$$B = \frac{\left(-1+t^2,\ -2t,\ 1+t^2\right)}{\sqrt{2}\,(1+t^2)}$$

$$K = \frac{1}{3(1+t^2)^2}$$

$$\tau = \frac{1}{3(1+t^2)^2}$$

Spherical Image

Defn: — Another application of Frenet Formulae.

Let $\beta : I \to \mathbb{R}^3$ be a unit speed curve. The spherical image of β is the curve $\sigma \approx T$; that is;

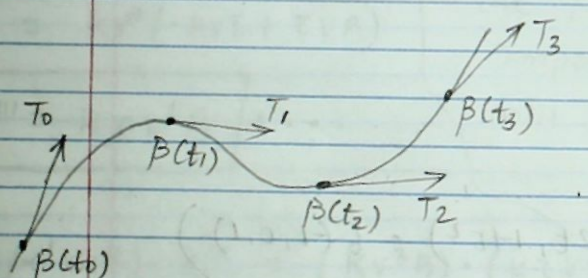
$$\sigma : I \to \mathbb{R}^3$$

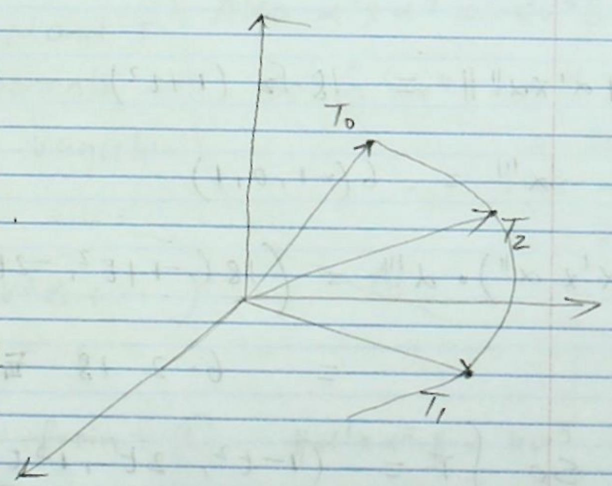
is given by;

$$\sigma(t) = (V_1(t),\ V_2(t),\ V_3(t))_{(0,0,0)}$$

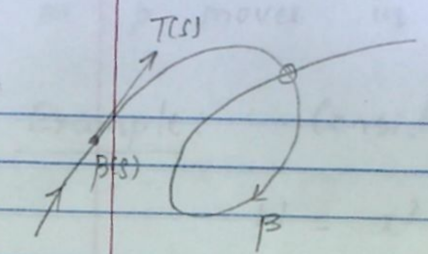
where;

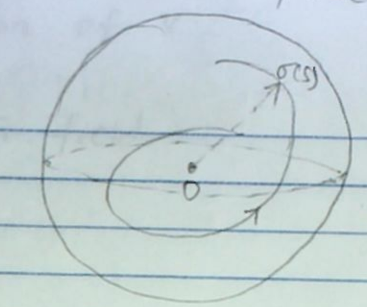
$$\beta'(t) = (V_1(t),\ V_2(t),\ V_3(t))_{\beta(t)}$$ is tangent at the point $\beta(t)$ of the curve.





Take the vector part of all these tangent vectors and translate them to the origin. Then join the tails of all these vectors. Then curve so formed lies on a sphere of unit radius.

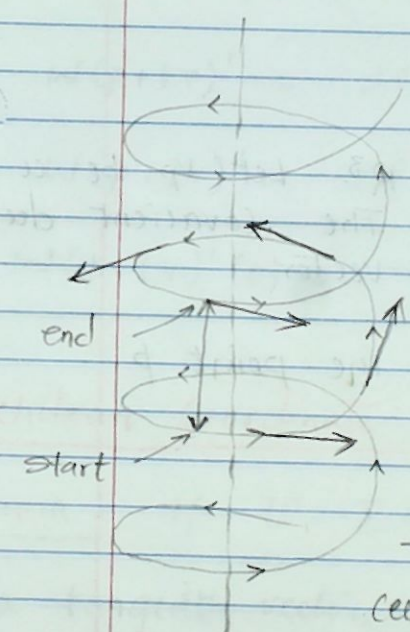


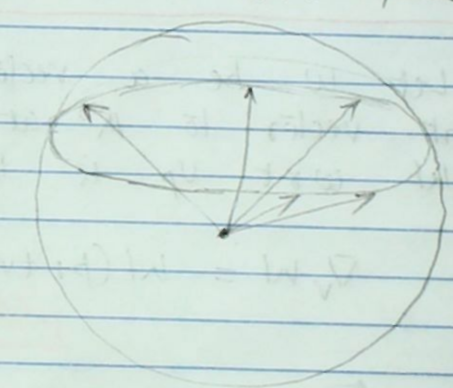


Example: The unit helix is given by

$$\beta(t) = \left(a\cos\frac{s}{t},\ a\sin\frac{c}{t},\ \frac{bs}{c}\right) \quad ; \text{ where } c = \sqrt{a^2+b^2}.$$ Then the spherical image σ of β is given by;

$\boxed{\text{A Circle on a unit sphere.}}$ — Tells us how tangents are changing directions.



