A brief guide to Randomized Benchmarking

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***Abstract*—In this review paper we discuss about Randomized Benchmarking, one of the important techniques to characterize noise present in the quantum devices. We also study briefly, some variants of RB protocol used for characterizing certain type of noise. We then study construction of Clifford group for single qubit by 2 methods. For multi-qubit Clifford we discuss the method of randomly generating stabilizer table and decomposing it to quantum circuit by two approaches. We implemented these approaches in QIBO which is a classical simulator framework for Quantum Computing.**

I. Introduction

For any quantum computation we perform on a Quantum Information Processor(QIP), we desire to have a quantification on the level of coherent control we have over the quantum system. We need to have a benchmark i.e. a test which could measure the performance of the system. Benchmarks have also been used to characterize the high performance classical computers. Similarly for quantum devices we would like to know if the fundamental operations performed through quantum gates are implemented correctly, if there are any new errors introduced when multiple operations are put together and how well will the quantum device implement new circuits. When dealing with quantum systems there are many types of error which may arise when performing any set of quantum operations. Noise could be induced from sources such as-

- Poor Calibrations of quantum gates: This could lead to systematic errors which are coherent errors, i.e. could be represented as a unitary operator. E.g. in case of superconducting quantum circuits, gates are implemented as tuned up microwave pulses with very specific shapes and parameters. If these parameters are a little offset, then rotation applied to the qubit would be slightly under-rotated or over-rotated, or the axis of rotation could be little shifted.
- State Preparation and Measurement Errors(SPAM): These are errors associated while preparing the initial quantum state and measurement errors associated with measurement of qubits. E.g. in superconducting quantum circuits, the qubits are brought to $|0\rangle^{\otimes n}$ state by cooling the superconducting device to very low temperatures. But due to ambient thermal noise, there would be some probability that the qubits aren't in $|0\rangle^{\otimes n}$ state.
- Decoherence: This is an incoherent error which is caused by interaction of qubit with its environment. In terms of Bloch sphere representation , this causes qubit to leave the Bloch sphere surface. This could be caused by dephasing, damping, longitudinal and transverse relaxation.
- Leakage: Quantum devices in reality are not true qubits i.e. they may have more than two computational levels. Thus we desire that the qubits be bounded in the $|0\rangle$,$|1\rangle$ subspace.
- Crosstalk: This is the coupling and interaction between qubits. If two qubits are coupled then applying a gate on one qubit could also have some effect on the other one.

These processes could reduce the fidelity of the quantum system and limit the performance. In order to characterize and quantify the noise present in the system, techniques such as Quantum Process Tomography(QPT) have been used. However, QPT suffers from two shortcomings: first being the assumption that the state preparation and measurement errors are much lower than the process which is being characterized; and second is that the number of experiments required for characterization grows exponential with the number of qubits. These two shortcomings are overcome by the technique Randomized Benchmarking. Randomized Benchmarking is the process of using randomization methods for benchmarking quantum gates.

II. Randomized Benchmarking Protocol

In this protocol, we measure the exponential decay rate of the fidelity as a function of sequence length for some random sequences of gates. This measured decay rate gives an estimate of the average error probability per gate. The gates for the sequences are chosen at random from the Clifford group which is a unitary-2 design. Clifford group is a group of quantum gates which have a generating set {H,S,CNOT}. The n qubit RB protocol consists of the following steps[1]:

1. Generate a sequence of $m+1$ quantum operators with the first m operators chosen uniformly at random from the n qubit Clifford group and the final operation chosen such that the net sequence(without considering errors) is an identity operation. Due to the property of a group the composition of the whole sequence and its inverse is also an element of the group .Every gate would be having an associated error operator which we would be considering as both gate and time independent. Let that error operator be Λ. Thus each operator C_i would have error Λ associated with it. Thus the entire sequence can be modelled by the operation

$$S = \prod_{i=1}^{i=m+1} \Lambda \circ C_i$$

2. We start with n qubits in the initial state $|0\rangle^{\otimes n}$. Then we apply the randomly generated sequence modelled by the

operation above. We run the circuit nshots number of times and as the whole sequence is an identity operation, the population of the $|0\rangle^{\otimes n}$ state should be ideally nshots for noiseless case. As we are modelling using the noise Λ, the population would be less than nshots. Mathematically, we would be calculating the survival probability $Tr[|0\rangle\langle 0|S(|0\rangle\langle 0|)]$. Here S is the operation modelled above acting on intial state $|0\rangle$.

