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Interior methods for constrained optimization

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Interior methods for optimization were widely used in the 1960s, primarily in the form of barrier methods. However, they were not seriously applied to linear programming because of the dominance of the simplex method. Barrier methods fell from favour during the 1970s for a variety of reasons, including their apparent inefficiency compared with the best available alternatives. In 1984, Karmarkar's announcement of a fast polynomial-time interior method for linear programming caused tremendous excitement in the field of optimization. A formal connection can be shown between his method and classical barrier methods, which have consequently undergone a renaissance in interest and popularity. Most papers published since 1984 have concentrated on issues of computational complexity in interior methods for linear programming. During the same period, implementations of interior methods have displayed great efficiency in solving many large linear programs of ever-increasing size. Interior methods have also been applied with notable success to nonlinear and combinatorial problems. This paper presents a self-contained survey of major themes in both classical material and recent developments related to the theory and practice of interior methods.

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1. Introduction to interior methods

1.1. The way we were

Before 1984, the question 'How should I solve a linear program?' would have been answered almost without exception by 'Use the simplex method'. In fact, it would have been extremely difficult to find serious discussion of any method for linear programming (LP) other than the famous simplex method developed by George B. Dantzig in 1947.

As most readers already know, the simplex method is an iterative procedure derived from a fundamental property of essentially all linear programs: an optimal solution lies at a vertex of the feasible region. Beginning with a vertex, the simplex method moves between adjacent vertices, decreasing the objective as it goes, until an optimal vertex is found.

Although nonsimplex strategies for LP were suggested and tried from time to time, such techniques had never approached the simplex method in overall speed and reliability. Hence the simplex method retained unquestioned pre-eminence as the linear programming method of choice for nearly 40 years. (We describe later the persistent unhappiness with the simplex method on grounds of its theoretical complexity.)

Such an exclusive focus on the simplex method had several effects on the field of optimization. Largely for historical reasons, the simplex method is surrounded by a bevy of highly specialized terminology ('basic feasible solution') and pedagogical constructs (the tableau) with little apparent connection to other continuous optimization problems. Many researchers and practitioners consequently viewed linear programming as philosophically distinct from nonlinear programming. This conceptual gap reinforced a tendency to develop 'new' linear programming methods only as variations on the simplex method.

In marked contrast, the field of *nonlinear* optimization was characterized not only by the constant development of new methods with differing flavours, but also by a shift over time in the preferred solution techniques. Since the late 1970s, for example, nonlinearly constrained optimization problems have been solved with sequential quadratic programming (SQP) methods, which involve a sequence of constrained subproblems based on the Lagrangian function. In the 1960s, however, constrained problems were most often converted to *unconstrained* subproblems. Penalty and barrier methods were especially popular, both motivated by minimizing a composite function that reflects the original objective function as well as the influence of the constraints. Classical barrier methods, intended for inequality constraints, include a composite function containing an impassable positive singularity ('barrier') at the boundary of the feasible region, and thereby maintain strict feasibility while approaching the solution.

Although barrier methods were widely used and thoroughly analysed during the 1960s (see Section 3 for details and references), they nonetheless suffered a severe decline in popularity in the 1970s for various reasons, including inherent ill-conditioning as well as perceived inefficiency compared to alternative strategies. By the late 1970s, barrier methods were considered for the most part an interesting but passé solution technique.

As we shall see, the situation today (1991) in both linear and nonlinear programming has altered dramatically since 1984, primarily as a result of dissatisfaction with the theoretical computational complexity of the simplex method.

1.2. Concerns about the simplex method

On 'real-world' problems, the simplex method is invariably extremely efficient, and consistently requires a number of iterations that is a small multiple (2–3) of the problem dimension. Since the number of vertices associated with any LP is finite, the simplex method is also guaranteed under quite mild conditions to converge to the optimal solution. The number of vertices, however, can be exponentially large. The well known 'twisted cube' example of Klee and Minty (1972) is a linear program with n variables and $2n$ inequality constraints for which the simplex method with the standard pivot-selection rule visits each of the 2^n vertices. The *worst-case* complexity of the simplex method (the number of arithmetic operations required to solve a general LP) is consequently *exponential* in the problem dimension. The gigantic gap between the observed and worst-case performance of the simplex method is still puzzling; the issue of whether an (undiscovered) simplex pivot rule could improve its complexity is also unresolved.

As the formal study of computational complexity increased in importance during the 1960s and 1970s, it became a strongly held article of faith among computer scientists that a 'fast' algorithm must be *polynomial-time*, meaning that the number of operations required to solve the problem should be bounded above by a polynomial in the problem size. The simplex method clearly does not satisfy this property. Although practitioners routinely and happily solved large linear programs with the simplex method, the existence of a provably polynomial algorithm remained a major open question.

In 1979, to the accompaniment of worldwide publicity, Leonid Khachian published the first polynomial algorithm for LP. The *ellipsoid method* of Khachian is based on earlier techniques for nonlinear programming developed by other mathematicians, notably Shor, Yudin and Nemirovsky. An interesting feature of Khachian's approach is that it does not rely on combinatorial features of the LP problem. Rather, it constructs a sequence of ellipsoids such that each successive ellipsoid both encloses the optimal solution and undergoes a strict reduction in volume. The ellipsoid method generates improving iterates in the sense that the region of uncertainty surrounding the solution is monotonically 'squeezed'. (Simplex iterates are also improving in the sense that the objective value is decreasing, but they provide no information about the closeness of the current iterate to the solution.)

The crucial elements in polynomiality of the ellipsoid method are what might be termed outer and inner bounds for the solution. The outer bound guarantees an initial enclosing ellipsoid, and the inner bound specifies the size of the final ellipsoid needed to ensure sufficient closeness to the exact solution. Similar features also figure prominently in the complexity analysis of interior methods, and are discussed in Section 6.

Despite its polynomial complexity, the ellipsoid method's performance was extremely disappointing. In practice, the number of iterations tended to be almost as large as the worst-case upper bound, which, although polynomial, is *very* large. The simplex method accordingly retained its position as the clear winner in any comparison of actual solution times. Creation of the ellipsoid method led to an unexpected anomaly in which an algorithm with the desirable theoretical property of polynomiality compared unfavourably in speed to an algorithm with worst-case exponential complexity. The quest therefore continued for an LP algorithm that was not only polynomial, but also efficient in practice.

