Defn: $(M, \mathcal{O})$ topological space.

$C \subseteq M$ is called CLOSED if $M \setminus C$ is Open

Ex: $[0,1]$ is closed in $(\mathbb{R}, \mathcal{O}_{st})$

because $\mathbb{R} \setminus [0,1] = \underbrace{\underbrace{(-\infty, 0)}_{\in \mathcal{O}_{st}} \cup \underbrace{(1, \infty)}_{\in \mathcal{O}_{st}}}_{\in \mathcal{O}_{st}}$

$[0,1]$ contains end points, hence we can say it is "Not Open" but we still cannot conclude that it is closed.

⚠ In General, a subset of a topological space can be;

(i) Open
or (ii) Closed
or (iii) open and closed ← M, ϕ - can be proved, in a connected top. space, they are the only sets that are both open & closed.
or (iv) open and not closed
or (v) not open and closed
(vi) not open and not closed.

Observation: For any $(M, \mathcal{O})$ topological space

(a) $\phi = M \setminus \underset{\in \mathcal{O}}{M}$ open $\Rightarrow$ M is closed

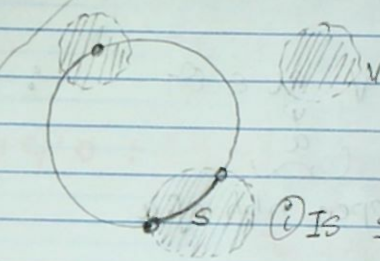
(b) $M = M \setminus \underset{\in \mathcal{O}}{\phi}$ open $\Rightarrow$ ϕ is closed.

Real Life Example:

Consider the set $S' := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

One way to establish a topology on S is to let

$$\mathcal{O} := \mathcal{O}_{std\,\mathbb{R}^2}\big|_{S'}$$



(i) Is s open?
$\mathcal{U} \in \mathcal{O}_{std\,\mathbb{R}^2}$
$s = \mathcal{U} \cap S'$ hence s is open.

(ii) Is ϕ open?
$\phi = S' \cap V$ hence open

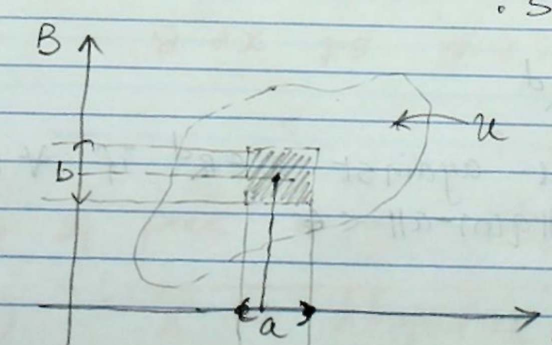
(iii) $T = \{\bullet\}$ is open? No, becoz we cannot generate T as an intersection of an open set in std topology $\mathbb{R}^2$ and circle S'.

Product Topology

Defn: $(A, \mathcal{O}_A)$ and $(B, \mathcal{O}_B)$ are topological spaces. Equip $A \times B$ with the so called Product Topology $\mathcal{O}_{A \times B}$ is defined implicitly by

$$\mathcal{U} \in \mathcal{O}_{A\times B} \iff \forall \underset{(a,b) \in A\times B}{p} \in \mathcal{U} \;\; \exists \underset{a}{S} \in \mathcal{O}_A,\ \underset{b}{T} \in \mathcal{O}_B : S \times T \subseteq \mathcal{U}$$

Intuitively



Can be checked that $\mathcal{O}_{A\times B}$ provides a topology on $A \times B$.

Remark: (i) One can do this for any finite $A_1 \times A_2 \times \cdots \times A_n$.

(ii) $\mathcal{O}_{\text{standard } \mathbb{R}^d} = \mathcal{O}_{\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d}}$

2.3 Convergence

Defn: A sequence q (i.e. a map $q: \mathbb{N} \to M$) on a topological space $(M, \mathcal{O})$ is said to converge against a "limit" point $a \in M$ if

$$\forall \underbrace{\mathcal{U} \in \mathcal{O}}_{\text{open nbh. of } a}, \; a \in \mathcal{U} : \exists N \in \mathbb{N} ; \forall n > N : q(n) \in \mathcal{U}$$

Example: (a) $(M, \{\phi, M\})$ – M with chaotic topology.

Let $q: \mathbb{N} \to M$ be some sequence.

Claim: Any sequence converges against every point.

In the above defn. of convergence ; $\mathcal{U} = M$ in this case becoz there is only one open set containing a. Since $q(n) \in \mathcal{U} = M$ for all n. the defn. of convergence goes through.

(b) $(M, \mathcal{P}(M))$ – M with discrete topology.

Claim: Only all almost constant sequences converge.

↳ non constant only at finite number of points.

(c) $M = \mathbb{R}^d$, $\mathcal{O} = \mathcal{O}_{\text{st}}$ in $\mathbb{R}^d$

Thm: $q: \mathbb{N} \to \mathbb{R}$ converges against $a \in \mathbb{R}^d$ if $\forall \varepsilon > 0$: $\exists N \in \mathbb{N}$ $\forall n \in N$: $\|q(n) - a\| < \varepsilon$

Ex: $q(n) = 1 - \frac{1}{n+1}$, $M = \mathbb{R}$ } not almost constant & hence not convergent in $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ but convergent in $(\mathbb{R}, \mathcal{O}_{\text{std}})$

0 1

2.4 Continuity

Defn: Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. Then a map $\phi: M \to N$ is called continuous if

$$\forall V \in \mathcal{O}_N : \text{preim}(V) \in \mathcal{O}_N \; ; \; \text{preim}(V) = \{m \in M \mid \phi(m) \in V\}$$

"ϕ continuous if preimage of open set is an open set".

Ex. (a) $\phi: M \to N$, $\mathcal{O}_M = \mathcal{P}(M)$, $\mathcal{O}_N$

Claim: Any map ϕ is continuous. Since $\text{preim}(V)$ $\forall V \in \mathcal{O}_N$ is a subset of M & hence $\in \mathcal{P}(M)$ & is open.

(b) $\phi: M \to N$, $\mathcal{O}_M$, $\mathcal{O}_N = \{\phi, N\}$

any map ϕ is continuous.

(c) $\phi: \mathbb{R}^d \to \mathbb{R}^f$, $\mathcal{O}_{\text{std}}$, $\mathcal{O}_{\text{std}}$ } $\Rightarrow$ recover the standard defn. of continuity.

~~Defn:~~

Homeomorphism

Defn: Let $\phi : M \to N$ be a bijection. Now equipping $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ we called ϕ a Homeomorphism if;

(a) $\phi : M \to N$ continuous

(b) $\phi^{-1} : N \to M$ continuous.

Remark: Homeo(morphism)s are structure preserving maps in topology.

If there exists homeomorphism

$$M \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} N$$

then ϕ provides a one-one pairing of the open sets of M with those of N.

In case homeo ϕ exists : $(M, \mathcal{O}_M) \underset{top.}{\cong} (N, \mathcal{O}_N)$ Isomorphic in topological sense.