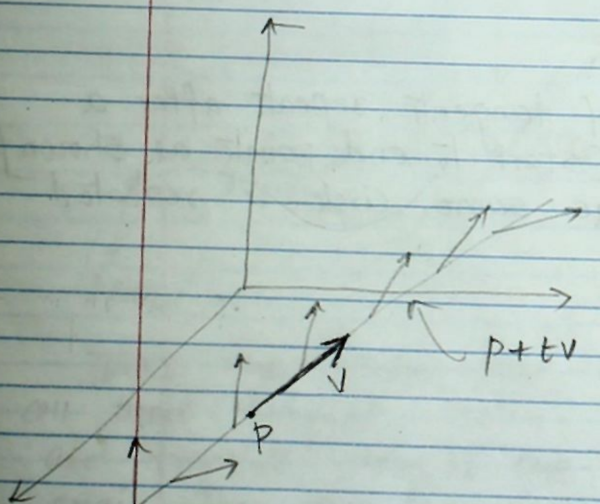


The direction of tangents repeats after a certain period [start to end points as shown]. After this the same circle is repeated.

Covarient Derivative

Defn: Let W be a vector field on $\mathbb{R}^3$. Let v_p be a tangent vector to $\mathbb{R}^3$ at point p. The covarient derivative of W w.r.t v_p is the tangent vector

$$\nabla_v W = W(p+tv)'(0) \quad \text{at the point } p.$$



$\nabla_v W$ measures the initial rate of change of $W(p)$ as p moves in the direction of v.

Example: Consider a vector field

$$W = x^2 U_1 + yz\, U_2$$

and $v = (-1,0,2)$ at $p = (2,1,0)$

To find the covarient derivative we must find $W(p+tv)'(0)$

$$p+tv = (2,1,0) + t(-1,0,2)$$
$$= (2-t,\ 1,\ 2t)$$

$$W(p+tv) = (2-t)^2 U_1 + (1\cdot 2t) U_2 = (2-t)^2 U_1 + 2t U_2$$

$$W(p+tv)' = 2(2-t)(-1) U_1 + 2 U_2$$

$$W(p+tv)' = -2(2-t) U_1 + 2 U_2$$

$$\boxed{W(p+tv)'(0) = -4 U_1 + 2 U_2}$$

Euclidean Coordinate Representation

Lemma: If $W = \sum w_i U_i$ is a vector field on $\mathbb{R}^3$, and v is a tangent vector at p, then ; $w_i : \mathbb{R}^3 \to \mathbb{R}$, $v[w_i]$ = directional derivative of w_i at p in the direction v.

$$\nabla_v W = \sum_{i=1}^{3} v[w_i]\, U_i(p)$$

Proof: By Defn.

$$\nabla_v W = W(p+tv)'(0).$$

$$W(p+tv) = \left(\sum_{i=1}^{3} w_i U_i\right)(p+tv)$$

$$W(p+tv) = \sum_{i=1}^{3} w_i(p+tv)\, U_i(p+tv)$$

$$W(p+tv) = \sum_{i=1}^{3} w_i(p+tv)\, U_i$$

Recall: $w_i : \mathbb{R}^3 \to \mathbb{R}$

$$v_p[w_i] = \left.\frac{d w_i(p+tv)}{dt}\right|_{t=0}$$

$$W(p+tv)' = \sum_{i=1}^{3} \frac{d w_i(p+tv)}{dt}\, U_i$$

$$\Rightarrow W(p+tv)' = \left(\sum_{i=1}^{3} \frac{d w_i(p+tv)}{dt}\, U_i\right)(0)$$

$$\boxed{W(p+tv)' = \sum_{i=1}^{3} v_p[w_i]\, U_i}$$

~~So we have defined/derived covarient derivative in terms of~~

So we have derived/defined Covarient Derivatives in terms of Directional Derivatives.

Properties of Covarient Derivatives: From the properties of directional derivative we can derive;

Theorem: Let v and w be tangent vectors to $\mathbb{R}^3$ at p, and let Y and Z be vector fields on $\mathbb{R}^3$. Then for numbers a, b & functions f.

(1) $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$

Covarient Derivative in the direction which is a linear combo of directions v and w is given by linear comb. of covarient derivatives in the directions of v and w.

