We now know that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

then

$$\left.\begin{aligned} df_1 &= \frac{\partial f_1}{\partial x_1}dx_1 + \frac{\partial f_1}{\partial x_2}dx_2 + \frac{\partial f_1}{\partial x_3}dx_3 \\ df_2 &= \frac{\partial f_2}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f_2}{\partial x_3}dx_3 \\ df_3 &= \frac{\partial f_3}{\partial x_1}dx_1 + \frac{\partial f_3}{\partial x_2}dx_2 + \frac{\partial f_3}{\partial x_3}dx_3 \end{aligned}\right\}$$

~~so~~ $\cancel{d\phi = \sum df_i \wedge dx_i}$

$= \cancel{\partial f_1 \wedge dx}$

So $d\phi = \sum df_i \wedge dx_i = df_1\wedge dx_1 + df_2\wedge dx_2 + df_3 \wedge dx_3$

$$= \frac{\partial f_1}{\partial x_1}\underbrace{dx_1\,dx_1}_{0} + \frac{\partial f_1}{\partial x_2}dx_2\,dx_1 + \frac{\partial f_1}{\partial x_3}dx_3\,dx_1$$

$$+ \frac{\partial f_2}{\partial x_1}dx_1\,dx_2 + \frac{\partial f_2}{\partial x_2}\underbrace{dx_2\,dx_2}_{0} + \frac{\partial f_2}{\partial x_3}dx_3\,dx_2$$

$$+ \frac{\partial f_3}{\partial x_1}dx_1\,dx_3 + \frac{\partial f_3}{\partial x_2}dx_2\,dx_3 + \frac{\partial f_3}{\partial x_3}\underbrace{dx_3\,dx_3}_{0}$$

$$d\phi = \left(\frac{\partial f_2}{\partial x_1}dx_1dx_2 - \frac{\partial f_1}{\partial x_2}dx_1dx_2\right) + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)dx_2dx_3$$

$$+ \left(\frac{\partial f_3}{\partial x_1}dx_1dx_3 - \frac{\partial f_1}{\partial x_3}\right)dx_1dx_3$$

So Exterior derivative $d\phi$ of a 1-form $\phi = f_1dx_1 + f_2dx_2 + f_3dx_3$ is given as;

$$\boxed{d\phi = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)dx_1dx_2 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right)dx_1dx_3 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)dx_2dx_3} \quad \text{2-form}$$

Example: $\phi = xy\,dx + x^2dz$; $\begin{cases}\phi = f_1dx_1 + f_2dx_2 + f_3dx_3 \\ d\phi = df_1\wedge dx_1 + df_2\wedge dx_2 + df_3\wedge dx_3\end{cases}$

$d\phi = d(xy)\wedge dx + d(x^2)\wedge dz$

$d\phi = (x\,dy + y\,dx)\wedge dx + (2x\,dx)\wedge dz$

$= x\,dy\,dx + y\,dx\,dx + 2x\,dx\,dz$

$\left\{ d\phi = x\,dy\,dx + 2x\,dx\,dz \right\}$ 2-form

(Exterior Derivative of 1-form is the 2-form.)

Exterior Derivative of 2-form gives 3-form : Let $\psi = f\,dx\,dy + g\,dx\,dz + h\,dy\,dz$ be a 2-form. Then the exterior derivative of ψ is the 3-form;

$$d\psi = df\wedge dx\hat{\ }dy + dg\wedge dx\hat{\ }dz + dh\wedge dy\hat{\ }dz$$

Example: Let $\psi = xy\,dx\,dy + x^2dx\,dz + xz\,dy\,dz$ then

$d\psi$ is given as

$$d\psi = d(xy)\wedge dx\,dy + d(x^2)\wedge dx\,dz + d(xz)\wedge dy\,dz$$

$$= (x\,dy + y\,dx) \wedge dx\,dy + 2x\,dx\,dx\,dz + (x\,dz + z\,dx) \wedge dy\,dz$$

$$= \cancel{x\,dy\,dx\,dy}^{\,0} + \cancel{y\,dx \wedge dx\,dy}^{\,0} + \cancel{2x\,dx\,dx\,dz}^{\,0} + \cancel{x\,dz\,dy\,dz}^{\,0} + z\,dx\,dy\,dz$$

$$= z\,dx\,dy\,dz \qquad \boxed{3\text{-Form}}$$

Theorem: Let f and g be functions, ϕ and ψ be 1-forms.

Then

(1) $d(fg) = df\,g + f\,dg$

(2) $d(f\phi) = df \wedge \phi + f\,d\phi$

(3) $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

Proof: (1) as seen earlier.

(2) $d(f\phi) = df \wedge \phi + f \wedge d\phi$

$\phi = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$

$$d(f\phi) = d(f f_1 dx_1 + f f_2 dx_2 + f f_3 dx_3)$$

Linearity of Exterior Derivative

$$= d(f f_1 dx_1) + d(f f_2 dx_2) + d(f f_3 dx_3).$$

Use defn. of Exterior Derivative of 1-Form

$$= d(f f_1) \wedge dx_1 + d(f f_2) \wedge dx_2 + d(f f_3) \wedge dx_3$$

$\wedge$ is distributive

$$= (f\,df_1 + f_1\,df) \wedge dx_1 + (f\,df_2 + f_2\,df) \wedge dx_2 + (f_3\,df + f\,df_3) \wedge dx_3$$

$$= (f\,df_1\,dx_1 + f_1\,df\,dx_1) + (f\,df_2\,dx_2 + f_2\,df\,dx_2) + (f_3\,df\,dx_3 + f\,df_3\,dx_3)$$

$$= (f\,df_1 dx_1 + f\,df_2\,dx_2 + f\,df_3 dx_3) + (f_1\,df\,dx_1 + f_2\,df\,dx_2 + f_3\,df\,dx_3)$$

$$= f(df_1 dx_1 + df_2 dx_2 + df_3 dx_3) + df(f_1 dx_1 + f_2 dx_2 + f_3 dx_3)$$

$$= f\,d\phi + df \wedge \phi \qquad \begin{cases} df \text{ is 1-form} \\ \phi \text{ is 1-form} \end{cases}$$

Proof (3) $\quad d(\phi \wedge \psi) = d\phi \wedge \psi$