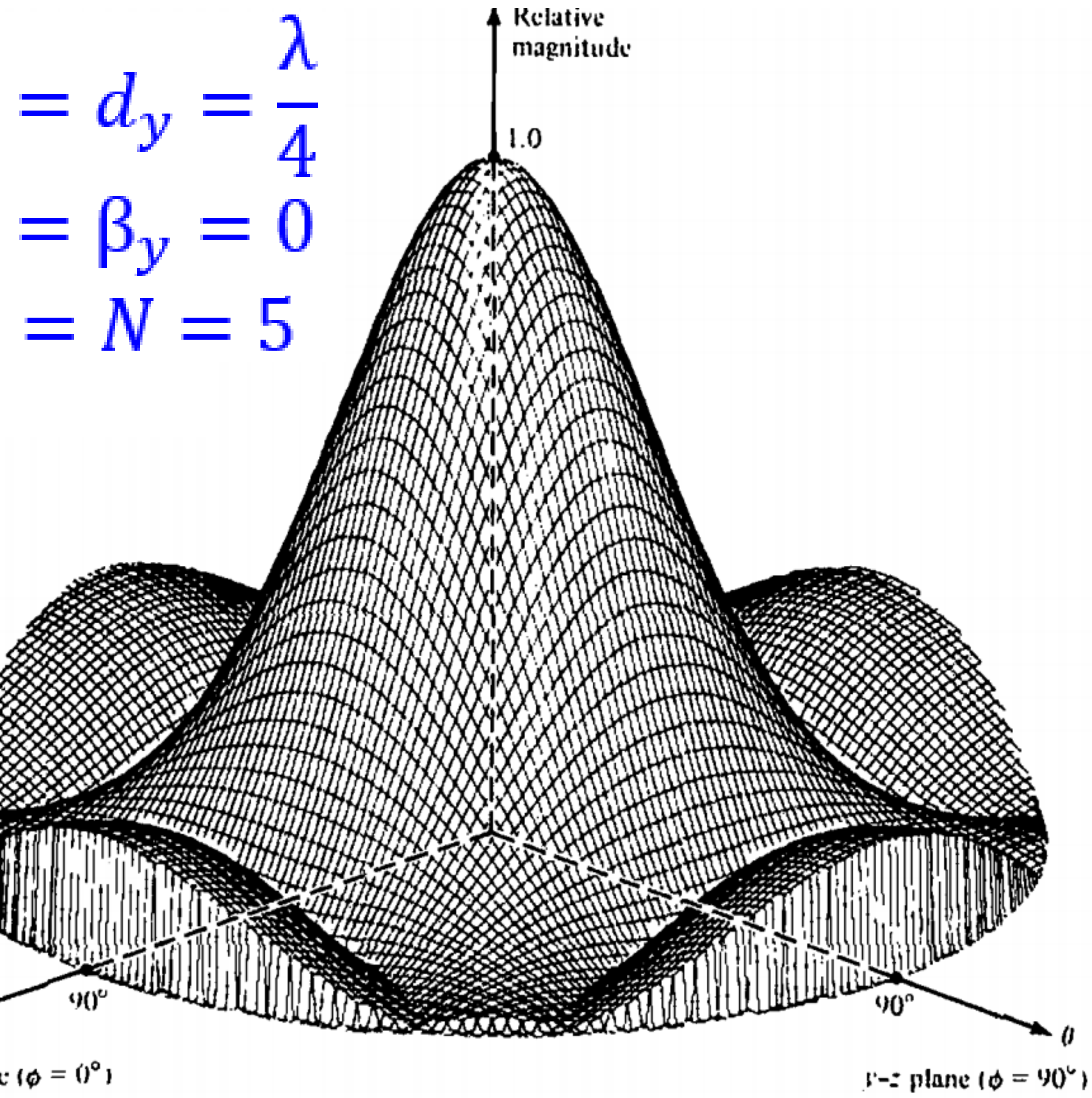
$kd_y(\sin\theta\sin\phi - \sin\theta_0\sin\phi_0) = \pm 2n\pi$ **n = 0, 1, 2,….**

