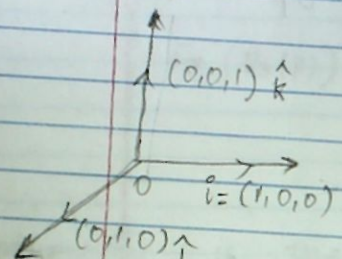
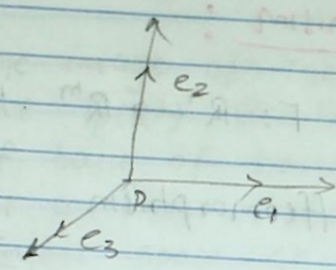
DOT PRODUCTS AND FRAMES



Import this structure to any tangent space at a point.



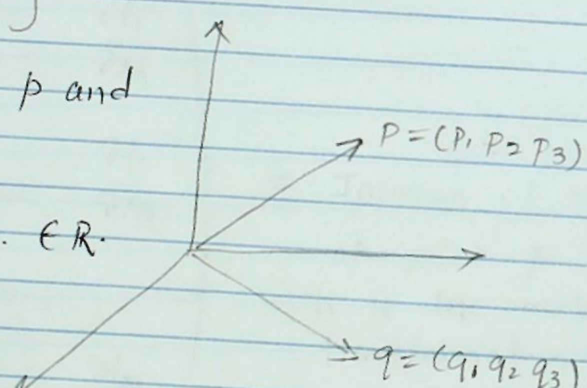
Dot Product:

$p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$ be two points in $\mathbb{R}^3$.

$$\left.\begin{aligned} OP &= p_1\hat{i} + p_2\hat{j} + p_3\hat{k} \\ OQ &= q_1\hat{i} + q_2\hat{j} + q_3\hat{k} \end{aligned}\right\} \text{Position vectors of } P \text{ and } Q.$$

Then the dot product of p and q is the real number

$$p \cdot q = p_1 q_1 + p_2 q_2 + p_3 q_3 \in \mathbb{R}.$$



Some Properties of Dot Products

Let p, q and r be points in $\mathbb{R}^3$ and a, b be real numbers. then we have the following properties that hold true for Dot Products.

(1) Bilinearity : (i) $r \cdot (ap + bq) = a(r \cdot p) + b(r \cdot q)$
(ii) $(ap + bq) \cdot r = a(p \cdot r) + b(q \cdot r)$. } can be checked easily.

(2) Symmetry : $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$.

Then $\boxed{p \cdot q = q \cdot p}$ Commutativity of Dot Products.

(3) Positive Definiteness : $p \cdot p \geq 0$; $p \cdot p = 0 \iff p = 0$.

Dot Product on Tangent Vectors

We know that tangent space any point in $\mathbb{R}^3$, that is $T_p(\mathbb{R}^3)$ is isomorphic to $\mathbb{R}^3$. We can use this relation to define Dot Product on any Tangent Space $T_p(\mathbb{R}^3)$.

$$\mathbb{R}^3 \begin{cases} p = (p_1, p_2, p_3) \text{ and } q = (q_1\ q_2\ q_3) \\ p \cdot q = p_1 q_1 + p_2 q_2 + p_3 q_3 \end{cases}$$

$T_p(\mathbb{R}^3) \cong \mathbb{R}^3$ as seen earlier.

Vector space.

Question: Can we define dot product on $T_p(\mathbb{R}^3)$?

Vectors in $T_p(\mathbb{R}^3)$ are of the form $v_p = (v_1\ v_2\ v_3)_p$ and $w_p = (w_1\ w_2\ w_3)_p$.

We define dot product $\boxed{v_p \cdot w_p = v_1 w_1 + v_2 w_2 + v_3 w_3.}$

Example : $v_p = (2,3,4)_{(1,1,1)}$ and $w_p = (2,4,6)_{(1,1,1)}$.

$$v_p \cdot w_p = (2,3,4)\cdot(2,4,6)$$
$$= 4 + 12 + 26 = 42.$$

On similar lines we can define norm of any tangent vector in the following way;

Norm of Tangent vector at $p \in \mathbb{R}^3$.