This search ended in 1984, when Narendra Karmarkar presented a novel interior method of polynomial complexity for which he reported solution times 50 times faster than the simplex method. Once again, international coverage in the popular press surrounded the event, which has had remarkable and lasting scientific consequences.

Karmarkar's announcement led to an explosion of interest among researchers and practitioners, with substantial progress in several directions. Interior methods are indeed 'fast'; extensive numerical trials have shown conclusively that a variety of interior methods can solve many very large linear programs substantially faster than the simplex method. After a formal relationship was shown between Karmarkar's method and classical barrier methods (Gill *et al.*, 1986), much research has concentrated on the common theoretical foundations of linear and nonlinear programming.

Unlike the simplex method, interior techniques can obviously be applied to nonlinear optimization problems. (In fact, they were devised more than 30 years ago for this purpose!) Interior methods have already been developed for quadratic and nonlinear programming, and extensions of the interior approach to difficult combinatorial problems have also been proposed; see Karmarkar (1990).

A fundamental theme permeating the motivation for interior methods is the creation of continuously parametrized families of approximate solutions that asymptotically converge to the exact solution. As the parameter approaches its limit, the paths to the solution trace smooth trajectories whose geometric properties can be analysed. Each iteration of a 'path-following' method constructs a step intended to follow one of these trajectories, moving both 'toward' and 'along' the path. In the first heyday of barrier methods, these ideas led to great interest in extrapolation. Today, they are being generalized and extended to new problem areas; for a discussion of such ideas in linear programming, see Megiddo (1987), Bayer and Lagarias (1989, 1991), and Karmarkar (1990). The field of interior methods seems to offer the continuing promise of original theory and efficient methods.

1.3. Overview

This article covers only a small part of the large and rapidly expanding number of topics related to interior methods. Although the term 'interior methods' is not precisely defined, several themes perceived as disparate before 1984 can now be placed in a unified framework. For reasons of space, we motivate interior methods only through a 'classical' barrier function. Karmarkar's original 1984 algorithm was based on nonlinear projection, a perspective that provides interesting geometric insights. See Gonzaga (1992), Nesterov and Nemirovsky (1989), and Powell (1990) for further interpretations.

Work in interior methods today is a melange of rediscovered as well as new methods, complexity analysis, and sparse linear algebra. The approach taken in this article is to present some initial background on optimization (Section 2), followed by a detailed treatment of the theory of classical barrier methods (Section 3). After reviewing Newton's method (Section 4), we turn in Section 5 to the special case of linear programming, and describe the structure of several interior methods. A particular interior LP method and its complexity analysis are given in detail (Section 6) to give the flavour of such proofs. The practical success of interior methods is dependent on efficient linear algebra; the relevant techniques for linear and nonlinear problems are described in Section 7. Finally, we close by mentioning selected directions for future research.

2. Background in optimization

2.1. Definitions and notation

Optimization problems, broadly speaking, involve finding the 'best' value of some function. A continuous optimization problem has three ingredients: a set of variables, usually denoted by the real n-vector x; an *objective* function $f(x)$ to be optimized (minimized or maximized); and *constraints* (equality and/or inequality) that restrict acceptable values of the variables.

Except for the linear programming case, our main interest is in inequality convex program only if each *negative* constraint function $-c_i(x)$ is convex, i.e. if $c_i(x)$ itself is concave. Hence minus signs appear throughout our discussion of the constraints in convex programs.

Using these definitions, it is easy to see that a linear function is convex (and also concave), and that a linear programming problem is a convex program. Two properties that are important in interior methods for linear programming are stated formally in the following theorems; see Fiacco and McCormick (1968) or Fletcher (1987) for details.

Theorem 1 If x^* is a local constrained minimizer of a convex programming problem, it is also a global constrained minimizer. Further, the set of minimizers of a convex program is convex.

Theorem 2 If the optimization problem (2.10) is a convex program, and if x^* satisfies the feasibility and first-order necessary conditions (2.13a–d), then x^* is a global constrained minimizer of (2.10).

3. Barrier methods

3.1. Intuition and motivation

Suppose that we wish to minimize $f(x)$ subject to a set of inequality constraints $c_i(x) \geq 0$, $i = 1, \ldots, m$. If the constraints affect the solution, either an unconstrained minimizer of $f(x)$ is infeasible (for example, when minimizing x^2 subject to $x \geq 1$), or else $f(x)$ is unbounded below when the constraints are removed (for example, when minimizing x^3 subject to $x \geq 1$). Consequently, if an optimization method tries to achieve a 'large' reduction in the objective function from its value at a feasible point, the iterates tend to move *outside* the feasible region. In fact, many popular algorithms for nonlinearly constrained optimization (such as SQP methods; see, for example, Fletcher (1987), and Gill *et al.* (1981)) typically produce infeasible iterates that approach feasibility only in the limit.

When feasibility at intermediate points is essential – for example, in practical problems where the objective function is meaningless unless the constraints are satisfied – it seems desirable for iterates to approach the constrained solution from the *interior* of the feasible region. *Barrier methods* constitute a well known class of methods with this property.

Barrier methods may be applied only to inequality constraints for which strictly feasible points exist. This property does not hold for all inequality constraints, even if the feasible region is nonempty; for example, consider the constraints $x_1 + x_2 \geq 0$ and $-x_1 - x_2 \geq 0$, for which the feasible region consists of the line $\{x_1 + x_2 = 0\}$.

Given an initial strictly feasible point and mild assumptions about the feasible region, strict feasibility can be retained by minimizing a composite function consisting of the original objective $f(x)$ plus a positive multiple of an infinite 'barrier' at the boundary of strict($\mathcal{F}$). The most effective methods for unconstrained optimization (such as Newton's method; see Section 4) require differentiability. A suitable barrier term is therefore composed of functions that are smooth at strictly feasible points, but contain a positive singularity if any constraint is zero. Under these conditions, a minimizer of the composite function must occur at a strictly feasible point.

When the barrier term is heavily weighted, a minimizer of the composite function will lie, informally speaking, 'far away' from the boundary. If the coefficient of the barrier term is reduced, the singularity becomes less influential, except at points near the boundary; minimizers of the composite function can then move closer (but not 'too close') to the boundary. The weight on the barrier term thus tends to regulate the distance from the iterates to the boundary. In the parlance of modern interior methods, the barrier term forces the iterates to remain *centred* in the strictly feasible region.

