Mappings And Tangent Maps

Introduction to Mappings:

Aim: To understand the mappings $f: \mathbb{R}^m \to \mathbb{R}^n$. different values of m and n.
In particular the cases $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f: \mathbb{R}^2 \to \mathbb{R}^3$ and $f: \mathbb{R}^3 \to \mathbb{R}^3$.

Examples: $f: \mathbb{R}^2 \to \mathbb{R}^2$; $f(x,y) = (x+y, x-y)$

$f: \mathbb{R}^2 \to \mathbb{R}^3$; $f(x,y) = (x^2, y^2, xy)$

$f: \mathbb{R}^3 \to \mathbb{R}^3$; $f(x,y,z) = (x, yz, y^2)$.

Defn: Given function $F: \mathbb{R}^n \to \mathbb{R}^m$, let $f_1, f_2, \cdots, f_m$ denote the real valued functions on $\mathbb{R}^n$ such that;

$$F(p) = \left(f_1(p), f_2(p), \cdots, f_m(p)\right) ; \quad f_i: \mathbb{R}^n \to \mathbb{R}.$$

for all points $p \in \mathbb{R}^n$.

These functions are called Euclidean coordinate functions of F, and we write $F = (f_1, f_2, \cdots, f_m)$

Example: $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = (xy, x^2+y^2, y^2) = (f_1, f_2, f_3)$$

$$\left.\begin{aligned} f_1 &= xy = f_1(x,y) \\ f_2 &= x^2+y^2 = f_2(x,y) \\ f_3 &= y^2 = f_3(x,y) \end{aligned}\right\} \begin{aligned} &f_1: \mathbb{R}^2 \to \mathbb{R} \\ &f_2: \mathbb{R}^2 \to \mathbb{R} \\ &f_3: \mathbb{R}^3 \to \mathbb{R}. \end{aligned}$$

Mappings : A function $F: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if all its Euclidean coordinate functions are differentiable. A differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a Mapping.

Example: $F: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x,y,z) = (x^2, yz, xy).$$

Here coordinate functions are $f_1: \mathbb{R}^3 \to \mathbb{R}$ given as $f_1(x,y,z) = x^2$

Similarly $f_2: \mathbb{R}^3 \to \mathbb{R}$ given as $f_2(x,y,z) = yz$ All

$f_3: \mathbb{R}^3 \to \mathbb{R}$ given as $f_3(x,y,z) = xy$.

All coordinate functions are differentiable & hence $F(x,y,z)$ is indeed a Mapping.

For $p = (p_1, p_2, p_3)$

$$F(p) = (f_1(p), f_2(p), f_3(p)) = (x(p)^2, y(p)z(p), x(p)y(p))$$

$$\boxed{F(p) = (p_1^2, p_2p_3, p_1p_2)}$$

$x: \mathbb{R}^3 \to \mathbb{R}$

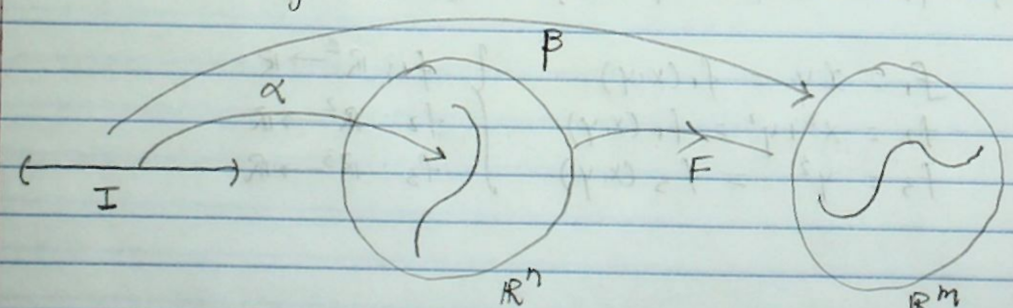
$x(p_1,p_2,p_3) = p_1$

$y(p_1,p_2,p_3) = p_2$

$z(p_1,p_2,p_3) = p_3$

Aim: To understand these mappings in detail and we will make use of simple kind of mappings i.e. curves in order to understand these complicated mappings.

Definition: If $\alpha: I \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ and $F: \mathbb{R}^n \to \mathbb{R}^m$ is a mapping, then the composite function $\beta = F(\alpha): I \to \mathbb{R}^m$ called the image of α under F.