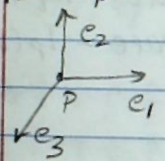
$r_p = (r_1\ r_2\ r_3)_p \in T_p(\mathbb{R}^3)$.

$\|r_p\| = \|r\|$

i.e. Norm of the vector part.

Frames

Defn: A set e_1, e_2 and e_3 of three mutually orthogonal unit vectors tangent to $\mathbb{R}^3$ at p is called a frame at point p.



e_1, e_2 and $e_3 \in T_p(\mathbb{R}^3)$.

Example:
$U_1(p) = (1,0,0)_p \in T_p(\mathbb{R}^3)$
$U_2(p) = (0,1,0)_p \in T_p(\mathbb{R}^3)$
$U_3(p) = (0,0,1)_p \in T_p(\mathbb{R}^3)$.

$U_1(p)\cdot U_2(p) = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$

And so on.

$\{U_1(p), U_2(p), U_3(p)\}$ is a frame at the point p.

Alternate way of defining frame at point p: The set of tangent vector e_1, e_2 and e_3 to $\mathbb{R}^3$ at p is a frame at p if and only if

$$e_i \cdot e_j = \delta_{ij} \quad \text{for all } i,j \in \{1,2,3\}$$

Example:
$e_1 = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)_p \in T_p(\mathbb{R}^3)$
$e_2 = \left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{-2}{\sqrt6}\right)_p \in T_p(\mathbb{R}^3)$
$e_3 = \left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}, 0\right)_p \in T_p(\mathbb{R}^3)$.

forms a frame at p.

Theorem: Let e_1, e_2 and e_3 be a frame at a point p of $\mathbb{R}^3$. If v is any tangent vector to $\mathbb{R}^3$ at p, then,

$$v = (v\cdot e_1)\,e_1 + (v\cdot e_2)\,e_2 + (v\cdot e_3)\,e_3.$$

Attitude Matrix of a Frame

Let $\{e_1, e_2, e_3\}$ be a frame at a point p in $\mathbb{R}^3$.

if
$e_1 = (a_{11}\ a_{12}\ a_{13})_p$
$e_2 = (a_{21}\ a_{22}\ a_{23})_p$
$e_3 = (a_{31}\ a_{32}\ a_{33})_p$

Then the matrix
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
is called the attitude matrix of the frame $\{e_1, e_2, e_3\}$.

Example: At any point p of $\mathbb{R}^3$
$U_1(p), U_2(p), U_3(p) \in T_p(\mathbb{R}^3)$

$U_1(p) = (1,0,0)_p$
$U_2(p) = (0,1,0)_p$
$U_3(p) = (0,0,1)_p$

Attitude Matrix corresponding to this frame ie

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rows can be interchanged to get a different attitude matrix.

Example: $e_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)_p$, $e_2 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)_p$ and

$e_3 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)_p$ $\{e_1, e_2, e_3\} \in T_p(\mathbb{R}^3)$ forms a frame at point p.

Attitude Matrix corresponding to this frame;

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$$

{Attitude Matrix is an Orthogonal Matrix since the rows are orthonormal.}

Cross Product:

Let v and w be tangent vector to $\mathbb{R}^3$ at p. The cross product of v and w is the determinant

$$v \times w = \begin{vmatrix} U_1(P) & U_2(P) & U_3(P) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example: $\bar{v} = (1, 3, -1)_p$, $\bar{w} = (-1, 0, 6)_p$

$$\bar{v} \times \bar{w} = \begin{vmatrix} U_1(P) & U_2(P) & U_3(P) \\ 1 & 3 & -1 \\ -1 & 0 & 6 \end{vmatrix}$$

$$\bar{v} \times \bar{w} = 18U_1(p) - 5U_2(p) + 3U_3(p).$$

Properties of Cross Product

Linearity: u, v and w be tangent vectors to $\mathbb{R}^3$ at the same point p and let a, b be scalers.

Then $(au + bv) \times w = a(u \times w) + b(v \times w)$

$u \times (av + bw) = a(u \times v) + b(u \times w)$.

