Differential Geometry Study Notes

- Based on lecture series delivered and made available on youtube by:

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Directional Derivative

Associated with each tangent vector v_p to $\mathbb{R}^3$ is the straight line

$$t \rightarrow p + tv.$$

$p = (1,1,1)$, $v = (2,3,4)$

$t \rightarrow (1,1,1) + t(2,3,4)$

$= (1+2t, 1+3t, 1+4t)$

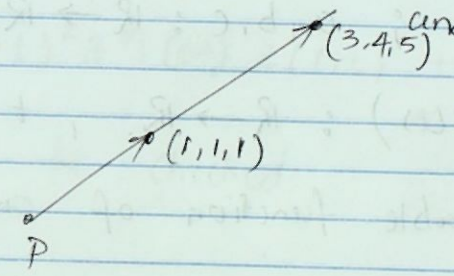
$\{t \rightarrow (1+2t, 1+3t, 1+4t)\}$ Means? $t=0 \rightarrow (1,1,1)$

$t=1 \rightarrow (3,4,5)$.

$\vdots$

and so on.

Essentially we have parametric equations of a line.



Defn. (Direc. Derivative)

f - real valued function on $\mathbb{R}^3$

v_p - tangent vector to $\mathbb{R}^3$ at p.

$$v_p(f) = \frac{d}{dt} f(p+tv)\Big|_{t=0} \quad \text{since } \{p \equiv t=0\}$$

is called directional derivative of f w.r.t. v_p.

Example 1. $f = x^2yz$ with $p = (1,1,0)$ and $v = (1,0,-3)$.

Then; $\underbrace{p+tv}_{\text{the line associated with } p \text{ and } v} = (1,1,0) + t(1,0,-3) = (1+t, 1, -3t)$

$$f(p+tv) = f(1+t, 1, -3t) = (1+t)^2(1)(-3t)$$
$$= -3t - 6t^2 - 3t^3.$$

Now $\frac{d}{dt}(f(p+tv)) = \frac{d}{dt}(-3t-6t^2-3t^3)$

$$= -3 - 12t - 9t^2.$$

$$v_p(f) = \frac{d}{dt}(f(p+tv))\Big|_{t=0} = (-3-12t-9t^2)_{t=0} = -3.$$

Recall: Chain Rule

$f: \mathbb{R}^3 \to \mathbb{R}$; $a, b, c : \mathbb{R} \to \mathbb{R}$

$f(a(t), b(t), c(t)) : \mathbb{R} \to \mathbb{R}$; $t \to f(a(t), b(t), c(t))$

is a differentiable function of one variable t, and

$$\frac{d}{dt} f(a,b,c) = \frac{\partial f}{\partial x_1}(a,b,c)\frac{da}{dt} + \frac{\partial f}{\partial x_2}(a,b,c)\frac{db}{dt} + \frac{\partial f}{\partial x_3}(a,b,c)\frac{dc}{dt}$$

$f = x^2yz$; $a \to \mathbb{R} \to \mathbb{R}$; $a(t) = t+1$
$b(t) = t^2$
$c(t) = t^3$

$$f(a,b,c) = (t+1)^2(t^2)(t^3)$$
$$= t^5(1+t)^2$$

$$\frac{d}{dt} f(a,b,c) = t^5\, 2(1+t) + 5t^4(1+t)^2 \quad \text{— (1)}$$

Now $f = x^2yz$; $f(a,b,c) = (1+t)^2(t^2)(t^3)$

$$\frac{\partial f}{\partial x} = 2xyz \qquad \frac{da}{dt} = 2(1+t)$$

$$\frac{\partial f}{\partial y} = x^2z \qquad \frac{db}{dt} = 2t$$

$$\frac{\partial f}{\partial z} = x^2y \qquad \frac{dc}{dt} = 3t^2.$$

$$\frac{d}{dt} f(a,b,c) = \frac{\partial f}{\partial x}\frac{da}{dt} + \frac{\partial f}{\partial y}\frac{db}{dt} + \frac{\partial f}{\partial z}\frac{dc}{dt}$$

$$= 2(1+t)^2(t^2)(t^3)\,2(1+t) + (1+t^2)^2(t^3)\cdot 2t + (1+t^2)^2 t^2 \cdot 3t^2$$

Same as (1) can be checked.

Lemma: $V_p = (V_1, V_2, V_3)$ is tangent vector to $\mathbb{R}^3$, then

$$\boxed{V_p[f] = \sum V_i \frac{\partial f}{\partial x_i}(p)} \equiv V_1 f_{x_1}(p) + V_2 f_{x_2}(p) + V_3 f_{x_3}(p) \text{ in } \mathbb{R}^3.$$

Example 2. $f = x^2yz$; $p = (1,1,0)$ and $v = (1,0,-3)$.

If $V_p = (V_1, V_2, V_3)$ is tangent vector to $\mathbb{R}^3$, then

$$V_p[f] = \sum V_j \frac{\partial f}{\partial x_j}(p)$$

$$V_p[f] = 1\left(\frac{\partial f}{\partial x}\right)_p + 0\left(\frac{\partial f}{\partial y}\right)_p + (-3)\left(\frac{\partial f}{\partial z}\right)_p$$

$$= 1(0) + 0(0) - 3(1) = -3.$$

$$\boxed{V_p[f] = -3}$$

Directional Derivatives (Properties)

f and g be functions on $\mathbb{R}^3$; V_p and W_p tangent vectors, a & b scalers.

(1) $(aV_p + bW_p)[f] = aV_p[f] + bW_p[f]$

(2) $V_p[af + bg] = aV_p[f] + bV_p[g]$

(3) $V_p[fg] = V_p[f]\cdot g(p) + f(p)\cdot V_p[g]$

Operation of a Vector Field on function $f: \mathbb{R}^3 \to \mathbb{R}$

Let V be vector field; $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$. Then $V[f]$ is defined as;

$$\boxed{V[f]: \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad V[f](p) = V(p)[f]}$$

$V[f]$ is a function which to each point p in $\mathbb{R}^3$ gives us the derivative of f w.r.t. tangent vector $V(p)$.