Proof: $\nabla_{av+bw} Y = \sum_{i=1}^{3} (av+bw)[y_i]\, U_i$ — (I) ; $Y = \sum y_i U_i$

Recall $(av+bw)[y_i] = a\,v[y_i] + b\,w[y_i]$

So I becomes

$$\nabla_{av+bw} Y = \sum_{i=1}^{3} \left(a\,v[y_i] + b\,w[y_i]\right) U_i$$

$$\boxed{\nabla_{av+bw} Y = a\sum v[y_i]\, U_i + b\sum w[y_i]\, U_i}$$

~~Using this property (1) we can calculate covarient derivative in any direction spanned by v and w.~~

(2) $\nabla_v (aY + bZ) = a\nabla_v Y + b\nabla_v Z$

$$\nabla_v (aY+bZ) = \nabla_v \left(a\sum y_i U_i + b\sum z_i U_i\right) \quad ; \quad Y = \sum y_i U_i,\ Z = \sum z_i U_i.$$

$$= \nabla_v \left[\left(a\sum y_i\right) + \left(b\sum z_i\right)\right] U_i$$

$$= \nabla_v \left(\sum (a y_i + b z_i)\right) U_i$$

$$= \sum v[a y_i + b z_i]\, U_i$$

~~From (1)~~ Recall: $v[a y_i + b z_i] = a\,v[y_i] + b\,v[z_i]$

$$= \left(a\sum v[y_i] + b\sum v[z_i]\right) U_i$$

$$\boxed{\nabla_v (aY+bZ) = a\nabla_v Y + b\nabla_v Z}$$

(3) $\nabla_v (f\nabla) = v[f]\, Y(p) + f(p) \nabla_v Y$
(4) $\nabla [Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v(Z)$ } Proofs in Barett O'Neill.

Vector Field of Covarient Derivatives

Defn: Let V and W be vector fields on $\mathbb{R}^3$.

The covarient derivative $\nabla_V W$ of W w.r.t V is a vector field defined by

$$\nabla_V W(p) = \nabla_{V(p)} W$$

Given a vector field W we are finding its covarient derivative at each point in a direction given by vector field V.

Corollary: $W = \sum W_i U_i$; then

$$\boxed{\nabla_V W(p) = \sum_i V(p)[W_i]\, U_i}$$

$$\boxed{\nabla_V W = \sum V[W_i]\, U_i}$$

Example: We always use $\nabla_V W = \sum V[W_i] U_i$ and $U_i[f] = \frac{\partial f}{\partial x_i}$

Given $W = xy\, U_1 - e^z U_3$ and $V = z U_1 + (x-y) U_2$; then

$$V[xy] = z U_1[xy] + (x-y) U_2[xy]$$

$$= zy + (x-y)x$$

$$V[e^z] = z U_1[e^z] + (x-y) U_2[e^z] = 0$$

$$\boxed{\nabla_V W = \left(yz + x(x-y)\right) U_1}$$

From Curves to Space

Question: Can we generalize the ideas & methods devised to study curves in $\mathbb{R}^3$ like the Frenet-Serret Formulas & the idea of frame fields, in order to study the space $\mathbb{R}^3/\mathbb{R}^n$ itself or surfaces in $\mathbb{R}^3/\mathbb{R}^n$.

Idea: ① Assign a frame (mutually orthogonal frame) to each point of the surface (just the way we devised a (T,N,B) frame to study curves)

② Calculate the changes in the frame using the frame i.e. in terms of the frame itself ; Like the Frenet-Serret Formulas.

Recall: Orthonormal Expansion of a Vector

Theorem: Let e_1, e_2, e_3 be a frame at a point p of $\mathbb{R}^3$. If v is any tangent vector to $\mathbb{R}^3$ at p, then

$$v = (v, e_1) e_1 + (v, e_2) e_2 + \cdots + (v, e_3) e_3$$

So we carry the idea for Euclidean space $\mathbb{R}^3$. Then the restriction of the idea to geometries in $\mathbb{R}^3$ is simple.

Frame Fields

Defn: Let V and W be vector fields in $\mathbb{R}^3$, then we can define their dot product $V \cdot W$ as ;

$$V \cdot W(p) = V(p) \cdot W(p).$$

Similarly we can define the following;

Defn: Let V be a vector field then we define $\|V\|$ to be a real valued function on $\mathbb{R}^3$

$$\|V\| : \mathbb{R}^3 \to \mathbb{R}$$

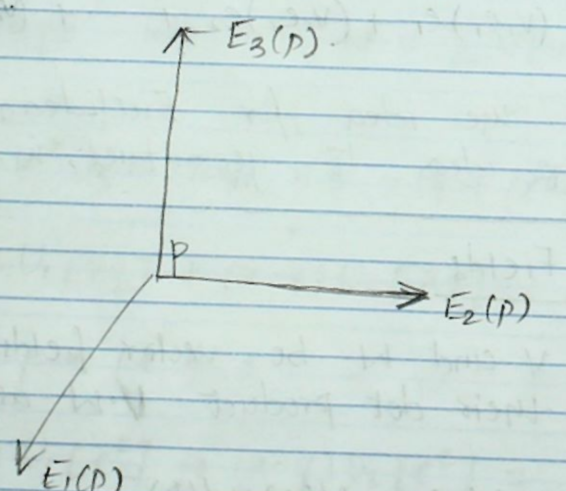
given by; $\|V(p)\| = \sqrt{V(p)\cdot V(p)}$

