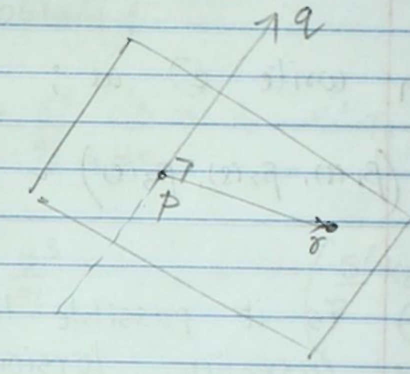
Frenet Approximation & Plane Curves

Recall :

Planes : A plane through p and orthogonal to $q \neq 0$ consists of all points r such that

$$(r-p)\cdot q = 0$$



Aim : To approximate a curve near an arbitrary point on the curve.

How to achieve this ?

(1) Use Taylor series
(2) Use Frenet Frame at a point.

We are essentially trying to approximate a curve using its geometric information like curvature, torsion etc which are given by Frenet Formulas.

We shall be dealing with unit speed curves to begin with

Frenet Approximation

Approximation of Euclidean coordinate functions.

Let $\beta = (\beta_1, \beta_2, \beta_3)$ be a unit speed curve. We can approximate each one of the βi ; $i = 1,2,3$ using Taylors Series in the following way ;

$$\beta i \sim \beta i(0) + \frac{d\beta i}{ds}(0)\, s + \frac{d^2\beta_i}{ds^2}(0)\,\frac{s^2}{2!} + \frac{d^3\beta_i}{ds^3}(0)\,\frac{s^3}{3!} + \cdots$$

① $\beta = (\beta_1, \beta_2, \beta_3) \approx \Big(\beta_1(0) + \beta_1'(0)s + \beta_1''(0)\frac{s^2}{2!} + \cdots,\ \beta_2(0) + \beta_2'(0)s + \beta_2''(0)\frac{s^2}{2!} + \cdots,$

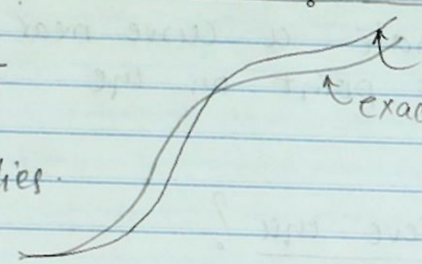
$\beta_3(0) + \beta_3'(0)s + \beta_3''(0)\frac{s^2}{2!} + \cdots\Big)$

② $$\beta \approx \underbrace{\big(\beta_1(0), \beta_2(0), \beta_3(0)\big)}_{\beta(0)} + s\underbrace{\big(\beta_1'(0), \beta_2'(0), \beta_3'(0)\big)}_{\beta'(0)} + \frac{s^2}{2!}\underbrace{\big(\beta_1''(0), \beta_2''(0), \beta_2''(0)\big)}_{\beta''(0)} + \frac{s^3}{3!}\underbrace{(\quad)}_{\beta'''(0)}$$

We can write ② as;

$$\boxed{\beta(s) = \big(\beta_1(s), \beta_2(s), \beta_3(s)\big) \approx \beta(0) + s\beta'(0) + \frac{s^2}{2!}\beta''(0) + \frac{s^3}{3!}\beta'''(0)} \quad (*)$$

From (*) is it possible to determine curvature, torsion and other geometric properties of the original curve?



Let us now apply Frenet Formulas to find $\beta'(0)$, $\beta''(0)$ & $\beta'''(0)$.

We know $T(s) = \beta'(s)$ so this gives

$\{\beta'(0) = T(0) = T_0 \text{ (denote)}\}$

We also know that;

$T'(s) = \kappa(s)N(s) \Rightarrow \beta''(s) = \kappa(s)N(s)$

$\Rightarrow \beta''(0) = \kappa(0)N(0)$

Denote $\beta''_0 = \kappa_0 N_0$.

Let us now evaluate $\beta'''(0)$.

We use Leibniz Rule; $\because \beta''(s) = \kappa(s)N(s)$

$$\left\{\beta'''(s) = \frac{d}{ds}(\kappa N) = \frac{d\kappa}{ds}N + \frac{dN}{ds}\kappa.\right\} \quad ③$$

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$T = \beta'$

$T' = \beta''$

From the Frenet Formulas we have $N'(s) = -\kappa T + \tau B$.

Substitute in ③ to obtain;

$$\left\{\beta''' = \frac{d\kappa}{ds}N - \kappa^2 T + \tau\kappa B.\right\}$$

So we get $\beta'''_0 = \beta'''(0) = \frac{d\kappa}{ds}(0)N_0 - \kappa_0^2 T_0 + \tau_0\kappa_0 B_0$.