Ex 3. For the natural frame field U_1, U_2 and U_3 we get;

$$U_1[f](p) = U_1(p)[f] = (1,0,0)_p[f]$$

$$U_1[f](p) = U_1(p)[f] = (1,0,0)_p[f] = 1\cdot\frac{\partial f}{\partial x}(p) + 0\cdot\frac{\partial f}{\partial y}(p) + 0\cdot\frac{\partial f}{\partial z}(p)$$

$$= \frac{\partial f}{\partial x}(p).$$

Similarly
$$\left.\begin{aligned} U_2[f](p) &= (0,1,0)_p[f] = \frac{\partial f}{\partial y}(p) \\ U_3[f](p) &= \frac{\partial f}{\partial z}(p). \end{aligned}\right\}$$

Basic Operations:

Let V and W be vector fields on $\mathbb{R}^3$ and f, g and h are real valued functions;

$f: \mathbb{R}^3 \to \mathbb{R}$ so is g & h.

(1) $(fV + gW)[h] = fV[h] + gW[h]$

(2) $V[af + bg] = aV[f] + bV[g]$.

(3) $V[fg] = V[f]\cdot g + fV[g]$.

Proof: $(fV + gW)[h](p) = \Big((fV+gW)(p)\Big)[h]$

$= \Big(f V(p) + g(W(p)\Big)[h]$

$= \Big[\underbrace{f(p)}_{\text{scaler}} V(p) + \underbrace{g(p)}_{\text{scaler}} W(p)\Big][h]$

$= f(p) V(p)[h] + g(p) W(p)[h]$

$= \Big(fV[h] + gW[h]\Big)(p)$

(2) $V[af + bg] = aV[f] + bV[g]$

Let p be any point in $\mathbb{R}^3$

$V[af+bg](p) \overset{def.}{=} V(p)[af+bg]$

$= aV(p)[f] + bV(p)[g]$

$= aV[f](p) + bV[g](p)$

$= (aV[f] + bV[g])(p)$

$V(p)[af+bg]$

$= \frac{d}{dt}(af+bg)(p+tV(p))\Big|_{t=0}$

$= \frac{d}{dt}\Big(af(p+tV(p)) + bg(p+tV(p))\Big)\Big|_{t=0}$

$= \frac{d}{dt} af(p+tV(p))\Big|_{t=0} + \frac{d}{dt} bg(p+tV(p))\Big|_{t=0}$

$= a\frac{d}{dt} f(p+tV(p))\Big|_{t=0} + b\frac{d}{dt} g(p+tV(p))\Big|_{t=0}$

$= aV(p)[f] + bV(p)[g]$

Example: Let V be vector field $V = x^2U_1 + yU_3$

$f = x^2y - y^2z$. Calculate $V[f]$.

Solution: $V[f] = \cancel{} \underbrace{(x^2U_1 + yU_3)(x^2y - y^2z)}_{V(1\cdot f - 1\cdot g)}$.

$= 1\cdot(x^2U_1 + yU_3)[f]$

$\quad - 1\cdot(x^2U_1 + yU_3)[g]$

$= x^2U_1[x^2y] + yU_3[f = x^2y] - x^2U_1[y^2z] - yU_3[y^2z]$

$= x^2\left(\frac{\partial f}{\partial x}\right) + y\left(\frac{\partial f}{\partial z}\right) - x^2\left(\frac{\partial f}{\partial x}\right) - y\left(\frac{\partial f}{\partial z}\right)$

$= x^2(2xy) + y\cdot 0 - x^2(0) - y(y^2)$

$\boxed{V[f] = 2x^3y - y^3}$

Curves in $\mathbb{R}^3$

Defn: Curve in $\mathbb{R}^3$ is a differentiable function $\alpha: I \to \mathbb{R}^3$; I open interval.

Ex 1. $\alpha: (0,2) \to \mathbb{R}^3$

$\alpha(t) = (t^2, t^3+1, 2t)$

$\underbrace{\alpha_1(t) \quad \alpha_2(t) \quad \alpha_3(t)}_{\text{all are differentiable}}$

is a curve in $\mathbb{R}^3$.

Ex 2. $\alpha : (0,3) \longrightarrow \mathbb{R}^3$

$$\left.\begin{array}{l} \alpha(t) : (\cos t, \sin t, 0) \\ \quad\quad\;\; \alpha_1(t) \;\; \alpha_2(t) \;\; \alpha_3(t) \end{array}\right\} \text{curve in } \mathbb{R}^3$$

Ex 3. Straight line in $\mathbb{R}^3$: line through p and in the direction q is a curve ;

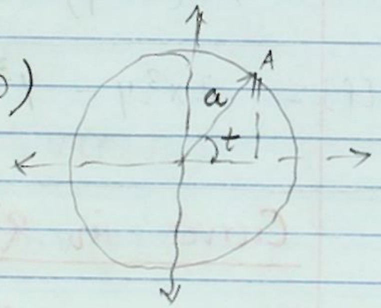
$$\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$$

defined as ;

$$\alpha(t) = p + tq = (\underbrace{p_1 + tq_1}_{\alpha_1(t)}, \underbrace{p_2 + tq_2}_{\alpha_2(t)}, \underbrace{p_3 + tq_3}_{\alpha_3(t)}).$$

$$\left.\begin{array}{l} p = (1,1,1). \\ v = (0,1,3) \\ \alpha(t) = (1 + 0t, 1 + 1\cdot t, 1 + 3t) \end{array}\right\} \text{curve in } \mathbb{R}^3.$$

Ex 4. Circle : $t \longrightarrow (a\cos t, a\sin t, 0)$



Ex 5: Helix : $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$

$$\alpha(t) = (a\cos t, a\sin t, bt)$$

where $a > 0$, $b \neq 0$.

* Some More Examples *

Velocity Vector :

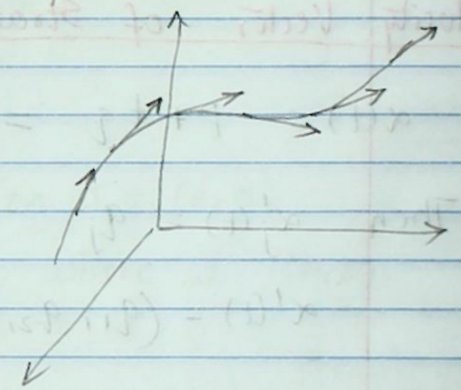
Defn: Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For each number t in I, the velocity vector of α at 't' is the tangent vector

$$\alpha'(t) = \left(\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)\right)_{\alpha(t)} \quad \text{at the point } \alpha(t) \text{ in } \mathbb{R}^3.$$

impt.