Example: Consider the mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F(u,v) = (u^2 - v^2, 2uv)$$

To see the effect of this mapping we consider images of special curve α in $\mathbb{R}^2$, namely circle of radius $= r$.

$$\alpha(t) = (r\cos t, r\sin t)\ ;\ 0 \le t \le 2\pi$$

Now find the image ~~of~~ $\beta = F(\alpha(t))$

$$\beta = F(\alpha(t)) = (r^2\cos^2 t - r^2\sin^2 t,\ 2(r\cos t)(r\sin t))$$

$$= (r^2\cos 2t,\ r^2\sin 2t)\ ;\ 0 \le t \le 2\pi$$

travel around circle twice.

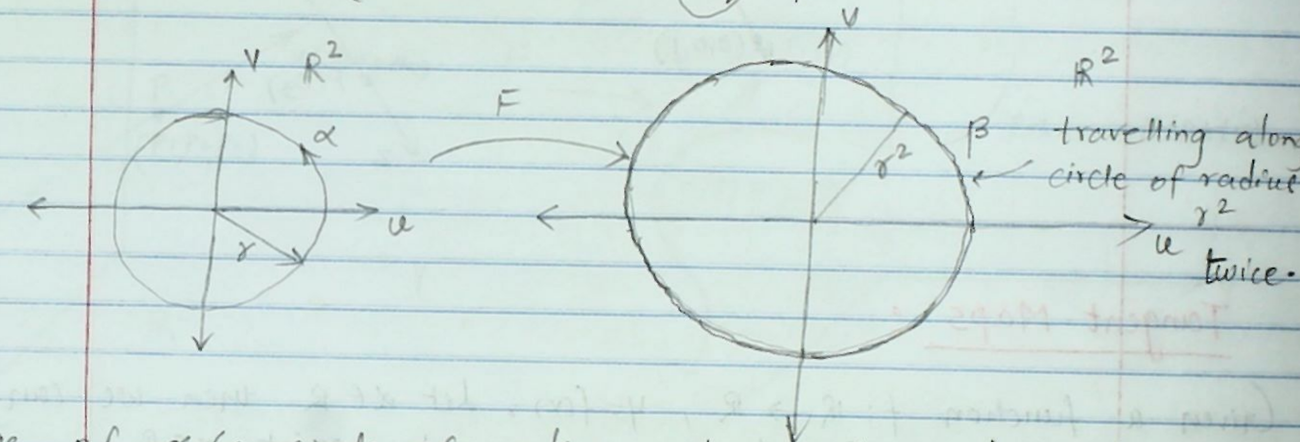
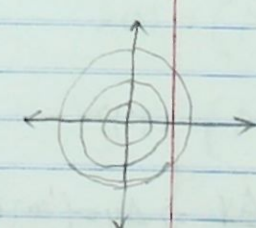


Image of α (a circle of radius $= r$ travelled once) under F is circle of radius r^2 travelled twice.



"$\mathbb{R}^2$ can be written as union of circles of all possible radii $\in \mathbb{R}$. Thus the ~~mapping of $\mathbb{R}^2$~~ mapping F will map $\mathbb{R}^2$ to itself wrapped twice."

Each point in $\mathbb{R}^2$ is getting mapped to another point in $\mathbb{R}^2$ but twice.

Example: Consider the mapping $F: \mathbb{R}^2 \to \mathbb{R}^3$

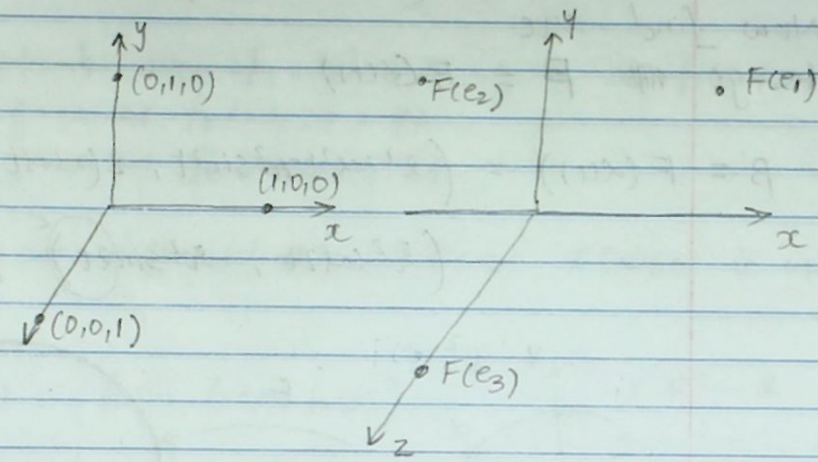
$$F(x,y,z) = (x-y,\ x+y,\ 2z).$$

$f_1 = x-y,\ f_2 = x+y,\ f_3 = 2z.$

Observe that F is a linear transformation.