3. We choose K_m randomly generated sequences and calculate the survival probability for each sequence and average over all K_m realizations to get average fidelity for that particular sequence length m. When an error process is averaged over the uniform space of unitaries, called the Haar measure, the result is a depolarizing channel. As Clifford group is a unitary-2 design, averaging an error process over the entire Clifford group(also called twirl) is equivalent to averaging the error over the Haar measure. A depolarizing error has a parameter α, such that with probability α the state remains same as before and with probability 1-α becomes maximally mixed state. Thus if ρ_i is the initial density matrix, after a depolarizing error it becomes,

$$\rho_f = \alpha\rho_i + \frac{(1-\alpha)}{2^n} * I$$

Now applying a sequence of m gates would have an effect, with probability α^m the state remains unchanged and with probability $1-\alpha^m$ changes to maximally mixed state thus giving,

$$\rho_f = \alpha^m\rho_i + \frac{(1-\alpha^m)}{2^n} * I$$

As we started with the initial state $|0\rangle^{\otimes n}$, we get the survival probability as,

$$\alpha^m + \frac{(1-\alpha^m)}{2^n} = \frac{2^n-1}{2^n}\alpha^m + \frac{1}{2^n} = A\alpha^m + B$$

The above exponential decay relation of survival probability is valid for any sequence of unitary gates so long we have a depolarizing channel. Specifically for Clifford gate this is true for any error which is a Completely Positive Trace Preserving (CPTP) map because of the twirl property discussed above.

4. Repeat steps 1-3 for different sequence length m in order to obtain the relation of average fidelity with the sequence length:

$$F(m) = A\alpha^m + B$$

Here α is the exponential decay rate. Coefficients A and B absorb the state preparation and measurement errors as well as the edge effect from the error on the final gate. The average error rate can be calculated by

$$r = (1-\alpha) - \frac{(1-\alpha^m)}{2^n} = \frac{(d-1)(1-\alpha)}{d}$$

where $d = 2^n$. All the calculations above is for the zeroth order model where the noise is gate and time independent. For higher order models, a perturbative expansion[2] is made for the error operators and we would get more than one exponentially decaying curves for the average fidelity relation with sequence length.

A. Variants of Randomized Benchmarking

The previous algorithm is the Standard Randomized Benchmarking protocol which as we saw gives an estimate of the average error rate per Clifford gate. There have been attempts to modify this protocol in order to obtain a more deeper quantification on the errors. The variants of Standard Randomized Benchmarking are-

- Interleaved Randomized Benchmarking[6]: This protocol is used to estimate the average error of an individual gate. In this setup, we run two experiments.
  - At first the standard RB experiment is ran for Clifford elements selected uniformly at random.
  - In the second step, the Clifford element 'C' for which the error rate is to be estimated is interleaved between the same set of gates used in first step.

  These 2 steps are repeated for different Clifford sequence lengths m to obtain exponential decay parameter p and p_C for steps 1 and 2 respectively. With these the average error rate for Clifford C is estimated as-

$$r_C = \frac{(d-1)(1-p_C/p)}{d}$$

- Purity Randomized Benchmarking[7]: As we saw in the last section that errors can be coherent as well as incoherent, Purity RB is devised to quantify how coherent the errors are. In the RB protocol we started with initial state $|0\rangle^{\otimes n}$ which is a pure quantum state and thus $Tr(\rho^2) = 1$. For coherent errors this trace would be preserved. Likewise if the errors are incoherent, $Tr(\rho^2)$ would decay exponentially with the Clifford sequence length m. The steps for the algorithm are-
  - Generate the random RB sequences
  - To calculate the purity $Tr(\rho^2)$, for every RB sequence run 3^n experiments for measuring each qubit in X, Y and Z bases. For each of these calculate expectation value of 2^n Pauli matrices. The purity is thus calculated as-

$$Tr(\rho^2) = \frac{1}{d}\sum_{k=1}^{k=4^n} < P_k >^2$$

  - The last step is to fit $Tr(\rho^2)$ to an exponentially decaying curve $Ap^m + B$, where p is a measure of how unitary the error is. For coherent errors $p = 1$, meaning the $Tr(\rho^2)$ doesn't vary with sequence length.
- Leakage Randomized Benchmarking[8]: This method as the name suggest is devised to quantify and characterize the leakage errors. Let us define the computational subspace χ_1 to be of dimension d_1 where number of qubits $n = log_2(d_1)$ and the leakage subspace χ_2 to be of dimension d_2. Therefore the total state space has a dimension $d = d_1 + d_2$. Also, let $\mathbb{1}_1$ and $\mathbb{1}_2$ be the projectors onto the subspaces χ_1 and χ_2. Now, the leakage error which transfers population to leakage subspace is called gate leakage and similarly error which

cause population transfer to computational subspace is called gate seepage. In order to quantify these metrics we define two quantities leakage rate L_1 and seepage rate L_2. The protocol to find these quantities is as follows:

- Perform the standard RB protocol, with Clifford gates chosen uniformly at random. Alongwith the survival probability p_0 of $|0\rangle^{\otimes n}$, we calculate probabilities of other remaining computational bases states p_i for $j = 1$ to $d_1 - 1$. Thus probability of being in the subspace χ_1 can be written as

$$p_{\mathbb{1}_1} = \sum_{i=0}^{d_1-1} p_i$$

  For non-leakage errors the above sum would be 1.
- We repeat the above procedure for different lengths m and fit $p_{\mathbb{1}_1}(m)$ to the decay model $A+B\lambda^m$. Once these parameters are estimated, average leakage rate L_1 and average seepage rate L_2 can be estimated as

$$L_1 = (1 - A)(1 - \lambda)$$

$$L_2 = A(1 - \lambda)$$

- Simultaneous Randomized Benchmarking[9]: In this method, the RB protocol is ran simultaneously on subsets of qubits and is used to characterize the amount of addressability between subsystems. Consider quantum system C with subsystems A and B between which we want to find the addressability. The protocol follows 3 steps-
  - Implement RB on subsystem A alone and find the decay parameter and error rate.
  - Implement RB on subsystem B alone and find the decay parameter and error rate.
  - Implement RB on both subsystems simultaneously and obtain the decay parameter and error rate.

  We can thus have a metric to quantify increase in error rate in Simultaneous case over the individual cases to quantify addressability errors.

III. Single Qubit RB

In order to run the RB protocol we need to generate the Clifford group. We start by constructing the Clifford group for a single qubit. We know that, Clifford group is a normalizer of the Pauli group i.e. for all elements $C \in C_n$, $CP_nC^\dagger = P_n$ where P_n is the Pauli group.

Here we will be ignoring the global phases on the Clifford elements and on the Pauli operators and thus would be considering $C_n/U(1)$ and $P_n/U(1)$. As Cliffords are normalizers of Pauli group, the Ad-action of a Clifford on the the I always maps to I. For the Pauli operator Z which has a spectral decomposition as $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, the Ad-action of a Clifford on the Z gate should map it to either X, Y or Z Pauli gate and thus maps the eigenstate $|0\rangle$ to any of the six axial states (6 eigenstates of the 3 pauli matrices) and the eigenstate $|1\rangle$ is mapped to the corresponding orthognal eigenstate. For each of these 6 possibilities, there are four possible orientations for the eigenstate $|+\rangle$ of the X pauli matrix as the orthogonality relation shoud be maintained. And similarly the $|-\rangle$ state would be mapped to the corresponding orthogonal state of the state to which $|+\rangle$ is mapped. Thus, there are 24 possible rotations which would preserve the Pauli group (ignoring the global phases on the elements). Considering a generating set $\{X, Y, X_{\frac{\pi}{2}}, Y_{\frac{\pi}{2}}, X_{\frac{-\pi}{2}}, Y_{\frac{-\pi}{2}}\}$ the 24 Clifford elements can be written as-

Map	Decomposition
$\|0\rangle \to \|0\rangle$	I
	XY
	$X_{\frac{-\pi}{2}} Y_{\frac{\pi}{2}} X_{\frac{\pi}{2}}$
	$X_{\frac{-\pi}{2}} Y_{\frac{-\pi}{2}} X_{\frac{\pi}{2}}$
$\|0\rangle \to \|1\rangle$	X
	Y
	$X_{\frac{\pi}{2}} Y_{\frac{\pi}{2}} X_{\frac{\pi}{2}}$
	$X_{\frac{-\pi}{2}} Y_{\frac{\pi}{2}} X_{\frac{-\pi}{2}}$
$\|0\rangle \to \|+\rangle$	$Y_{\frac{\pi}{2}}$
	$Y_{\frac{-\pi}{2}} X$
	$X_{\frac{\pi}{2}} Y_{\frac{\pi}{2}}$
	$X_{\frac{-\pi}{2}} Y_{\frac{\pi}{2}}$
$\|0\rangle \to \|-\rangle$	$Y_{\frac{-\pi}{2}}$
	$Y_{\frac{\pi}{2}} X$
	$X_{\frac{\pi}{2}} Y_{\frac{-\pi}{2}}$
	$X_{\frac{-\pi}{2}} Y_{\frac{-\pi}{2}}$
$\|0\rangle \to \frac{\|0\rangle + i\|1\rangle}{\sqrt{2}}$	$X_{\frac{\pi}{2}}$
	$Y X_{\frac{\pi}{2}}$
	$Y_{\frac{\pi}{2}} X_{\frac{\pi}{2}}$
	$Y_{\frac{-\pi}{2}} X_{\frac{\pi}{2}}$
$\|0\rangle \to \frac{\|0\rangle - i\|1\rangle}{\sqrt{2}}$	$X_{\frac{-\pi}{2}}$
	$Y X_{\frac{-\pi}{2}}$
	$Y_{\frac{\pi}{2}} X_{\frac{-\pi}{2}}$
	$Y_{\frac{-\pi}{2}} X_{\frac{-\pi}{2}}$

IV. Generalization to Multi Qubit RB

The approach we used earlier for writing Clifford elements by considering rotations on eigenstates of Pauli operators is only feasible for single qubit case. The number of Clifford elements for $n = 1$, 2 and 3 are 24, 11,520 and 92,897,280 respectively. Thus we can see that generating the entire Clifford group for $n > 1$ isn't feasible. Moreover for RB we just need a way to randomly choose a Clifford element rather than generating the entire group. This can done by randomly generating a stabilizer table corresponding to a certain Clifford element and then decompose the stabilizer table to corresponding quantum circuit.