Radiation Pattern of 5x5 Planar Array

$= d_y = \frac{\lambda}{4}$

$= \beta_y = 0$

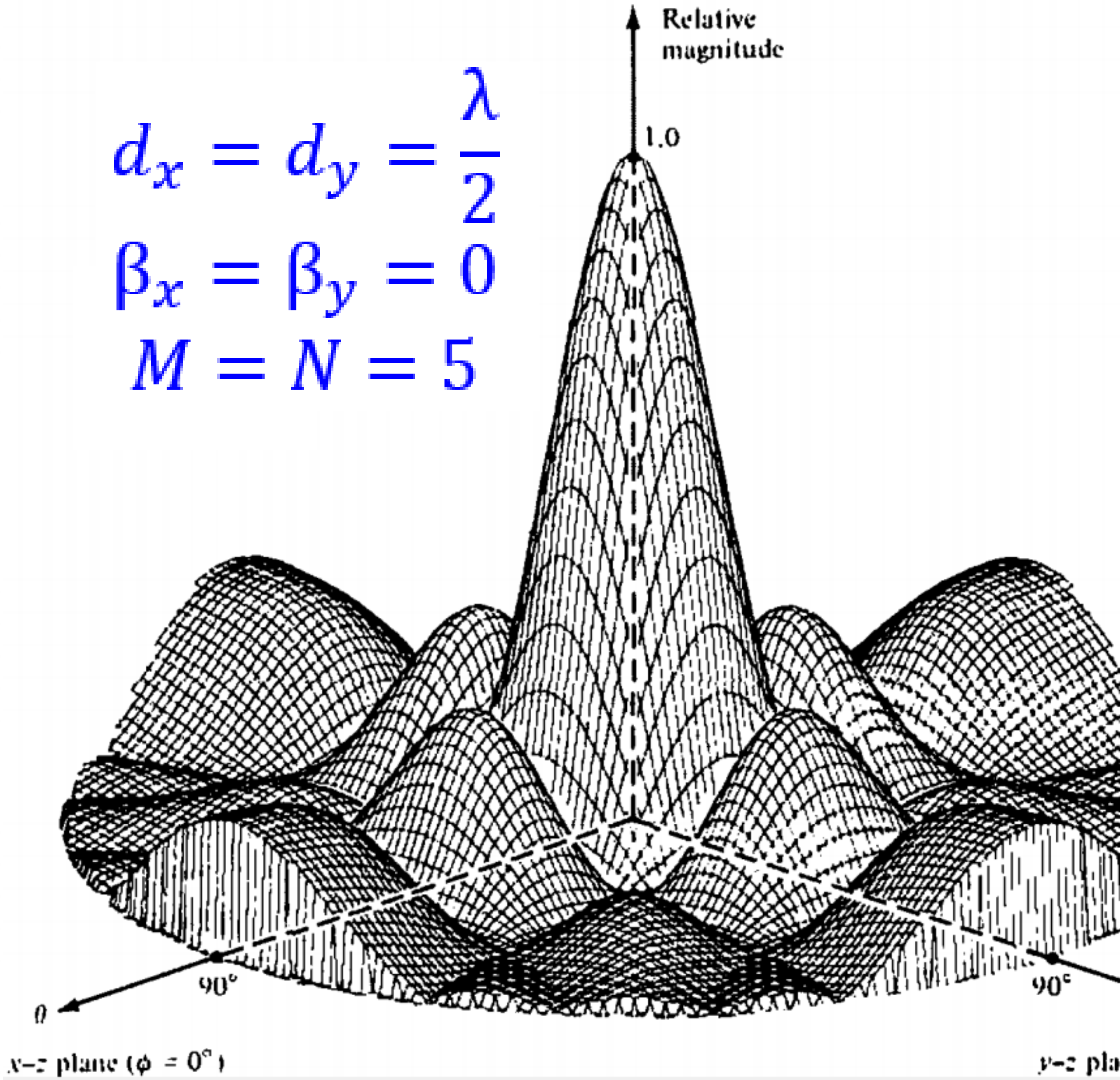
$= N = 5$



$d_x = d_y = \frac{\lambda}{2}$

$\beta_x = \beta_y = 0$

$M = N = 5$



Directivity of Planar Array

Directivity of Rectangular Array

$$\boxed{D = \pi D_x D_y \cos\theta_0}$$

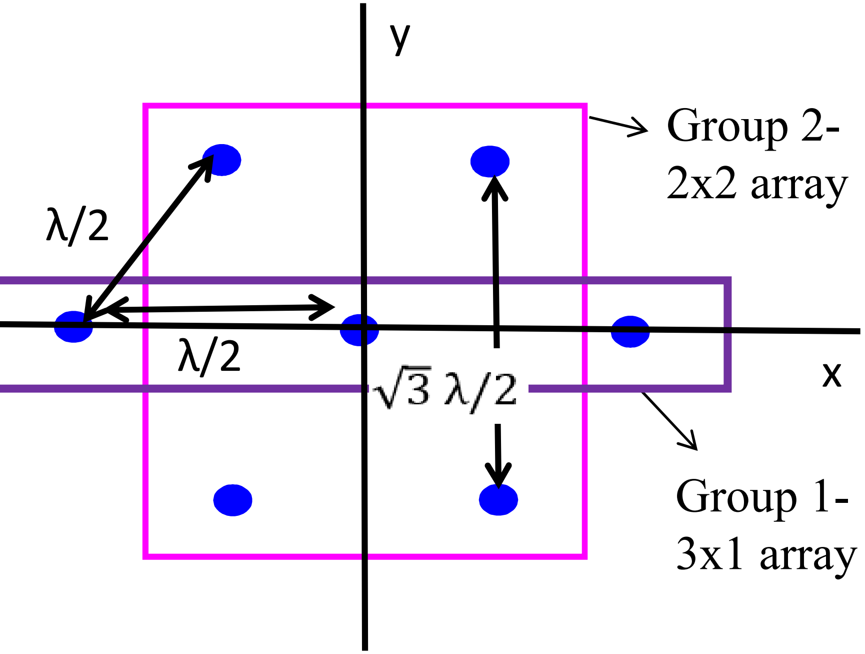
For Broadside Array: $\boxed{D = \pi D_x D_y}$

Directivity of Circular Array

$$G = \frac{4\pi A}{\lambda^2}, \qquad A = \pi a^2 \qquad \boxed{D = \left(\frac{2\pi a}{\lambda}\right)^2}$$

Hexagonal Array – 7 Elements

Example: Calculate the array factor of a 7-elements hexagonal array (2 elements in first and third rows, 3 elements in the second row).



$$AF_n(\theta,\phi) = \left\{\frac{1}{M}\frac{\sin\left(\frac{M}{2}\psi_x\right)}{\sin\left(\frac{\psi_x}{2}\right)}\right\}\left\{\frac{1}{N}\frac{\sin\left(\frac{N}{2}\psi_y\right)}{\sin\left(\frac{\psi_y}{2}\right)}\right\}$$

$$\psi_x = \frac{2\pi}{\lambda} d_x \sin\theta\cos\phi$$

$$\psi_y = \frac{2\pi}{\lambda} d_y \sin\theta\sin\phi$$

Total Array Factor = Array Factor of (Group 1 + Group 2)

AF of Hexagonal Array – 7 Elements

Array factor of Group 2: M = 3, N = 1

$$AF_1(\theta,\phi) = \left\{ \frac{1}{3} \frac{\sin\left(\frac{3}{2}\psi_x\right)}{\sin\left(\frac{\psi_x}{2}\right)} \right\} \qquad d_x = \frac{\lambda}{2}$$