Alteration Rule: If u and v are tangent vectors to $\mathbb{R}^3$ at a point p, then $u \times v = -v \times u$.

Speed of Curve

Defn: Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) : I \longrightarrow \mathbb{R}^3$ be a curve. The speed of α is defined as the length of the velocity of α:

$$v = \|\alpha'\| = \sqrt{\left(\frac{d\alpha_1}{dt}\right)^2 + \left(\frac{d\alpha_2}{dt}\right)^2 + \left(\frac{d\alpha_3}{dt}\right)^2}$$

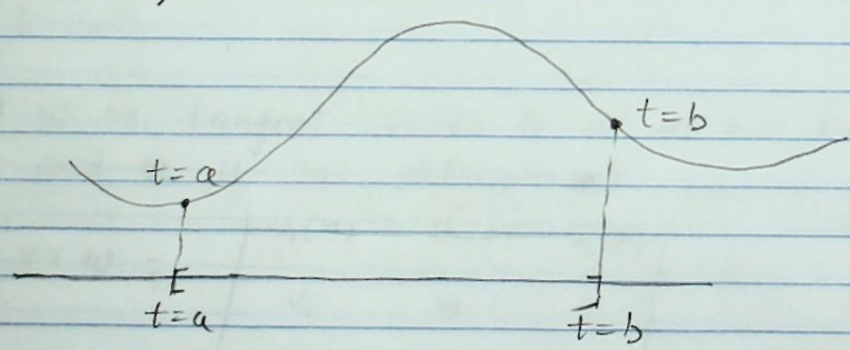
Arc Length: Arc length of α from $t=a$ to $t=b$ is equal to;

$$\int_a^b \|\alpha'\| \, dt = \int_a^b v(t) \, dt$$

Curve Segment: Restriction of a curve $\alpha : I \to \mathbb{R}^3$ to a closed interval $[a,b]$ is called a curve segment. If σ is a curve segment of α,

$$\sigma : [a,b] \longrightarrow \mathbb{R}^3$$

and the legth of σ is written as $L(\sigma)$.



Unit Speed Curve: **Defn:** A curve β has unit speed if its speed is 1 at every point, that $\|\beta'\| = 1$.

Example: Let $\beta : I \to \mathbb{R}^3$ be defined by,

$$\beta(t) = (\cos t, \sin t, t)$$

$$\beta'(t) = (-\sin t, \cos t, 1)$$

$$\|\beta'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \Rightarrow \beta(t) \text{ is not a unit speed curve.}$$

Ex. $\beta(t) = (\cos t, \sin t, 1) \Rightarrow \beta'(t) = (-\sin t, \cos t, 0)$

$$\|\beta'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1 \Rightarrow \beta(t) \text{ is a unit speed curve.}$$

Arc Length Reparametrization

Theorem: Let $\alpha : I \to \mathbb{R}^3$ be a regular curve. Then there exists a reparametrization β of α such that

$$\|\beta'\| = 1$$

Recall: Regular curve ; $\alpha'(t) \neq 0 \; \forall t$

Reparametrization ; $\alpha : I \to \mathbb{R}^3$

$h : J \to I$ differentiable

$\beta(s) = \alpha(h(s))$ reparametrization

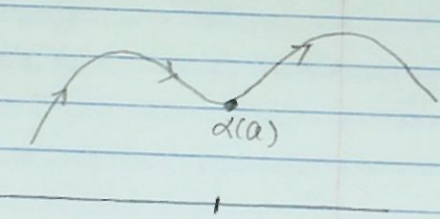
Same curve followed at different speed.

$\alpha: I \to R^3$; $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ $\Big\} t = a$

$\alpha(a) = (\alpha_1(a), \alpha_2(a), \alpha_3(a))$

Proof: Let $t=0$ be the value of the parameter corresponding to the base point.

We define $s(t) = \int_a^t \|\alpha'(u)\| du$



Recall: (Fundamental Theorem of Calculus)

Let $f(x)$ be a continuous function, we define $F(x) = \int_a^x f(u)\,du$. $F(x)$ is continuous and differentiable

and $F'(x) = f(x)$.