A. Stabilizer Tables

We define the group P_n of n qubit Pauli operators to consist of all tensor products of n Pauli matrices including multiplicative factors ± 1 or $\pm i$ and thus contains a total of 4^{n+1} elements. Consider a pure quantum state $|\psi\rangle$. It is stabilized by unitary U if $U|\psi\rangle = |\psi\rangle$. The single qubit pauli matrices and the states which they stabilize are- $Z : |0\rangle$, $-Z : |1\rangle$, $X : |+\rangle$, $-X : |-\rangle$, $Y : |0\rangle + i|1\rangle$ and $-Y : |0\rangle - i|1\rangle$. The I matrix stabilizes all states whereas $-I$ stabilizes no state. A stabilizer state is a quantum state which can be obtained from $|0\rangle^{\otimes n}$ by applying CNOTs, Hadamards and Phase gates only i.e. can be obtained by application of a Clifford operator to the state $|0\rangle^{\otimes n}$. A given n-qubit stabilizer state $|\psi\rangle$ is stabilized by exactly 2^n Pauli operators. The group of Pauli operators that stabilize the state thus has a generating set of size at most $\log_2 2^n = n$. We now build a Stabilizer Tableau using the n stabilizer generators and n destabilizer generators which alongwith stabilizer generators form the complete Pauli group upto a global phase. Thus a stabilizer state can be represented by such a table given below-[3]

$$\left(\begin{array}{ccc|ccc|c} x_{11} & \cdots & x_{1n} & z_{11} & \cdots & z_{1n} & r_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n1} & \cdots & x_{nn} & z_{n1} & \cdots & z_{nn} & r_n \\ \hline x_{(n+1)1} & \cdots & x_{(n+1)n} & z_{(n+1)1} & \cdots & z_{(n+1)n} & r_{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(2n)1} & \cdots & x_{(2n)n} & z_{(2n)1} & \cdots & z_{(2n)n} & r_{2n} \end{array}\right)$$

Here rows 1 to n represent the destabilizer generators and rows n to $2n$ represent the stabilizer generators. Consider i^{th} row, bits x_{ij} and z_{ij} determine the j^{th} Pauli matrix: 00 means I, 01 means Z, 10 means X and 11 means Y. Finally r_i denotes the phase which is 1 if the Pauli has negative phase and 0 if positive phase. One possible stabilizer tableau of $|00\rangle$ is given below which has stabilizer generators YI and IY and is called as the standard initial tableau.

$$\left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Consider a stabilizer state $|\psi\rangle$ having a stabilizer table Stab($|\psi\rangle$). Let g be one of the stabilizer generators that stabilizes the state. Now let us apply a unitary U on the state $|\psi\rangle$

$$U|\psi\rangle = Ug|\psi\rangle = UgU^\dagger U|\psi\rangle$$

Thus we can see that the state $U|\psi\rangle$ is stabilized by generators $UgU^\dagger$. Using this observation we create updation rules of stabilizer tables on application of a unitary U. For all $i \in \{1, 2, ..., 2n\}$:

CNOT from control a to target b: Set $r_i := r_i \oplus x_{ia} z_{ib} (x_{ia} \oplus z_{ib} \oplus 1)$, $x_{ib} := x_{ib} \oplus x_{ia}$ and $z_{ia} := z_{ia} \oplus z_{ib}$.

Hadamard on qubit a: Set $r_i := r_i \oplus x_{ia} z_{ia}$ and swap x_{ia} and z_{ia}.

S on qubit a: Set $r_i := r_i \oplus x_{ia} z_{ia}$ and $z_{ia} := z_{ia} \oplus x_{ia}$.

$S^\dagger$ on qubit a: Set $r_i := r_i \oplus x_{ia}(\tilde{z}_{ia})$ and $z_{ia} := z_{ia} \oplus x_{ia}$.