Substitute the values of $\beta'(0)$, $\beta''(0)$ and $\beta'''(0)$ in (*) to obtain;

$$\beta(s) \approx \beta(0) + s(T_0) + \frac{s^2}{2!}(\kappa_0 N_0) + \frac{s^3}{3!}\left(\frac{d\kappa}{ds}(0)N_0 - \kappa_0^2 T_0 + \tau_0\kappa_0 B_0\right)$$

$$\beta(s) \approx \beta(0) + T_0\Big(s + \underbrace{\frac{s^3}{3!}(-\kappa_0^2)}_{\text{ignore}}\Big) + B_0\left(\frac{s^3}{3!}\tau_0\kappa_0\right) + N_0\Big(\frac{s^2}{2}\kappa_0 + \underbrace{\frac{s^3}{3!}\frac{d\kappa}{ds}(0)}_{\text{ignore}}\Big)$$

We finally get the following approximation.

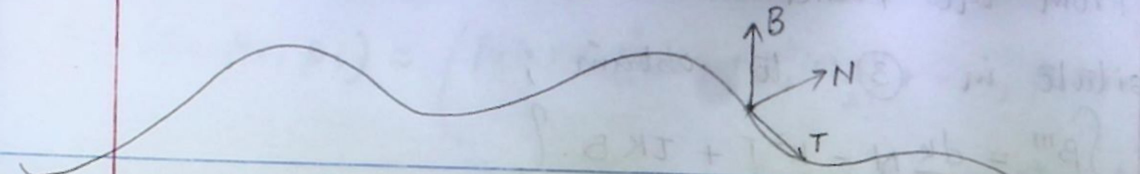
Defn (Frenet Approximation) Let β be a unit speed curve then; $\hat{\beta}(s) = \beta(0) + sT_0 + \kappa_0\frac{s^2}{2}N_0 + \tau_0\kappa_0\frac{s^3}{6}B_0$ is called Frenet approximation of β near $s = 0$.

Question: Can we do the same process for a point s_0 other than $s = 0$.

Frenet Approximation at point s_0.

Using Taylor series expansion around $s = s_0$ we can derive;

$$\boxed{\hat{\beta}_{s_0}(s) = \beta(s_0) + (s - s_0)T_{s_0} + \kappa_{s_0}\frac{(s-s_0)^2}{2!}N_{s_0} + \kappa_{s_0}\tau_{s_0}\frac{(s-s_0)^3}{3!}B_{s_0}}$$



Given a curve, corresponding to every point on the curve we have the Frenet Frame (T, B, N) and using this Frenet Frame and Frenet Formulas we can approximate every point on the curve.

Some Geometric Interpretations of Frenet Approximations

$$\hat{\beta}(0) = \underbrace{\beta(0) + s T_0}_{\text{Consider only the linear terms in } s} + \underbrace{k_0 \frac{s^2}{2} N_0 + \tau_0 k_0 \frac{s^3}{6} B_0}_{\text{ignore}}$$

The curve defined by $s \to \beta(0) + s T_0$ is the Tangent line to β at $s=0$.

It is the best linear approximation to β near $\beta(0)$.