So we have $s(t) = \int_a^t \|\alpha'(u)\| du$

$$\left\{ \begin{aligned} \frac{ds}{dt} &= \|\alpha'(t)\| \\ &> 0 \end{aligned} \right\}$$; Since curve is Regular $\alpha'(t) \neq 0$ $\Rightarrow \|\alpha'(t)\| > 0$.

Now note that function $s(t)$ is given with derivative of s w.r.t t strictly greater than 0; then $s(t)$ is a strictly increasing function and hence invertible.

Thus there exists $t = t(s)$; $\frac{dt}{ds} \cdot \frac{ds}{dt} = 1$

We shall now show that $\|\beta'(s)\| = \|\alpha'(t(s))\| = 1$

From the lemma seen in the section on we have $\beta'(s) = \frac{dh}{ds}(s)\,\alpha'(h(s))$ where β is the reparametrization of α by h.

$\|\beta'(s)\| = \|\alpha'(t(s))\| \cdot \left\|\frac{dt}{ds}\right\|$

But $\alpha'(t(s)) = \frac{ds}{dt}$ as proved earlier.

So $\left\{ \|\beta'(s)\| = \left\|\frac{ds}{dt}\right\| \cdot \left\|\frac{dt}{ds}\right\| = 1 \right\}$.

Example: Consider a helix $\alpha: \mathbb{R} \to \mathbb{R}$ defined by

$$\alpha(t) = (a\cos t, a\sin t, bt)$$

$\alpha'(t) = (-a\sin t, a\cos t, b)$

$\|\alpha'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} \neq 1$.

New Parameter:

$s = \int_0^t \|\alpha'(u)\| du = \int_0^t \sqrt{a^2+b^2}\, du$

$\Rightarrow s(t) = \sqrt{a^2+b^2}\,(t-0) = \sqrt{a^2+b^2}\cdot t$

$\boxed{s(t) = \sqrt{a^2+b^2}\, t}$ $\Rightarrow \left\{ t = \frac{s(t)}{\sqrt{a^2+b^2}} \right\}$

Substitute $t = \frac{s}{\sqrt{a^2+b^2}}$ in the original form of the curve $\alpha(t)$.

$$\alpha(t) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right)$$

Note that we are essentially following the proof.

It can now be easily checked that $\|\alpha'(s)\| = 1$.

Orientation of Reparametrization

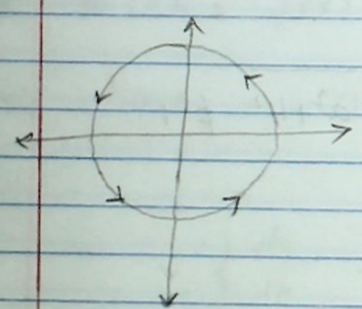
A reparametrization of a curve α given by

$$\beta(t) = \alpha(h(s)).$$

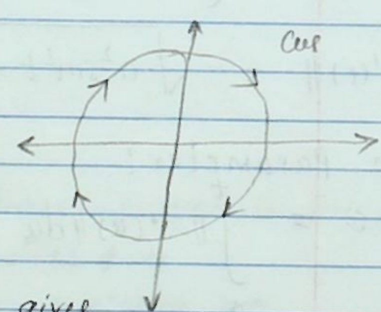
is orientation preserving if $h'(s) \geq 0$

is orientation reversing if $h'(s) \leq 0$

Example: $\alpha(t) = (\cos t, \sin t)$ gives a circle traversed in anti-clockwise sense.



$t = 2\pi - \tau$



$\alpha(\tau) = (\cos(2\pi-\tau), \sin(2\pi-\tau))$ gives circle traversed in clockwise sense.

We have $t = 2\pi - \tau$

$\frac{dt}{d\tau} = -1 < 0$ hence orientation reversing.

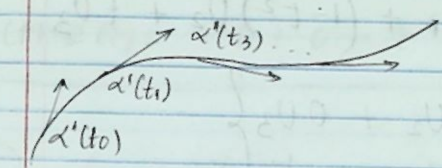
Fact: Unit Speed Reparametrization is always orientation preserving. Since $\frac{ds}{dt} > 0$ as seen in the proof!

