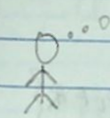
Lecture 1: Topological Spaces

Motivation: ① At the coarsest level, spacetime is a set, not enough to talk about continuity of maps.

② Weakest structure that can be established on a set which allows definition of ~~maps~~ continuity of maps is a topology.



Defn: Let M be a set; a Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ (the power set of M: set of all subsets of M) satisfying the following axioms;

(i) $\phi \in \mathcal{O}$, $M \in \mathcal{O}$

(ii) $U \in \mathcal{O}$, $V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$

(iii) $U_\alpha \in \mathcal{O} \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

$\alpha \in A$ (arbitrary index set)

Examples: (1) $M = \{1,2,3\}$

(a) $\mathcal{O}_1 = \{\phi, \{1,2,3\}\}$ is a topology.
(Satisfies all axioms and can be easily checked)

(b) $\mathcal{O}_2 = \{\phi, \{1\}, \{2\}, \{1,2,3\}\}$ not a topology

$\{1\} \cup \{2\} = \{1,2\} \notin \mathcal{O}_2$

(2) M any set :

Utterly Useless! Extreme examples; smallest and biggest topologies

(a) $\mathcal{O}_{chaotic} = \{\emptyset, M\}$ is a topology

(b) $\mathcal{O}_{discrete} = \mathcal{P}(M)$ Union and intersection of subsets of a set is a subset and hence always contained in $\mathcal{P}(M)$.

(3) $M = \mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(p_1, p_2, \ldots, p_d) \mid p_i \in \mathbb{R}\}$

<u>Defn</u>: $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R}^d)$; $\mathcal{O}_{standard}$ cannot be written explicitly now but will have to be written implicitly.

We define $\mathcal{O}_{standard}$ in two steps.

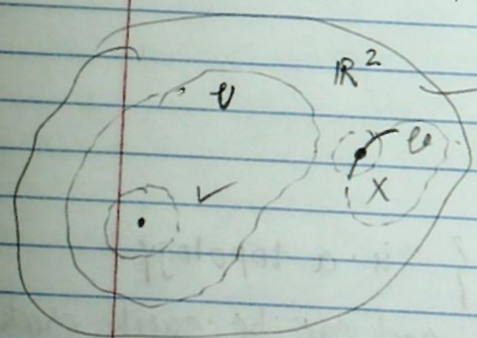
(α) <u>Soft Ball</u> :

Euclidean open ball really!

$$B_r(p) = \left\{ (q_1, q_2, \ldots q_d) \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} (q_i - p_i)^2 < r^2 \right\}$$

$r \in \mathbb{R}^+$, $p \in \mathbb{R}^d$

(β) $\mathcal{U} \in \mathcal{O}_{standard} \Leftrightarrow \forall p \in \mathcal{U} : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$



$\mathcal{U} \subseteq \mathbb{R}^n$ should not contain its boundary.

Some Terminology

M set

$\mathcal{O}$ topology

$(M, \mathcal{O})$ topological space

OPEN $\not\Rightarrow$ CLOSED, CLOSED $\not\Rightarrow$ OPEN

$\mathcal{U} \in \mathcal{O} \Leftrightarrow$ call $\mathcal{U} \subseteq M$ an open set

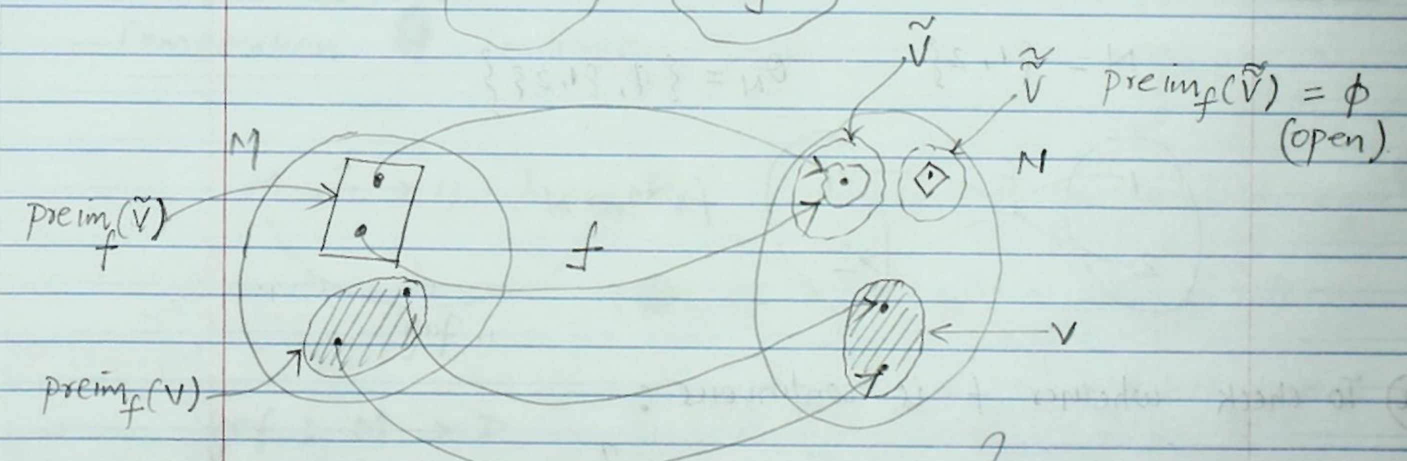
$M \setminus A \in \mathcal{O} \Leftrightarrow$ call $A \subseteq M$ closed set.