X on qubit a: Set $r_i := r_i \oplus z_{ia}$

Z on qubit a: Set $r_i := r_i \oplus x_{ia}$

Y on qubit a: Set $r_i := r_i \oplus x_{ia} \oplus z_{ia}$

B. Generation of a random Stabilizer Table

Now that we have defined the formalism of a stabilizer table, let us see how we can generate stabilizer tables corresponding to a Clifford chosen uniformly at random. We define symmetric group over n qubits S_n having generating set $\{SWAP\}$ where for $S \in S_n$, $S(j) = i$ if S maps qubit j to qubit i. We also define an n qubit Borel group B_n which has a generating set $\{X, CZ, P, CNOT^{\downarrow}\}$. Here $CNOT^{\downarrow}$ implies a $CNOT$ gate from control c to target t where $c < t$. For an element $F(O, \Gamma, \Delta) \in B_n$, Γ is a symmetric matrix and Δ is a lower triangular unit diagonal matrix ($\Delta_{i,i} = 1$) and O is a Pauli operator. $F(O, \Gamma, \Delta)$ have a canonical quantum circuit representation given by-

$$F(O, \Gamma, \Delta) = O \prod_{i=1}^{n} P_i^{\Gamma_{i,i}} \prod_{1 \le i < j \le n} CZ_{i,j}^{\Gamma_{i,j}} \prod_{1 \le i < j \le n} CNOT_{i,j}^{\Delta_{j,i}}$$

As given in [4], any Clifford operator $U \in C_n$ can be uniquely written as,

$$U = F_1(Id, \Gamma, \Delta).(\prod_{i=1}^{n} H_i^{h_i}) S . F_2(O', \Gamma', \Delta')$$

where $h_i \in \{0, 1\}$, S is a permutation layer of n qubits and $F_1(Id, \Gamma, \Delta)$ and $F_2(O', \Gamma', \Delta')$ are elements of Borel group B_n such that the matrices Γ and Δ obey the following rules for all $i, j \in [1, .., n]$:

1. if $h_i = 0$ and $h_j = 0$, then $\Gamma_{i,j} = 0$;
2. if $h_i = 1$ and $h_j = 0$ and $S(i) > S(j)$, then $\Gamma_{i,j} = 0$;
3. if $h_i = 0$ and $h_j = 0$ and $S(i) > S(j)$, then $\Delta_{i,j} = 0$;
4. if $h_i = 1$ and $h_j = 1$ and $S(i) < S(j)$, then $\Delta_{i,j} = 0$;
5. if $h_i = 1$ and $h_j = 0$, then $\Delta_{i,j} = 0$;

The first step is to sample the Hadamard layer h and permutation layer S. As shown in [4], for randomly generating generating a Clifford element we need to sample h and S from the Quantum Mallows distribution given by-

$$P_n(h, S) = \frac{2^{I_n(h,S)}}{\prod_{i=1}^{n}(4^i - 1)}$$

where,

$$I_n(h, S) = n(n-1)/2 + |h| + \sum_{1 \le i < j \le n, S(i) < S(j)} (-1)^{1+h_i}$$

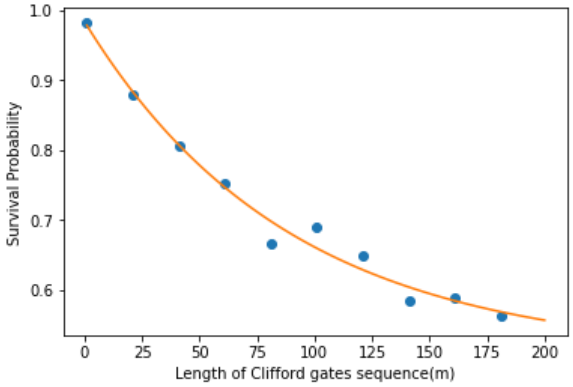

Fig. 1. Exponential Decay Curve for 1-qubit RB using method from section 5.A. Blue dots are the data points and orange line is the exponential fit. We used nshots=200 and $K_m = 3$.

The operator $F = F(O, \Gamma, \Delta)$ defined above has a stabilizer tableau given by

$$\begin{bmatrix} \Delta & 0 \\ \Gamma\Delta & (\Delta^{-1})^T \end{bmatrix}$$

Once h and S are sampled, using conditions 1-5 stated above and the fact that Γ is symmetric and Δ is lower triangular and unit diagonal, Γ and Δ are sampled. We can see that in the conditions 1-5 there are no restrictions on entries of Γ for cases $h_i = h_j = 1$ or ($h_i = 1$ and $h_j = 0$ and $S(i) < S(j)$) or ($h_i = 0$ and $h_j = 1$). Similarly for Δ, for cases ($h_i = 0$ and $h_j = 1$) or ($h_i = 1$ and $h_j = 1$ and $S(i) > S(j)$) or ($h_i = 0$ and $h_j = 0$ and $S(i) < S(j)$) the entries have no restriction. For all these cases entries are filled uniformly at random from $\{0, 1\}$. The entries of Γ' and Δ' are filled uniformly at random either 0 or 1 provided the conditions that Γ' is symmetric and Δ' is lower triangular and unit diagonal is satisfied. Now, the stabilizer tableau of F_1 and F_2 can be determined. Starting with stabilizer tableau of F_2 permutation layer S is applied, followed by Hadamard layer h using the updation rule discussed in the previous section. Finally stabilizer tableau of F_1 is multiplied to get the stabilizer tableau of the Clifford. The last element missing is the phase column. The phases are generated uniformly at random either 0 or 1. This gives a stabilizer table which would correspond to a unique Clifford element.