Frame Field (Defn): Three vector fields E_1, E_2 and E_3 on $\mathbb{R}^3$ form a Frame Field on $\mathbb{R}^3$ if they satisfy;

$$E_i \cdot E_j = \delta_{ij} \quad \text{for } 1 \leq i,j \leq 3$$

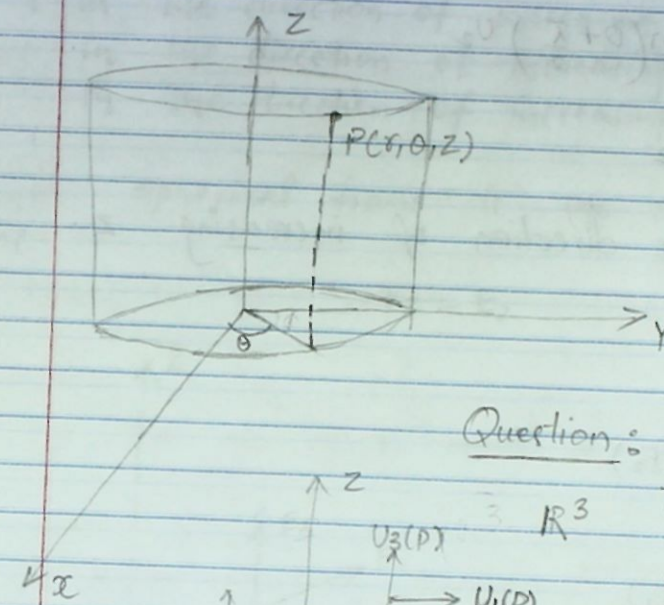
$$= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Thus at each point p the vectors $E_1(p)$, $E_2(p)$ and $E_3(p)$ do in fact form a Frame since they have unit length and are mutually orthogonal.



Examples of Frame Fields;

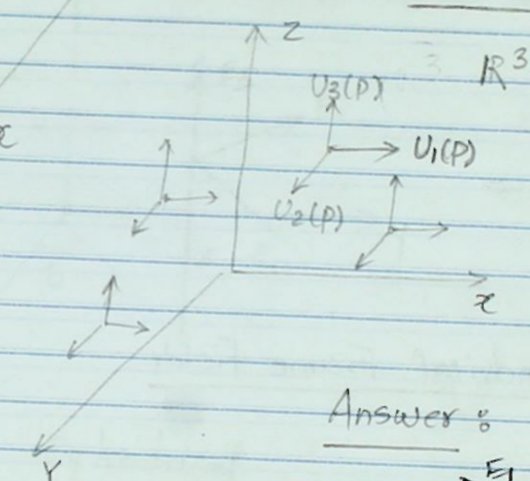
Cylendrical Frame Field



Cylendrical coordinate system

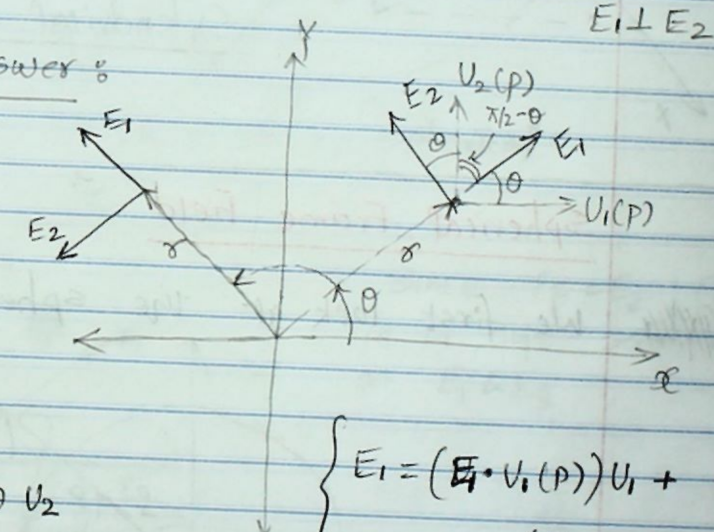
$(x,y,z) \equiv P \equiv (r,\theta,z)$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$



Question: Can we find out vector field in the direction of increasing r, θ and z just the way we have U_1, U_2 & U_3 in the direction of increasing x, y and z.

Answer:



E_1 – in the direction of increasing r

E_2 – in the direction of increasing θ

$$E_1 = \cos\theta\, U_1 + \sin\theta\, U_2$$

Similarly $E_2 = \cos\left(\frac{\pi}{2}+\theta\right) U_1 + \sin\left(\frac{\pi}{2}+\theta\right) U_2$.