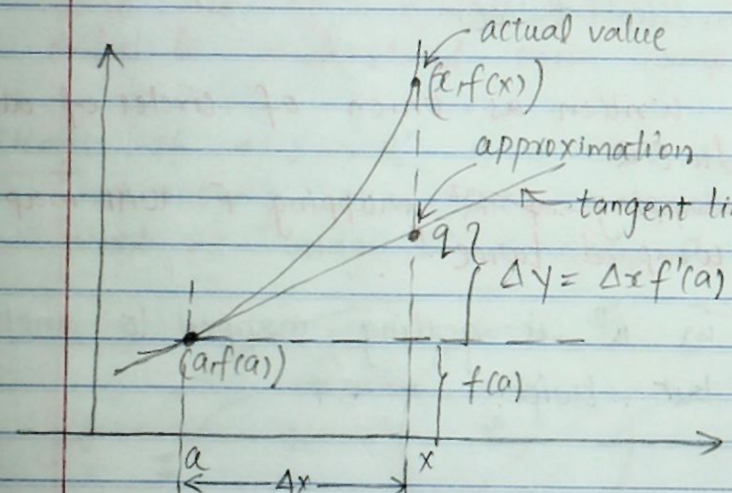
$e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

$F(e_1) = (1,1,0)$
$F(e_2) = (-1,1,0)$
$F(e_3) = (0,0,2)$



Tangent Maps:

Given a function $f: \mathbb{R} \to \mathbb{R}$, $y = f(x)$. Let $a \in \mathbb{R}$ then we can approximate functional value at a nearby point $x \in \mathbb{R}$.



$$f'(a) = \frac{\Delta y}{\Delta x} \Rightarrow \Delta y = f'(a)\Delta$$

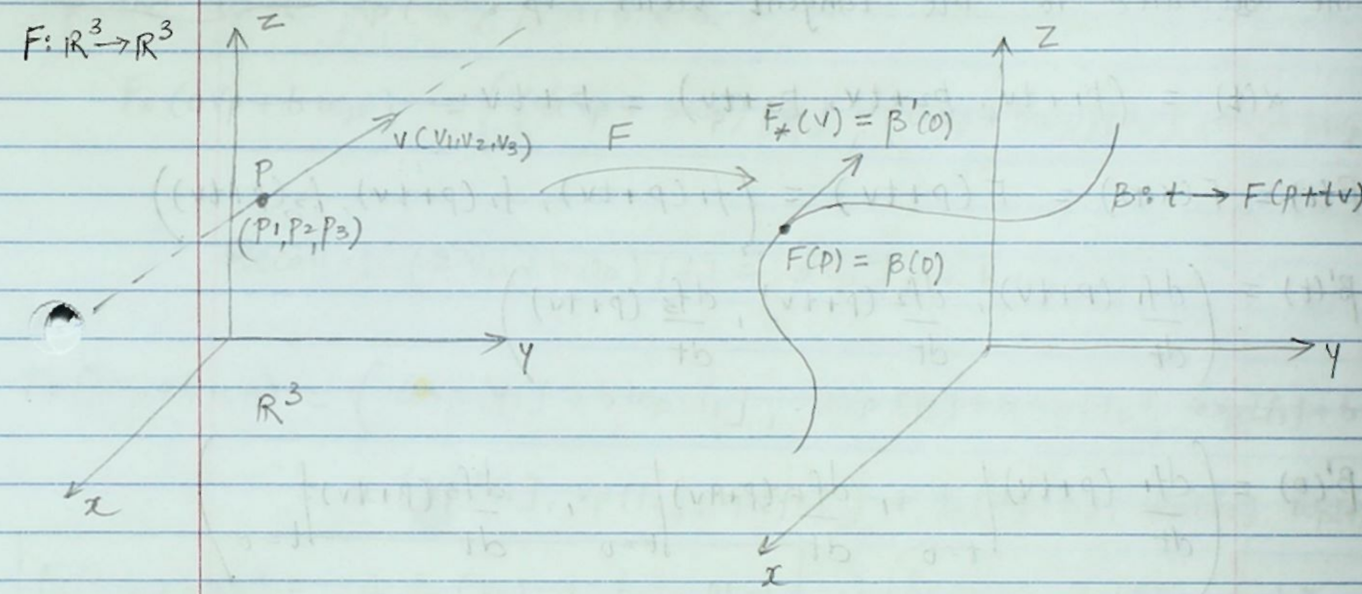
$$f(x) \approx f(a) + \Delta x f'(a).$$

So we used the tangent line to approximate a function in the vicinity of a point 'a'.