$M \setminus A \in \mathcal{O} \Rightarrow M \setminus A$ is open. ie. complement of M is open.

Continuous Maps

A map $f: M \longrightarrow N$

domain, target



∃ ◇ : map is not surjective.

∃ ⬡ : map is not injective.

When is the map continuous?

Answer to the question whether a map $f: M \to N$ is continuous depends (by defn.) on which topologies are chosen on M and N.

Defn. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f: M \to N$ is continuous (w.r.t $\mathcal{O}_M$ and $\mathcal{O}_N$) if

$\forall\, V \in \mathcal{O}_N$: $\text{pre-image}_f(V) \in \mathcal{O}_M$

Defn: Pre-Image

$f: M \to N \,(\supseteq V)$ then

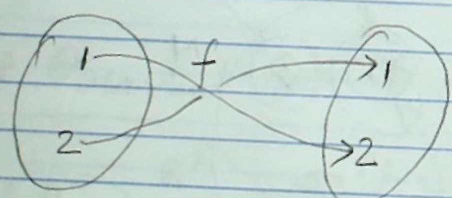
$\text{preim}_f(V) := \{m \in M \mid f(m) \in V\}$

Note the difference b/w preimage and inverse. Even where the map is not invertible the preimage can always be defined.

Example

$M = \{1, 2\}$ $\quad \mathcal{O}_M = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$N = \{1, 2\}$ $\quad \mathcal{O}_N = \{\phi, \{1,2\}\}$



$f: M \to N$

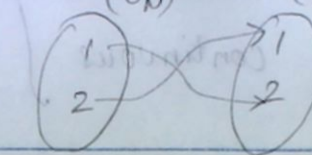
(a) To check whether f is continuous :

$\text{preim}_f(\phi) = \phi \in \mathcal{O}_M$

$\text{preim}_f(\{1,2\}) = M \in \mathcal{O}_M$

Thus f is continuous

Let us look at $f^{-1}: N_{(\mathcal{O}_N)} \to M_{(\mathcal{O}_M)}$



$\text{preim}_f(\{1\}) = \{2\} \notin \mathcal{O}_N$; hence f^{-1} is not continuous.

Notice if we hadn't exchanged 1's & 2's in the map, ie. $f(1) = 1$ and $f(2) = 2$; we would have had an identity map and the inverse would also be the identity map. But then under the same topologies $\mathcal{O}_M$ & $\mathcal{O}_N$ the identity map from M to N would have been continuous and the inverse of identity map (still an identity map) would not be continuous.

Remark on Closed set

$\{2\}$ closed

$\because \quad M \setminus \{2\} = \{1\} \in \mathcal{O}_M$

Composition of Continuous Maps

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & N \arrow[r, "g"] & P \end{tikzcd}$$

$g \circ f : M \to P$

$m \to (g \circ f)(m) = g(f(m))$

Key theorem : $\left.\begin{array}{l} f \text{ continuous} \\ g \text{ continuous} \end{array}\right\} \Rightarrow g\circ f$ is continuous.

Proof: Let $V \in \mathcal{O}_P$

$$\begin{aligned} \text{preim}_{g\circ f}(V) &= \{m \in M \mid (g\circ f)(m) \in V\} \\ &= \{m \in M \mid f(m) \in \text{preim}_g(V)\} \\ &= \underbrace{\text{preim}_f(\underbrace{\text{preim}_g(V)}_{\in \mathcal{O}_N \to g \text{ is continuous}})}_{\mathcal{O}_M \longrightarrow f \text{ is continuous}} \end{aligned}$$

Inheritance of Topology :

Many useful ways to inherit a topology from some given topological space (S).

Given $S \subseteq M \leftarrow \mathcal{O}_M$

Question: Can one construct on S a topology from $\mathcal{O}_M$ on M.

Answer: Yes. Defn. $\mathcal{O}|_S \subseteq \mathcal{P}(S)$ {Subset Topology inherited from topology on the superset M.