We can thus calculate E_1 & E_2 at each point in $\mathbb{R}^3$ in terms of natural frame field

$$\begin{cases} E_1 = (E_1 \cdot U_1(P))U_1 + (E_1 \cdot U_2(P))U_2 \\ E_1 \cdot U_1(P) = \|E_1\| \, \|U_1(P)\| \cos\theta \\ E_1 \cdot U_1(P) = \cos\theta \\ E_1 \cdot U_2(P) = \sin\theta \end{cases}$$

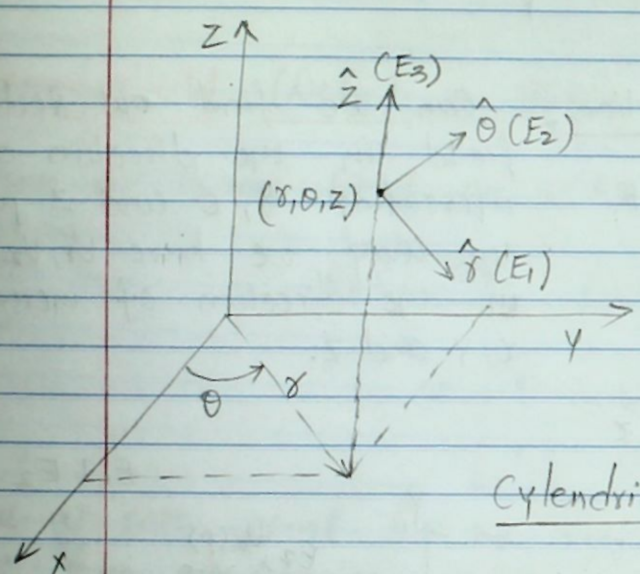
E_3 and U_3 are in the same direction. So we have the following Frame Fields.

$$E_1 = \cos\theta\, U_1 + \sin\theta\, U_2$$

$$E_2 = \cos\left(\theta+\frac{\pi}{2}\right) U_1 + \sin\left(\theta+\frac{\pi}{2}\right) U_2$$

$$E_2 = -\sin\theta\, U_1 + \cos\theta\, U_2$$

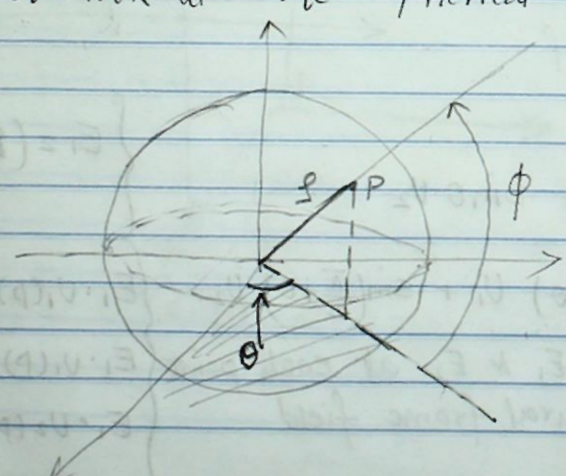
$E_3 = U_3$; in the direction of increasing z.



Cylendrical Frame Fields

Spherical Frame Field

We first look at the spherical coordinates.



So the spherical coordinates are given by ρ, θ and $\phi \equiv (\rho, \theta, \phi)$

From Cylendrical Frame to Spherical Frame

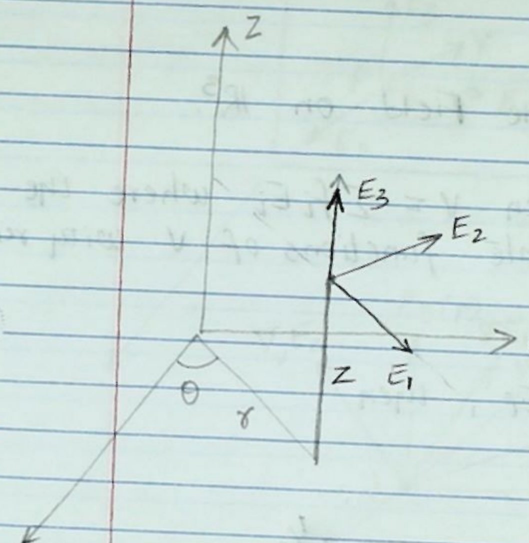
F_1 : in the direction of increasing ρ

F_2 : in the direction of increasing θ.

F_3 : in the direction of increasing ϕ

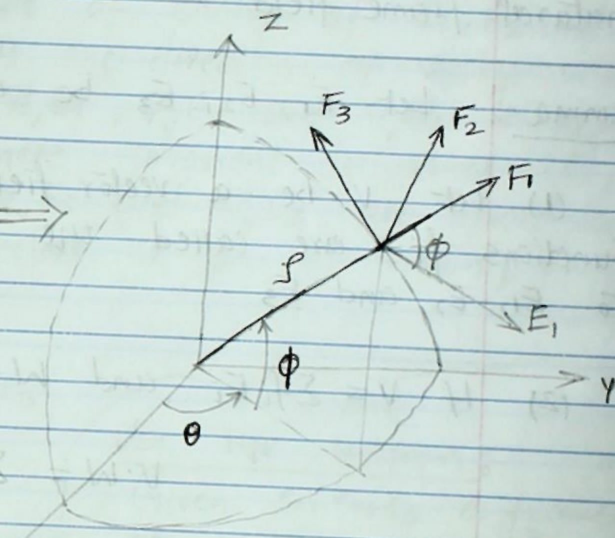
F_2 in spherical frame is the same as E_2 in the cylendrical frame.