Question: Can we generalize this process to mappings $f: \mathbb{R}^n \to \mathbb{R}^m$?

Next Goal: To find analogous linear approximation for the mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ near a point p in $\mathbb{R}^n$.

For this we use special curves in $\mathbb{R}^n$, namely lines associated with tangent vector v_p.



$$\alpha(t) = (p_1+tv,\ p_2+tv,\ p_3+tv) \quad ; \quad \beta = F(\alpha(t))$$

$$\alpha(0) = (p_1, p_2, p_3) \qquad \beta(0) = F(\alpha(0)) = F(p).$$

Tangent Map (Defn.) Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If v is $p = (p_1,p_2,p_3)$, $v = (v_1,v_2,v_3)$ a tangent vector to $\mathbb{R}^n$ at p. Let $F_*(v)$ be the initial velocity vector of the curve $t \to F(p+tv)$. The resulting function F_* sends tangent vectors to $\mathbb{R}^n$ to tangent vectors to $\mathbb{R}^m$, and is called Tangent Map of F.

How to calculate Tangent Maps

Proposition: Let $F = (f_1, f_2, \cdots, f_m)$ be a mapping from $\mathbb{R}^n \to \mathbb{R}^m$. If v is a tangent vector to $\mathbb{R}^n$ at p, then

$$F_*(v) = \left(v[f_1], v[f_2], \cdots, v[f_m]\right) \text{ at } F(p).$$

$v[f_i]$ = directional derivative of real valued function f_i in the direction of v_p

Proof: For simplicity take $m = 3$. $F = (f_1, f_2, f_3)$.

Line associated to the tangent vector v_p:

$$\alpha(t) = (p_1 + tv, p_2 + tv, p_3 + tv) = p + tv.$$

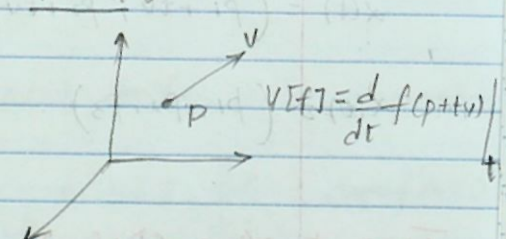
$$\beta(t) = F(\alpha(t)) = F(p+tv) = \left(f_1(p+tv), f_2(p+tv), f_3(p+tv)\right)$$

$$\beta'(t) = \left(\frac{df_1}{dt}(p+tv), \frac{df_2}{dt}(p+tv), \frac{df_3}{dt}(p+tv)\right)$$

$$\beta'(0) = \left(\frac{df_1}{dt}(p+tv)\Big|_{t=0}, \frac{df_2}{dt}(p+tv)\Big|_{t=0}, \frac{df_3}{dt}(p+tv)\Big|_{t=0}\right)$$

$$\boxed{\beta'(0) = \left(v[f_1], v[f_2], v[f_3]\right)}$$

Recall:

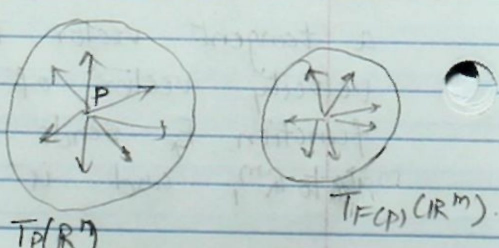


Tangent Map of a Mapping at a point p.

The tangent map F_* sends tangent vectors at p to tangent vectors at $F(p)$. Thus for each p in $\mathbb{R}^n$, the map gives rise to a function

$$F_*: T_p(\mathbb{R}^n) \longrightarrow T_{F(p)}(\mathbb{R}^m).$$

called the Tangent map of F at p.



Linearity of Tangent Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

If $F: \mathbb{R}^n \to \mathbb{R}^m$ is a mapping, then at each point p of $\mathbb{R}^3$ the tangent map $F_*: T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ is a linear transformation.

