Chapter 1

An Overview

This work is an attempt on my part to create a sound base for myself in selected topics in optimization theory and theory of smooth manifolds, with an aim to pursue research in optimization on manifolds. My goal has been to study some classical topics from constrained optimization theory, differential geometry and understand generalizations of some notions from constrained optimization theory to the Smooth/Riemannian manifold setting that have been done recently by Dr Ronny Bergmann, Dr Roland Herzog and Dr Nicholas Boumal. To this end a repository of notes has been created which consolidates my study of these topics. The write ups include a study of the classical Karush Kuhn Tucker conditions and all the crucial notions leading to the derivation of these conditions. Particular emphasis has been given to the study of constraint qualifications and the relationship between these constraint qualifications. This is followed by a study of convex analysis with the aim to understand the theory of convex optimization problems and relate it to the study of KKT conditions, constraint qualifications and duality theory. I then study some classical algorithms to solve constrained optimization problems with a particular focus on the barrier method and the primal dual interior point methods. Since the eventual aim is to study optimization on smooth manifolds, I studied differential geometry and theory of smooth manifolds from various sources. These primarily included lectures by Professor Shoaib Iqbal of COMSATS Islamabad, Pakistan, Professor Frederic Schuller of University of Twente, Netherland and Professor Harish Seshadri of Indian Institute of Sciences Bangalore, India made available on youtube and the recent book by Dr Nicholas Boumal on Optimization on manifolds. A set of handwritten notes from these lectures have also been prepared. With this background it was possible to go ahead and study the papers by Bergmann and Herzog and Nicholas Boumal.

These write ups have been written as an exercise leading to the candidacy examination. The write ups are self contained to a large extent and can be used as a first introduction to these topics. A brief summary and list of topics included in each write up is given below.

Constrained Optimization - KKT conditions and Duality
1. Motivation - Examples, Basic definitions, Farkas' lemma.
2. Detailed derivation of KKT conditions subject to Abadie's constraint qualification.
3. A discussion of constraint qualifications primarily the linear independence constraint qualification (LICQ), Mangasarian Fromovitz constraint qualification (MFCQ), Abadie's constraint qualification (ACQ) and Guinard constraint qualification (GCQ). The proof of LICQ MFCQ ACQ GCQ. Counterexamples showing that the implications do not necessarily hold in the reverse order. A proof sketch of the fact that GCQ is the weakest constraint qualification.
4. The dual problem, weak and strong duality and the saddle point criterion for strong duality

Convex analysis and optimization
1. Basics of convex sets, convex functions and some fundamental results about convex functions and their characterizations. 2. A brief motivation and discussion of some crucial notions in convex analysis including

the notions of relative interior, distance function, separation theorems and subgradients. 3. The Slater's constraint qualification, KKT conditions and duality results for convex optimization problem with inequality and affine equality constraints. 4. An introduction to Fenchel conjugates.

Algorithms for constrained optimization problems
1. External penalty function method. Its motivation, some theoretical results including a convergence theorem. A computational example with a MATLAB code that involves solving the corresponding unconstrained penalty problem using Modified Newtons method with line search.
2. A brief discussion of exact penalty function method and augmented lagrangian method.
3. A general discussion of logarithmic barrier function method including a general convergence theorem and its discussion based on SIAM review and Acta numerica papers by M. Wright et el.
4. Some basics of linear programming problems.
5. Logarithmic barrier method applied to convex programming problems in particular the linear programming problem.
6. Primal dual central path algorithms as a simple modification of the logarithmic barrier method. The application of primal dual algorithms to solve linear programming problems.
7. Some numerical results. Solving randomly generated linear programming instances using the logarithmic barrier method and primal dual interior point methods.

Constrained optimization and constraint qualifications on the smooth manifolds.
1. Some preliminaries from differential geometry and the theory of smooth manifolds.
2. Formulation of the KKT conditions for a constrained optimization problem on smooth manifolds using intrinsic concepts.
3. Discussion of the constraint qualifications and the generalization of LICQ, MFCQ, ACQ and GCQ for constrained optimization problem on the smooth manifolds.

Basic Convex Analysis on Smooth Manifolds.
1. Study of connections and basics definitions of convexity on smooth manifolds. Simple generalizations of some results about convex functions on manifolds.

The write ups 1 and 2 have been uploaded. The writes ups 3 and 4 will be uploaded in the next two weeks. The scanned notes will also be posted soon.