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A NECESSARY AND SUFFICIENT QUALIFICATION FOR CONSTRAINED OPTIMIZATION*

F. J. GOULD† AND JON W. TOLLE‡

Abstract. A weak qualification is given which insures that a broad class of constrained optimization problems satisfies the analogue of the Kuhn–Tucker conditions at optimality. The qualification is shown to be necessary and sufficient for these conditions to be valid for any objective function which is differentiable at the optimum.

1. Introduction. Consider the general optimization problem.

Problem 1. Maximize $f(x)$ subject to

$$g_i(x) \leqq 0, \qquad i \in I = \{1, \cdots, m\},$$
$$r_i(x) = 0, \qquad i \in E = \{1, \cdots, k\},$$
$$x \in D \subseteq R^n.$$

Much research in mathematical programming has been devoted to determining necessary conditions for a given point to be a solution to this problem. These necessary conditions usually include special assumptions which qualify the problem under consideration. A familiar example is the classical Lagrange multiplier theorem for the equality-constrained problem (I empty, $D = R^n$). The usual assumption in this theorem is that the Jacobian matrix of the constraints r_i has full row rank at a local optimum. Such a condition is called a qualification, in this case a constraint qualification. Note that this condition is independent of the objective function f. Therefore, if the constraint qualification holds, it is implicit in the classical theorem that the Lagrange multiplier rules are valid for the same constraints and any other differentiable objective function. Numerous authors have obtained different qualifications for various special cases of Problem 1 in order to insure that similar optimality criteria are valid. There has been interest in both determining the weakest such qualifications and in extending the results to increasingly general problems. This paper will be concerned with both of these efforts.

Historically, the interest in optimality criteria for the inequality-constrained problem (E empty, $D = R^n$) has been comparatively recent. The first necessary conditions appear to have been presented in 1948 by Fritz John [10] with no qualification other than the assumption that all functions are continuously differentiable.

FRITZ JOHN'S NECESSARY OPTIMALITY CONDITIONS. If x_0 is a solution to Problem 1 with $E = \varnothing$, $D = R^n$, then there exists a $u = (u_0, \cdots, u_m) \in R^{m+1}$

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† Department of Statistics, University of North Carolina, Chapel Hill, North Carolina 27514.
‡ Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27514.

such that

$$
\begin{aligned}
&u_0 \nabla f(x_0) - \sum_{i=1}^{m} u_i \nabla g_i(x_0) = 0, \\
(1.1) \qquad &u_i g_i(x_0) = 0, \qquad i = 1, \cdots, m, \\
&u \neq 0, \quad u_i \geqq 0, \qquad i = 0, \cdots, m.
\end{aligned}
$$

Stronger necessary conditions for the same inequality-constrained problem were obtained in 1951 by Kuhn and Tucker [11]. They produced a qualification for the constraint functions which when satisfied implies that the u_0 in (1.1) must be positive. Thus the set of candidates for a local optimum is narrowed by the introduction of the constraint qualification. It is a simple exercise to find constraint qualifications which allow the strengthening of Fritz John's result to that of Kuhn and Tucker.[1]

In 1961, Arrow, Hurwicz and Uzawa [2], considering the inequality-constrained problem, gave a constraint qualification for the Kuhn–Tucker conditions which was weaker than that given by Kuhn and Tucker. In particular, their qualification was shown to be the weakest possible if the constraint set $S_g = \{x : g_i(x) \leqq 0, i \in I\}$ is convex. In 1967, Abadie [1] introduced a new qualification which neither implies nor is implied by the qualification of Arrow, Hurwicz and Uzawa. Recently, Evans [6] has exhibited a qualification which is weaker than any of those mentioned above.

In 1967, Mangasarian and Fromovitz [12] considered Problem 1 with $D = R^n$. They proved the analogue of the Fritz John result for this problem and then presented a constraint qualification which validated the extension of the Kuhn–Tucker conditions to the case of mixed constraints.

Other significant papers in this area include those of Canon, Cullum and Polak [4], Cottle [5] and Varaiya [14].

In this paper a necessary and sufficient qualification is exhibited for the extension of the Kuhn–Tucker conditions to Problem 1 and for its special forms, the problem with inequality constraints and the classical problem with equality constraints. It is important to note that the new qualification, in its general form, is dependent upon the set D as well as the constraint functions.