Subset Topology : $\mathcal{O}|_S := \{U \cap S \mid U \in \mathcal{O}_M\}$

Check: ① $\emptyset = \underset{\mathcal{O}_M}{\underset{\cap}{\emptyset}} \cap S \Rightarrow \emptyset \in \mathcal{O}|_S$

$S = \underset{\mathcal{O}_M}{\underset{\cap}{M}} \cap S \Rightarrow S \in \mathcal{O}|_S$

Lecture 2: Manifolds (Topological Manifolds)

Motivation: General Comments

There exist so many topological spaces that mathematicians cannot even classify them. For spacetime physics, we may focus on topological spaces $(M, \mathcal{O})$ that can be charted, analogous to how the surface of the Earth is charted on the atlas.

Topological Manifold

Defn: A topological space $(M, \mathcal{O})$ is called a $\boxed{d}$ dimensional topological manifold if; (so topological space plus extra conditions.)

$$\forall p \in M : \exists \underset{\substack{\Psi \\ p}}{\mathcal{U}} \in \mathcal{O} : \exists x : \mathcal{U} \to x(\mathcal{U}) \subseteq \mathbb{R}^{\boxed{d}}$$

$\mathcal{U} \subseteq M \longleftrightarrow \mathcal{O}$; $\mathbb{R}^d \longleftrightarrow \mathcal{O}_{\text{standard}}$

(1) x invertible

$x^{-1} : x(\mathcal{U}) \longrightarrow \mathcal{U}$

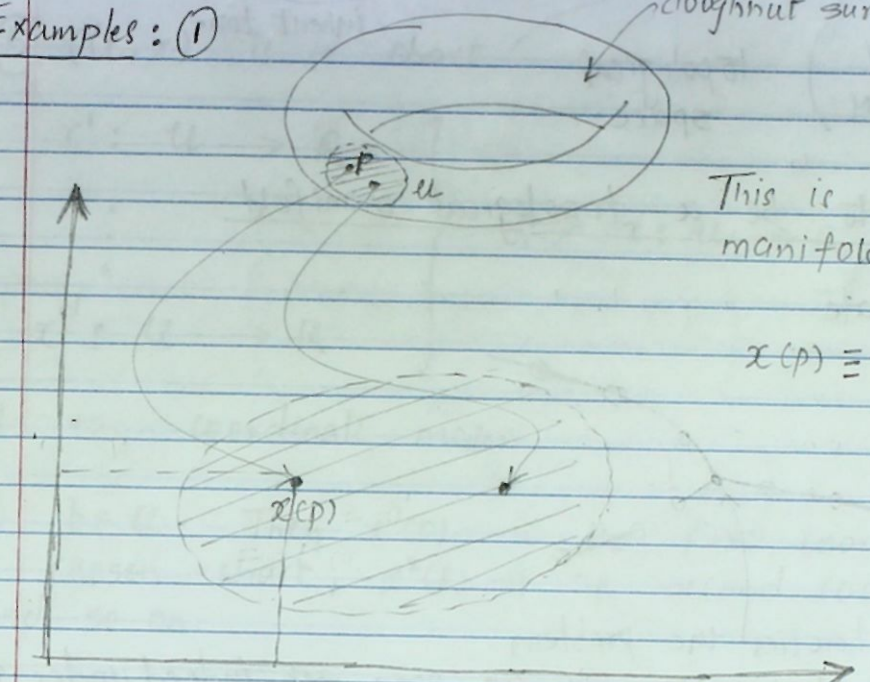
continuity of x and x^{-1} w.r.t these two topologies.

(2) x continuous

(3) x^{-1} continuous.

We now look at some examples;

Examples: (1)



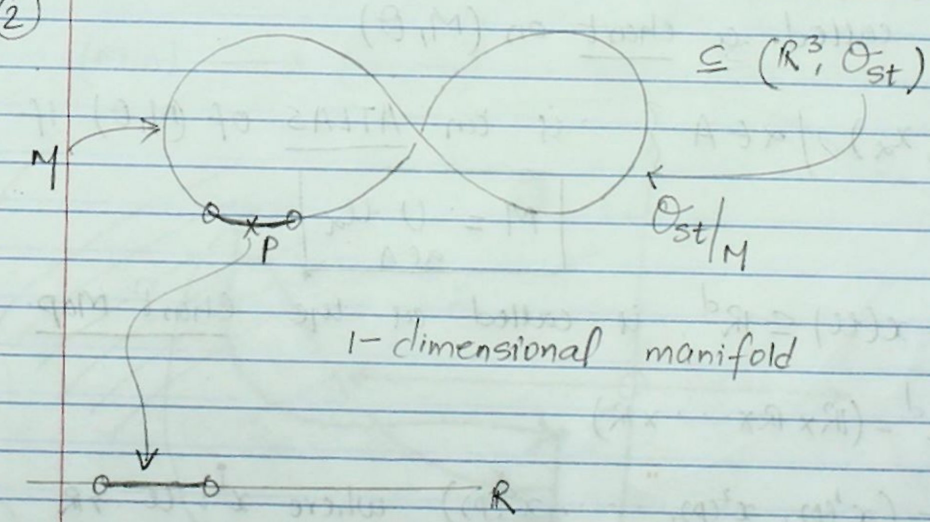
This is a 2-dimensional manifold.

$$x(p) \equiv (x^1(p), x^2(p))$$

Terminology : $(\mathcal{U}, x)$ is called a CHART.