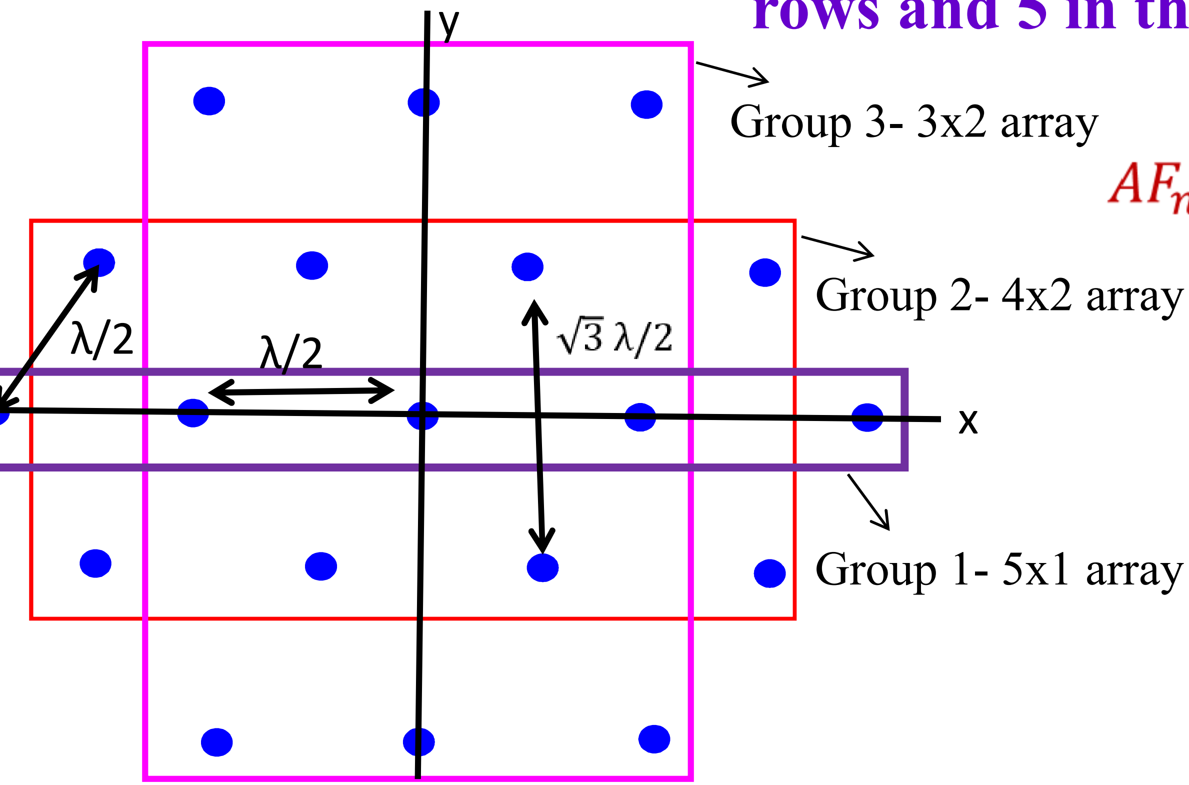
Array factor of Group 2: M = 2, N = 2

$$AF_2(\theta,\phi) = \left\{ \frac{1}{2} \frac{\sin\left(\frac{2}{2}\psi_x\right)}{\sin\left(\frac{\psi_x}{2}\right)} \right\} \left\{ \frac{1}{2} \frac{\sin\left(\frac{2}{2}\psi_y\right)}{\sin\left(\frac{\psi_y}{2}\right)} \right\} \qquad d_x = \frac{\lambda}{2} \quad d_y = \sqrt{3}\frac{\lambda}{2}$$

Total Array Factor = $AF_1 + AF_2$

Hexagonal Array – 19 Elements

Example: Calculate the array factor of a 19-elements hexagonal array (3 elements in first and fifth rows, 4 elements in the second and fourth rows and 5 in the third row)



$$AF_n(\theta,\phi) = \left\{ \frac{1}{M} \frac{\sin\left(\frac{M}{2}\psi_x\right)}{\sin\left(\frac{\psi_x}{2}\right)} \right\} \left\{ \frac{1}{N} \frac{\sin\left(\frac{N}{2}\psi_y\right)}{\sin\left(\frac{\psi_y}{2}\right)} \right\}$$

$$\psi_x = \frac{2\pi}{\lambda} d_x \sin\theta\cos\phi$$

$$\psi_y = \frac{2\pi}{\lambda} d_y \sin\theta\sin\phi$$

Total Array Factor = Array Factor of (Group 1 + Group 2 + Group 3)