Vector Fields on Curves:

Defn: A vector field Y on a curve α is a function that assigns to each number t in I a tangent vector $Y(t)$ to R^3 at the point $\alpha(t)$.

We are essentially restricting the defn of usual vector field which assigns a tangent vector to a point in R^3 to a curve in R^3

Example: For curve α the velocity α' satisfies the defn of vector field on a curve — Velocity vector field on α.



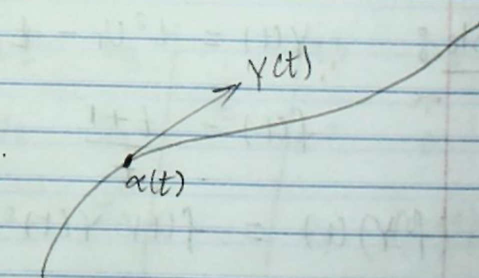
Properties of Vector Fields on Curves:

① If Y is a vector field on $\alpha: I \to R^3$, then for each t in I we can write

$$Y(t) = (y_1(t), y_2(t), y_3(t))_{\alpha(t)} = \sum y_i(t)\, U_i(\alpha(t)).$$

The real valued functions y_1, y_2 and $y_3: R \to R$ are called Euclidean coordinate functions.

Note that; $U_1(\alpha(t)) = (1,0,0)_{\alpha(t)}$
$U_2(\alpha(t)) = (0,1,0)_{\alpha(t)}$
$U_3(\alpha(t)) = (0,0,1)_{\alpha(t)}$.



(2) Addition of Vector Fields : Let $Y(t) = y_1 U_1 + y_2 U_2 + y_3 U_3$ and $Z(t) = z_1 U_1 + z_2 U_2 + z_3 U_3$ be two vector valued functions on $\alpha : I \to \mathbb{R}^3$. Then the addition of $Y(t)$ and $Z(t)$ is defined as;

$$(Y(t) + Z(t)) = (y_1 U_1 + y_2 U_2 + y_3 U_3) + (z_1 U_1 + z_2 U_2 + z_3 U_3)$$

$$Y(t) + Z(t) = (y_1 + z_1) U_1 + (y_2 + z_2) U_2 + (y_3 + z_3) U_3 .$$

Example :

$$Y(t) = t^2 U_1 - t U_3 \quad , \quad Z(t) = (1-t^2) U_2 + t U_3$$

$$Y(t) = t^2 U_1 + 0 U_2 - t U_3$$

$$Z(t) = (1-t^2) U_2 + t U_3 = 0 U_1 + (1-t^2) U_2 + t U_3$$

$$\left\{ Y(t) + Z(t) = t^2 U_1 + (1-t^2) U_2 + 0 U_3 \right\}$$

(3) Scaler Multiplication : Let $Y(t) = y_1 U_1 + y_2 U_2 + y_3 U_3$ be a vector field on $\alpha : I \to \mathbb{R}^3$ and $f(t)$ be a real valued functions. Then we define fY

$$(fY)(t) = f(t) Y(t) = f(t) (y_1 U_1 + y_2 U_2 + y_3 U_3)$$

$$(fY)(t) = y_1 f(t) U_1(t) + y_2 f(t) U_2(t) + y_3 f(t) U_3(t).$$

Example : $Y(t) = t^2 U_1 - t U_3$

$$f(t) = \frac{t+1}{t}$$

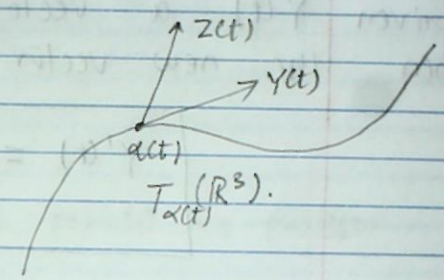
$$(fY)(t) = f(t) \cdot Y(t) = \left(\frac{t+1}{t}\right) (t^2 U_1 - t U_3)$$