2. Notation and terminology. Let D be an arbitrary set in R^n and let $x_0 \in D$ be a local solution to the optimization problem, Problem 1. The objective function $f : R^n \to R$ and the constraint functions $g : R^n \to R^m$, $r : R^n \to R^k$ are assumed to be continuous on some open set containing D and differentiable at x_0. We shall assume that at least one of the sets I, E is nonempty.[2]

Define the following sets in terms of the constraint functions g and r:

$$
\begin{aligned}
S_g &= \{x \in D : g_i(x) \leqq 0, i \in I\}, \\
S_r &= \{x \in D : r_i(x) = 0, i \in E\}, \\
S &= S_g \cap S_r,
\end{aligned}
$$

[1] For example, suppose the vectors $\nabla g_1(x_0), \cdots, \nabla g_m(x_0)$ are linearly independent. Then if $u_0 = 0$, it follows that $u_i = 0$, $i = 1, \cdots, m$, which contradicts the fact that $u \neq 0$.

[2] Cf. Remark 3.

$$L_0 = \text{subspace of } R^n \text{ spanned by } \nabla r_i(x_0),\ i \in E,$$

$$L_0^\perp = \text{orthogonal complement of } L_0,$$

$$I_0 = \{i \in I : g_i(x_0) = 0\},$$

$$C_0 = \{x \in R^n : \langle x, \nabla g_i(x_0)\rangle \leqq 0,\ i \in I_0\}.$$

$d^T h \leqq 0$

If $I = \emptyset$, set $S_g = R^n$; if $I_0 = \emptyset$, set $C_0 = R^n$; and if $E = \emptyset$, set $S_r = R^n$. The set S is called the constraint set, g and r are called constraints, I_0 is the set of indices of active inequality constraints, and C_0 is a nonempty closed convex polyhedral cone determined by the active inequality constraints.

Let A be an arbitrary set in R^n.

DEFINITION 1. $x \in R^n$ is said to be in the *polar cone* of A, denoted A', if and only if $\langle x, y\rangle \leqq 0$ for all $y \in A$.

Properties of the polar cone relevant to this work are:

(i) A' is a closed convex cone,

(ii) $A_1 \subseteq A_2$ implies $A_2' \subseteq A_1'$,

(iii) $A'' = A$ if and only if A is a closed convex cone,

(iv) $A' = [\text{closure of the convex hull of } A]'$,

(v) if A is a subspace, $A^\perp = A'$.

For a discussion of polar cones and their properties, see [3], [8] and [13].

DEFINITION 2. If A is nonempty, then the *cone of tangents* $T(A, x_0)$ to A at $x_0 \in A$ is the set of all $x \in R^n$ such that there exist a sequence $\{x_n\} \in A$, converging to x_0, and a nonnegative sequence $\{\lambda_n\} \in R$ such that $\{\lambda_n(x_n - x_0)\}$ converges to x.

The set $T(A, x_0)$ is a nonempty closed cone determined by the geometric configuration of A. It need not be convex. The cone of tangents to certain constraint sets was apparently first employed in the mathematical programming context in concurrent publications of Abadie [1] and Varaiya [14].

If Problem 1 has a local solution at x_0 for a given objective function f, then f is said to have a constrained local maximum at x_0. The following definition extends the notion of Lagrange regularity given in [2].

DEFINITION 3. The triple (g, r, D) of Problem 1 is said to be *Lagrange regular* at x_0 if and only if for every objective function f, with a constrained local maximum at x_0, there exist vectors $\psi \in R^k$ and $u \in R^m$ such that

$$\nabla f(x_0) = \sum_{i=1}^{k} \psi_i \nabla r_i(x_0) + \sum_{i=1}^{m} u_i \nabla g_i(x_0), \tag{2.1}$$

$$u_i g_i(x_0) = 0, \qquad i = 1, \cdots, m, \tag{2.2}$$

$$u_i \geqq 0, \qquad i = 1, \cdots, m. \tag{2.3}$$

The conditions (2.1), (2.2), (2.3) represent the analogue of the Kuhn–Tucker conditions for Problem 1. Conditions (2.1) and (2.2) can be written in the more convenient form

$$\nabla f(x_0) = \sum_{i=1}^{k} \psi_i \nabla r_i(x_0) + \sum_{i \in I_0} u_i \nabla g_i(x_0). \tag{2.4}$$

3. Statement of the theorem. The major result of this paper is now given.

THEOREM. *The triple* (g, r, D) *is Lagrange regular at* x_0 *if and only if* $(C_0 \cap L_0^{\perp})' = T'(S, x_0)$.