V. Decomposition of Stabilizer Table to a Quantum Circuit

We obtained a stabilizer table corresponding to a Clifford element chosen uniformly at random. We now discuss three methods to decompose the constructed Clifford Stabilizer Table to a quantum circuit.

A. Decomposing 1-qubit Stabilizer tables

We start by looking at the phases of the stabilizer table. For $n = 1$, we would have $destabilizer\_phase$ and $stabilizer\_phase$. We recall that the standard initial tableau is given by

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

Step 1: We start by considering the following 3 conditions-

- If $destabilizer\_phase = 1$ and $stabilizer\_phase = 0$, it means Z gate is applied, as applying Z to destabilizer generator of standard initial tableau would give $ZXZ^\dagger = -X$, thus giving the phase.
- If $stabilizer\_phase = 1$ and $destabilizer\_phase = 0$, it means X gate is applied, as applying X to stabilizer generator of standard initial tableau would give $XZX^\dagger = -Z$, thus giving the phase.
- If $destabilizer\_phase = 1$ and $destabilizer\_phase = 1$, it means Y gate is applied.

Step 2: After dealing with phases we move to the first 2 columns. For $n = 1$ there are 6 distinct stabilizer tables we can construct(ignoring the phase column).

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right)$$

These can thus be achieved by application of I, H, S, SH, SHS and $HS^\dagger$ respectively. These gates are chosen such that after applying they do not change the phases anymore. (Note that the rightmost gate is applied first in SH)

Using this method we created Clifford sequences of length 1 to 200 in steps of 20 and implemented the standard RB protocol in Qibo using a depolarizing channel with depolarizing parameter $\alpha = 0.995$. Fig. 1 shows the exponential decaying fit of survival probability vs. sequence length. After exponential fitting we get the parameter $p = 0.98881014$. The accuracy could be increased if the number of random instances of Clifford sequences(K_m) is increased. The average error per Clifford comes out to be 0.005594932097917671

B. Aaronson-Gottesmann Approach

In this approach we start by applying certain gates to the stabilizer table until we get the standard initial tableau. Consider stabilizer table of the stabilizer state $|\psi\rangle$. If g is one of the generator of this table, then the state $U|\psi\rangle$ has the generator $UgU^\dagger$. Thus if a unitary U transforms stabilizer table of $|\psi\rangle$ to standard initial tableau, $U|\psi\rangle = |0\rangle$ which means $|\psi\rangle = U^{-1}|0\rangle$ and thus the desired Clifford element is

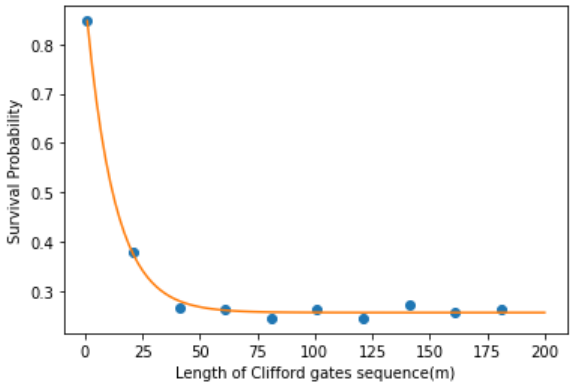

Fig. 2. Exponential decay curve using Aaronson-Gottesmann approach for $n = 2$ qubits RB protocol. Blue dots are the data points and orange line is the exponential fit. We used nshots=200 and $K_m = 3$.

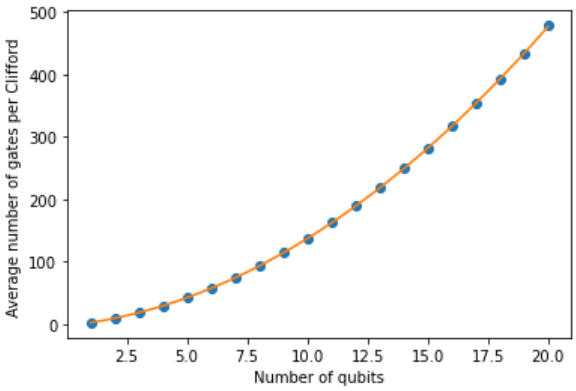

Fig. 3. Complexity of gates for Aaronson-Gottesmann Approach. Blue dots are the data points and orange line is the quadratic fit. We can see that the average number of gates per Clifford is $O(n^2)$.

the quantum circuit corresponding to U^{-1}. The terminology used below is, $destabilizer.x_{i,j}$ corresponds to the x bit of the j^{th} qubit of the i^{th} $destabilizer$ generator. The steps to convert a random stabilizer table to standard initial tableau are:

For $i = 1$

1. Set $destabilizer.x_{i,i} = 1$: If $destabilizer.x_{i,j} = 1$ for $j > i$ then use swap gate between qubits i and j. Otherwise if $destabilizer.z_{i,j} = 1$, a swap gate can be applied between qubits i and j to swap $z_{i,i}$ and $z_{i,j}$ for $i \neq j$, and then a hadamard gate can be applied to swap $x_{i,i}$ and $z_{i,i}$.