$$F_2 = E_2$$

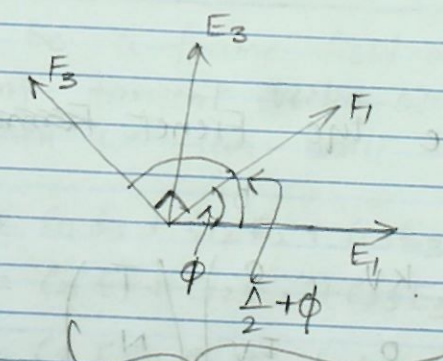


Cylendrical

$\Longrightarrow$



Spherical



Since $F_2 = E_2$; F_1 & F_3 lie in the same plane as E_1 & E_3

Similar calculation as in cylendrical frame field derivation yields ;

$$\left.\begin{aligned} F_2 &= E_2 \\ F_1 &= \cos\phi\, E_1 + \sin\phi\, E_3 \\ F_3 &= \cos\left(\phi+\tfrac{\pi}{2}\right) E_1 + \sin\left(\phi+\tfrac{\pi}{2}\right) E_3 \\ F_3 &= -\sin\phi\, E_1 + \cos\phi\, E_3 \end{aligned}\right\} \begin{aligned} &\text{Can be easily Checked;} \\ &F_2\cdot F_3 = 0 \\ &F_1\cdot F_3 = 0 \\ &F_1\cdot F_2 = 0 \\ &\|F_1\| = \|F_2\| = \|F_3\| = 1 \end{aligned}$$

We have thus expressed spherical frame field in terms of cylendrical frame field.

We can similarly express spherical frame field in terms of the natural frame field.

<u>Lemma</u>: Let E_1, E_2, E_3 be a Frame Field on $\mathbb{R}^3$.

(1) Let V be a vector field, then $V = \sum f_i E_i$ where the functions f_i are called the coordinate functions of V with respect to E_1, E_2 and E_3

(2) If $V = \sum f_i E_i$ and $W = \sum g_i E_i$, then

$$V\cdot W = \sum f_i g_i$$

and $$\|V\| = \|V\cdot V\|^{1/2} = \left(\sum f_i\right)^{1/2}.$$

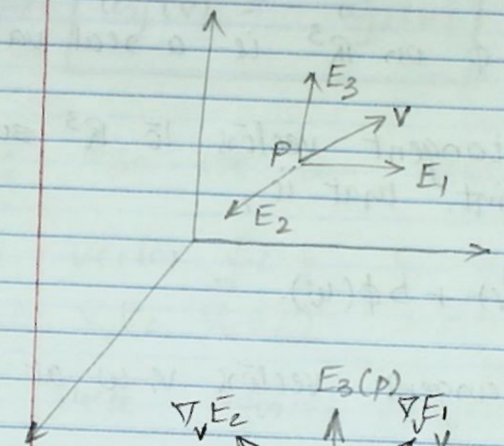
Connection Forms

For the curve we have the Frenet ~~Formulas~~ Frame and the Frenet Formulas;

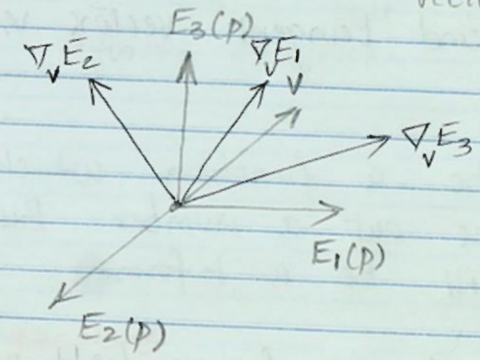
$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & kv & 0 \\ -kv & 0 & \tau v \\ 0 & -\tau v & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

<u>Question</u>: Can we do the same for any Frame Field?

<u>Answer</u>: Yes, we can express the covarient derivative of a frame field $\{E_1, E_2, E_3\}$ in terms of vector fields E_1, E_2, E_3 themselves.



At point p, we can find the rate of change of E_1, E_2 and E_3 in the direction of v by $\nabla_v E_1$, $\nabla_v E_2$ and $\nabla_v E_3$. Now these covarient derivatives $\nabla_v E_1$, $\nabla_v E_2$ and $\nabla_v E_3$ are itself tangent vectors at p as shown;



<u>Recall</u>: <u>Use the theorem</u>: Given e_1, e_2, e_3 a frame at a point p of $\mathbb{R}^3$. If ~~vector~~ v is a tangent vector to $\mathbb{R}^3$ at p, then $v = (v, e_1)e_1 + (v, e_2)e_2 + (v, e_3)e_3$.