The condition $(C_0 \cap L_0^{\perp})' = T'(S, x_0)$ represents a qualification which enjoys the generality of Problem 1. For the special inequality-constrained problem with $D = R^n$, $E = \varnothing$, the qualification reduces to $C_0' = T'(S_g, x_0)$. The set T' appears to have been first introduced and discussed by Varaiya [14], but the weak qualification $C_0' = T'(S_g, x_0)$ was explicitly discussed by Evans [6]. The above theorem shows that this latter qualification is in fact necessary, i.e., the weakest possible, for validating the Kuhn–Tucker conditions. For the equality-constrained problem with $D = R^n$, $I = \varnothing$, the new qualification reduces to $L_0 = T'(S_r, x_0)$. This condition is therefore necessary and sufficient for the classical Lagrange multiplier rule to be valid.

4. Proof of the theorem. Several lemmas will be given which will be employed in the development of the proof.

LEMMA 4.1 (Farkas' lemma). *Of the two systems*

$$Ax = b, \quad x \geqq 0$$

and

$$xA \leqq 0, \quad \langle x, b\rangle > 0,$$

one and only one has a solution, where the $m \times n$ *matrix* A *and the element* b *in* R^m *are given.*

A proof of this lemma can be found in [7], [9, p. 44].

LEMMA 4.2. *The conditions* (2.4), (2.3) *hold if and only if* $\nabla f(x_0) \in (C_0 \cap L_0^{\perp})'$.

Proof. Let R be the $n \times k$ matrix whose columns are $\nabla r_i(x_0)$, $i \in E$, and G be the matrix whose columns are $\nabla g_i(x_0)$, $i \in I_0$. Then (2.4) can be written in the form

$$\nabla f(x_0) = (R, G)\begin{pmatrix} \psi \\ u \end{pmatrix}. \tag{4.1}$$

Suppose (4.1) has a solution (ψ, u) with $u_i \geqq 0$, $i \in I_0$. Define variables η_i, v_i, $i \in E$, by

$$\eta_i = \begin{cases} 2\psi_i, & \psi_i \geqq 0, \\ -\psi_i, & \psi_i < 0, \end{cases}$$

$$v_i = \begin{cases} \psi_i, & \psi_i \geqq 0, \\ -2\psi_i, & \psi_i < 0. \end{cases}$$

Then (η, v, u) is a solution of the system

$$\nabla f(x_0) = (R, -R, G)\begin{pmatrix} \eta \\ v \\ u \end{pmatrix} \tag{4.2}$$

with $\eta_i \geqq 0$, $v_i \geqq 0$ for $i \in E$ and $u_i \geqq 0$ for $i \in I_0$. Conversely, if (η, v, u) is a nonnegative solution to (4.2) then $(\eta - v, u)$ will be a solution to (4.1) with $u_i \geqq 0$, $i \in I_0$. Now from Lemma 4.1, it follows that (4.2) has a nonnegative solution if

$\nabla r_i^T x = 0$

$\nabla g_i^T x \leq 0$

and only if $\langle x, \nabla f(x_0)\rangle \leqq 0$ whenever $\langle x, \nabla r_i(x_0)\rangle = 0$ for $i \in E$ and $\langle x, \nabla g_i(x_0)\rangle \leqq 0$ for $i \in I_0$. That is, (4.1) has a solution (ψ, u) with $u_i \geqq 0$, $i \in I_0$, if and only if $\langle x, \nabla f(x_0)\rangle \leqq 0$ whenever $x \in C_0 \cap L_0^\perp$. Therefore conditions (2.4), (2.3) are equivalent to $\nabla f(x_0) \in (C_0 \cap L_0^\perp)'$, completing the proof.

The following lemma was proved by Varaiya [14]. A proof is included here for completeness.