Ex 1. $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$

$$\alpha'(t) = (3 - 3t^2, 6t, 3 + 3t^2)_{\alpha(t)}$$

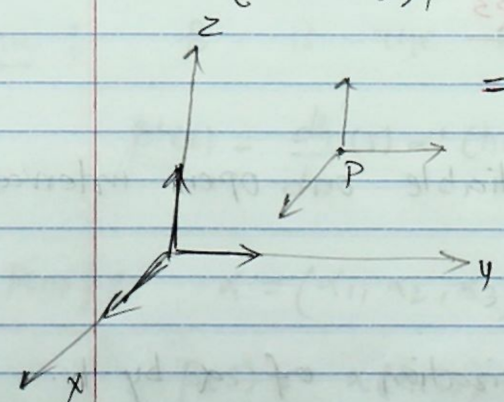


An Identity : We know the natural frame field defined by $U_1(p) = (1,0,0)_p$, $U_2(p) = (0,1,0)_p$, $U_3(p) = (0,0,1)_p$

We can always write for any tangent vector of $\mathbb{R}^3$ at p:

$$(v_1, v_2, v_3)_p = \sum v_j U_j(p).$$

$$= v_1 U_1(p) + v_2 U_2(p) + v_3 U_3(p).$$



For for $\alpha'(t) \equiv \quad \alpha'(t) = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right)_{\alpha(t)}$

$$= \frac{d\alpha_1}{dt} U_1(\alpha(t)) + \frac{d\alpha_2}{dt} U_2(\alpha(t)) + \frac{d\alpha_3}{dt} U_3(\alpha(t)).$$

Velocity Vector of Straight line :

$\alpha(t) = p + tq = (p_1 + tq_1, p_2 + tq_2, p_3 + tq_3)$

Then $\alpha_j'(t) = q_j$

$\alpha'(t) = (q_1, q_2, q_3)_{\alpha(t)} = \vec{q}_{\alpha(t)}$

Velocity Vector Helix

$\alpha(t) = (a\cos t, a\sin t, bt)$

$\alpha'(t) = (-a\sin t, a\cos t, b)_{(\alpha(t))}$ velocity remains const in z- direction.

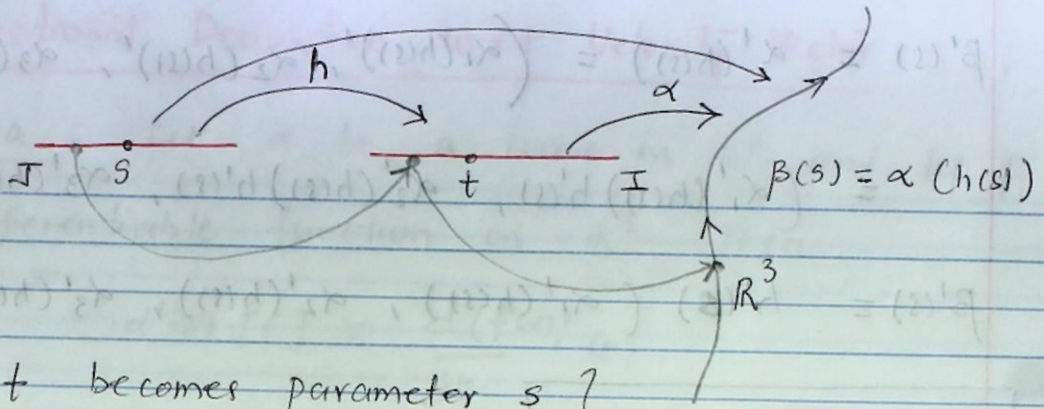
Reparametrization of Curves in R^3

Defn: $\alpha : I \to R^3$ be a curve.

$h : J \to I$ is a differentiable on open interval J, then the composite function

$$\beta = \alpha(h) : J \to R^3$$

is a curve called reparametrization of α by h.



Parameter t becomes parameter s which gets mapped to the curve
Domain now becomes different and image is the same curve
} – Reparametrization.

Example : $\alpha(t) = (\sqrt{t}, t\sqrt{t}, 1-t)$ on $I = (0,4)$.

$h : J \to I$ as $h(s) = s^2$ where $J = (0,2)$.

Then the reparametrization is

$$\beta(s) = \alpha(h(s)) = \alpha(s^2) = (s, s^3, 1-s^2)$$

$$\sqrt{s^2} \quad s^2\sqrt{s^2} \quad 1-s^2$$

~~Velocity of Reparametrizati~~

~~Lemma :~~

Velocity Vector of Reparametrization of the curve :

Lemma : β is the reparametrization of α by h, then

$$\beta'(s) = \frac{dh}{ds}(s)\,\alpha'(h(s)).$$

Proof : $\alpha = (\alpha_1, \alpha_2, \alpha_3)$

$$\beta(s) = \alpha(h(s)) = (\alpha_1(h(s)), \alpha_2(h(s)), \alpha_3(h(s)))$$

$$\beta'(s) = \alpha'(h(s)) = \left(\alpha_1(h(s))', \alpha_2(h(s))', \alpha_3(h(s))'\right)$$

$$= \left(\alpha_1'(h(s))\,h'(s), \alpha_2'(h(s))\,h'(s), \alpha_3'(h(s))\,h'(s)\right)$$

$$\beta'(s) = h'(s)\left(\alpha_1'(h(s)), \alpha_2'(h(s)), \alpha_3'(h(s))\right)$$

$$\boxed{\beta'(s) = h'(s)\,\alpha'(s)}$$

Exercise (BON) (1.4) $\quad \alpha(t) = \left(1+\cos t, \sin t, 2\sin\frac{t}{2}\right)$

$h(s) = \cos^{-1}(s) \quad ; \quad J: 0 < s < 1$

Find coordinate function

$$\begin{cases} \beta = \alpha(h) \\ \beta(s) = \alpha(h(s)) \end{cases}$$

$$= \alpha(\cos^{-1}(s))$$

$$= \alpha\left(1+\cos(\cos^{-1}(s)), \sin(\cos^{-1}s), 2\sin\left(\frac{\cos^{-1}s}{2}\right)\right)$$