$$\beta(s) \sim \beta(0) + s T_0$$

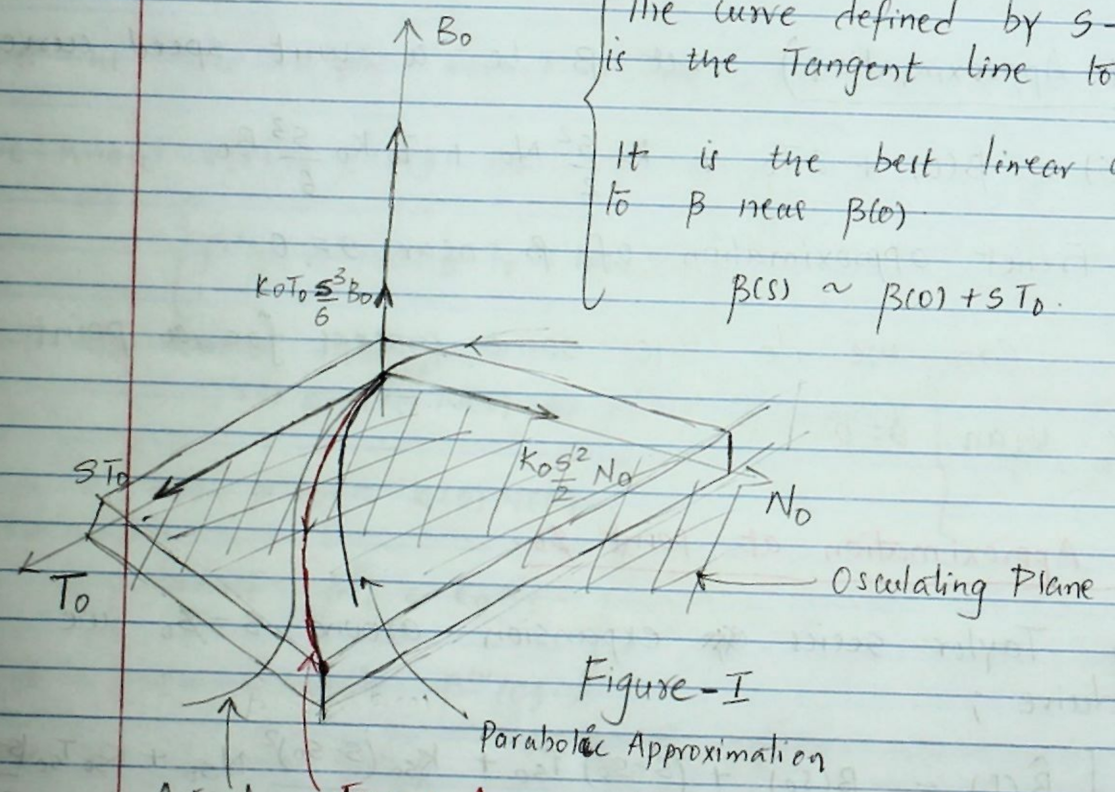


Figure-I

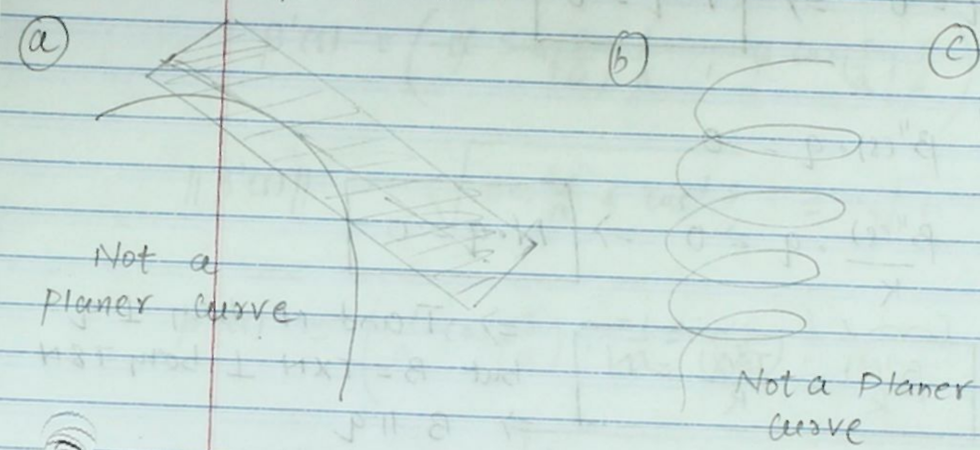
Next consider the first three terms of the approximation

$$\hat{\beta}(s) = \beta(0) + s T_0 + k_0 N_0 \frac{s^2}{2}$$

– This gives a quadratic approximation & hence a parabola in the T_0 & N_0 plane. This is also called the $\boxed{\text{Osculating Plane.}}$

Planer Curve:

A curve in $\mathbb{R}^3$ is the set of all points which lie in the same plane in $\mathbb{R}^3$.



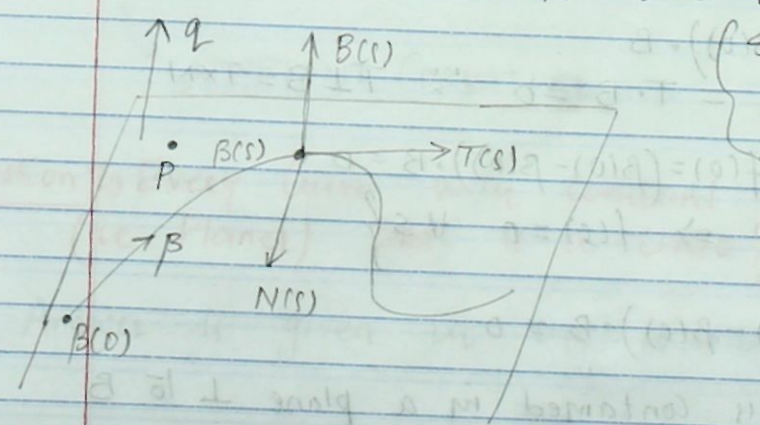
(c) The parabola of quadratic approximation lies in the osculating Plane. This is an example of Planer Curve.

