An Overview

With an aim to pursue research in optimization on manifolds, this work is an attempt on my part to create a sound base for myself in select topics in optimization theory and the theory of smooth manifolds. My goal has been to study some classical topics from the general theory of constrained optimization, theory and algorithms from convex optimization and the basics of differential geometry. An effort has been made to understand the generalizations of some classical notions from constrained optimization theory to the Smooth/Riemannian manifold setting. These generalizations have recently been proposed by Bergmann and Herzog (2019) at TU Chemnitz, Germany and Boumal and Liu (2019) at Princeton University. Bergmann and Herzog have proposed generalizations to the KKT conditions and constraint qualifications using intrinsic notions on the smooth manifold. Boumal and Liu on the other hand have proposed the penalty method and augmented Lagrangian method to the constrained optimization problem on the Riemannian manifold. I view the results in these papers as a motivation for my original research leading to a dissertation .

To this end a repository of notes (write ups) has been created which consolidates my study of these topics. The write ups include a study of the classical Karush-Kuhn-Tucker(KKT) conditions and all the crucial ideas leading to the derivation of these conditions. A particular emphasis has been given to the study of constraint qualifications and the relationship between these constraint qualifications. The general theory of duality has also been touched upon including the saddle point criterion for strong duality. This is followed by a study of convex analysis with the aim to understand the theory of convex optimization problems and relate them to the study of KKT conditions, constraint qualifications and duality theory. I then study some classical algorithms to solve constrained optimization problems with a particular focus on the barrier method and the primal dual interior point methods to solve the linear programming problems. Since the eventual aim is to study optimization on smooth manifolds, I have been studying differential geometry and theory of smooth manifolds from various resources. These resources primarily include the lectures on these topics by Professor Shoaib Iqbal of COMSATS Islamabad, Pakistan, Professor Frederic Schuller of University of Twente, Netherland and Professor Harish Seshadri of Indian Institute of Sciences Bangalore, India. These lectures have been made available on Youtube by the respective universities. A set of handwritten notes from these lectures have also been prepared. With this background it has been possible to study the papers by Bergmann and Herzog and Nicholas Boumal.

The write ups have been written as an exercise leading to the candidacy examination. They are self contained to a large extent and can be used as a first introduction to these topics. A brief summary and list of topics included in each write up is given below.

1. Constrained Optimization - KKT conditions, constraint qualifications and duality

1.1. Motivation and examples, basic definitions and the Farkas' lemma.

1.2. A detailed derivation of the KKT conditions subject to Abadie's constraint qualification.

1.3. A discussion of constraint qualifications, primarily the Linear independence constraint qualification (LICQ), Mangasarian Fromovitz constraint qualification (MFCQ), Abadie's constraint qualification (ACQ) and Guinard constraint qualification (GCQ). A complete proof of the implication LICQ $\Rightarrow$ MFCQ $\Rightarrow$ ACQ $\Rightarrow$ GCQ has been included.

1.4. Counterexamples showing that the implications do not necessarily hold in the reverse order. A proof sketch of the fact that the GCQ is the weakest constraint qualification. An observation about the GCQ and its failure in a classical example found in literature. This example is usually cited as a counterexample for the failure of LICQ at the local optimal which turns out to be a KKT point.

1.5. The dual problem, weak and strong duality and the saddle point criterion for strong duality.

2. Convex analysis and optimization

2.1. Basics of convex sets, convex functions and some fundamental results about convex functions and their characterizations.

2.2. A brief motivation and discussion of some crucial topics in convex analysis including the notions of the relative interior, the distance function, separation theorems and subgradients.

2.3. The Slater's constraint qualification, the KKT conditions and strong duality results for convex optimization problem with inequality and affine equality constraints.

2.4. An introduction to the Fenchel conjugate.

3. Algorithms for constrained optimization problems

3.1. The external penalty function method. Its motivation, some theoretical results including a convergence theorem. A computational example with a MATLAB code that involves solving the corresponding unconstrained penalty problem using Modified Newtons method with line search.

3.2. A brief discussion of exact penalty function method and augmented lagrangian method.

3.3. A general discussion of logarithmic barrier function method including a general convergence theorem and its discussion based on SIAM review and Acta numerica papers by M. Wright et el.

3.4. Some basics of linear programming problems.

3.5. Logarithmic barrier method applied to convex programming problems, in particular the linear programming problem.

3.6. The primal dual central path algorithm as a simple modification of the logarithmic barrier method.

3.7. The two classical neighborhoods of the central path leading to the short and long step algorithms.

3.8. A detailed analysis of the short and long step algorithms with a proof of the polynomial complexity of these algorithms.

3.9. Some numerical results as a proof of concept. Solving randomly generated linear programming instances using the logarithmic barrier method and primal dual interior point methods.

4. Constrained optimization and constraint qualifications on the smooth manifolds.

4.1. Some preliminaries from differential geometry and the theory of smooth manifolds.

4.2. Formulation of the KKT conditions for a constrained optimization problem on smooth manifolds using intrinsic concepts.

4.3. Discussion of the constraint qualifications and the generalization of LICQ, MFCQ, ACQ and GCQ for constrained optimization problem on the smooth manifolds.

5. Basic Convex Analysis on Smooth Manifolds.

5.1. Study of connections and basics definitions of convexity on smooth manifolds. Simple generalizations of some results about convex functions on manifolds.

Current Status. The write ups 1 and 2 have been uploaded. The writes ups 3 and 4 will be uploaded in the next two weeks. The scanned notes will also be posted soon.