LEMMA 4.3. *If f has a constrained local maximum at x_0, then $\nabla f(x_0) \in T'(S, x_0)$.*

Proof. We show that $\langle \nabla f(x_0), x\rangle \leqq 0$ for all $x \in T(S, x_0)$. Suppose $x \in T(S, x_0)$. Then there exist a sequence $\{x_n\} \in S$ converging to x_0 and a nonnegative sequence $\{\lambda_n\} \in R$ such that $\{\lambda_n(x_n - x_0)\}$ converges to x. Since f is differentiable at x_0, we have, for each n,

$$f(x_n) = f(x_0) + \langle x_n - x_0, \nabla f(x_0)\rangle + \varepsilon\|x_n - x_0\|,$$

where $\varepsilon \to 0$ as $n \to \infty$. This implies that

$$\langle \lambda_n(x_n - x_0), \nabla f(x_0)\rangle = \lambda_n(f(x_n) - f(x_0)) - \varepsilon\lambda_n\|x_n - x_0\|.$$

Letting $n \to \infty$, we have $\langle \lambda_n(x_n - x_0), \nabla f(x_0)\rangle \to \langle x, \nabla f(x_0)\rangle$ and $\varepsilon\lambda_n\|x_n - x_0\| \to 0$. Thus $\lambda_n(f(x_n) - f(x_0))$ has a limit which must be nonpositive since $\lambda_n(f(x_n) - f(x_0)) \leqq 0$ for all n sufficiently large. Consequently

$$\langle x, \nabla f(x_0)\rangle = \lim \lambda_n(f(x_n) - f(x_0)) \leqq 0,$$

which proves the lemma.

Lemma 4.3 furnishes a necessary condition for a differentiable objective function to attain a local maximum over any set A. In case x_0 is interior to A, $T(A, x_0) = R^n$ and $T'(A, x_0) = \{0\}$. Whence $\nabla f(x_0) = 0$.

LEMMA 4.4. $(L_0^\perp \cap C_0)' \subseteq T'(S, x_0)$.

Proof. It is sufficient to show that $T(S, x_0) \subseteq L_0^\perp \cap C_0$. Let $x \in T(S, x_0)$. Then there exist a sequence $\{x_n\} \in S$ converging to x_0 and a nonnegative sequence $\{\lambda_n\} \in R$ such that $\{\lambda_n(x_n - x_0)\}$ converges to x. Since each r_i is differentiable,

$$r_i(x_n) = r_i(x_0) + \langle \nabla r_i(x_0), x_n - x_0\rangle + \varepsilon_i\|x_n - x_0\|,$$

where $\varepsilon_i \to 0$ as $n \to \infty$. Now $x_n \in S$, so $r_i(x_n) = r_i(x_0) = 0$ and

$$\langle \nabla r_i(x_0), \lambda_n(x_n - x_0)\rangle = -\varepsilon_i\lambda_n\|x_n - x_0\|.$$

Letting $n \to \infty$, we obtain

$$\langle \nabla r_i(x_0), x\rangle = 0,$$

i.e., $x \in L_0^\perp$. Hence $T(S, x_0) \subseteq L_0^\perp$. Taking $D = R^n$, Abadie [1] has shown that $T(S_g, x_0) \subseteq C_0$. Since $S \subseteq S_g$ it follows that $T(S, x_0) \subseteq C_0$, which completes the proof of the lemma.

We now turn to a proof of the theorem. Let f be any objective function, differentiable at x_0, with a constrained local maximum at x_0. By Lemma 4.3, $\nabla f(x_0) \in T'(S, x_0)$. If $T'(S, x_0) = (C_0 \cap L_0^\perp)'$, then by Lemma 4.2 conditions (2.4), (2.3) hold and hence (g, r, D) is Lagrange regular at x_0. It remains to show that if (g, r, D) is Lagrange regular at x_0, then $T'(S, x_0) = (C_0 \cap L_0^\perp)'$.

Assume (g, r, D) is Lagrange regular at x_0. We shall show that for every $y \in T'(S, x_0)$ there exists an objective function f, which is differentiable at x_0, which has a constrained local maximum at x_0, and for which $\nabla f(x_0) = y$. The Lagrange regularity hypothesis together with Lemma 4.2 then yields $y \in (C_0 \cap L_0^\perp)'$, and hence the result $T'(S, x_0) \subseteq (C_0 \cap L_0^\perp)'$. That equality actually holds follows from Lemma 4.4.

Let $y \in T'(S, x_0)$. In order to make the notation less cumbersome, we shall assume, without loss of generality, that x_0 is the origin and that y is the unit vector $\hat{e}_n$, i.e., $y_i = 0, i = 1, \cdots, n-1, y_n = 1$. The symbol N_ε will denote an open ball of radius ε about the origin, and S_ε will represent the set $S \cap N_\varepsilon$. A sequence of sets $\{C_l\}_{l=1}^\infty$ is defined as follows:

unit unit

$$C_l = \left\{x \in R^n : \|x\| \neq 0, \cos^{-1}(\langle y, x/\|x\|\rangle) \leqq \left(\frac{\pi}{2} - \frac{\pi}{l+2}\right)\right\}.$$

Note that $C_l \cup \{0\}$ is a closed circular convex cone with axis y and boundary rays forming an angle $\pi/2 - \pi/(l+2)$ with y.