Proof: For simplicity $m = 3$, $F = (f_1, f_2, f_3)$

$$F_*(v_p) = \left(v[f_1], v[f_2], v[f_3]\right)_{F(p)} \in T_{F(p)}(\mathbb{R}^m) \equiv T_{F(p)}(\mathbb{R}^3).$$

To show $F_*: T_p(\mathbb{R}^n) \longrightarrow T_{F(p)}(\mathbb{R}^m)$ is linear we consider v_p and $w_p \in T_p(\mathbb{R}^n)$; $a, b \in \mathbb{R}$.

$$F_*(av_p + bw_p) = \left((av_p + bw_p)[f_1], (av_p + bw_p)[f_2], (av_p + bw_p)[f_3]\right)$$

Recall: $(av_p + bw_p)[f] = av_p[f] + b\,w_p[f]$

$$F_*(av_p + bw_p) = \left(av_p[f_1] + bw_p[f_1], av_p[f_2] + bw_p[f_2], av_p[f_3] + bw_p[f_3]\right)$$
$$= a\left(v_p[f_1], v_p[f_2], v_p[f_3]\right) + b\left(w_p[f_1], w_p[f_2], w_p[f_3]\right)$$

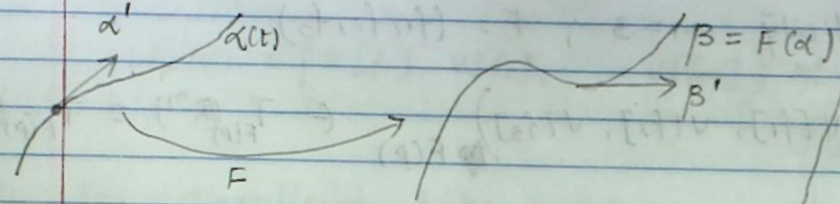
$$\boxed{F_*(av_p + bw_p) = a\,F_*(v_p) + b\,F_*(w_p)}$$; Recall: $T_p(\mathbb{R}^n)$ and $T_{F(p)}(\mathbb{R}^m)$ are vector spaces.

Thus we have shown that the transformation F_* between two vector spaces is a linear transformation.

We now look at another property of ~~these~~ the tangent map. We are deriving these properties using the proposition where we have explicitly constructed the tangent map.

Velocity of an Image Curve :

Mappings Preserve Velocities : Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\beta = F(\alpha)$ is the image of a curve in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$



α' - a tangent vector on the curve α. Its image β under the mapping F. Calculate velocity vector at any point on β $(=\beta')$. Then β' and α' can be related via the transformation F_*

Proof : for simplicity take $m=3$

If $F = (f_1, f_2, f_3)$, then $\beta = F(\alpha)$

$$= (f_1(\alpha), f_2(\alpha), f_3(\alpha))$$

From the previous proposition ;

directional derivatives of f_j in the direction of $\alpha'(t)$

$$F_*(\alpha'(t)) = (\alpha'(t)[f_1], \alpha'(t)[f_2], \alpha'(t)[f_3])_{F(\alpha(t))}$$

Recall :
We have from a previous lemma ; For α a curve in $\mathbb{R}^3$ let f be a differentiable function on $\mathbb{R}^3$. Then $\alpha'(t)[f] = \frac{d}{dt}(f(\alpha))(t)$.

So we have $\alpha'(t)[f_i] = \frac{d}{dt} f_i(\alpha)(t)$

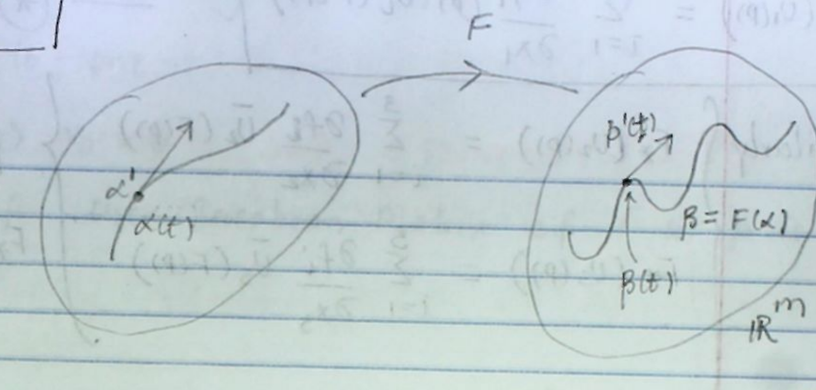
Hence we have
$$F_*(\alpha'(t)) = \left(\frac{df_1(\alpha)}{dt}(t), \frac{df_2(\alpha)}{dt}(t), \frac{df_3(\alpha)}{dt}(t)\right)_{\beta(t) = F(\alpha(t))}$$
$$= \beta'(t).$$

