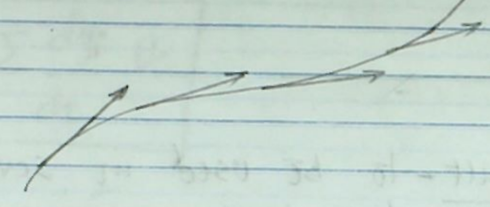
Curvature Initially we deal with curves with unit speed ie $\|\beta'\| = 1$

Unit Tangent: Let $\beta : I \longrightarrow \mathbb{R}^3$ be a unit speed curve. The velocity $T = \beta'$ is called the unit tangent vector field on β.



$\|T\| = 1$

Example:

$\alpha(t) = (\sin t, \cos t, 1) \Rightarrow \alpha'(t) = (+\cos t, -\sin t, 0)$

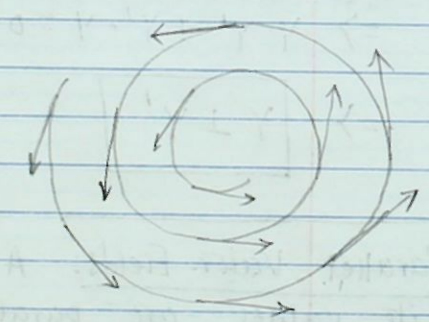
$\|\alpha'(t)\| = \sqrt{\sin^2 t + \cos^2 t}$

$= 1$

Curvature Vector Field: Rate of change of tangent directions captures the curvature of the curve.

Defn: The rate of change of tangent directions that is;

$T' = \beta''$ is called the curvature vector field.



Since we have that

$\|T\| = 1 \Rightarrow \|T\|^2 = 1$

So $T \cdot T = 1$

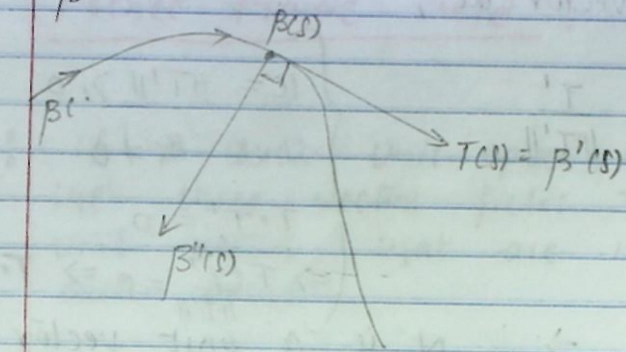
Differentiating both sides $T \cdot T' + T' \cdot T = 0$

$\Rightarrow 2T \cdot T' = 0$

$\boxed{T \cdot T' = 0}$

$\Rightarrow T'$ is normal to T.

A vector field V on β is called normal to β if $V \cdot \beta' = 0$. Thus the curvature vector field T' is normal to β.



As proved $T \cdot T' = 0$

$\Rightarrow T \perp T'$

$\Rightarrow \beta' \perp \beta''$

Curvature: The function $k : I \longrightarrow \mathbb{R}$ defined by

$$k(t) = \|T'(t)\| = \left\| \frac{dT}{ds} \right\|$$ is called the curvature function of β.

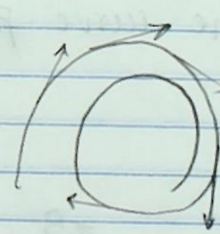
$\left\| \frac{dT}{ds} \right\|$ essentially captures the rate of change of direction of tangent vectors. So it records the change in the direction of tangent vectors & in a sense captures how much a curve is curved.

The curvature function satisfies $k(t) \geq 0$ for all $s \in I$.

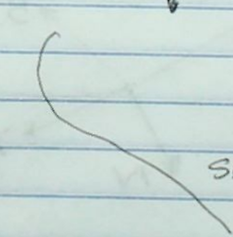
Note:

① When k is small then the curve is close to straight.

② When K is large the turn in the curve is sharp.



greater curvature.



smaller curvature.