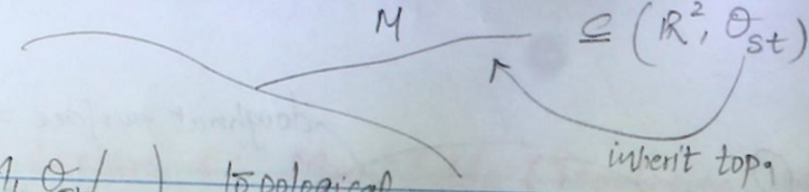
So for every point of the manifold, there exists a chart that contains the point.

(2)



1-dimensional manifold

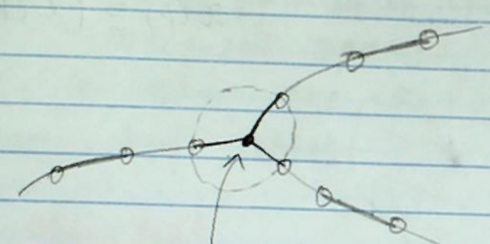
③



$\left(M, \mathcal{O}_{st}\big|_M\right)$ topological space

but fails to be a topological manifold.

Let us investigate



here lies the problem

the black region is an open set indeed under restricted topology but due to "bifurcation" it is not possible to find an invertible; in both directions continuous map into $\mathbb{R}^1$ or $\mathbb{R}^2$.

Terminology

① $(\mathcal{U}, x)$ is called a chart on $(M, \mathcal{O})$

② $\mathcal{A} = \{(\mathcal{U}_\alpha, x_\alpha) \mid \alpha \in A\}$ is an ATLAS of $(M, \mathcal{O})$ if

$$\boxed{M = \bigcup_{\alpha \in A} \mathcal{U}_\alpha}$$

③ $x : \mathcal{U} \longrightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$ is called a the Chart Map.

④ $x(\mathcal{U}) \subseteq \mathbb{R}^d = (\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R})$

so $x(\mathcal{U}) = (x^1(p), x^2(p), \cdots, x^d(p))$ where $x^i : \mathcal{U} \to \mathbb{R}$, $i = 1, 2, \cdots, d$.

⑤ $(\mathcal{U}, x)$ is a chart

$$\left.\begin{array}{l} x^1 : \mathcal{U} \longrightarrow \mathbb{R} \\ \vdots \\ x^d : \mathcal{U} \longrightarrow \mathbb{R} \end{array}\right\} \Longleftrightarrow x : \mathcal{U} \to \mathbb{R}^d$$

called the coordinate maps.

⑥ $p \in \mathcal{U}$. Then $x^1(p)$ is the first coordinate of point p w.r.t chosen chart; $x^2(p)$ is the second coordinate of point p and so on.