As the factor multiplying the barrier term decreases to zero, intuition suggests that minimizers of the composite function will converge to a constrained solution x^* that lies on the boundary of strict($\mathcal{F}$). We shall see later (Sections 3.3 and 3.4) that this intuition can be verified rigorously under reasonably mild conditions.

We stress that there is ample room for many formulations of a 'barrier function', as indicated by the range of definitions in Fiacco and McCormick (1968) and in Nesterov and Nemirovsky (1989). Other varieties of composite functions – called 'potential' and 'centering' functions – have also been proposed for use in interior methods; see, for example, Sonnevend (1986) and Gonzaga (1992). Karmarkar's original (1984) LP algorithm included a logarithmic potential function. The method of centres of Huard (1967) imposes an additional constraint at each iteration based on the current value of the objective function; see Renegar (1988) for an LP method based on this idea.

In all cases, the composite functions display a common motivation of simultaneously reflecting the objective function (thereby encouraging its reduction) as well as forcing iterates to stay 'nicely centred' in the feasible region. They differ, however, in the balance of these sometimes conflicting aims.

3.2. The logarithmic barrier function

For simplicity, we discuss only the simplest barrier function based on a logarithmic singularity, which was not only the most popular in the 1960s, but also has received substantial attention since 1984. The logarithmic barrier function was first defined by Frisch in 1955, and was extensively studied and analysed during the 1960s. Detailed theoretical discussions of classical barrier methods, along with historical background, are given in Fiacco and McCormick (1968) and Fiacco (1979).

The *logarithmic barrier function* associated with minimizing $f(x)$ subject to $c(x) \geq 0$ is

$$B(x, \mu) = f(x) - \mu \sum_{i=1}^{m} \ln c_i(x), \tag{3.1}$$

where the *barrier parameter* μ is strictly positive. (When the meaning is clear, we may write B with a single argument μ or without arguments.) Since the logarithm is undefined for nonpositive arguments, the logarithmic barrier function is defined only in strict($\mathcal{F}$).

Simply stating the definition (3.1) does not give an adequate impression of the dramatic effects of the imposed barrier. Figure 1 depicts the one-dimensional variation of a barrier function for two values of μ. Even for the modest value $\mu = 0.1$, the (visually) extreme steepness of the singularity is evident.

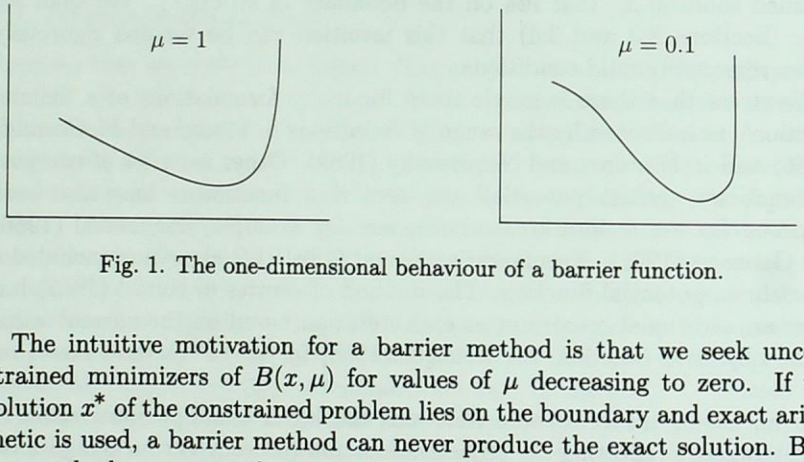

Fig. 1. The one-dimensional behaviour of a barrier function.

The intuitive motivation for a barrier method is that we seek unconstrained minimizers of $B(x, \mu)$ for values of μ decreasing to zero. If the solution x^* of the constrained problem lies on the boundary and exact arithmetic is used, a barrier method can never produce the exact solution. Barrier methods consequently terminate when the current iterate satisfies some approximation to the desired optimality conditions. 'Classical' barrier algorithms as well as many recent interior methods have the following form:

Generic Barrier Algorithm

0. Set x_0 to a strictly feasible point, so that $c(x_0) > 0$, and set μ_0 to a positive value; $k \leftarrow 0$.

1. Check whether x_k qualifies as an approximate local constrained minimizer for the original problem (2.10). If so, stop with x_k as the solution.

2. Compute an unconstrained minimizer $x(\mu_k)$ of $B(x, \mu_k)$.

3. $x_{k+1} \leftarrow x(\mu_k)$; choose $\mu_{k+1} < \mu_k$; $k \leftarrow k + 1$; return to Step 1.

In practice, the calculation of $x(\mu_k)$ in Step 2 is carried out approximately, and only a few iterations of an unconstrained method may be performed before the barrier parameter is updated. In the theoretical results given here, we assume that $x(\mu_k)$ is an exact unconstrained minimizer.

We illustrate the behaviour of the generic algorithm on a simple two-variable example:

$$\begin{array}{ll} \text{minimize} & x_1x_2 - \frac{1}{2}x_1^2 - x_2 \\ \text{subject to} & x_1^2 + x_2^2 \le 2 \\ & x_1^2x_2^2 \quad \le 10. \end{array}$$

The first constraint is satisfied inside the circle of radius $\sqrt{2}$ centred at the origin; although the second constraint is redundant, it nonetheless affects each minimizer of the barrier function. The point $x^* = (-1,1)^T$ is an isolated local constrained minimizer at which only the first constraint is active. Figure 2 depicts selected barrier minimizers converging to x^*, which lies on the boundary of the feasible region (depicted as a dashed curve).

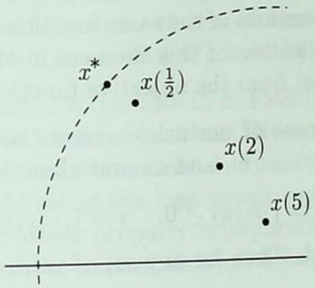

Fig. 2. Convergence of barrier minimizers to $x^* = (-1,1)^T$.

The next two sections provide a rigorous foundation for the generic approach, including the assumptions necessary to make it succeed in converging to a solution x^* of the original constrained problem. After establishing local convergence properties, we return in Section 3.5 to a more detailed analysis of the sequence of barrier minimizers.