Simplification = yields

Directional Derivative w.r.t Velocity Vector

Lemma: Let α be a curve in $\mathbb{R}^3$ and let f be a differentiable function on $\mathbb{R}^3$. Then

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

Proof: $v_p = (v_1, v_2, v_3)$ is tangent vector to $\mathbb{R}^3$, then

$$v_p[f] = \sum v_j \frac{\partial f}{\partial x_j}(p)$$

$$\alpha' = \left(\overset{v_1}{\frac{d\alpha_1}{dt}}, \overset{v_2}{\frac{d\alpha_2}{dt}}, \overset{v_3}{\frac{d\alpha_3}{dt}}\right)_{\alpha(t)}$$

So ; $\alpha'(t)[f] = \sum_j \frac{\partial f}{\partial x_j}(\alpha(t))\frac{d\alpha_j}{dt}(t)$

But by Chain Rule ; $\qquad f(\alpha(t)) = f(\alpha_1(t), \alpha_2(t), \alpha_3(t))$

$$\left\{\alpha'(t)[f] = \frac{d}{dt}(f(\alpha))(t)\right\}$$

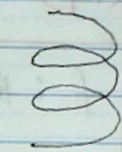
Properties of Curves :

① **One to One Curves** : $\alpha : I \to \mathbb{R}^3$

one to one if

$\alpha(t_1) \neq \alpha(t_2)$ for $t_1 \neq t_2$.

- Does not Intersect Itself
- Repeat Itself over after some time.



② **Periodic Curves** : $\alpha : \mathbb{R} \to \mathbb{R}^3$ (Periodic)

Periodic if $\exists$ positive number p s.t.

$$\alpha(t+p) = \alpha(t) \quad \text{for all } t \in \mathbb{R}.$$

Circle : $\alpha(t+2\pi) = \alpha(t)$.

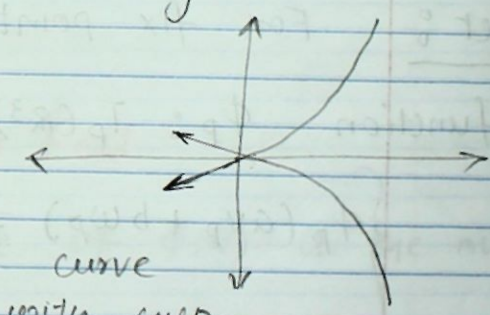
③ **Regular Curve** : $\alpha : I \to \mathbb{R}^3$ is called Regular if

$$\alpha'(t) \neq (0,0,0)_{\alpha(t)} \quad \text{for all } t \in I.$$

Example : $\alpha(t) = (t^2, t^3, 0)$ is not regular.

$\alpha'(t) = (2t, 3t^2, 0)$

$\alpha'(0) = (0,0,0)$.



DIFFERENTIALS

1- FORMS

Given $f: \mathbb{R}^3 \to \mathbb{R}$; differential is defined as;

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$ } what does this mean?

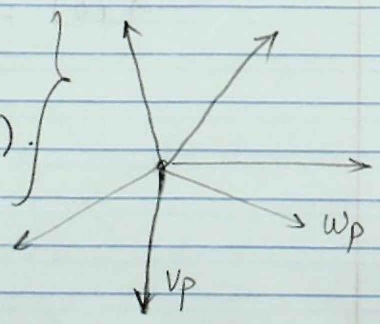
Defn: A 1-form ϕ on $\mathbb{R}^3$ is a real valued function on the set of all tangent vectors to $\mathbb{R}^3$ such that ϕ is linear at each point, that is;

$$\phi(av_p + bw_p) = a\phi(v_p) + b\phi(w_p).$$

where a, b are scalers and v, w are tangent vectors to $\mathbb{R}^3$ at the same point.

Fact: For fix point p in $\mathbb{R}^3$ the resulting function $\phi_p : T_p(\mathbb{R}^3) \to \mathbb{R}$ is linear.

$$\phi_p(av_p + bw_p) = a\,\phi(v_p) + b\,\phi(w_p).$$



Sum of two 1-forms:

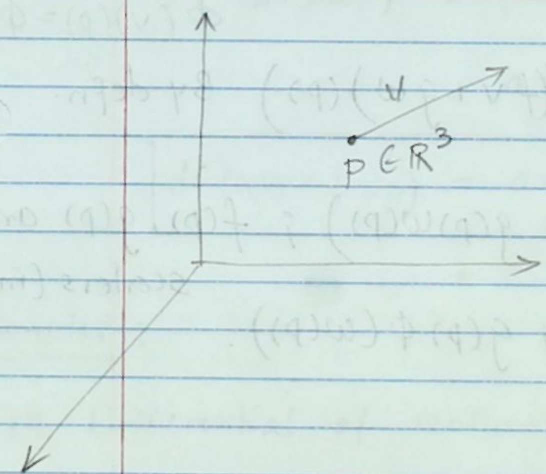
Let ϕ and ψ be two 1-forms, then sum is defined as;

$$(\phi + \psi)(v) = \phi(v) + \psi(v).$$ for all tangent vectors v.

Product of 1-Forms with a function $f: \mathbb{R}^3 \to \mathbb{R}$

$f: \mathbb{R}^3 \to \mathbb{R}$ and ϕ - 1 form For any tangent vector v_p to $\mathbb{R}^3$ we define

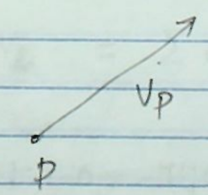
$$(f\phi)(v_p) = f(p)\cdot\phi(v_p).$$



Evaluating 1-form on a vector field.

At each point $p \in \mathbb{R}^3$ the value of $\phi(v)$ is the number $\phi(V(p))$

$\phi(v_p)$



Differentiable 1-Form :

1-Form ϕ is differentiable ; if $\phi(V)$ is differentiable whenever V is differentiable.

$$V = f_1 U_1 + f_2 U_2 + f_3 U_3$$

V-differentiable when all components are differentiable

Properties of $\phi(V)$; $f, g : \mathbb{R}^3 \to \mathbb{R}$.

(1) $\phi(fV + gW) = f\phi(V) + g\phi(W)$

(2) $(f\phi + g\psi)(V) = f\phi(V) + g\psi(V)$

$\phi(V)(p) = \phi(V(p))$

Proof : $\phi(fV + gW)(p) = \phi((fV+gW)(p))$ By defn.