Characterization of Planer Curves in terms of the Geometric parameter studied as part of Frenet Formulas.

– $\boxed{\text{Planer Curves have 0-Torsion.}}$

Theorem: Let β be a curve in $\mathbb{R}^3$ with positive curvature $k > 0$. Then β is a Planer curve $\Leftrightarrow$ $\tau = 0$ i.e. a torsion free curve.

So torsion measures how far away from being a planer is a given curve.



Proof :($\Rightarrow$) Let β be a planer curve, to show $\tau = 0$

Since β lies in a plane; $\exists$ a point p on the plane and vector q $\perp$ to the plane such that $(\beta(s)-p)\cdot q = 0$

$$(\beta(s)-p)\cdot q = 0$$

Differentiating $\beta'(s)\cdot q - (\beta(s)-p)\cdot q'' = 0$

$$\beta'(s)\cdot q = 0 \Rightarrow \boxed{T\cdot q = 0}$$

Differentiating again;

$$\beta''(s)\cdot q = 0$$

$$\Rightarrow \frac{\beta''(s)}{K}\cdot q = 0 \Rightarrow \boxed{N\cdot q = 0}$$

$$\boxed{\frac{\beta''(s)}{K} = \frac{T'(s)}{K} = N}$$

$\Rightarrow$ T and N both $\perp$ q.

but $B = T\times N \perp$ both T & N

$\Rightarrow B \parallel q$

$\Rightarrow B = \frac{q}{\|q\|} \Rightarrow B' = 0$

$$\Rightarrow B' = 0 \Rightarrow B' = \tau N = 0 \Rightarrow \boxed{\tau = 0 \because N \neq 0}$$

Proof ($\Leftarrow$) Let $\tau = 0$; to show that the curve is planer.

$\tau = 0 \Rightarrow B' = -\tau N = 0 \Rightarrow B' = 0$

Consider $f(s) = (\beta(s)-\beta(0))\cdot B$

$f'(s) = \beta'(s)\cdot B = T\cdot B = 0 \quad \because T\perp B = T\times N$

So $f'(s) = 0$; also $f(0) = (\beta(0)-\beta(0))\cdot B = 0$

Thus $\{f'(s) = 0,\ f(0) = 0 \Rightarrow f(s) = 0 \ \forall s\}$

$\therefore f(s) = 0 \Rightarrow (\beta(s)-\beta(0))\cdot B = 0$

$\Rightarrow \beta(s)$ is contained in a plane $\perp$ to B and containing $\beta(0)$

$\Rightarrow \beta$ is Planer

Example: Consider the following circle

$$\beta(s) = \left(a\cos\frac{s}{a},\ a\sin\frac{s}{a}\right)$$

Then its curvature is given as; $K = \left\|\frac{dT}{ds}\right\| = \|\beta''\|$

$$\beta'(s) = \left(-\cancel{a}\sin\left(\frac{s}{a}\right)\frac{1}{\cancel{a}},\ \cancel{a}\cos\left(\frac{s}{a}\right)\frac{1}{\cancel{a}}\right)$$

$$\|\beta'(s)\| = \sqrt{\sin^2\frac{s}{a} + \cos^2\frac{s}{a}} = \sqrt{1} = 1$$; hence unit speed curve

$$\beta''(s) = \left(\frac{1}{a}\cos\left(\frac{s}{a}\right),\ \frac{-1}{a}\sin\left(\frac{s}{a}\right)\right)$$

$$K = \|\beta''(s)\| = \sqrt{\left(\frac{1}{a}\cos\left(\frac{s}{a}\right)\right)^2 + \left(-\frac{1}{a}\sin\left(\frac{s}{a}\right)\right)^2}$$

$$= \sqrt{\frac{1}{a^2}(1)} = \frac{1}{a}$$

$\boxed{K = \frac{1}{a}}$ For a unit speed circle with radius $= a$

CONSTANT CURVATURE.

Question: Is Every Curve with constant curvature and Zero Torsion (ie Planer) is a circle? — Partially True.

Answer is given in the following lemma.

Lemma: If β is a unit speed curve with constant curvature k and torision $\tau = 0$, then β is part of circle of radius $\frac{1}{k}$.

Proof: We must show that there exists a point c (center of circle) such that its distance from $\beta(s)$ remains $= \frac{1}{k}$

Since $\tau = 0 \Rightarrow$ Planer curve

$k = c \Rightarrow$ curvature is constant.