2. Set $destabilizer.row[i] = 0$ except $destabilizer.x_{i,i}$: Make $destablizer.x_{i,j} = 0$ by applying CNOTs between qubit i and j with i as control and j as target for all $j > i$.

Then $destabilizer.z_{i,i}$ is made 1(if not already) by applying S gate and then reverse CNOTs are applied from qubit j as control to qubit i as target to make $destabilizer.z_{i,j} = 0$ for all $j > i$. Once the whole row is made 0, $destabilizer.z_{i,i}$ can be made 0 by applying an S gate or $S^\dagger$ gate.

3. With steps 1 and 2, we would automatically have $stabilizer.z_{i,i} = 1$. Thus we just need to set $stabilizer.row[i] = 0$ except for $z_{i,i} = 1$. As seen previously $stabilizer.z_{i,j}$ for $j > i$ can be made 0 by applying reverse CNOTs between control qubit j and target qubit i. In order to make $stabilizer.X[i]$ row 0, a hadamard is applied so that $x_{i,i}$ becomes 1(if not already) by swapping $x_{i,i}$ and $z_{i,i}$. Then $x_{i,j}$ can be made 0 by applying CNOT gates from control qubit i to target qubit j for all $j > i$. If the swapped $z_{i,i}$ is 1 then S gate is applied to make it 0 and finally hadamard gate is applied again to swap $x_{i,i}$ and $z_{i,i}$ so that whole row is zero except $z_{i,i}$.

4. Now the row number i is incremented by 1 and steps 1-3 are repeated. This is done until $i = n$.

With these 4 steps, the only thing remaining is to make the phases 0.

1. We now have an identity matrix at $destabilizer.X$. Thus if $destabilizer.phase$ is 1 for the i^{th} $destabilizer$ generator, then we can apply Z gate on the i^{th} qubit so that $ZX_iZ^\dagger = -X_i$ would give an extra phase thus nullifying the already existing phase.

2. Similarly we have an identity matrix at $stabilizer.Z$. Thus if $stabilizer.phase$ is 1 for the i^{th} stabilizer generator, then we can apply X gate on the i^{th} qubit so that $XZ_iX^\dagger = -Z_i$ would give an extra phase thus nullifying the already existing phase.

This whole procedure would transform the stabilizer tableau to the standard initial tableau. Thus the required circuit as we discussed already is the inverse of the whole circuit. This procedure has an undue advantage in RB protocol as the last element we need in the RB sequence is the inverse of the whole sequence. Thus once we know the stabilizer table corresponding to the first m elements combined, we can run the above scheme to find the inverse element.

Using this method we created random Clifford sequences of length 1 to 200 in steps of 20 and implemented the standard RB protocol in Qibo for $n = 2$ qubits using a depolarizing channel with depolarizing parameter $\alpha = 0.99$. Fig. 2 shows the exponential decaying fit of survival probability vs. sequence length. After exponential fitting we get the parameter $p = 0.92265341$. The average error per Clifford comes out to be 0.058009940359752016. Fig. 3 shows the gate complexity of Clifford elements obtained through this method which is $O(n^2)$.

C. Optimal $CNOT$ Cost Decomposition Method

This method follows the approach of "unentangling" the Clifford operator by reducing the $CNOT$ cost of a Clifford table by applying certain combinations of gates and then decomposing the remainder stabilizer table by single qubit decomposition method discussed earlier. We will be discussing this approach for $n = 2$ qubits.

The first step is to devise a method to calculate the $CNOT$ cost of a given Clifford operator stabilizer table. Given any

stabilizer table we will ignore the phase column for a moment.

$$\left(\begin{array}{cc|cc} x_{11} & x_{12} & z_{11} & z_{12} \\ x_{21} & x_{22} & z_{21} & z_{22} \\ \hline x_{31} & x_{32} & z_{31} & z_{32} \\ x_{41} & x_{42} & z_{41} & z_{42} \end{array}\right)$$

Now consider submatrices A and B where-

$$A = \begin{pmatrix} x_{11} & z_{11} \\ x_{31} & z_{31} \end{pmatrix}$$

$$B = \begin{pmatrix} x_{12} & z_{12} \\ x_{32} & z_{32} \end{pmatrix}$$

For any quantum operation we apply from Section.4.A to the stabilizer table, we notice that the rank of matrices A and B changes only on application of a $CNOT$ gate. Thus there exist a relation between $CNOT$ cost and the rank of matrices A and B. We will denote rank of matrices A and B as r_A and r_B respectively. Let us start by considering the standard initial tableau for which the matrices A and B are-