$= \phi(f(p)V(p) + g(p)W(p))$; $f(p), g(p)$ are scalers (in $\mathbb{R}$).

$= f(p)\phi(V(p)) + g(p)\phi(W(p))$.

$= (f\phi(V))(p)$

$(fV + gW)(p) = f(p)V(p) + g(p)W(p)$.

fV is a vector field (Pg 8 BON)

Differentials :

Defn : If f is a differentiable function from $\mathbb{R}^3 \to \mathbb{R}$, the differential df of f is the 1-form such that ;

$$df(v_p) = \underbrace{v_p[f]}_{} \quad \text{for all tangent vectors } v_p.$$

directional derivative of f in the direction of v_p.

1-form - df satisfies all conditions of 1-form ;

(1) df maps every tangent vector to a real number.
(2) df is linear because of the properties of directional derivative

$$(av_p + bw_p)[f] = av_p[f] + bw_p[f]$$

So $df(av_p + bw_p) = (av_p + bw_p)[f]$

$= av_p[f] + bw_p[f]$

$$\boxed{df(av_p + bw_p) = av_p[f] + bw_p[f]} \boxed{= a\,df(v_p) + b\,df(w_p)}$$

Examples :

(1) Differential of Natural Coordinate Functions :

Natural coordinate function from $\mathbb{R}^3 \to \mathbb{R}$; $x_1(p_1,p_2,p_3) = p_1$
$x_2(p_1,p_2,p_3) = p_2$
$x_3(p_1,p_2,p_3) = p_3$.

$$dx_j(v_p) = v_p[x_j] = \sum_{i=1}^{3} \frac{dx_j}{dx_i}(p)\, v_i = \sum \delta_{ji} v_i = v_j$$

dx_j depends on v and not on the point of application p.

$$\begin{cases} dx_1(v_p) = v_1 \\ dx_2(v_p) = v_2 \\ dx_3(v_p) = v_3 \end{cases} \qquad \begin{cases} v = (v_1, v_2, v_3) \\ p = (p_1, p_2, p_3) \\ v_p = (v_1, v_2, v_3)_{(p_1, p_2, p_3)} \end{cases}$$

Ex 2. Linear Combination of 1-forms : Let $f_1, f_2, f_3 : R^3 \to R$ be three functions let $\psi = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$.

$$\left.\begin{array}{l} f_1 dx_1 \\ f_2 dx_2 \\ f_3 dx_3 \end{array}\right\} \text{All 1-forms added together.}$$

Seen earlier

$$f\phi(v_p) = f(p)\,\phi(v_p)$$

$$(\phi+\psi)(v_p) = \phi(v_p) + \psi(v_p).$$

So ψ is a 1-form and for any vector v_p we have;

$$\psi(v_p) = (f_1 dx_1 + f_2 dx_2 + f_3 dx_3)(v_p)$$

$$= f_1 dx_1(v_p) + f_2 dx_2(v_p) + f_3 dx_3(v_p)$$

$$=$$

product of function with 1-form

$$= f_1(p)\,dx_1(v_p) + f_2(p)\,dx_2(v_p) + f_3(p)\,dx_3(v_p)$$

$$= f_1(p)\,v_1 + f_2(p)\,v_2 + f_3(p)\,v_3 \qquad \left.\begin{array}{l} dx_1(v_p) = v_1 \\ dx_2(v_p) = v_2 \\ dx_3(v_p) = v_3 \end{array}\right\}$$

$$\boxed{\psi(v_p) = \sum_{i=1}^{3} f_i(p)\, v_i}$$

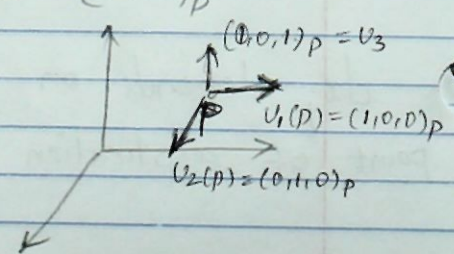
Next we show that in fact each 1-form can be written in the form of ψ above.

Euclidean Coordinate Functions of a 1-form

Lemma: ϕ is a 1-form on R^3, then $\phi = \sum f_i dx_i$, where $f_i = \phi(U_i)$. These ~~coordinate~~ functions ~~are~~ f_1, f_2 and f_3 are called Euclidean coordinate functions of ϕ.

Proof: ϕ and $\sum f_i dx_i$ are equal if and only if they have same value on every tangent vector $v_p = \sum v_i U_i(p)$

$$\left\{\begin{array}{l} \phi = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \\ U_1(p) = (1,0,0)_p \end{array}\right.$$



$$\cancel{\Sigma(f_i dx_i)}\ (\Sigma f_i dx_i)(v_p) = \Sigma f_i(p)\, v_i \quad — ①$$

On the other hand; $\phi(v_p) = \phi(\Sigma v_i U_i(p))$

linearity of 1-form ϕ

$$\phi(v_p) = \phi(v_1 U_1(p) + v_2 U_2(p) + v_3 U_3(p))$$

$$\phi(v_p) = v_1\,\phi(U_1(p)) + v_2\,\phi(U_2(p)) + v_3\,\phi(U_3(p)).$$

So $\phi(v_p) = \phi(\Sigma v_i U_i(p)) = \Sigma v_i\, \phi(U_i(p))$

② $= \Sigma v_i f_i(p)$; $\left\{ \because \phi(U_i) = f_i \text{ - form assumption} \right.$

From ① and ②.

So ϕ and $\Sigma f_i dx_i$ have same value on every tangent vector.

Corollary: The Classic Differentials

$$df(v_p) = v_p[f]$$

Let $f : R^3 \to R$ be a differentiable function. Then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Proof: $v_p = (v_1, v_2, v_3)_p$ is any tangent vector to R^3, then

$$df(v_p) = v_p[f] = \Sigma v_j \frac{\partial f}{\partial x_j}(p)$$

Saw in previous lemma

$$= \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(p)\, v_j = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(p)\, dx_j(v_p) \qquad \left\{\begin{array}{l} dx_1(v_p) = v_1 \\ dx_2(v_p) = v_2 \\ dx_3(v_3) = v_3 \end{array}\right.$$

$$= \left(\Sigma \frac{\partial f}{\partial x_j} dx_j\right)(v_p)$$

$$(f\phi)(v_p) = f(p)\,\phi(v_p)$$

hence the two 1-forms df and $\Sigma \frac{\partial f}{\partial x_j} dx_j$ are the same.