Covarient Derivative of Frame Field

Let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$. Let v be any tangent vector at point p of $\mathbb{R}^3$. Then we can write

$$\left.\begin{aligned} \nabla_v E_1 &= C_{11} E_1 + C_{12} E_2 + C_{13} E_3 \\ \nabla_v E_2 &= C_{21} E_1 + C_{22} E_2 + C_{23} E_3 \\ \nabla_v E_3 &= C_{31} E_1 + C_{32} E_2 + C_{33} E_3 \end{aligned}\right\} \quad C_{ij} = \nabla_v E_i \cdot E_j(p), \quad 1 \le i, j \le 3.$$

Also note that the coefficients C_{ij} depend only on the vector v, so we can write them as;

$$c_{ij}(v) = \omega_{ij}(V) = \nabla_v E_i \cdot E_j(p) \; ; \; 1 \le i,j \le 3.$$

The function $\omega_{ij}(v)$ as given above is a function of tangent vectors on $\mathbb{R}^3$, and the output is a real number.

Question (Recall) Have we met any function that satisfy this property?

Yes; Recall the defn. of 1-form;

Recall Defn. (1-Form) A 1-form ϕ on $\mathbb{R}^3$ is a real valued function on the set of all tangent vectors to $\mathbb{R}^3$ such that ϕ is linear at each point, that is,

$$\phi(av + bw) = a\phi(v) + b\phi(w).$$

For any numbers a, b and tangent vectors v, w at the same point of $\mathbb{R}^3$.

We have $\omega_{ij}(v)$ to be a function which takes in a tangent vector and gives out a number. But is it linear too?? If yes, then it will be a 1-form.

Lemma: Let E_1, E_2, E_3 be a frame field on $\mathbb{R}^3$. For each tangent vector v to $\mathbb{R}^3$ at p, let

$$\omega_{ij}(v) = \nabla_v E_i \cdot E_j(p) \text{ for } 1 \le i,j \le 3$$

Then each ω_{ij} is a one-form and $\omega_{ij} = -\omega_{ji}$.

Proof: $\omega_{ij}(av + bw) = (\nabla_{av+bw} E_i) \cdot E_j(p)$

$= (a \nabla_v E_i + b \nabla_w E_i) \cdot E_j(p)$

$= a \nabla_v E_i \cdot E_j(p) + b \nabla_w E_i \cdot E_j(p)$

(2) Now; $E_i \cdot E_j = \delta_{ij}$

Recall; $v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$

Differentiating we get

$$0 = v[\delta_{ij}] = v[E_i \cdot E_j] = \nabla_v E_i \cdot E_j(p) + E_i(p) \cdot \nabla_v E_j$$

$$0 = \nabla_v E_i \cdot E_j(p) + E_i(p) \cdot \nabla_v E_j = \omega_{ij}(v) + \omega_{ji}(v) = 0$$

$$\Rightarrow \boxed{\omega_{ij}(v) = -\omega_{ji}(v)}$$

Remark: So we can now find the covariant derivatives of the vector fields E_1, E_2 and E_3 in any direction v $(\nabla_v E_1, \nabla_v E_2, \nabla_v E_3)$ and express in terms of E_1, E_2 and E_3 and this information is contained in these 1-forms $\omega_{ij}(v)$.

∇

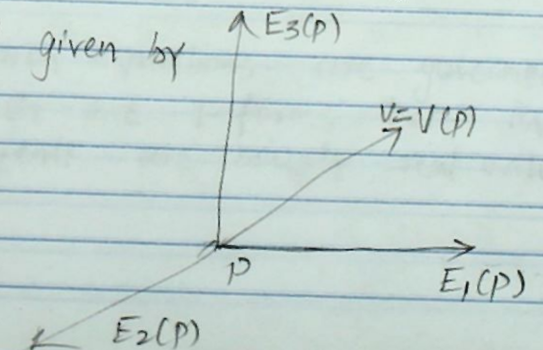
Connection Equations

Theorem: Let ω_{ij} be the connection forms of the frame field E_1, E_2 and E_3 on $\mathbb{R}^3$. Then for any vector field V on $\mathbb{R}^3$

$$\nabla_V E_i = \sum \omega_{ij}(V) E_j \; ; \; \sum \omega_{ij}(v) E_j(p)$$

$$= \sum \omega_{ij}(V(p)) E_j(p)$$

vector 'v' given by $V(p)$.