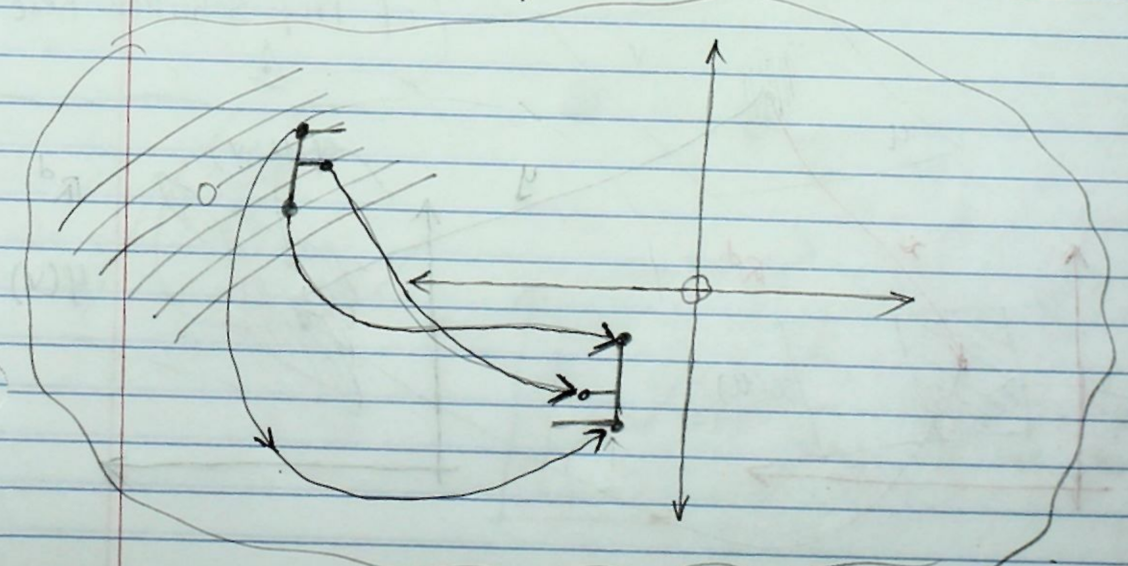
Example: $M = \mathbb{R}^2$

$\mathcal{U} = \mathbb{R}^2 \setminus \{(0,0)\}$

$x : \mathcal{U} \longrightarrow \mathbb{R}^2$

$(m, n) \longrightarrow (-m, -n)$

Listen in for an amazing analogy between $\mathbb{R}^2$ and $\mathbb{R}^2$ in "powdered" form ☺



~~another chart map on $\mathcal{U} \subseteq M$~~

~~$(m,n) \longrightarrow \left(\sqrt{m^2+n^2},\ \arctan\left(\frac{n}{m}\right)\right)$ — polar coordinates.~~

Now let $\mathcal{U} = \mathbb{R}^2 \setminus \{(a,0)\}$; $a \in \mathbb{R}_0^+$

define another chart map on $\mathcal{U}$.

$$(m,n) \xmapsto{y} \left(\sqrt{m^2+n^2},\ \arctan\left(\frac{n}{m}\right)\right) \quad (r,\theta) \text{ polar map.}$$

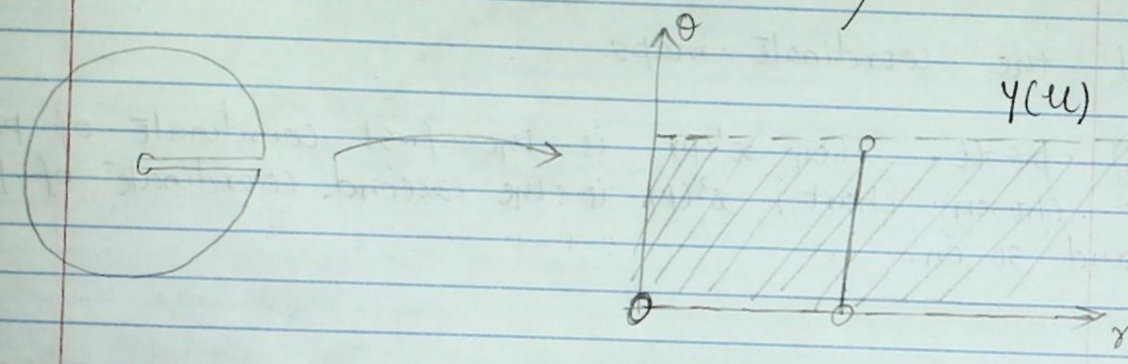
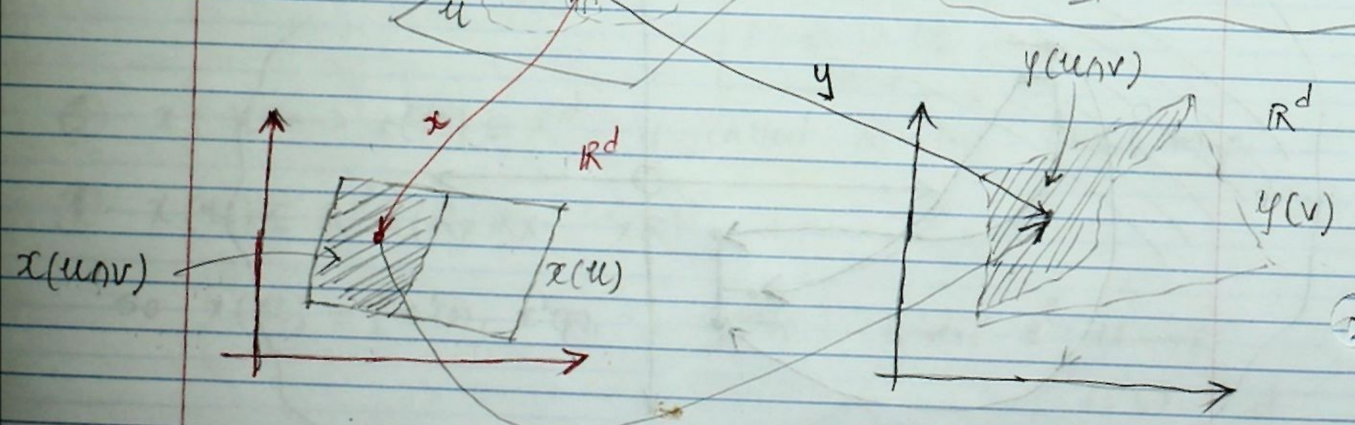