AF of Hexagonal Array – 19 Elements

Array Factor of Group 1: M=5, N=1

$$d_x = \frac{\lambda}{2}$$

$$AF_1(\theta,\phi) = \left\{ \frac{1}{5} \frac{\sin\left(\frac{5}{2}.\frac{2\pi}{\lambda}.d_x \sin\theta\cos\phi\right)}{\sin\left(\frac{\psi_x}{2}\right)} \right\}$$

Array Factor of Group 2: M=4, N=2

$$d_x = \frac{\lambda}{2} \quad d_y = \sqrt{3}\frac{\lambda}{2}$$

$$AF_2(\theta,\phi) = \left\{ \frac{1}{4} \frac{\sin\left(\frac{4}{2}.\frac{2\pi}{\lambda}.d_x \sin\theta\cos\phi\right)}{\sin\left(\frac{\psi_x}{2}\right)} \right\} \left\{ \frac{1}{2} \frac{\sin\left(\frac{2}{2}\frac{2\pi}{\lambda}.d_y \sin\theta\sin\phi\right)}{\sin\left(\frac{\psi_y}{2}\right)} \right\}$$

AF of Hexagonal Array – 19 Elements (Contd.)

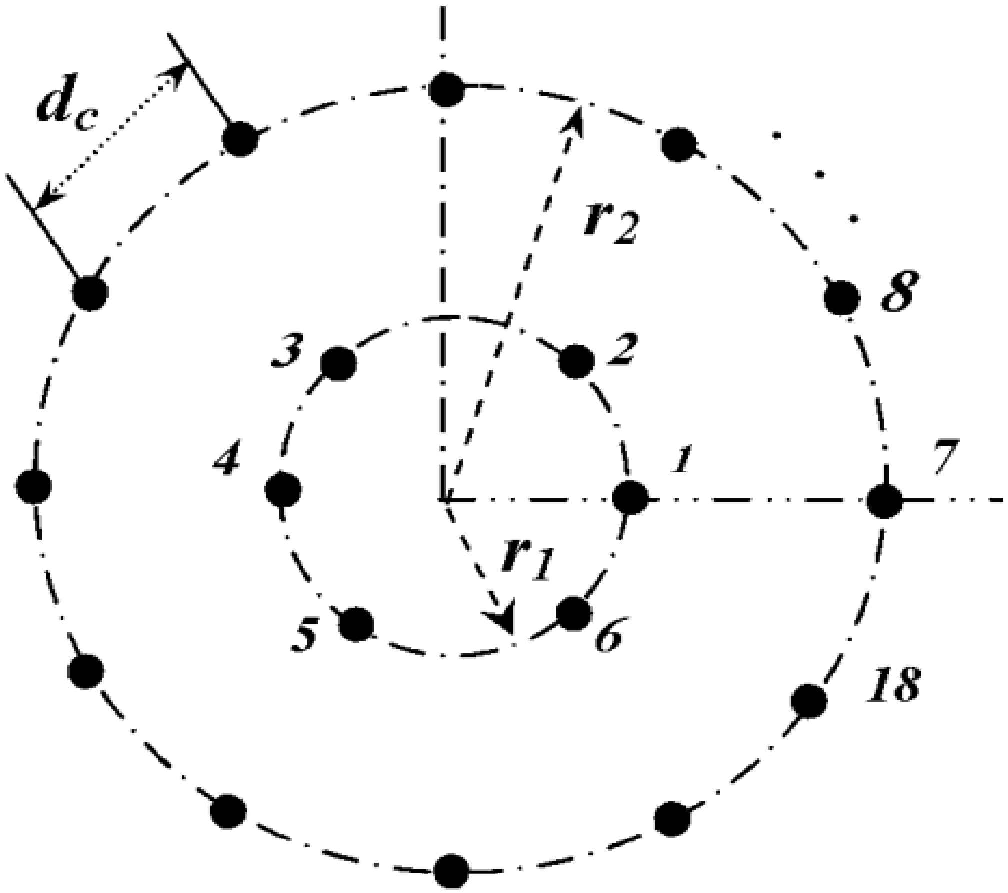
Array Factor of Group 3: M=3, N=2 $d_x = \frac{\lambda}{2}$ $d_y = \sqrt{3}\,\lambda$