Let us define a new curve $\gamma = \beta + \frac{1}{k} N$ $\quad \gamma(s) = \beta(s) + \frac{1}{k} N(s)$.

$$\gamma' = \beta' + \frac{1}{k} N' \quad \{ k \text{ is const, comes out of derivative.}$$

From Frenet Formulas $N' = -kT + \underbrace{\tau}_{=0} B^{0}$ ~~~~ torsion $= 0$

So $\gamma' = \beta' + \frac{1}{\cancel{k}}(-\cancel{k}T)$

$= \beta' - T = T - T = 0 \equiv (0,0,0)$

$\Rightarrow \gamma =$ constant $= (c_1, c_2, c_3) = c$; $c_i \ \forall i$ are constants

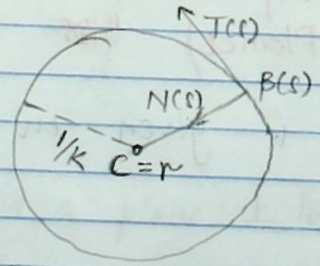
Now $d(\beta, \gamma) = d(\beta, \underbrace{\beta + \frac{1}{k}N}_{=c}) = d(\beta, c)$

$= \| \cancel{\beta} - \cancel{\beta} - \frac{1}{k} N \|$

$= \| \frac{-1}{k} N \| = \frac{1}{k}$; $\because \|N\| = 1$.

$$\boxed{d(\beta, \gamma) = \frac{1}{k} ; \text{const.}}$$

So γ gives us the point we were looking for and we have shown $\exists$ a point $c (\equiv \gamma)$ s.t. its distance from $\beta(s)$ remains constant.

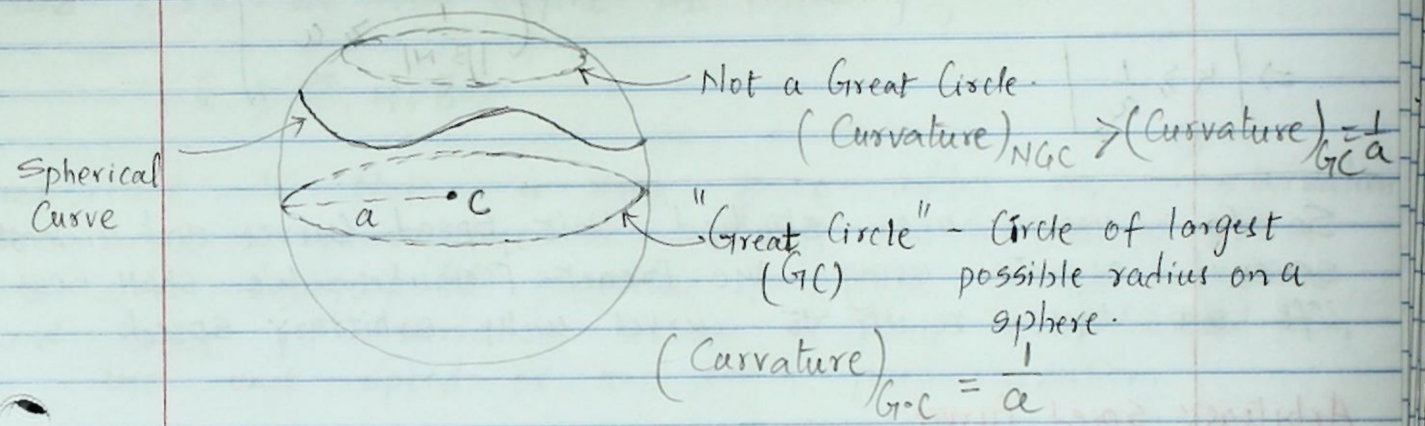


Planer Curve ($\tau = 0$) with constant curvature is a CIRCLE or part of it.

Remark: So far in theorems O & O we have made use of the Frenet Formulas. We shall now look at another application of Frenet Formulas in understanding the so called Spherical Curves.

Spherical Curves

Roughly, these are curves on a sphere.



From the above figure we can make a conjecture.

Conjecture: A spherical curve has curvature $k \geq \frac{1}{a}$, where 'a' is the radius of the ~~sphere~~ sphere.

Proof: Let β be a curve on a sphere of radius $= a$

Since β lies on the ~~curve~~ sphere $\Rightarrow \|\beta\| = a$.