3.3. *Theoretical results for convex programs*

Pre-1984 presentations of barrier methods for nonlinear problems typically begin with general results, which are then specialized to convex programs. We have chosen instead to give a self-contained presentation of the convex results first. Readers whose primary interest is in interior methods for linear and convex programming can read this section only and skip to Section 3.5.

Consider the convex programming problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad c_i(x)\geq 0, \quad i=1,\ldots,m, \tag{3.2}$$

where f and $\{-c_i\}$ are convex. In this section, $\mathcal{F}$ denotes the feasible region for the constraints of (3.2). Recall from Theorem 1 that every local minimizer of a convex program is a global minimizer; hence, if any minimizer exists, the optimal value of f in $\mathcal{F}$ is unique.

An obvious fundamental question involves the conditions under which a solution x^* of (3.2) is the limit of a sequence of unconstrained minimizers of the barrier function. The main assumption needed to prove convergence results is that the set $\mathcal{M}$ of minimizers of (3.2) is *bounded.* (We know already from Theorem 1 that $\mathcal{M}$ is convex.) Boundedness of the set of minimizers holds automatically under the much stronger assumption that the feasible region itself is bounded.

The major results of this section are given in Theorem 5. Two other theorems serve as a prelude.

Theorem 3 (a version of Theorem 24 in Fiacco and McCormick (1968)) shows that, if a set of convex functions defines a *bounded* feasible region, then suitably perturbed versions of the same functions also define a bounded feasible region. The application of this theorem in the proof of Theorem 4 involves a level set derived from the objective function.

Theorem 3 (Boundedness of perturbed convex sets.) Let $-\varphi_i(x)$ be a convex function for $i=1, \ldots, m$, and assume that the convex set

$$\mathcal{N}=\{x \mid \varphi_i(x)\geq 0, \quad i=1,\ldots,m\}$$

is nonempty and bounded. Then for any set of values $\{\Delta_i\}$, where $\Delta_i\geq 0$, $i=1, \ldots, m$, the set

$$\{x \mid \varphi_i(x)\geq -\Delta_i, \quad i=1,\ldots,m\}$$

is bounded.

Proof. The result will follow in an obvious way if verified for $\Delta_1>0$ and $\Delta_i=0$, $i\neq 1$. Given $\Delta_1>0$, let $\mathcal{N}_1$ denote the set

$$\mathcal{N}_1=\{x \mid \varphi_1(x)\geq -\Delta_1 \ \text{ and } \ \varphi_i\geq 0, \ i=2,\ldots,m\}.$$

Because $\mathcal{N}_1$ is the intersection of a finite number of convex sets, $\mathcal{N}_1$ is convex.

To prove by contradiction that $\mathcal{N}_1$ is bounded, we assume the contrary: for any point $x_1\in\mathcal{N}$, there exists a ray emanating from x_1 that does not intersect the boundary of $\mathcal{N}_1$, so that $x_1+\alpha p$ lies in $\mathcal{N}_1$ for some direction p and any $\alpha\geq 0$. (The fact that any unbounded convex set must contain a ray is standard; see, for example, Grünbaum (1967).)

Because $\mathcal{N}$ is bounded by assumption, there must be a point x_2 on this ray that does *not* lie in $\mathcal{N}$. Let x_2 be such a point, given by $x_2=x_1+\alpha_2 p$ for some $\alpha_2 > 0$, for which φ_1 assumes a *negative* value, say $\varphi_1(x_2) = -\delta < 0$, where $\delta < \Delta_1$.

Let x_3 denote a point on the ray that lies *beyond* x_2, i.e. $x_3 = x_1 + \alpha_3 p$, where $\alpha_3 > \alpha_2$. The point x_2 can then be written as

$$x_2 = x_1 + \theta(x_3 - x_1) = (1-\theta)x_1 + \theta x_3, \tag{3.3}$$

where $0 < \theta < 1$.

Applying Definition 9 of a convex function to the expression (3.3) for x_2, we obtain

$$(1-\theta)\varphi_1(x_1) + \theta\varphi_1(x_3) \le \varphi_1(x_2) = -\delta,$$

which gives

$$\theta\varphi_1(x_3) \le -\delta - (1-\theta)\varphi_1(x_1).$$

Because $\varphi_1(x_1) \ge 0$ and $0 < \theta < 1$, it follows that

$$\varphi_1(x_3) \le \frac{-\delta}{\theta}.$$

If θ is sufficiently small, namely $\theta < \delta/\Delta_1$, the value of $\varphi_1(x_3)$ must be strictly less than $-\Delta_1$, which shows that x_3 cannot lie in $\mathcal{N}_1$. This gives the desired contradiction, and shows that $\mathcal{N}_1$ must be bounded. □

The next result is related to Lemma 12 in Fiacco and McCormick (1968), which applies to a general barrier function. Given a convex program with a nonempty strict interior and a bounded set of minimizers, the theorem states that any particular level set of the logarithmic barrier function is bounded and closed. The boundedness property is important because it implies that the set of *minimizers* of the barrier function is bounded.

Theorem 4 (Compactness of barrier function level sets.) Consider the convex program of minimizing $f(x)$ subject to $c_i(x) \ge 0$, $i = 1, \ldots, m$. Let $\mathcal{F}$ denote the (convex) feasible region. Assume that strict($\mathcal{F}$) is nonempty and that the set of minimizers $\mathcal{M}$ for the convex program is nonempty and bounded. Then for any $\mu_k > 0$ and any constant τ, the level set

$$S(\tau) = \{x \in \text{strict}(\mathcal{F}) \mid B(x, \mu_k) \le \tau\}$$

is bounded and closed, where $B(x, \mu_k)$ is the logarithmic barrier function.

Proof. Boundedness of $S(\tau)$ will be established by showing that, under the stated assumptions, the barrier function cannot remain bounded above while its argument becomes unbounded.

Let $\hat{x}$ denote any point in strict($\mathcal{F}$) (which is assumed to be nonempty). Given any $\epsilon > 0$, let $\hat{D}$ denote the level set defined by the values of $f(\hat{x})$ and ϵ:

$$\hat{D} = \{x \in \mathcal{F} \mid f(x) \le f(\hat{x}) + \epsilon\}. \tag{3.4}$$

Convexity of f implies that $\hat{D}$ is convex (see Section 2.3). The functions f and $\{c_i\}$ are smooth, so that $\hat{D}$ is closed. The first step in proving the theorem is to show that $\hat{D}$ is bounded, from which it will follow that $\hat{D}$ is compact.