We have $F = (f_1, f_2, f_3) =$

$$\beta(t) = F(\alpha(t)) = (f_1(\alpha(t)), f_2(\alpha(t)), f_3(\alpha(t)))$$

$$\beta'(t) = \left(\frac{df_1(\alpha(t))}{dt}, \frac{df_2(\alpha(t))}{dt}, \frac{df_3(\alpha(t))}{dt}\right)$$

Thus $\boxed{F_*(\alpha') = \beta'}$



Tangent Maps on Vector Fields

Let $\{U_j\}$ $(1 \le j \le n)$ and $\{\bar{U}_i\}$ $(1 \le i \le m)$ be the natural frame fields of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then ;

If $F = (f_1, f_2, \dots, f_m)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$F_*(U_j(p)) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(p)\, \bar{U}_i(p) \quad ; 1 \le j \le n.$$

Proof : For simplicity take $m = n = 3$ Recall : $U_1(p) = (1,0,0)_p$

$$F = (f_1, f_2, f_3)$$

$$F_*(U_1(p)) = (U_1(p)[f_1], U_1(p)[f_2], U_1(p)[f_3])$$

Recall :
$$v_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(p) = v_1 \frac{\partial f}{\partial x_1}(p) + v_2 \frac{\partial f}{\partial x_2}(p) + v_3 \frac{\partial f}{\partial x_3}(p)$$

$$U_1(p)[f_1] = 1 \cdot \frac{\partial f_1}{\partial x_1}(p) + 0 \cdot \frac{\partial f_1}{\partial x_2}(p) + 0 \cdot \frac{\partial f_1}{\partial x_3}(p) = \frac{\partial f_1}{\partial x_1}(p)$$

$$U_1(p)[f_2] = \frac{\partial f_2}{\partial x_1}(p)$$

$$U_1(p)[f_3] = \frac{\partial f_3}{\partial x_1}$$

Thus
$$F_*(U_1(p)) = \left(\frac{\partial f_1}{\partial x_1}(p), \frac{\partial f_2}{\partial x_1}(p), \frac{\partial f_3}{\partial x_1}(p)\right)_{F(p)} =$$
$$= \frac{\partial f_1}{\partial x_1}(p)\, \bar{U}_1(F(p)) + \frac{\partial f_2}{\partial x_1}(p)\, \bar{U}_2(F(p)) + \frac{\partial f_3}{\partial x_1}(p)\, \bar{U}_3(F(p))$$

$$\boxed{F_*(U_1(p)) = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_1}(p)\,\bar{U}_i(F(p))} \quad \text{— } (*)$$

Similarly

$$F_*(U_2(p)) = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_2}\,\bar{U}_2(F(p))$$

$$F_*(U_3(p)) = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_3}\,\bar{U}_3(F(p))$$

In General

$$F_*(U_j(p)) = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_j}(p)\,\bar{U}_i(F(p))$$

$j = 1, 2, 3.$

Notice $(*)$ is the expression of the image of the first basis vector $(1,0,0)_p$ of the Tangent Space $T_p(\mathbb{R}^3)$. Similarly $F_*(U_2(p))$ is the image of second element of basis of tangent space $T_p(\mathbb{R}^3)$ and so on. Now note that F_* is a linear transformation applied to the basis vectors of tangent space at point $p = T_p(\mathbb{R}^3)$. In other words, we have actually found matrix of linear transformation - which is the tangent map.

Matrix Representation of the Tangent Map:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

= Jacobian of the mapping F at point p.

It is the matrix of linear transformation $F_*: T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$

We now study the mappings F using the tangent maps.

Defn: A mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ is "Regular" provided that at every point p of $\mathbb{R}^n$ the tangent map F_{*p} is one-one

$$F_{*p}: T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m).$$

From Linear Algebra:

For $p \in \mathbb{R}^n$ the following are equivalent;

(1) F_{*p} is one-to-one.

(2) $F_{*p}(v_p) = 0 \Rightarrow v_p = 0$

(3) The rank of the Jacobian Matrix of F at p is n.

Diffeomorphism:

Defn: If $F: \mathbb{R}^n \to \mathbb{R}^m$ has an inverse mapping, then F is called Diffeomorphism.

$F: \mathbb{R}^n \to \mathbb{R}^m$

$\exists\, G: \mathbb{R}^m \to \mathbb{R}^n$ s.t. $F \circ G = G \circ F = I$