$\frac{\partial f}{\partial x_j}$ is a function realvalued.

Exercise 1.5 (BON)

Q1(a) $v = (1,2,-3)$ and $p = (0,-2,1)$. Evaluate the following 1-form on tangent vectors v_p.

(a) $y^2 dx$

Solution: $f\phi(v_p) = f(p)\,\phi(v_p)$

$dx(v_p) = dx\left((1,2,-3)_{(0,-2,1)}\right) = 1$

$y^2(0,-2,1) = (-2)^2 = 4$

$(y^2 dx)\left((1,2,-3)_{(0,-2,1)}\right) = y^2(0,-2,1)\,dx(v_p)$

$= 4(1) = 4.$

Q2. $\phi = \Sigma f_i dx_i$ and $V = \Sigma v_i U_i$ vector field.

Show $\phi(V) = \Sigma f_i v_i$

Proof: $\phi(V)$

$\phi(fV + gW) = f\phi(V) + g\phi(W)$ ①

$(f\phi + g\psi)(V) = f\phi(V) + g\psi(V)$. ②

$= (\Sigma f_i dx_i)(\Sigma v_i U_i)$

$= (f_1 dx_1 + f_2 dx_2 + f_3 dx_3)(v_1U_1 + v_2U_2 + v_3U_3)$ ①

$= (f_1 dx_1 + f_2 dx_2 + f_3 dx_3)\, v_1U_1 + (f_1 dx_1 + f_2 dx_2 + f_3 dx_3)\, v_2 U_2 +$

$(f_1 dx_1 + f_2 dx_2 + f_3 dx_3)(v_3U_3)$

$= (f_1 dx_1)(v_1U_1) + (f_2 dx_2) v_1 U_1 + (f_3 dx_3) v_1 U_1 + \dots$

$= f_1 v_1 dx_1(U_1) + f_2 v_1 \underbrace{dx_2(U_1)}_{0} + f_3 v_1 \underbrace{dx_3(U_1)}_{0} + \dots$

$dx_1(U_1)(p) = dx_1(U_1(p))$
$= dx_1((1,0,0)_p)$
$= 1$

$dx_2(U_2)(p) = dx_2(U_1(p))$
$= dx_2((1,0,0)_p)$
$= 0$

$dx_3(U_3)(p) = 0.$

$= f_1 v_1 + f_2 v_2 + f_3 v_3$

Properties of Differentials

$df(v_p) = v_p[f]$.

① Lemma: fg be product of differentiable functions f and g on $\mathbb{R}^3$. Then $d(fg) = g\,df + f\,dg$.

Proof: $d(fg) = \sum_i \frac{\partial(fg)}{\partial x_i} dx_i$

from corollary $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$.

$= \Sigma\left(f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}\right)dx_i$

$= f\,\Sigma \frac{\partial g}{\partial x_i} dx_i + g\,\Sigma\frac{\partial f}{\partial x_i} dx_i$

$= f\,dg + g\,df$

② Chain Rule:

Lemma: $f: \mathbb{R}^3 \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ differentiable functions.

$h(f): \mathbb{R}^3 \to \mathbb{R}$ is also differentiable. Then

$d(h(f)) = h'(f)\,df$.

Proof: Chain Rule for $h(f)$ is

$\frac{\partial(h(f))}{\partial x_i} = h'(f)\frac{\partial f}{\partial x_i}$

$$dh(f) = \sum \frac{\partial h(f)}{\partial x_i} dx_i$$

$$= \sum h'(f) \frac{\partial f}{\partial x_i} dx_i = h'(f) \sum \frac{\partial f}{\partial x_i} dx_i$$

$$\boxed{dh(f) = h'(f)\, df}$$

Example: Calculate df; where $f = (x^2-1)y + (y^2+2)z$.

Using product rule and chain rule instead $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$

$$df = d\big((x^2-1)y + (y^2+2)z\big)$$

$$= d\big((x^2-1)y\big) + d\big((y^2+2)z\big)$$

$$= d(x^2-1)\, y + d(y)\cdot(x^2-1) + d(y^2+2)\cdot z + d(z)(y^2+2)$$

$$= 2x\,dx\, y + dy(x^2-1) + 2y\,dy\cdot z + dz\,(y^2+2)$$

$$= \underbrace{(2xy)}_{\frac{\partial f}{\partial x}} dx + \underbrace{(x^2-1+2yz)}_{\frac{\partial f}{\partial y}} dy + \underbrace{(y^2+2)}_{\frac{\partial f}{\partial z}} dz.$$

Now we evaluate df on tangent vector v_p where $v_p = (v_1, v_2, v_3)$ and $p = (p_1, p_2, p_3)$

$$df(v_p) = v_p[f] = \qquad \left.\begin{array}{l} dx(v_p) = v_1 \\ dy(v_p) = v_2 \\ dz(v_p) = v_3 \end{array}\right\}$$

$$\| $$

$$(2xy\,dx)(v_p) + (x^2-1+2yz)\,dy(v_p) + (y^2+2)\,dz(v_p)$$

$$= (2xy)(p)\, dx(v_p) \quad (x^2-1+2yz)(p)\, dy(v_p) + (y^2+2)(p)\, dz(v_p)$$

$$= \cancel{2p_1p_2}\ 2p_1p_2 v_1 + (p^2-1+2p_2p_3) v_2 + (p_2^2+2) v_3.$$

Q5 (Exer 1.5 BON) Express the differentials of the following functions in the standard form $\sum f_i dx_i$.

(a) $(x^2+y^2+z^2)^{1/2} : R^3 \longrightarrow R$

Chain Rule: $d(h(f)) = h'(f)\, df$.

$$d(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} d(x^2+y^2+z^2) \; ; \quad d(x^2) = d(x\cdot x) = x\,dx + x\,dx = 2x\,dx$$

$$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2} (2x\,dx + 2y\,dy + 2z\,dz)$$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{(x^2+y^2+z^2)}} dx + \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}} dy + \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}} dz$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} dx + \frac{y}{\sqrt{x^2+y^2+z^2}} dy + \frac{z\,dz}{\sqrt{x^2+y^2+z^2}}$$

$$= f_1 dx + f_2 dy + f_3 dz$$

Q7 (Exer. 1.5 BON)

Differential Forms :

1-Forms are part of a large system called Differential forms.