Chart Transition Maps

Imagine two charts $(\mathcal{U},x)$ and (V,y) with overlapping regions;

Listen to the discussion of Prof Schuller here ⚠



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More Formally:

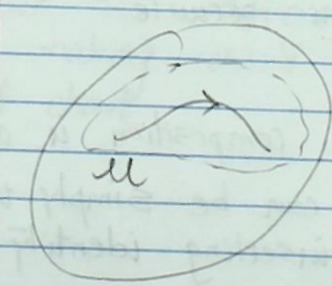
$$\begin{tikzcd}
 & \mathcal{U} \cap V \arrow[dl, "x"'] \arrow[dr, "y"] & \\
\mathbb{R}^d \supseteq x(\mathcal{U}\cap V) \arrow[rr, "y \circ x^{-1}"'] & & y(\mathcal{U}\cap V) \subseteq \mathbb{R}^d
\end{tikzcd}$$

called the chart transition map.

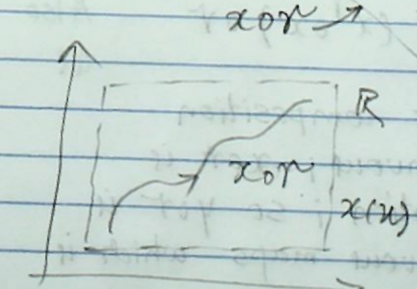
> Chart Transition Maps essentially contains the information/instruction on how to glue together the charts of an Atlas.

Manifold Philosophy

Often it is desirable or indeed the way to define properties (like continuity) of real world objects (say curve $\mathbb{R} \xrightarrow{\gamma} M$) by judging suitable conditions not on the real world object itself but on chart representation of that real world object.



$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & \mathcal{U} \arrow[d, "x"] \\
 & x(\mathcal{U}) \subseteq \mathbb{R}^d
\end{tikzcd}$$



$x \circ \gamma$ is the curve in the chart.

Disadvantage :

The notion we define in the chart representation may be ill defined (because an arbitrarily chosen chart is employed) We need to make sure that the property does not change if we can afford another "fantasy" i.e. another map.

Formally :

$$\begin{tikzcd}
 & & y(\mathcal{U}) \\
\mathbb{R} \arrow[rr, "\gamma"] \arrow[urr, "y\circ\gamma"] \arrow[drr, "x\circ\gamma"'] & & \mathcal{U} \arrow[u, "y"'] \arrow[d, "x"] \\
 & & x(\mathcal{U}) \subseteq \mathbb{R}^d \arrow[u, bend right=80, dashed, "(y\circ x^{-1})\circ(x\circ\gamma)"']
\end{tikzcd}$$

Look at $x\circ\gamma$ and say whatever properties this object $x\circ\gamma$ has say continuity, call it to be continuity of object $\mathbb{R} \xrightarrow{\gamma} \mathcal{U}$. Now this may be a problem unless there is another "fantasy", another chart $y(\mathcal{U})$. Then we have the object $y\circ\gamma$. Now if we can ensure that in some chart some property holds & it also holds in any other chart too, then it is independent of the choice of fantasy. Suppose we know that $x\circ\gamma$ is continuous, can we conclude that with respect to any other chart its also continuous? Yes we can. This is because

$$y\circ\gamma = (y\circ x^{-1})\circ(x\circ\gamma)$$

Since composition is associative.

$$= y\circ(x^{-1}\circ x)\circ\gamma$$

Also can be simply thought of as inserting identity $x^{-1}\circ x$.

Now $y\circ x^{-1}$ is chart composition map which is continuous; $x\circ\gamma$ is continuous by assumption; so $y\circ\gamma$ is composition of continuous maps which is continuous.

So the continuity of chart transition map makes the chart dependent definition; chart independent. [fantasy independent]

What about differentiability ?