To show that the set $\hat{D}$ is bounded, we invoke Theorem 3. Let f^* denote the minimum value of $f(x)$ for $x \in \mathcal{F}$. Because every local minimizer of a convex program is a global minimizer (see Theorem 1), the quantity $\Delta = f(\hat{x}) + \epsilon - f^*$ must be positive. By assumption, the set $\mathcal{M}$, which may be written as $\{x \in \mathcal{F} \mid f(x) \le f^*\}$, is nonempty and bounded; further, $\mathcal{M}$ is convex because f is convex. We now define the function $\phi(x)$ as $f^* - f(x)$, and observe that $-\phi$ is convex. Theorem 3 then applies to ϕ and the positive perturbation Δ, and implies *boundedness* of the set

$$\{x \in \mathcal{F} \mid \phi(x) \ge -\Delta\} = \{x \in \mathcal{F} \mid f^* - f(x) \ge f^* - f(\hat{x}) - \epsilon\},$$

which is simply a rearranged definition of $\hat{D}$. Consequently, $\hat{D}$ is compact. It is straightforward to see that its boundary, $\mathrm{bnd}(\hat{D})$, is also compact. The definition (3.4) of $\hat{D}$ shows that $\hat{x}$ does not lie on the boundary of $\hat{D}$.

Having established the compactness of $\hat{D}$ and its boundary, we can now prove boundedness of $S(\tau)$ by contradiction. Assume the contrary of the desired result, namely that for some $\mu_k > 0$, there is an *unbounded* sequence $\{y_j\}$ of points in $\mathrm{strict}(\mathcal{F})$ for which the barrier function values $B(y_j, \mu_k)$ remain bounded above.

For such a sequence, let j be sufficiently large so that y_j lies outside $\hat{D}$. By definition of $\hat{D}$, it must hold that

$$f(y_j) > f(\hat{x}) + \epsilon.$$

Let z_j be the point on the boundary of $\hat{D}$ where the line connecting $\hat{x}$ and y_j intersects the boundary. (Because $\hat{D}$ is convex, z_j is unique.) Let λ_j be the scalar satisfying $0 < \lambda_j < 1$ such that

$$z_j = (1 - \lambda_j)\hat{x} + \lambda_j y_j. \tag{3.5}$$

We have assumed that $\|y_j\|$ is unbounded for sufficiently large j. Since $\|z_j\|$ is finite, (3.5) shows that

$$\lambda_j \to 0 \quad \text{as} \quad j \to \infty. \tag{3.6}$$

Because $\hat{x}$ and y_j are both in $\mathrm{strict}(\mathcal{F})$, we know that $c_i(\hat{x}) > 0$ and $c_i(y_j) > 0$ for $i = 1, \ldots, m$. Convexity of $-c_i(x)$ combined with (3.5) gives

$$c_i(z_j) \ge (1 - \lambda_j)c_i(\hat{x}) + \lambda_j c_i(y_j) > 0, \tag{3.7}$$

which shows that $z_j \in \mathrm{strict}(\mathcal{F})$. Since z_j is by definition in $\mathrm{bnd}(\hat{D})$, we conclude from (3.4) that $f(z_j) = f(\hat{x}) + \epsilon$. Because f is convex (see Definition 9), (3.5) implies

$$f(z_j) \le (1-\lambda_j)f(\hat{x}) + \lambda_j f(y_j).$$

Dividing by λ_j and substituting $f(z_j) = f(\hat{x}) + \epsilon$, we obtain a *lower bound* on $f(y_j)$:

$$f(y_j) \ge f(\hat{x}) + \frac{\epsilon}{\lambda_j}. \tag{3.8}$$

It then follows from (3.6) that

$$f(y_j) \to \infty \quad \text{as} \quad j \to \infty,$$

so that the objective function values at $\{y_j\}$ become unbounded.

Turning back to the constraint functions, positivity of λ_j means that the first inequality in (3.7) can be rewritten as

$$c_i(y_j) \le c_i(\hat{x}) + \frac{c_i(z_j) - c_i(\hat{x})}{\lambda_j}. \tag{3.9}$$

Since the set $\mathrm{bnd}(\hat{D})$ is compact, the function $c_i(x) - c_i(\hat{x})$ achieves its maximum for some $x \in \mathrm{bnd}(\hat{D})$. Let d_i denote

$$d_i = \max\{c_i(x) - c_i(\hat{x}) \mid x \in \mathrm{bnd}(\hat{D})\}.$$

We now wish to demonstrate that $d_i \ge 0$. Because $z_j \in \mathrm{bnd}(\hat{D})$ and $c_i(y_j) > 0$, we apply the definition of d_i and relation (3.9) to show that

$$c_i(\hat{x}) + \frac{d_i}{\lambda_j} \ge c_i(y_j) > 0, \quad i = 1, \ldots, m. \tag{3.10}$$

If d_i were negative, the first expression in (3.10) would eventually become negative as $\lambda_j \to 0$, which is impossible. It follows that $d_i \ge 0$ for $i = 1$, $\ldots$, m.

Finally, the barrier function $B(y_j, \mu_k)$ is formed. Using (3.8), (3.10), monotonicity of the logarithm function, and positivity of μ_k, we have:

$$\begin{aligned} B(y_j, \mu_k) &= f(y_j) - \mu_k \sum \ln c_i(y_j) \\ &\ge f(\hat{x}) + \frac{\epsilon}{\lambda_j} - \mu_k \sum \ln(c_i(\hat{x}) + (d_i/\lambda_j)) \\ &= f(\hat{x}) + \frac{\epsilon - \mu_k \lambda_j \sum \ln(c_i(\hat{x}) + (d_i/\lambda_j))}{\lambda_j}. \end{aligned} \tag{3.11}$$

The logarithm function has the property that, for a positive constant ν and $\delta \ge 0$,

$$\lim_{\lambda \to 0_+} \lambda \ln\Big(\nu + \frac{\delta}{\lambda}\Big) = 0.$$

Thus the limit of the numerator in (3.11) is ϵ, and the quotient in (3.11) is unbounded above as $\lambda_j \to 0$. It follows that $B(y_j, \mu_k)$ is unbounded above as $j \to \infty$, thereby contradicting our assumption that the barrier function values $\{B(y_j, \mu_k)\}$ are bounded above for an unbounded sequence $\{y_j\}$. This proves that $S(\tau)$ is bounded.