$\Rightarrow \beta \cdot \beta = \|\beta\|^2 = a^2$

Differentiating; $\beta' \cdot \beta + \beta \cdot \beta' = 0 \Rightarrow 2\beta \cdot \beta' = 0 \Rightarrow \beta \cdot \beta' = 0$

$\Rightarrow \{ T \cdot \beta = 0 \}$ (*)

Differentiate (*) again; $T \cdot \beta' + T' \cdot \beta = 0$; $\beta' = T$

$\Rightarrow T \cdot \beta' = T \cdot T = \|T\|^2 = 1$

$\Rightarrow T' \cdot \beta + T \cdot \beta' = T' \beta + T \cdot T = 0$

$\Rightarrow \boxed{T' \cdot \beta + 1 = 0}$

From Frenet formulas we have $T' = \kappa N$

So we have $1 + \beta\cdot(\kappa N) = 0$

$\Rightarrow \beta\cdot(\kappa N) = -1 \Rightarrow \boxed{\kappa = \frac{-1}{\beta\cdot N}}$ Use Schwarz Inequality on $\beta\cdot N$.

Using Schwarz Inequality we get $|\beta\cdot N| \leq \|\beta\|\,\|N\| \leq a$ $\quad \|N\| = 1$.

$\Rightarrow \kappa = |\kappa| = \left|\frac{-1}{\beta\cdot N}\right| = \frac{1}{|\beta\cdot N|} \geq \frac{1}{a}$ $\left\{ |\beta\cdot N| \leq a \quad \frac{1}{|\beta\cdot N|} \geq a \right\}$

$\Rightarrow \boxed{\kappa \geq \frac{1}{a}}$

So far we have studied Unit Speed Curves and understood several aspects using the Frenet Formulas. We shall now lift all these results to curves with arbitrary speed.

Arbitrary Speed Curves

The idea is to reparametrize the curve using arc length parametrization which is a unit speed parametrization.

Recall: Theorem: Let $\alpha : I \to \mathbb{R}^3$ be a regular curve. Then there exists a reparametrization β of α such that $\|\beta'\| = 1$.

Also recall the example ⓪ from the section on reparametrization;

We had $\alpha(t) = (a\cos t, a\sin t, bt)$ and obtained $s = \frac{t\sqrt{a^2+b^2}}{\cancel{\sqrt{a^2+b^2}}}$

We reparametrized as;

$$\alpha(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\frac{s}{\sqrt{a^2+b^2}}\right)$$

Now $\alpha(s(t)) = \left(a\cos\frac{s(t)}{\sqrt{a^2+b^2}}, a\sin\frac{s(t)}{\sqrt{a^2+b^2}}, b\frac{s(t)}{\sqrt{a^2+b^2}}\right)$

$$\alpha(s(t)) = (a\cos t, a\sin t, bt) = \alpha(t).$$

So we have $\boxed{\alpha(s(t)) = \alpha(t)}$ — To be used in subsequent discussions = reparametrized

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Seen from previous discussions; If $\bar{\alpha}$ is a unit speed curve and $\bar{\kappa} > 0$ we can define the following;

$$\bar{\kappa}, \bar{\tau}, \bar{T}, \bar{N}, \bar{B}$$

Important to Note is that since after reparametrization the curve remains the same, (only the method to obtain the curve changes) its geometric properties remain the same and hence its curvature, torsion etc can be calculated from the unit speed or arc length parametrization.

Defn: Let α be a regular curve and let $\bar{\alpha}$ be its arc-length parametrization. Then we define;

- Curvature function of α : $\kappa = \bar{\kappa}(s)$
- Torsion function of α : $\tau = \bar{\tau}(s)$
- Unit Tangent vector field of α : $T = \bar{T}(s)$
- Principle Normal vector field of α : $N = \bar{N}(s)$
- Binormal vector field of α : $B = \bar{B}(s)$.

Discussion: Torsion τ : We know $\tau = 0$ iff curve (unit curve) is planer.

α arbitrary speed curve. Let $\bar{\alpha}$ be its arc length reparametrization. Since $\bar{\alpha}$ be unit speed curve (arc length repara. is a unit speed curve!) then we can define torsion of $\bar{\alpha}$.

Torsion of $\bar{\alpha} = 0$ $\Big\{$ if curve is planer
Torsion of $\alpha = 0$

Frenet Formulas (For any regular curves in $\mathbb{R}^3$).

Remark: For arbitrary speed curves α; the speed $v(t) = \|\alpha'(t)\|$ becomes an important factor in deriving General Frenet Formulae.