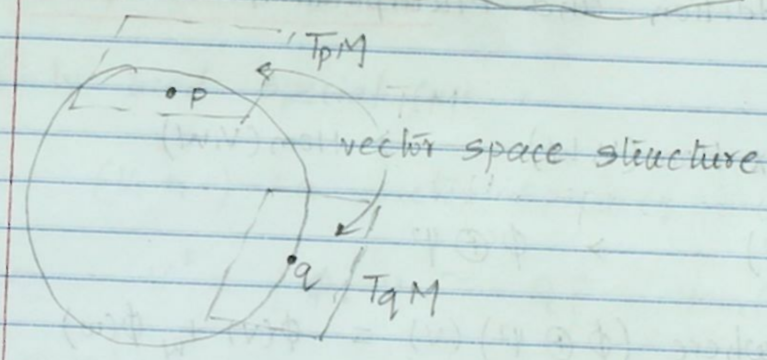
Suppose we call curve γ in real world differentiable if $x\circ\gamma : \mathbb{R} \to \mathbb{R}^d$ is differentiable. Because differentiability of $\gamma : \mathbb{R} \to \mathcal{U}$ can't be decided immediately because $\mathcal{U}$ is simply a set with some topology and we may not have enough structure. For example we may not do sums/divisions etc needed to define differentiation quotient. So while we could talk about continuity of manifolds level in order to talk abt differentiability we must define differentiability at the chart level because for a map from $\mathbb{R} \to \mathbb{R}^d$ we can establish addition & multiplication to define differentiation. Now suppose we do this, could we still say that if $x\circ\gamma$ is differentiable, are we guaranteed $y\circ\gamma$ is diff'ble too? We can't! Because $y\circ x^{-1}$ is only continuous & composition of a differentiable and a continuous func. gives only continuity not differentiability.

So on a topological manifold with continuous chart transition maps, we cannot define notion of differentiability by defining it in the charts. But then chart is our only possibility. And we would like to lift the notion of chart to $\gamma : \mathbb{R} \to \mathcal{U}$ s.t. the notion remains independent of chart. Now where to charts come from. From an Atlas. So we rip off the charts which have only continuous transition maps but not differentiable. We take away all the trouble making making charts; so we have a restricted Atlas. Maximal chart

Lecture 3: Multilinear Algebra

We will <u>not</u> equip spacetime with a vector space structure.

However, the tangent spaces T_pM {to be defined in Lecture 5} to smooth manifolds {to be defined in lecture 4.} have vector space structure.



It is benificial to first study vector spaces abstractly for two reasons;

(1) For Construction of T_pM, one needs an intermediate vector space $C^\infty(M)$.

(2) Tensor techniques are most easily understood in an abstract setting.

3. Vector Space of Homomorphism

Fun Fact: $(V,+,\cdot)$ $(W,+,\cdot)$ vector spaces

Defn: $Hom(V,W) = \{\phi : V \xrightarrow{\sim} W\}$

collection of all linear maps from V to W. Denoted as $\phi : V \xrightarrow{\sim} W$

Is $Hom(V,W)$ a vector space?

We first define Addition and Multiplication operations

$\oplus$ $Hom(V,W) \times Hom(V,W) \longrightarrow Hom(V,W)$

$$(\phi, \psi) \longrightarrow \phi \oplus \psi$$

$$\text{where } (\phi \oplus \psi)(v) = \phi(v) +_W \phi(w)$$

$\odot$ $c\,Hom(V,W) \longrightarrow Hom(V,W)$; c from scaler field.

$$(c,\phi) \longrightarrow c\odot\phi$$

$$\text{where } (c\odot\phi)(v) = c \cdot_W \phi(v)$$

$\boxed{(Hom(V,W), \oplus, \odot) \text{ is a vector space.}}$

Examples: $Hom(P,P)$ is a vector space

$\delta \in Hom(P,P)$

Since δ is a linear map, its composition with itself is a linear map and hence;

$$\delta \circ \delta \in Hom(P,P)$$

$$\vdots$$

$$\underbrace{\delta \circ \cdots \circ \delta}_{m\text{-times}} \in Hom(P,P)$$

Also since $Hom(P,P)$ is a vector space

$$c \odot \delta \underset{Hom(P,P)}{\oplus} \delta\circ\delta \in Hom(P,P)$$

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4. Dual Vector Space

Heavily used special case.

Given $(V,+,\cdot)$ a vector space:

Defn: $V^* = \{\phi : V \xrightarrow{\sim} \mathbb{R}\} = Hom(V,\mathbb{R})$

$(V^*, \oplus, \odot)$ is a vector space and called the Dual vector space to vector space V.

Terminology: $\phi \in V^*$ is called informally a covector.

Example: $I : P \xrightarrow{\sim} \mathbb{R}$

ie: $I \in P^*$

Define $I(p) := \int_0^1 p(x)\,dx$

Check Linearity; $I(p) = \int_0^1 p(x)\,dx$

$$I(p+q) = \int_0^1 (p+q)(x)\,dx$$

$$= \int (p(x) + q(x))\,dx$$

$$= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx = I(p) + I(q).$$

Similarly $I(\lambda p) = \lambda I(p)$

ie. $\boxed{I = \int_0^1 dx}$ — Integral operator.

5. Tensors

Defn: Let $(V,+,\cdot)$ be a vector space.

An (r,s) tensor T over V is a multilinear map; $r, s \in \mathbb{N}_0$

$$T: \underbrace{V^* \times V^* \times \cdots \times V^* \times V \times \cdots \times V}_{} \xrightarrow{\sim\, r+s} \mathbb{R}$$

$r+s$ tuple; r-entries are covectors and last s-entries are vectors.