To show that $S(\tau)$ is closed, we prove that it contains all its accumulation points. Let $\{x_j\}$ be a convergent sequence in $S(\tau)$, with limit point $\bar{x}$. It follows from the continuity of f and $\{c_i\}$ in strict($\mathcal{F}$) that $\bar{x}$ must satisfy $B(\bar{x}, \mu_k) \leq \tau$. Further, $\bar{x}$ must either be in strict($\mathcal{F}$) or else have the property that $c_i(\bar{x}) = 0$ for at least one index i.

If $\bar{x}$ is in strict($\mathcal{F}$), by definition $\bar{x}$ is in $S(\tau)$. Suppose that $\bar{x}$ is not in strict($\mathcal{F}$). Then, because $c_i(\bar{x}) = 0$ for some index i, unboundedness of the logarithm for a zero argument and convergence of $\{x_j\}$ to $\bar{x}$ together imply that, for sufficiently large j, the barrier term $-\sum_{i=1}^{m} \ln c_i(x_j)$ cannot be bounded above. In particular, for any constant γ and sufficiently large j,

$$-\sum_{i=1}^{m} \ln c_i(x_j) > \gamma. \tag{3.12}$$

We now define γ as $\gamma = (\tau - f^*)/\mu_k$; the value of γ is finite because f^* is finite. Since x_j lies in strict($\mathcal{F}$), we know from the convexity of f that $f(x_j) \geq f^*$, which means that $-f^* \geq -f(x_j)$. Applying this inequality and the definition of γ in (3.12), we obtain

$$-\sum_{i=1}^{m} \ln c_i(x_j) > \frac{\tau - f(x_j)}{\mu_k}.$$

After rearrangement, this relation implies that $B(x_j, \mu_k) > \tau$, i.e., that $x_j \notin S(\tau)$, a contradiction. We conclude that any accumulation point of a sequence in $S(\tau)$ must lie in $S(\tau)$, which means that $S(\tau)$ is closed.

We have shown that $S(\tau)$ is both bounded and closed; its compactness is immediate. □

We are now ready to give the main theorem concerning barrier methods for convex programs. The most important result is (vi), which shows that limit points of a minimizing sequence for the barrier function converge to constrained minimizers of the convex program.

Theorem 5 (Convergence of barrier methods on convex programs.) Consider the convex program of minimizing $f(x)$ subject to $c_i(x) \geq 0$, $i = 1, \ldots, m$. Let $\mathcal{F}$ denote the feasible region for this problem, and assume that strict($\mathcal{F}$) is nonempty. Let $\{\mu_k\}$ be a decreasing sequence of positive barrier parameters such that $\lim_{k\to\infty} \mu_k = 0$. Assume that the set $\mathcal{M}$ of constrained local minimizers of the convex program is nonempty and bounded, and let f^* denote the optimal value of f. Then

(i) the logarithmic barrier function $B(x, \mu_k)$ is convex in strict($\mathcal{F}$);

(ii) $B(x, \mu_k)$ has a finite unconstrained minimizer in strict($\mathcal{F}$) for *every* $\mu_k > 0$, and the set $\mathcal{M}_k$ of unconstrained minimizers of $B(x, \mu_k)$ in strict($\mathcal{F}$) is convex and compact for every k;

(iii) any unconstrained local minimizer of $B(x, \mu_k)$ in strict($\mathcal{F}$) is also a *global* unconstrained minimizer of $B(x, \mu_k)$;

(iv) let y_k denote an unconstrained minimizer of $B(x, \mu_k)$ in strict($\mathcal{F}$); then, for all k,

$$f(y_{k+1}) \le f(y_k) \quad \text{and} \quad -\sum_{i=1}^{m} \ln c_i(y_k) \le -\sum_{i=1}^{m} \ln c_i(y_{k+1});$$

(v) there exists a compact set S such that, for all k, every minimizing point y_k of $B(x, \mu_k)$ lies in $S \cap \text{strict}(\mathcal{F})$;

(vi) any sequence $\{y_k\}$ of unconstrained minimizers of $B(x, \mu_k)$ has at least one convergent subsequence, and every limit point of $\{y_k\}$ is a local constrained minimizer of the convex program;

(vii) let $\{x_k\}$ denote a convergent subsequence of unconstrained minimizers of $B(x, \mu_k)$; then $\lim_{k\to\infty} f(x_k) = f^*$;

(viii) $\lim_{k\to\infty} B_k = f^*$, where B_k denotes $B(x_k, \mu_k)$.

Proof. It is straightforward to prove convexity of $B(x, \mu_k)$ using the convexity of f and $\{-c_i\}$, monotonicity of the logarithm function and Definition 9 of a convex function. Thus (i) is established.

The assumptions of this theorem are the same as those of Theorem 4. Let x_0 denote the strictly feasible point at which the barrier iterations are initiated. For the barrier parameter μ_k and some $\epsilon > 0$, we define the set S_0 as:

$$S_0 = \{x \in \text{strict}(\mathcal{F}) \mid B(x, \mu_k) \le B(x_0, \mu_k) + \epsilon\}.$$

Theorem 4 implies that S_0 is *compact* for all $\mu_k > 0$. It follows that the smooth function $B(x, \mu_k)$ assumes its minimum in S_0, necessarily at an interior point of S_0. We then apply Definition 7 and conclude that $B(x, \mu_k)$ has at least one finite *unconstrained* minimizer.

Because $B(x, \mu_k)$ is convex, any local minimizer is also a global minimizer, so that *every* unconstrained minimizer of $B(x, \mu_k)$ must be in the set S_0. Thus the set $\mathcal{M}_k$ of unconstrained minimizers of $B(x, \mu_k)$ is bounded. The set $\mathcal{M}_k$ is closed because the minimum value of $B(x, \mu_k)$ is unique, and it follows that $\mathcal{M}_k$ is compact. Convexity of $\mathcal{M}_k$ follows from Theorem 1, and result (ii) has been verified.

Result (iii) follows from Theorem 1, and results (i) and (ii).

