Candidacy Statement

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With an aim to pursue research in optimization on manifolds, my work leading up to this candidacy examination is an exercise to create a foundation for myself in select topics in optimization theory and the theory of smooth manifolds. My goal has been to study some classical topics from the general theory of constrained optimization, theory and algorithms from convex optimization and the basics of differential geometry.

Research Objectives

There have been some recent contributions to the constrained optimization theory on Riemannian manifolds. These include the works of Bergmann and Herzog (2019) at TU Chemnitz, Germany and by Boumal and Liu (2019) at Princeton University. Bergmann and Herzog have proposed generalizations of the classical Karush Kuhn Tucker conditions and constraint qualifications for a constrained optimization problem on a smooth manifold using intrinsic concepts. They have also proposed a generalization of the Fenchel conjugate from classical convex analysis to the Riemannian manifold. Boumal and Liu on the other hand have proposed the penalty method and augmented Lagrangian method to solve the constrained optimization problem on the Riemannian manifold. I view the results in these papers as a motivation and a starting point for my own work where I would like to expand on the theory of constrained optimization and convex analysis on Riemannian manifold.

Candidacy Objectives

As mentioned in the beginning, the goal has been to study and understand classical topics from constrained optimization theory, convex analysis and differential geometry. This includes a study of the Karush-Kuhn-Tucker (KKT) conditions and all the crucial ideas leading to the derivation of these conditions. A particular emphasis has been given to the study of constraint qualifications and the relationship between these constraint qualifications. The general theory of duality has also been touched upon including the saddle point criterion for strong duality. This is followed by a study of convex analysis with an aim to understand the theory of convex optimization problems and the role of KKT conditions, constraint qualifications and duality theory in convex optimization. Some classical algorithms to solve constrained optimization problems are studied. A particular focus has been on the barrier method and the primal dual interior point methods to solve the linear programming problems. A study of basics of differential geometry and theory of smooth manifolds has been conducted as a pre-requisite to work at the intersection of optimization theory and differential geometry.

To consolidate my study, a set of notes (write-ups) have been written as an exercise in the learning process. These notes have been written with the intention to simplify and consolidate the material studied from various books, papers and online lectures in one place. All the original resources have been cited and included in the bibliography at the end of each write-up. These notes are self contained to a large extent and can be used as a first introduction to these topics. A brief summary and a list of topics included in each write-up is given on the next page. The write-ups can be found on my github page for which the link is given below.

Github: https://github.com/TejasNatuOpt/Candidacy-Material

1. Constrained Optimization: KKT conditions, constraint qualifications and duality

Motivation and examples, basic definitions and the Farkas' lemma. A detailed derivation of the KKT conditions subject to Abadie's constraint qualification. A discussion of constraint qualifications, primarily the linear independence constraint qualification (LICQ), Mangasarian Fromovitz constraint qualification (MFCQ), Abadie's constraint qualification (ACQ) and Guinard constraint qualification (GCQ). A complete proof of the implication LICQ $\Rightarrow$ MFCQ $\Rightarrow$ ACQ $\Rightarrow$ GCQ has been included. Counterexamples showing that the implications do not necessarily hold in the reverse order. A proof sketch of the fact that the GCQ is the weakest constraint qualification. An observation about the GCQ and its failure in a classical example in optimization theory (this example and its variations are common in optimization literature, however this particular observation was not found anywhere to the best of our knowledge and survey). The dual problem, weak and strong duality and the saddle point criterion for strong duality.

2. Convex analysis and optimization

Basics of convex sets, convex functions and some fundamental results about convex functions and their characterizations. A brief motivation and discussion of some crucial topics in convex analysis including the notions of the relative interior, the distance function, separation theorems and subgradients. The Slater's constraint qualification, the KKT conditions and strong duality results for convex optimization problem with inequality and affine equality constraints. An introduction to the Fenchel conjugate.

3. Algorithms for constrained optimization problems

The external penalty function method. Its motivation, some theoretical results including a convergence theorem. A computational example with a Matlab code that involves solving the corresponding unconstrained penalty problem using modified Newton's method with line search. A general discussion of logarithmic barrier function method including a general convergence theorem and its discussion based on SIAM review and Acta numerica papers by M. Wright et el. Some basics of linear programming problems. The logarithmic barrier method applied to convex programming problems, in particular the linear programming problem. The primal dual central path algorithm as a simple modification of the logarithmic barrier method. A classical neighborhood of the central path called the $N_2(\theta)$ neighborhood leading to the short step algorithm. A detailed analysis of the short step algorithm with a proof of polynomial complexity. Some numerical results as a proof of concept. Also available on the github page are the codes written as part of some computational experiments.

4. Basics of differential geometry and theory of smooth manifolds

A set of handwritten notes on differential geometry and the theory of smooth manifolds from various resources. These resources primarily include the lectures on these topics by Professor Shoaib Iqbal of COMSATS Islamabad, Pakistan, Professor Frederic Schuller of University of Twente, Netherland and Professor Harish Seshadri of Indian Institute of Sciences Bangalore, India. These lectures have been made available on youtube by the respective universities.

5. Constrained optimization and constraint qualifications on the smooth manifolds.

Some preliminaries from differential geometry and the theory of smooth manifolds. Formulation of the KKT conditions for a constrained optimization problem on smooth manifolds using intrinsic concepts. Discussion of the constraint qualifications and the generalization of LICQ, MFCQ, ACQ and GCQ for constrained optimization problem on the smooth manifolds.