Lemma: If α is a regular curve in $\mathbb{R}^3$ with $k>0$, then.

$$\left.\begin{aligned} T' &= kvN \\ N' &= -kvT + \tau v B \\ B' &= -\tau N \end{aligned}\right\} \text{ where } v = \|\alpha(t)\|$$

In matrix notation we have;

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & kv & 0 \\ -kv & 0 & \tau v \\ 0 & -\tau v & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proof: $T = T(s)$; by definition.

$$\Rightarrow T' = T'(s)\frac{ds}{dt} \quad ; \quad s = \int_{t_0}^{t} \|\alpha'(u)\|\,du$$

$$\Rightarrow T' = T'(s)\,v \qquad \frac{ds}{dt} = \|\alpha'(u)\| = v(t).$$

$\boxed{T' = kNv}$; $T'(s) = kN$ for the reparametrized curve — from Frenet Formulas.

Similarly $N = N(s)$

$$N' = N'(s)\frac{ds}{dt}$$

$$N' = (-kT + \tau B)\,v(t) = -kvT + \tau v B.$$

$$\boxed{N' = -kvT + \tau v B}$$

Similarly it can be shown $\boxed{B' = -\tau v N}$.

So we have now derived the Frenet Formulas for arbitrary speed curves.

Velocity And Acceleration

Recall

Defn: Let $\alpha : I \to \mathbb{R}^3$ be a curve with coordinates $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ then the acceleration of α is defined as;

$$\alpha'' = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^3\alpha_3}{dt^2}\right)$$

Let α be such that it has constant speed; then $\|\alpha'(t)\| = c$

$$\Rightarrow \alpha'(t)\cdot\alpha'(t) = c$$

$$\Rightarrow \boxed{\alpha'\cdot\alpha'' = 0} \Rightarrow \text{Velocity and acceleration are perpendicular}$$

Lemma: Let α be a curve (regular) with speed function v, then its velocity and acceleration are expressed by;

$$\left\{\alpha' = vT, \quad \alpha'' = \frac{dv}{dt}T + kv^2N\right\}.$$

Proof: $\alpha(t) = \alpha(s(t))$

$$\alpha'(t) = \alpha'(s(t))\cdot\frac{ds}{dt} \Rightarrow \boxed{\alpha'(t) = vT}$$

Note: T is a unit tangent vector to the arc length reparametrized, which itself is a unit speed curve.

Differentiating again;

$$\alpha'' = \frac{dv}{dt}T + vT'\frac{ds}{dt}$$

$$= \frac{dv}{dt}T + v(\kappa N)\cdot v$$

T is tangent vector to curve with unit speed parametrization So $T' = \kappa N$ from Frenet Formulas.

$$\boxed{\alpha'' = \frac{dv}{dt}T + v^2(\kappa N)}$$

Frenet Apparatus for Regular Curve

Theorem: Let α be a curve in $\mathbb{R}^3$. Then

(1) $T = \dfrac{\alpha'}{\|\alpha'\|}$

(2) $N = B \times T$

(3) $B = \dfrac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$

(4) $\kappa = \dfrac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$

(5) $\tau = \dfrac{(\alpha' \times \alpha'')\cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$

Proof: (1) $\left(T = \dfrac{\alpha'}{\|\alpha'\|}\right)$

$\alpha' = vT$ from previous lemma

Now $v(t) = \|\alpha'\| \Rightarrow \alpha' = \|\alpha'\|\, T \Rightarrow \boxed{T = \dfrac{\alpha'}{\|\alpha'\|}}$

(4) $\left(\kappa = \dfrac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}\right)$

From previous lemma; $\alpha' = vT$; $\alpha'' = \dfrac{dv}{dt}T + \kappa v^2 N$

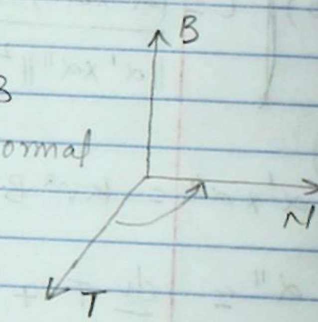
Then $\alpha' \times \alpha'' = (vT) \times \left(\dfrac{dv}{dt}T + \kappa v^2 N\right)$

$$\alpha' \times \alpha'' = \underbrace{v\frac{dv}{dt}(T \times T)}_{=0 \ \because\ T\times T = 0} + \kappa v^3 (T \times N)$$

$\alpha' \times \alpha'' = \kappa v^3 (T \times N)$; Now $T \times N = B$

$\alpha' \times \alpha'' = \kappa v^3 B$

B- unit bi-normal vector



$\|\alpha' \times \alpha''\| = \|\kappa v^3 B\|$

$= \kappa \|v^3 B\| = \kappa v^3 \|B\| = \kappa v^3$; v is speed.