To show result (iv), let y_k and y_{k+1} denote global minimizers of the barrier function for the barrier parameters μ_k and μ_{k+1}. By definition of y_k and be considered here, except for the following important result: the objective function value at any unconstrained minimizer x_k of $B(x, \mu_k)$ satisfies the inequality $f(x_k) - f^* \le m\mu_k$, where m is the number of constraints. We know from results (iv) and (vii) of Theorem 5 that $f^* \le f(x_k)$. Combining these bounds, we have

$$0 \le f(x_k) - f^* \le m\mu_k. \tag{3.26}$$

This somewhat surprising property implies that, when a barrier method is applied to a convex program, the deviation of $f(x_k)$ from optimality is always bounded by $m\mu_k$, independently of the particular problem functions. For comments about duality in linear programming, see Section 5.1.

3.4. Results for general nonlinear programs

Once we move from a convex program to a general nonlinear program, matters become far more complicated. In particular, certain topological assumptions are required to avoid pathological cases. Furthermore, the results apply only in a neighbourhood of a constrained minimizer, and involve convergence of subsequences of *global* minimizers of the barrier function. The general approach in this section follows that in Fiacco and McCormick (1968).

At the most basic level, the nice property given by Theorem 4 that the level sets of the barrier function are bounded if the set of constrained minimizers is bounded does not hold for the nonconvex case. If the feasible region is bounded, the barrier function is obviously bounded below. The following example of Powell (1972), however, shows that difficulties may arise when the feasible region is unbounded:

$$\text{minimize} \quad \frac{-1}{x^2+1} \quad \text{subject to} \quad x \ge 1. \tag{3.27}$$

The objective function is bounded below in the feasible region, and the unique solution is $x^* = 1$. In contrast, the barrier function

$$B(x, \mu) = \frac{-1}{x^2+1} - \mu \ln(x-1)$$

is *unbounded below* in the feasible region, although it has a local minimizer that approaches x^* as $\mu \to 0$.

The major local convergence results will be given in Theorem 7. To build up to the statement of this theorem, several preliminary results are required.

The following lemma, an adaptation of Corollary 8 from Fiacco and McCormick (1968), plays the role of Theorem 4 for the convex case. The general result is that, if a continuous function is unbounded above for all sequences of points in strict($\mathcal{F}$) and converging to its boundary, then the function must achieve its minimum value at a *strictly interior* point. The obvious application of Lemma 1 is when $B(x,\mu)$ plays the role of φ.

Lemma 1 Given a set of m smooth constraint functions $\{c_i(x)\}$, $i=1$, ..., m, let strict($\mathcal{F}$) denote the set defined by (2.3). Let S be a compact set, and assume that the set strict$(\mathcal{F})\cap S$ is nonempty. Consider any convergent sequence $\{y_k\}\in$ strict$(\mathcal{F})\cap S$ whose limit point $\bar{y}$ lies on the boundary of strict($\mathcal{F}$), i.e. such that

$$\lim_{k\to\infty} y_k = \bar{y}, \quad \text{where} \quad \bar{y}\in \mathrm{bnd}(\mathrm{strict}(\mathcal{F}))\cap S. \tag{3.28}$$

Suppose that φ is a continuous function on strict$(\mathcal{F})\cap S$ with the property that $\varphi(y_k)$ is unbounded above as $k\to\infty$ for every sequence $\{y_k\}$ satisfying (3.28). Then the global minimum value of φ in strict$(\mathcal{F})\cap S$, denoted by φ^*, is finite, and is achieved at some point x^* in strict$(\mathcal{F})\cap S$:

$$\min\{\varphi(x)\mid x\in \mathrm{strict}(\mathcal{F})\cap S\} = \varphi(x^*) = \varphi^*.$$

Proof. Given any point $\hat{x}$ in strict$(\mathcal{F})\cap S$, define the associated level set $\hat{W}$ as

$$\hat{W} = \{x\in \mathrm{strict}(\mathcal{F})\cap S \mid \varphi(x)\le\hat{\varphi}\},$$

where $\hat{\varphi}=\varphi(\hat{x})$. Because S is compact, $\hat{W}$ is bounded. Compactness of $\hat{W}$ will follow if we show that $\hat{W}$ is closed, i.e., contains all its accumulation points.

Let R denote the closed set

$$R = \mathrm{strict}(\mathcal{F})\cup \mathrm{bnd}(\mathrm{strict}(\mathcal{F})).$$

Because S is compact and R is closed, the set $R\cap S$ is compact. Consider any convergent sequence $\{x_k\}$ such that $x_k\in\hat{W}$ for all k, with limit point $\bar{x}$. Since $x_k\in$ strict$(\mathcal{F})\cap S$, $\bar{x}$ must lie in $R\cap S$. Hence $\bar{x}$ must lie in either strict$(\mathcal{F})\cap S$ or bnd(strict$(\mathcal{F}))\cap S$.

If $\bar{x}$ is in bnd(strict$(\mathcal{F}))\cap S$, then $\{x_k\}$ is a sequence satisfying (3.28), which means that $\varphi(x_k)\to\infty$. Since $\hat{\varphi}$ is an upper bound on the value of φ at any point in $\hat{W}$, we conclude that $x_k\notin\hat{W}$ for sufficiently large k, which is a contradiction. Any limit point $\bar{x}$ of a sequence in $\hat{W}$ therefore cannot be in bnd(strict$(\mathcal{F}))\cap S$, and must lie in strict$(\mathcal{F})\cap S$.

Because x_k is in $\hat{W}$, the relation $\varphi(x_k)\le\hat{\varphi}$ holds for all k. Continuity of φ in strict$(\mathcal{F})\cap S$ then implies that the limit point $\bar{x}$ satisfies $\varphi(\bar{x})\le\hat{\varphi}$, so that $\bar{x}$ possesses both properties required for membership in $\hat{W}$. Since $\{x_k\}$ is an arbitrary convergent sequence in $\hat{W}$, it follows that $\hat{W}$ contains all its accumulation points and is closed.

We know already that $\hat{W}$ is bounded, so that $\hat{W}$ is compact. Because φ is continuous in the compact set $\hat{W}$, it attains its global minimum in $\hat{W}$ at some point x^*. By definition of $\hat{W}$, the value of φ at any point in the closure of strict($\mathcal{F}$), i.e. is either strictly feasible or else an accumulation point of strict($\mathcal{F}$).