Principle Normal (N)

Let β be a curve. Let $K > 0$, that is curve is never straight. We define principle normal vector as;

$$N = \frac{T'}{K} = \frac{T'}{\|T'\|}$$

$$\|N\| = \left\|\frac{T'}{K}\right\| = \left\|\frac{T'}{\|T'\|}\right\|$$

$$= \frac{\|T'\|}{\|T'\|} = 1 \Rightarrow N \text{ is a unit vector.}$$

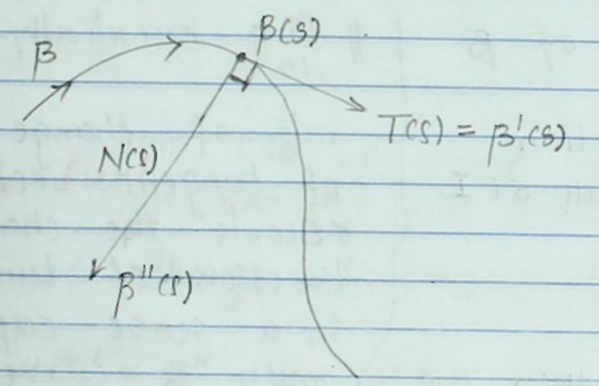
$K = \|T'\| \geq 0$

but $K \neq 0$

$T \cdot T' = 0$

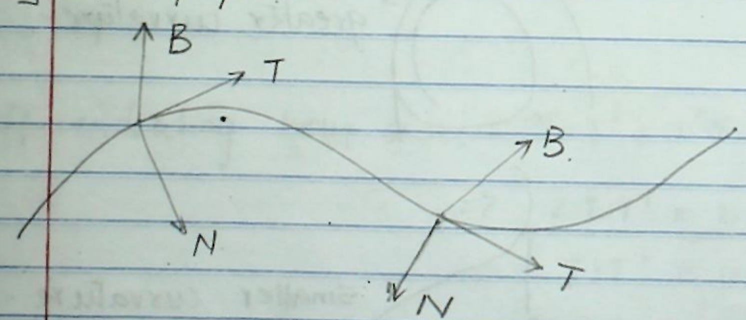
$\Rightarrow T \cdot \frac{T'}{\|T'\|} = 0 \Rightarrow T \cdot N = 0.$

The principle normal vector tells us ~~that~~ the direction in which the curve turns.



Binormal Vector Field (B)

The vector field on the curve $\beta = T \times N$ is called the Bi-Normal vector field of β.



Frenet Frame Field :

Lemma: If β is a unit speed curve in $\mathbb{R}^3$ with $K > 0$. Then the three vector fields T, N and B on β are unit vectors that are mutually orthogonal ~~to~~ at each point.

Proof: $\|T\| = \|\beta'\| = 1 \quad \because \beta$ is unit speed curve.

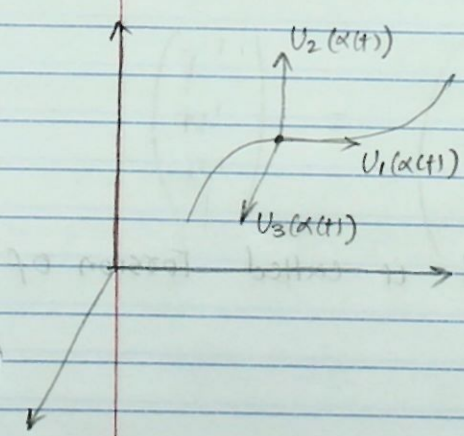
$$\|N\| = \left\|\frac{T'}{K}\right\| = \left\|\frac{T}{\|T\|}\right\| = \frac{\|T\|}{\|T\|} = 1$$

$$\|B\| = \|T \times N\| = \|T\|\,\|N\| \sin\frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1$$

Also $B = T \times N \Rightarrow B \perp T$ and $B \perp N$.

The Triplet (T, B, N) is called the Frenet Frame Field on β.

Earlier we had seen the natural frame field ~~$U_1(t), U_2(t)$~~ and ~~$U_3(t)$~~ $U_1(\alpha(t))$, $U_2(\alpha(t))$ and $U_3(\alpha(t))$.