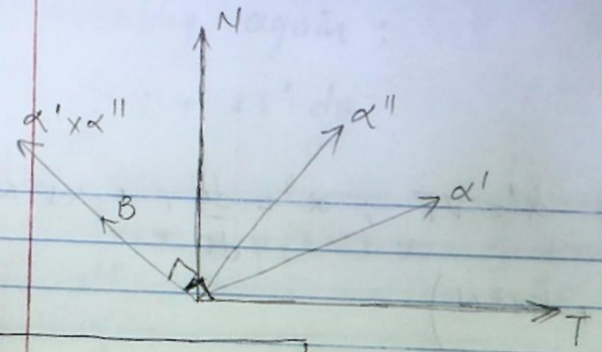
$$\boxed{\kappa = \frac{\|\alpha' \times \alpha''\|}{v^3} = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}}$$

(3) $\left(B = \dfrac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}\right)$

Assumption $\kappa = \dfrac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} > 0$; Also note since curve is regular $\|\alpha'\| \neq 0$

Now $\kappa = 0 \Leftrightarrow \|\alpha' \times \alpha''\| = 0$; therefore $\|\alpha' \times \alpha''\| > 0$

Since the cross product (norm) $\neq 0 \Rightarrow \alpha'$ and α'' are non collinear



$$\left.\begin{aligned} \alpha' &= vT \\ \alpha'' &= \frac{dv}{dt}T + kv^2 N \end{aligned}\right\} \text{From previous lemma.}$$

So α' and α'' lie in the same plane spanned by T and N.

Thus $\boxed{B = \dfrac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}}$

(5) $\left(\tau = \dfrac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2} \right)$

(i) $\alpha' \times \alpha'' = kv^3 B$ (see proof (4))

$\alpha'' = \dfrac{dv}{dt}T + kv^2 N$

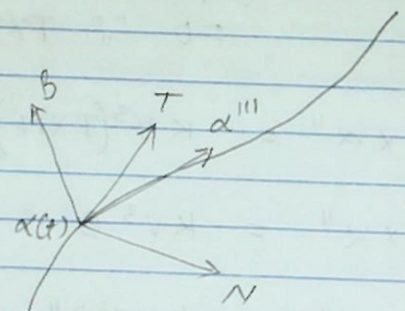
$\alpha''' = \left(\dfrac{dv}{dt}T + kv^2 N\right)'$

$= kv^2 N' + \dots$

$= kv^2(-kvT + \tau v B)$

$\boxed{\alpha''' = k\tau v^3 B} + \dots$

Note $T' = kvN$ is ignored becoz in the dot product the N and T components will vanish.



Note that α''' is a tangent vector at and since B, T, N form a frame;

$\alpha''' = aT + bN + cB$

Also $\alpha' \times \alpha'' = kv^3 B$

So $(\alpha' \times \alpha'') \cdot \alpha'''$ = some scaler multiple of B.

A1

So $(\alpha' \times \alpha'') \cdot \alpha''' = (kv^3 B) \cdot (k\tau v^3 B + \dots)$

$= k^2 v^3 \cdot v^3 \tau$

$\left\{ (\alpha' \times \alpha'') \cdot \alpha''' = k^2 v^6 \tau \right\}$ from (i) $\|\alpha' \times \alpha''\| = kv^3$

$\Rightarrow \|\alpha' \times \alpha''\|^2 = k^2 v^6$

Thus we have $\boxed{\tau = \dfrac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}}$

Example: Consider α a curve in $\mathbb{R}^3$ given by,

$$\alpha(t) = (3t - t^3,\ 3t^2,\ 3t + t^3)$$

Find the Frenet apparatus for α.

Solution: $\alpha'(t) = (3 - t^2,\ 6t,\ 3 + 3t^2)$

$\|\alpha'(t)\| = \sqrt{(3-t^2)^2 + (6t)^2 + (3+3t^2)^2}$

$v(t) = \|\alpha'(t)\| = \sqrt{18}\,(1+t^2)$

$\alpha'' = (-6t,\ 6,\ 6t)$

$$\alpha' \times \alpha'' = \begin{vmatrix} u_1 & u_2 & u_3 \\ 3-3t^2 & 6t & 3+3t^2 \\ -6t & 6 & 6t \end{vmatrix} = 18\,(-1+t^2,\ -2t,\ 1+t^2)$$

$\|\alpha' \times \alpha''\| = 18\sqrt{2}\,(1+t^2)$

$\alpha''' = 6(-1, 0, 1)$

$(\alpha' \times \alpha'') \cdot \alpha''' = \left(18(-1+t^2, -2t, 1+t^2) \cdot 6(-1, 0, 1)\right)$

$= 6 \cdot 2 \cdot 18$

So $\left\{ T = \dfrac{(1-t^2,\ 2t,\ 1+t^2)}{\sqrt{2}\,(1+t^2)} \right\}$