Construction of Differential Forms

$v_p = (v_1, v_2, v_3)_{(p_1,p_2,p_3)}$

$dx(v_p) = v_1$

$dy(v_p) = v_2$

$dz(v_p) = v_3$

Take real valued functions $f_1, f_2, f_3, \dots, f_n : \mathbb{R}^3 \to \mathbb{R}$

Take differential 1-forms of coordinate function dx, dy, dz.

Take sum and product of the above.

Examples: (1) $x^2\,dx\,dy + xy\,dy\,dz + (xz+y^2)\,dz\,dy$

(2) $xyz\,dx + (x^2z^2+y)\,dy + xyz\,dz$

(3) $x^2y^2\,dx\,dy\,dz$.

Multiplication of forms $dx\,dy$ is not commutative

Instead ; $dx\,dy = -dy\,dx$ called alteration rule.

Wedge Product : $dx \wedge dy = -dy \wedge dx$ { $\wedge$ can be ignored while writing.

Associative law : $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

$\alpha \wedge (\beta + \gamma) = (\alpha \wedge \beta) + \alpha \wedge \gamma$.

Important Consequence : Repeats are zero.

$dx \wedge dx = 0$

$\because dx \wedge dx = -dx \wedge dx \Rightarrow 2(dx \wedge dx) = 0$

$\Rightarrow dx \wedge dx = 0$.

Ex. Calculate $(x^2 dx\, dy + xy\, dy\, dz) \wedge (xz\, dx + (z^2+y)\, dx\, dy)$

$(x^2 dx dy + xy dy dz) \wedge (xz dx)$ $\quad \alpha \wedge (\beta + \gamma) = \alpha\wedge\beta + \alpha\wedge\gamma$

$+ (x^2 dx dy + xy dy dz) \wedge (z^2+y) dx dy$

$\Rightarrow (x z x^2) \overset{0}{dx dy dx} + (xy)(xz)\, dy\, dz\, dx + (z^2+y)\, x^2 \overset{0}{dx dy\, dx dy}$

$+ (xy)(z^2+y) \overset{0}{dy dz dx dy}$

$= x^2 y z\, dy dz dx$ → No brackets since wedge product is associative $\alpha \wedge (\beta \wedge \gamma) = (\alpha\wedge\beta)\wedge\gamma$

P-Forms

(1) 0-form is just a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$.

(2) 1-form is an expression of the form $f dx + g dy + h dz$ { Since every 1-form ϕ can be written as $\sum f_i dx_i$.

(3) 2-form is an expression of the form
$f\, dx \wedge dy + g\, dx \wedge dz + h\, dy \wedge dz \equiv f dx dy + g dx dz + h dy dz$.

(4) 3-form is an expression of the form $f\, dx \wedge dy \wedge dz$
$\equiv f dx dy dz$.

On $\mathbb{R}^3$, the 4-forms or p-forms $(p \geq 4)$ are all 0. Due to alternation rule;

$dx \wedge dy \wedge dz \wedge dy = 0$
$dx \wedge dy \wedge dx \wedge dx = 0$ } Repitition.
etc.

Example: Computation of Wedge Product

1-Form { $\phi = x dx - y dy$; $\psi = z dx + x dz$ } - 1 Form.

$\theta = z dy$ $\qquad$ Using associativity and distributivity

$\phi \wedge \psi = (x dx - y dy) \wedge (z dx + x dz)$

$= (x dx - y dy) \wedge (z dx) + (x dx - y dy) \wedge (x dz)$

$= \cancel{(xz) dx \wedge dx} - yz\, dy \wedge dx + x^2\, dx \wedge dz - yx\, dy \wedge dz$

$\boxed{\phi \wedge \psi = -yz\, dy dx + x^2 dx dz - yx\, dy dz}$ { - 2-Form. Product of two 1-forms turns out to be a 2-form. Is it true in General?

Let us calculate $\theta \wedge \phi \wedge \psi$.

$\theta \wedge \phi \wedge \psi = (z dy) \wedge (x dx - y dy) \wedge (z dx + x dz)$

$\phi \wedge \psi = -yz\, dy dx + x^2 dx dz - xy\, dy dz$

$\theta \wedge (\phi \wedge \psi) = (z dy) \wedge (-yz\, dy dx + x^2 dx dz - xy\, dy dz)$

$= -z^2 y\, \overset{0}{dy dy} dx + z x^2\, dy dx dz - zxy\, \overset{0}{dy dy} dz$

$\boxed{\theta \wedge \phi \wedge \psi = z x^2\, dy dx dz}$

$\boxed{= -z x^2\, dx dy dz}$ { 3-form. Product of three 1-forms turns out to be a 3-form

Ex. Calculate $(x^2 dx\,dy + xy\,dy\,dz) \wedge (xz\,dx + (z^2+y)\,dx\,dy)$

$(x^2dx\,dy + xy\,dy\,dz) \wedge (xz\,dx)$ $\qquad \alpha \wedge (\beta+\gamma) = \alpha\wedge\beta + \alpha\wedge\gamma$

$+ (x^2dx\,dy + xy\,dy\,dz) \wedge (z^2+y)\,dx\,dy$

$\Rightarrow (xz\,x^2)\cancelto{0}{dx\,dy\,dx} + (xy)(xz)\,\underline{dy\,dz\,dx} + (z^2+y)\,x^2\cancelto{0}{dx\,dy\,dx\,dy}$

$+ (xy)(z^2+y)\cancelto{0}{dy\,dz\,dx\,dy}$

$= x^2yz\,\underline{dy\,dz\,dx}$ No brackets sice wedge product is associative $\alpha\wedge(\beta\wedge\gamma) = (\alpha\wedge\beta)\wedge\gamma$

P-Forms

(1) 0-form is just a differentiable function $f: \mathbb{R}^3 \to \mathbb{R}$.

(2) 1-form is an expression of the form $f\,dx + g\,dy + h\,dz$ $\left\{\right.$ Since every 1-form ϕ can be written as $\sum f_i\,dx_i$.