LEMMA 4.5. *For every l, there exists an $\varepsilon(l) > 0$ such that $S_{\varepsilon(l)} \subseteq R^n - C_l$.*

Proof. Suppose otherwise. Then for some l there would exist a sequence $\{x_p\} \in C_l$ with $x_p \in S_{1/p}$ for every p. This implies

$S_{\varepsilon(l)} = S \cap N_{\varepsilon(l)}$

$$\langle y, x_p/\|x_p\|\rangle \geqq \cos\left(\frac{\pi}{2} - \frac{\pi}{l+2}\right) = \gamma > 0.$$

The sequence $\{x_p/\|x_p\|\}$ has a convergent subsequence, say $\{x_{p_j}/\|x_{p_j}\|\}$. Let $x_{p_j}/\|x_{p_j}\| \to \tilde{x}$. Then since $x_{p_j} \in S_{1/p_j} \subseteq S$, $x_{p_j} \to 0$. It follows that $\tilde{x} \in T(S, 0)$. But

$$\langle y, \tilde{x}\rangle = \lim_{p_j \to \infty} \langle y, x_{p_j}/\|x_{p_j}\|\rangle \geqq \gamma > 0.$$

Hence $y \notin T'(S, 0)$ which is a contradiction. This completes the proof.

Define the nonincreasing sequence $\{\varepsilon_k\}$ by

$$\varepsilon_k = \sup\{\varepsilon > 0 : S_\varepsilon \subseteq R^n - C_k\}. \tag{4.3}$$

Note that some ε_k may be infinite. Now define $\{\hat{\varepsilon}_k\}$ by

$$\hat{\varepsilon}_k = \begin{cases} \min(\varepsilon_1, 1), & k = 1, \\ \min\left(\varepsilon_k, \dfrac{\hat{\varepsilon}_{k-1}}{2}\right), & k > 1. \end{cases}$$

It is clear from the definition that $\hat{\varepsilon}_{k+1} < \hat{\varepsilon}_k$ and $\hat{\varepsilon}_k \to 0$.

Let z denote points in R^{n-1} and let $P: R^{n-1} \to R$ be defined as follows:

$$P(z) = \begin{cases} \left(\tan\dfrac{\pi}{3}\right)\hat{\varepsilon}_2 & \text{for } \|z\| \geqq \hat{\varepsilon}_2, \\ \left(\tan\dfrac{\pi}{k+2}\right)\hat{\varepsilon}_{k+1} + \dfrac{(\tan \pi/(k+1))\hat{\varepsilon}_k - (\tan \pi/(k+2))\hat{\varepsilon}_{k+1}}{\hat{\varepsilon}_k - \hat{\varepsilon}_{k+1}} \cdot (\|z\| - \hat{\varepsilon}_{k+1}) \\ \qquad \text{for } \hat{\varepsilon}_{k+1} \leqq \|z\| < \hat{\varepsilon}_k, \quad k \geqq 2, \\ 0 & \text{for } z = 0. \end{cases}$$

LEMMA 4.6. *P is continuous on R^{n-1} and differentiable at $z = 0$ with $\nabla P(0) = 0$. In addition, if $x = (z, x_n) \neq 0$, $x \in S_\varepsilon$, $\varepsilon \leqq \hat{\varepsilon}_1$, then $x_n < P(z)$.*

Proof. Since P is linear except at the points $\hat{\varepsilon}_k$, the reader can easily establish the continuity of P for $x \neq 0$ by verifying that $\lim_{z \to \hat{\varepsilon}_k} P(z) = P(\hat{\varepsilon}_k)$. Now observe that for $\hat{\varepsilon}_{k+1} \leqq \|z\| < \hat{\varepsilon}_k$ we have

$$P(z) = \frac{1}{(\hat{\varepsilon}_k - \hat{\varepsilon}_{k+1})}\left\{\left(\tan\frac{\pi}{k+1}\right)\hat{\varepsilon}_k(\|z\| - \hat{\varepsilon}_{k+1}) + \left(\tan\frac{\pi}{k+2}\right)\hat{\varepsilon}_{k+1}(\hat{\varepsilon}_k - \|z\|)\right\}$$

and hence

$$\left(\tan\frac{\pi}{k+2}\right)\|z\| \leqq P(z) \leqq \left(\tan\frac{\pi}{k+1}\right)\|z\|. \tag{4.4}$$