Assumption (b) disallows minimizers that occur at isolated feasible points (points in a neighbourhood containing no other feasible points). For example, consider the constraints $x \geq 1$ and $x^2 - 5x + 4 \geq 0$. The function $x^2 - 5x + 4$ is nonnegative if $x \leq 1$ and if $x \geq 4$, so that the feasible points lie in two separated regions. The constraint $x \geq 1$ eliminates all of the region $\{x \leq 1\}$ except the single point $x = 1$. The feasible region for *both* constraints therefore consists of the isolated point $\{x = 1\}$ and the set of points $\{x \geq 4\}$. Hence strict($\mathcal{F}$) is the set $\{x > 4\}$, and the point $x = 1$ does not lie in the closure of strict($\mathcal{F}$).

Barrier methods can be viewed as finding the infimum of f subject to $c(x) > 0$, and consequently cannot converge to minimizers occurring at isolated points. Isolated minimizers do not arise in the convex case because a convex set with a nonempty interior cannot contain an isolated point.

Theorem 7 (Local convergence for barrier methods.) Consider the problem of minimizing $f(x)$ subject to $c_i(x) \geq 0$, $i = 1, \ldots, m$. Let $\mathcal{F}$ denote the feasible region, and let $\mathcal{M}$ denote the set of minimizers corresponding to the objective function value f^*. Let $\{\mu_k\}$ be a decreasing sequence of positive barrier parameters such that $\lim_{k\to\infty} \mu_k = 0$. Assume that

(a) there exists a nonempty compact set $\mathcal{M}^*$ of local minimizers that is an isolated subset of $\mathcal{M}$;
(b) at least one point in $\mathcal{M}^*$ is in the closure of strict($\mathcal{F}$).

Then the following results hold:

(i) there exists a compact set S strictly containing $\mathcal{M}^*$ such that for any feasible point $\bar{x}$ in S but not in $\mathcal{M}^*$, $f(\bar{x}) > f^*$;
(ii) for all sufficiently small μ_k, $B(x, \mu_k)$ has at least one unconstrained minimizer in strict($\mathcal{F}$) $\cap$ int(S), and any sequence of global unconstrained minimizers of $B(x, \mu_k)$ in strict($\mathcal{F}$) $\cap$ int(S) has at least one convergent subsequence;
(iii) let $\{x_k\}$ denote any convergent subsequence of global unconstrained minimizers of $B(x, \mu_k)$ in strict($\mathcal{F}$) $\cap$ int(S); then the limit point of $\{x_k\}$ is in $\mathcal{M}^*$;
(iv) $\lim\limits_{k\to\infty} f(x_k) = f^* = \lim\limits_{k\to\infty} B(x_k, \mu_k)$.

Proof. Result (i) follows immediately from Theorem 6, which implies the existence of a strictly enclosing compact set S within which all points in $\mathcal{M}^*$ are global constrained minimizers.

Consider the behaviour of the barrier function $B(x, \mu_k)$ in the bounded set strict($\mathcal{F}$) $\cap$ S. Continuity of f and $\{c_i\}$ in $\mathcal{F}$ implies that $B(x, \mu_k)$ is continuous in strict($\mathcal{F}$) $\cap$ S. The barrier function possesses the properties of φ in Lemma 1, which then implies that $B(x, \mu_k)$ achieves a finite global minimum value at some point in strict$(\mathcal{F}) \cap S$. (This result is close but not equivalent to (ii), which states that the minimizing point lies in int(S).) Let y_k be any point in strict$(\mathcal{F}) \cap S$ for which the minimum value is achieved.

The sequence $\{y_k\}$ is bounded and hence has at least one limit point. Let $\hat{x}$ denote a limit point of $\{y_k\}$. Because y_k is strictly feasible for all k and the set S is compact, it follows that $\hat{x} \in \mathcal{F} \cap S$, so that $\hat{x}$ is feasible.

We wish to show that $\hat{x}$ lies in the set $\mathcal{M}^*$ of constrained minimizers, with $f(\hat{x}) = f^*$. The result will be proved by contradiction, and we accordingly assume the contrary, that $\hat{x} \notin \mathcal{M}^*$.

Since $\hat{x}$ is feasible and in S, result (i) implies that $f(\hat{x}) > f^*$. We next prove that this inequality implies the existence of a strictly feasible point x_{int} in S such that

$$f(\hat{x}) > f(x_{\text{int}}). \tag{3.31}$$

The point x_{int} can be found as follows. We know from assumption (b) that at least one point in $\mathcal{M}^*$ is in the closure of strict$(\mathcal{F})$. Let x^* denote such a point, which must either lie in strict$(\mathcal{F})$ or else be an accumulation point of strict$(\mathcal{F})$. Because $\mathcal{M}^*$ is contained in int(S), x^* is also in the interior of S.

If x^* itself is strictly feasible, x_{int} may be taken as x^*. If x^* is not strictly feasible, x^* is an accumulation point of strict$(\mathcal{F})$, which means that every neighbourhood of x^* contains strictly feasible points. Further, every neighbourhood of x^* contains points in S. We know that: f is continuous; $\hat{x}$ is feasible and lies in S; $f(\hat{x}) > f(x^*)$; and x^* is a global constrained minimizer of f for all feasible points in S. Hence there must be a strictly feasible point x_{int} in a neighbourhood of x^* for which $f(x_{\text{int}}) < f(\hat{x})$.

Let $\{x_k\}$ denote a convergent subsequence of $\{y_k\}$ with limit $\hat{x}$. The relation $f(\hat{x}) > f(x_{\text{int}})$ then implies that, for sufficiently large k,

$$f(x_k) > f(x_{\text{int}}). \tag{3.32}$$

Since x_{int} is in strict$(\mathcal{F}) \cap S$, our definition of x_k as a global minimizer of $B(x, \mu_k)$ in strict$(\mathcal{F}) \cap S$ implies the inequality

$$f(x_k) - \mu_k \sum_{i=1}^{m} \ln c_i(x_k) \le f(x_{\text{int}}) - \mu_k \sum_{i=1}^{m} \ln c_i(x_{\text{int}}). \tag{3.33}$$

Strict feasibility of x_{int} means that the barrier term involving x_{int} in (3.33) is *finite*, and

$$\lim_{k\to\infty} B(x_{\text{int}}, \mu_k) = f(x_{\text{int}}).$$

Suppose that the limit point $\hat{x}$ of $\{x_k\}$ is also strictly feasible, namely $\hat{x} \in \text{strict}(\mathcal{F}) \cap S$. Then the barrier term involving x_k is finite as $k \to \infty$