We can write the connection forms ω_{ij} in the matrix form;

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}$$

But since $\omega_{ij} = -\omega_{ji}$; so for $i=j$ we have $\omega_{ii} = -\omega_{ii} \Rightarrow \boxed{\omega_{ii} = 0}$

$$\omega_{ii} = 0 \; ; \; 1 \leq i \leq 3.$$

Hence the matrix of connection becomes

$$\omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}$$

Hence we can write connection equations as ;

$$\begin{cases} \nabla_V E_1 = 0 + \omega_{12}(V) E_2 + \omega_{13}(V) E_3 \\ \nabla_V E_2 = \omega_{21}(V) E_1 + 0 + \omega_{23}(V) E_3 \\ \nabla_V E_3 = \omega_{31}(V) E_1 + \omega_{32}(V) E_2 + 0 \end{cases}$$

Comparison : Frenet Formulas vs Connection Equations

Frenet Formulas

$$\left.\begin{aligned} T' &= 0 + \kappa v N + \boxed{0} \\ N' &= -\kappa v T + 0 + \tau v B \\ B' &= \boxed{0} + -\tau N + 0 \end{aligned}\right\}$$

Two additional 0's in the $\boxed{}$ positions as compared with connections formulas

Connection Equations

$$\left.\begin{aligned} \nabla_V E_1 &= 0 + \omega_{12}(V) E_2 + \overset{\neq 0}{\boxed{\omega_{13}(V) E_3}} \\ \nabla_V E_2 &= \omega_{21}(V) E_1 + 0 + \omega_{23}(V) E_3 \\ \nabla_V E_3 &= \boxed{\omega_{31}(V) E_1} \overset{\neq 0}{+} \omega_{32}(V) E_2 + 0 \end{aligned}\right\}$$

V gives us the direction to take covarient derivative.

(1) Why additional 0's in the positions denoted by "box" in the Frenet Formulas ?

Answer: Since Frenet Formulas are calculated in the very particular direction of T. Note that connections formulas are the rate of change of Frame Fields in any given direction as dictated by V; but Frenet Formulas give the rate of change of T, N, B in the direction of T at each point on the curve.

(2) The connection equations are governed by the coefficients $\omega_{ij}(V)$ which are 1-forms but in case of Frenet Formulas the coefficients are simply real valued functions and not 1-forms.

V - vector Field.

(3) We can deduce Frenet Formulas from Connection Equations

Connection Equations $\Longrightarrow$ Frenet Formulas.

Connection Equations give rate of change of Frame Fields in any direction given by $V(p)$ at p in $\mathbb{R}^3$. So if in $\mathbb{R}^3$ we take a curve and restrict the directions to tangent directions at each point; i.e. $V = T$; then the connection equations will reduce to Frenet Formulas.

So connection equations can give rate of change of Frame Field on any geometrical shape in $\mathbb{R}^3$ and Frenet Formulas are just a particular case of these equations.

This allows us to study any Geometric Problem in $\mathbb{R}^3$.

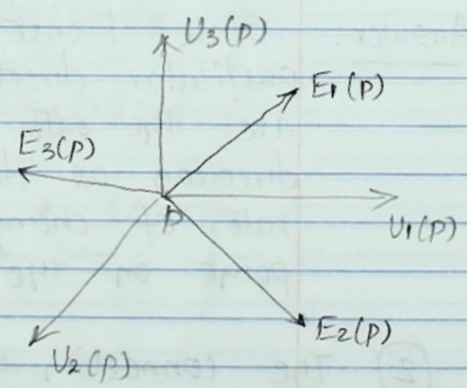
How to Calculate Connection Forms

Let $\{E_1, E_2, E_3\}$ be an arbitrary frame field and let $\{U_1, U_2, U_3\}$ be a natural field. We can write;

$$E_1 = a_{11} U_1 + a_{12} U_2 + a_{13} U_3$$

$$E_2 = a_{21} U_1 + a_{22} U_2 + a_{23} U_3$$

$$E_3 = a_{31} U_1 + a_{32} U_2 + a_{33} U_3.$$

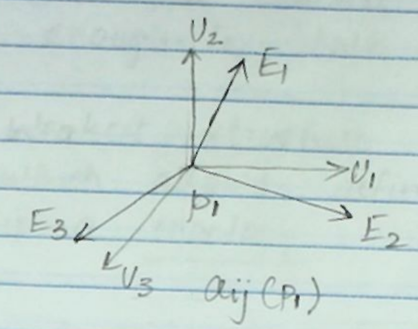


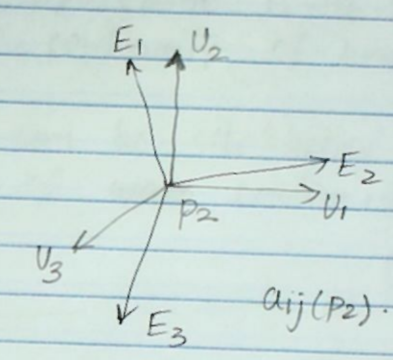
here the coefficients can be calculated using orthonormal expansion.

$$\boxed{a_{ij} = E_i \cdot U_j}$$

Here a_{ij} are real valued functions on $\mathbb{R}^3$

As the point p is changed, the coefficients a_{ij} will also change.





From here we can define the attitude matrix of the frame field.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$