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Here $r_A = 2$ and $r_B = 0$. Now we apply $CNOT(0,1)$ (where 0 is control qubit and 1 is target) on the stabilizer table and matrices A and B become-

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $r_A = 2$ and $r_B = 1$. Similarly for $CNOT(1,0)$, we get $r_A = 2$ and $r_B = 1$. Now we consider the case where $CNOT(0,1)$ is followed by $CNOT(1,0)$ on the standard initial tableau which gives-

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives $r_A = 1$ and $r_B = 2$. Similarly, for $CNOT(1,0)$ followed by $CNOT(0,1)$ we would get different A and B matrices but with same rank $r_A = 1$ and $r_B = 2$. Lastly we consider, application of $SWAP$ which is nothing but 3 $CNOT$ gates either $CNOT(0,1)CNOT(1,0)CNOT(0,1)$ or $CNOT(1,0)CNOT(0,1)CNOT(1,0)$. For this case we get A and B as-

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives $r_A = 0$ and $r_B = 2$. From the cases discussed above we can now infer that-

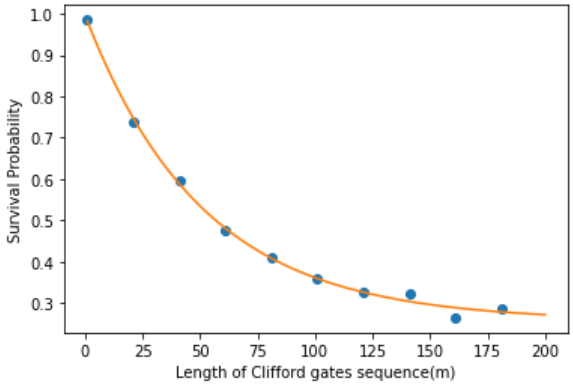

Fig. 4. Exponential decay curve using Optimal-CNOT Cost approach for $n = 2$ qubits RB protocol. Blue dots are the data points and orange line is the exponential fit. We used nshots=200 and $K_m = 3$.

- For $r_A = 2$, Cost of $CNOT = r_B$.
- For remaining cases Cost of $CNOT = r_B - r_A + 1$.

Now we move to discuss the algorithm to decompose the stabilizer table to a quantum circuit. As per [4], for every Clifford operator C applied to a computational basis state $|x\rangle$, there exists a Clifford circuit D such that $D(C|x\rangle)$ is a computational basis state. This means it removes all the superposition and entanglement. We are interested to just remove the entanglement and then the qubits can be treated independently.

Say we have a stabilizer table corresponding to a stabilizer state $|\psi\rangle$. Thus,

$$C|0\rangle = |\psi\rangle$$

Now we apply an operator D which unentangles the Clifford C,

$$D(C|0\rangle) = D|\psi\rangle$$

As the state $D|\psi\rangle$ is unentangled we can decompose stabilizer table corresponding to individual qubits separately by methods discussed in Section.5.A. Let us assume that decomposition gives Clifford circuit V for the stabilizer state $D|\psi\rangle$. We note here that as $|\psi\rangle$ is a stabilizer state and D is some Clifford operator and thus $D|\psi\rangle$ is also a stabilizer state. Thus we get,

$$D(C|0\rangle) = V|0\rangle$$

which will give the decomposition of C as,

$$C = D^{-1}V$$

Now the only thing remaining is to construct D. For this we start by applying different combinations of powers of $HS^\dagger$ which are $HS^\dagger$, $HS^\dagger HS^\dagger$ which is equal to SH and $HS^\dagger HS^\dagger HS^\dagger$ which is equal to I followed by $CNOT(0,1)$ to the stabilizer table for which the $CNOT$ cost reduces. We apply these 2 qubit blocks until we make the $CNOT$ cost 0. If we can find a way to calculate $CNOT$ cost of stabilizer table for $n > 2$, we can extend the approach for larger number of qubits.

Using this method we created random Clifford sequences of length 1 to 200 in steps of 20 and implemented the standard RB protocol in Qibo for $n = 2$ qubits using a depolarizing channel with depolarizing parameter $\alpha = 0.999$. Fig. 4 shows the exponential decaying fit of survival probability vs. sequence length. After exponential fitting we get the parameter $p = 0.98059569$. The average error per Clifford comes out to be 0.014553229412564095.

VI. Conclusion

In this paper we started by discussing various types of errors occuring in quantum devices and briefly discussed about how the standard Randomized Benchmarking and its variants help to quantify and characterize these errors. We discussed the standard RB protocol in detail and studied the decomposition of single qubit Clifford operators using π and $\pi/2$ rotation about X and Y axes. Then we discussed about generating random Clifford stabilizer tables and studied 3 methods for decomposing it to quantum circuits.

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