To show that P is differentiable at the origin with $\nabla P(0) = 0$, it suffices to show that

$$\lim_{\|z\| \to 0} \frac{P(z)}{\|z\|} = 0.$$

This follows immediately from the last half of (4.4). Now let $x = (z, x_n)$ be any point in S, with $x \neq 0$. If $\|z\| = 0$, then $x_n < P(z)$ immediately. Now consider $\hat{\varepsilon}_{k+1} \leqq \|z\| < \hat{\varepsilon}_k$, $k \geqq 1$. Since $\hat{\varepsilon}_k \leqq \varepsilon_k$ it follows from (4.3) and Lemma 4.5 that

$$\frac{x_n}{\|z\|} < \tan\frac{\pi}{k+2},$$

or

$$x_n < \left(\tan\frac{\pi}{k+2}\right)\|z\|.$$

This inequality together with (4.4) yields

$$x_n < P(z),$$

which completes the proof.

We shall now define the objective function $f: R^n \to R$ which is distinguished with a local maximum on S at 0, which is differentiable at 0, and for which $\nabla f(0) = y$. Let

$$f(x) = x_n - P(z)$$

where $x = (z, x_n)$. Note that f is continuous since P is continuous. Similarly, f is differentiable at the origin, and since $\nabla P(0) = 0$, $\nabla f(0) = (0, 1) = y$. Let $x = (z, x_n) \in S_{\hat{\varepsilon}_1}$, $x \neq 0$. Then from Lemma 4.6, $x_n < P(z)$ so $f(z, x_n) < 0$. Since $f(0) = 0$, f attains the promised local maximum on S at the origin. This completes the proof of the theorem.

5. Remarks.

Remark 1. The function P constructed in the proof of the theorem could be modified so as to be differentiable on R^n. Then $f = x_n - P(z)$ would likewise be differentiable on R^n. Thus Theorem 1 holds if we restrict Problem 1 to objective functions differentiable on R^n.

Remark 2. If $D = R^n$, it is known [4], [12] that either of the following qualifications guarantee Lagrange regularity at x_0:

(i) the vectors $\nabla r_i(x_0)$ are linearly independent and there is an $h \in R^n$ such that $h \in L_0^\perp$ and $\langle h, \nabla g_i(x_0)\rangle < 0$ for $i \in I_0$;

(ii) the vectors $\nabla r_i(x_0)$, $i \in E$, and $\nabla g_i(x_0)$, $i \in I_0$, are all linearly independent.

It follows from the theorem that both of these qualifications imply $T'(S, x_0) = (C_0 \cap L_0^\perp)'$. An example which shows that the converse implication does not hold is provided by the constraints

$$g(x, y, z) = z - x^2 - y^2,$$

$$r(x, y, z) = -z.$$

In this case, neither (i) nor (ii) hold at 0 but $(L_0^\perp \cap C_0)' = T'(S, 0)$.

Remark 3. The assumption that at least one of the index sets I and E is not empty was made only for convenience. If both sets are empty, the conditions (2.1), (2.2) and (2.3) should be replaced with the condition $\nabla f(x_0) = 0$. In this case the theorem correctly states that D is Lagrange regular at x_0 if and only if $T'(D, x_0) = (C_0 \cap L_0^\perp)' = \{0\}$. This condition can be satisfied without x_0 being an interior point of D.

Remark 4. The inequality constraints $r_i(x) = 0$ can be replaced by equivalent pairs of inequality constraints $r_i(x) \leqq 0$, $-r_i(x) \leqq 0$, and so every problem with mixed constraints can be considered as an inequality-constrained problem. If Problem 1 is transformed to an inequality-constrained problem in this way, and $\hat{C}_0$ is the cone determined by the new active inequality constraints, then it is easily verified that $L_0^\perp \cap C_0 = \hat{C}_0$. Therefore it is possible to conclude from the theorem that the two problems are equivalent in the sense that the Lagrange regularity of one implies the Lagrange regularity of the other. This result is not apparent from some of the previously known constraint qualifications (cf. the qualifications in Remark 2 above).

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