$$AF_3(\theta,\phi) = \left\{ \frac{1}{3} \frac{\sin\left(\frac{3}{2}.\frac{2\pi}{\lambda}.d_x \sin\theta\cos\phi\right)}{\sin\left(\frac{\psi_x}{2}\right)} \right\} \left\{ \frac{1}{2} \frac{\sin\left(\frac{2}{2}\frac{2\pi}{\lambda}.d_y \sin\theta\sin\phi\right)}{\sin\left(\frac{\psi_y}{2}\right)} \right\}$$

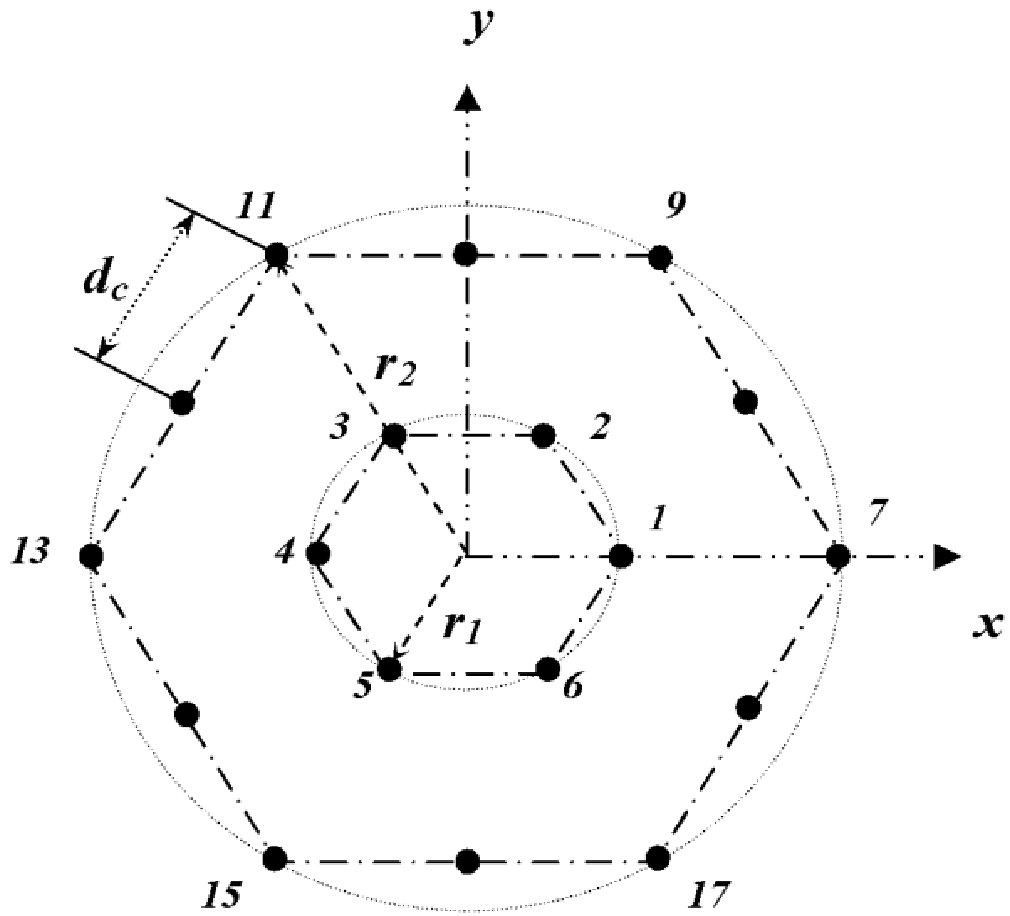
Total Array Factor:

$$AF_{\text{Total}} = AF_1 + AF_2 + AF_3$$

Circular Vs Hexagonal Array



Planar Circular Array



Planar Hexagonal Array