(3) 2-form is an expression of the form
$f\,dx\wedge dy + g\,dx\wedge dz + h\,dy\wedge dz \equiv f\,dx\,dy + g\,dx\,dz + h\,dy\,dz$.

(4) 3-form is an expression of the form $f\,dx\wedge dy\wedge dz$
$\equiv f\,dx\,dy\,dz$.

On $\mathbb{R}^3$, the 4-forms or p-forms $(p \geq 4)$ are all 0. Due to alternation rule;

$dx\wedge dy\wedge dz\wedge dy = 0$
$dx\wedge dy\wedge dx\wedge dx = 0$ $\Big\}$ Repitition.
etc.

Example: Computation of Wedge Product

1-Form $\{ \phi = x\,dx - y\,dy \;;\; \psi = z\,dx + x\,dz \}$ - 1Form.

$\theta = z\,dy$ $\qquad$ Using associativity and distributivity

$\phi\wedge\psi = (x\,dx - y\,dy)\wedge(z\,dx + x\,dz)$

$= (x\,dx - y\,dy)\wedge(z\,dx) + (x\,dx - y\,dy)\wedge(x\,dz)$

$= \cancel{(xz)\,dx\wedge dx} - yz\,dy\wedge dx + x^2\,dx\wedge dz - yx\,dy\wedge dz$

$\boxed{\phi\wedge\psi = -yz\,dy\,dx + x^2\,dx\,dz - yx\,dy\,dz}$ $\Big\}$ - 2-Form. Product of two 1-forms turns out to be a 2-form. Is it true in General?

Let us calculate $\theta\wedge\phi\wedge\psi$.

$\theta\wedge\phi\wedge\psi = (z\,dy)\wedge(x\,dx - y\,dy)\wedge(z\,dx + x\,dz)$

$\phi\wedge\psi = -yz\,dy\,dx + x^2\,dx\,dz - xy\,dy\,dz$

$\theta\wedge(\phi\wedge\psi) = (z\,dy)\wedge(-yz\,dy\,dx + x^2\,dx\,dz - xy\,dy\,dz)$

$= -z^2y\,\cancelto{0}{dy\,dy}\,dx + zx^2\,dy\,dx\,dz - zxy\,\cancelto{0}{dy\,dy}\,dz$

$\boxed{\theta\wedge\phi\wedge\psi = zx^2\,dy\,dx\,dz}$
$\boxed{= -zx^2\,dx\,dy\,dz}$ $\Big\}$ 3-form. Product of three 1-forms turns out to be a 3-form

Example $\phi = x\,dx - y\,dy$ (1-form)

$\eta = y\,dx\,dz + x\,dy\,dz$ (2-form)

Then $\phi \wedge \eta$

$= (x\,dx - y\,dy) \wedge (y\,dx\,dz + x\,dy\,dz)$

$= xy\,\underbrace{dx\,dx}_{0}\,dz + x^2\,dx\,dy\,dz - y^2\,dy\,dx\,dz - yx\,\underbrace{dy\,dy}_{0}\,dz$

$= x^2\,dx\,dy\,dz + y^2\,dx\,dy\,dz$ $dy\,dx = -dx\,dy$

$= (x^2+y^2)\,dx\,dy\,dz$ } ~~3-form~~ ~~1-form product 2-form = 3-form~~

Observation:

Product of a p-form and a q-form is a $(p+q)$-form

Lemma: If ϕ and ψ are 1-forms, then

$$\phi \wedge \psi = -\psi \wedge \phi$$

Proof: $\phi = \sum f_i\,dx_i$

$\psi = \sum g_i\,dx_i$

$\phi \wedge \psi = (\sum f_i\,dx_i) \wedge (\sum g_i\,dx_i)$

$= (f_1 dx_1 + f_2 dx_2 + f_3 dx_3) \wedge (g_1 dx_1 + g_2 dx_2 + g_3 dx_3)$

$= f_1 g_1 \underbrace{dx_1\,dx_1}_{=0} + f_1 g_2\,dx_1 dx_2 + f_1 g_3\,dx_1 dx_3$

$+ f_2 g_1\,dx_2\,dx_1 + f_2 g_2 \underbrace{dx_2\,dx_2}_{=0} + f_2 g_3\,dx_2 dx_3$

$+ f_3 g_1\,dx_3 dx_1 + f_3 g_2\,dx_3 dx_2 + f_3 g_3 \underbrace{dx_3 dx_3}_{=0}$

$= (f_1 g_2\,dx_1 dx_2 + f_1 g_3\,dx_1 dx_3 + f_2 g_1\,dx_2 dx_1 + f_2 g_3\,dx_2 dx_3$
$+ f_3 g_1\,dx_3 dx_1 + f_3 g_2\,dx_3 dx_2)$

$= (-g_2 f_1\,dx_2 dx_1 - g_3 f_1\,dx_3 dx_1 - g_1 f_2\,dx_1 dx_2 - g_3 f_2\,dx_3 dx_2$
$- g_1 f_3\,dx_1 dx_3 - g_2 f_3\,dx_2 dx_3)$

$= -(g_2 f_1\,dx_2 dx_1 + g_3 f_1\,dx_3 dx_1 + g_1 f_2\,dx_1 dx_2 + g_3 f_2\,dx_3 dx_2$
$+ g_1 f_3\,dx_1 dx_3 + g_2 f_3\,dx_2 dx_3$

$= -(g_1 dx_1 + g_2 dx_2 + g_3 dx_3)(f_1 dx_1 + f_2 dx_2 + f_3 dx_3)$

$= -(\sum g_i dx_i)(\sum f_i dx_i) = -\psi \wedge \phi.$

Exterior Derivative: Note f when differentiable is a 0-form.

For $f: \mathbb{R}^3 \to \mathbb{R}$, the differential df of f is the 1-form such that

$$df(v_p) = v_p[f] \quad \text{for all tangent vectors } v_p.$$

We can generalize the concept to define and apply it to a p-form and get a $(p+1)$-form.

Definition: If $\phi = \sum f_i dx_i$ is a 1-form on $\mathbb{R}^3$, the exterior derivative is the 2-form $d\phi = \sum df_i \wedge dx_i$

1form $\wedge$ 1form = 2-form.

$= \underbrace{df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3}_{\text{2 form}}$