$$(fY)(t) = (t+1) t \, U_1 - (t+1) U_3$$

(4) Dot Product of Two Vector Fields : Let $Y(t) = y_1 U_1 + y_2 U_2 + y_3 U_3$ and $Z(t) = z_1 U_1 + z_2 U_2 + z_3 U_3$ be two vector valued functions on $\alpha : I \to \mathbb{R}^3$. Then the dot product of $Y(t)$ and $Z(t)$ is defined ;

$$Y(t) \cdot Z(t) = (y_1 U_1 + y_2 U_2 + y_3 U_3) \cdot (z_1 U_1 + z_2 U_2 + z_3 U_3)$$

$$Y(t) \cdot Z(t) = y_1 z_1 + y_2 z_2 + y_3 z_3$$



Example: $Y(t) = t^2 U_1 - t U_3$
$Z(t) = (1-t^2) U_2 + t U_3$

$$Y(t) \cdot Z(t) = (t^2 U_1 + 0 U_2 - t U_3) \cdot ((1-t^2) U_2 + t U_3 + 0 U_1)$$

$$\boxed{Y(t) \cdot Z(t) = 0 + 0 - t^2 = -t^2}$$

(5) Cross Product of Vector Fields ; $Y(t)$ & $Z(t)$ as in (4)

$$Y(t) \times Z(t) = \begin{vmatrix} U_1 & U_2 & U_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Example: $Y(t) = t^2 U_1 + 0 U_2 - t U_3$

$Z(t) = (1-t^2) U_2 + t U_3$

$$Y(t) \times Z(t) = \begin{vmatrix} U_1 & U_2 & U_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} U_1 & U_2 & U_3 \\ t^2 & 0 & -t \\ 0 & (1-t^2) & t \end{vmatrix}$$

$= U_1(t(1-t^2)) - U_2(t^3) + U_3(t^2(1-t^2))$

$= t(1-t^2)U_1 - t^3U_2 + t^2(1-t^2)U_3$.

(6) Differentiation of Vector Fields on a Curve:

Given $Y(t)$ a vector field on a curve α, with $Y(t) = \sum y_i U_i$. Then the new vector field on α

$$\boxed{Y'(t) = \sum \frac{dy_i}{dt} U_i}$$

Example: Let $Y(t) = 2tU_1 - tU_3$

$Y'(t) = 2U_1 - U_3$

$Y''(t) = 0U_1 - 0U_3 = 0 \equiv (0,0,0)_{\alpha(t)}$

Properties of Differentiation

Let $Y(t) = y_1U_1 + y_2U_2 + y_3U_3$; $Z(t) = z_1U_1 + z_2U_2 + z_3U_3$ vector valued functions on $\alpha: I \to R^3$. Let $f: R \to R$ be real valued functions.

Linearity : $(aY + bZ)' = aY' + bZ'$

Leibnizian Property : $(Y \cdot Z)' = Y' \cdot Z + Y \cdot Z'$

$(fY) = f'Y + fY'$

Acceleration of α : Let $\alpha: I \to R^3$ be a curve with coordinates $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ then the acceleration of α is defined as;

$$\alpha'' = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^2\alpha_3}{dt^2}\right)$$

An Important Result - to be used in several results in further discussions.

Let $Y(t) = y_1U_1 + y_2U_2 + y_3U_3$ and $Z(t) = z_1U_1 + z_2U_2 + z_3U_3$ be two vector valued functions on $\alpha: I \to R^3$.
If $Y \cdot Z = \text{constant } (= c)$ then

$Y \cdot Z = c$

Differentiating $Y' \cdot Z + Z' \cdot Y = 0 \Rightarrow \boxed{Y' \cdot Z = -Z' \cdot Y}$

Now let Y be a vector field s.t $\|Y\| = 1$

then $Y \cdot Y = 1$

$\Rightarrow Y \cdot Y' + Y' \cdot Y = 0 \Rightarrow 2Y \cdot Y' = 0 \Rightarrow \boxed{Y \cdot Y' = 0}$

$\Rightarrow \boxed{Y \perp Y'}$

Parallel Vector Fields: A vector field on a curve is parallel if all its values are parallel tangent vectors.
That is;

$Y(t) = (c_1, c_2, c_3)_{\alpha(t)} = \sum c_i U_i(\alpha(t))$