Derivatives of a Frenet Field

We can express the derivatives T, N, B in terms of T, N and B.

Derivative of T: By Defn: $N = \frac{T'}{\kappa}$; $\kappa = \|T'\|$

$$\boxed{T' = N\kappa = \|T'\| N}$$

Derivative of B: We can write B' in the following form;

$$B' = (B'\cdot T)\,T + (B'\cdot N)\,N + (B'\cdot B)\,B \qquad \text{— (i)}$$

Now $\|B\| = 1$

$\Rightarrow \|B\|^2 = B\cdot B = 1$

$\Rightarrow B\cdot B' + B'\cdot B = 0$

$\Rightarrow 2B\cdot B' = 0$

$\Rightarrow B\cdot B' = 0$

$B\cdot T = 0$

$\Rightarrow (B\cdot T)' = 0$

$\Rightarrow B'\cdot T + T'\cdot B = 0$

$\Rightarrow B'\cdot T = -(T'\cdot B) = -(\kappa N\cdot B)$

$\Rightarrow B'\cdot T = -\kappa\underbrace{(N\cdot B)}_{=0}$ $\because B \perp N \& T.$

$\Rightarrow B'\cdot T = -\kappa\cdot 0$

$\Rightarrow B'\cdot T = 0$

Thus $B'\cdot B = 0$ and $B'\cdot T = 0$ in (i)

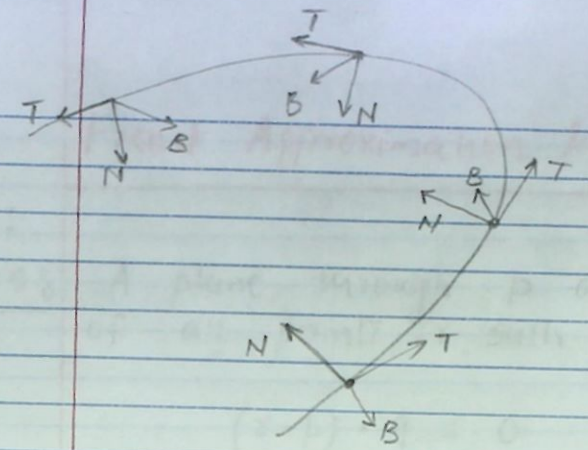
$\therefore \boxed{B' = (B'\cdot N)\,N}$

Torsion of a Curve

The function $\tau : I \to \mathbb{R}$ s.t. $B' = -\tau N$ is called torsion of the curve.

$$B' = (B'\cdot N)N = -\underbrace{(-B'\cdot N)}_{\tau}N$$

Torsion Essentially measures the twist in the curve.



Derivative of N:

$$N' = \underbrace{(N'\cdot T)}_{=-\kappa}T + \underbrace{(N'\cdot N)}_{=0}N + (N'\cdot B)B$$

$N\cdot T = 0 \Rightarrow N'\cdot T + N\cdot T' = 0$

$N'\cdot T = -N\cdot T' = -N\cdot(\kappa N)$

$= -\kappa\underbrace{N\cdot N}_{=1} = -\kappa$

$N'\cdot T = -\kappa$

$\|N\| = 1 \Rightarrow N\cdot N = 1 \Rightarrow 2N'\cdot N = 0$

$\Rightarrow N'\cdot N = 0$

$N\cdot B = 0 \Rightarrow N'\cdot B + N\cdot B' = 0$

$\Rightarrow N'\cdot B = -N\cdot B'$

$= -(N\cdot(-\tau N))$

$= (+\tau)(N\cdot N)$

$N'\cdot B = +\tau$

Frenet Formulas:

If β is a unit speed curve with curvature κ and torsion τ then we have;

$$T' = \kappa N$$

$$N' = -\kappa T + \tau B$$

$$B' = -\tau N$$

In the matrix notation we have

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} \kappa N \\ -\kappa T + \tau B \\ -\tau N \end{pmatrix}$$