Example: T $(1,1)$-tensor

① $T(\phi+\psi, v) = T(\phi, v) + T(\psi, v)$
② $T(\lambda\phi, v) = \lambda \cdot T(\phi, v)$
③ $T(\phi, v+w) = T(\phi, v) + T(\phi, w)$
④ $T(\phi, \lambda v) = \lambda T(\phi, v)$

Finally we can also write
$T(\phi+\psi, v+w) = T(\phi, v) + T(\phi, w) + T(\psi, v) + T(\psi, w)$.

~~Excursion: Given map $T: V^* \times V \xrightarrow{\sim} \mathbb{R}$ (multilinear)~~
~~define $\phi_T: V \xrightarrow{\sim} V$~~

Example: $g: P \times P \to \mathbb{R}$
$(p,q) \to \int_{-1}^{1} p(x)q(x)\,dx$

is a $(0,2)$ tensor. Solinner Product is really a $(0,2)$ tensor

6. Vectors and Cofactors as Tensors

Theorem (including proofs): (1) $\phi \in V^* \iff \phi: V \xrightarrow{\sim} \mathbb{R} \iff \phi$-$(0,1)$ tensor

(2) $v \in V = (V^*)^* \iff v: V^* \xrightarrow{\sim} \mathbb{R} \iff v - (1,0)$ tensor.
$(\dim V < \infty)$

Similarly Linear maps are $(1,1)$ tensor
Inner product is $(0,2)$ tensor.

$\boxed{\text{Discussion on Basis for Vector Space}}$

8. Basis for the Dual Space

Choose Basis $e_1, e_2, \ldots, e_n$ for V
We can choose basis $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$ for V^*.

However it is more economical to require:

Once $e_1, \ldots, e_n$ on V have been chosen, then

(*) $$\epsilon^a(e_b) = \delta^a_b = \begin{cases} 1 & a=b \\ 0 & a \neq b. \end{cases}$$

This uniquely determines the choice of $\epsilon^1, \epsilon^2, \cdots, \epsilon^n$ from the choice of $e_1, e_2, \cdots, e_n$.

Defn: If a basis $\epsilon^1, \epsilon^2, \cdots, \epsilon^n$ of V^* satisfies ⊛, it is called the Dual basis (of the dual basis).

Example: P $(N=3)$

e_0, e_1, e_2, e_3 basis if
$$\left.\begin{aligned} e_0(x) &= 1 \\ e_1(x) &= x \\ e_2(x) &= x^2 \\ e_3(x) &= x^3 \end{aligned}\right\} \text{Notation } e_a(x) = x^a.$$

What is the dual basis $\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3$.

Define $\epsilon^a := \frac{1}{a!} \partial^a \big|_{x=0}$ — Take polynomial; calculate $\frac{1}{a!} \times (a^{th}$ derivative) of the polynomial evaluated at 0.

Claim: the above defines a dual basis. So dual basis are not polynomials but derivative operators evaluated at $x=0$.

Proof: $\epsilon^a(e_b) = \frac{1}{a!} \partial^a \big|_{x=0} x^b$

(1) $a=b$; $= \frac{1}{a!} \partial^a \big|_{x=0} x^a$

$= \frac{1}{a!} (a!) \big|_{x=0} = 1$

(2) $a \neq b$; $= \frac{1}{a!} \partial^a \big|_{x=0} x^b = 0$

$$\Rightarrow \epsilon^a(e_b) = \begin{cases} 1 & a=b \\ 0 & a \neq b \end{cases} = \delta^a_b$$

8 Components of Tensors

$e_1, e_2, \cdots, e_n$ basis for V.
$\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ basis for V^*.

Defn: Let T be an (r,s)-tensor over a finite dimensional vector space V. Then define the $(r+s)^{\dim V}$ many real numbers

$$\underbrace{T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}}_{\in \mathbb{R}} = T(\epsilon^{i_1}, \epsilon^{i_2}, \cdots, \epsilon^{i_r}, e_{j_1}, e_{j_2}, \cdots, e_{j_s})$$

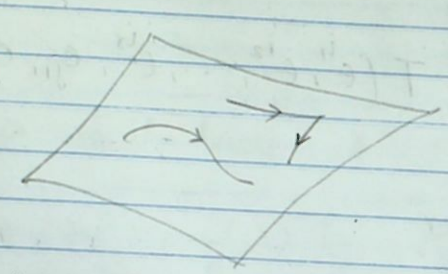
$i_1 \cdots i_r, j_1, \cdots, j_s \in \{1, \cdots, \dim V\}$

components of the tensor w.r.t the chosen basis.

Useful: Knowing these components one can reconstruct an entire tensor.

Lecture 4: Differentiable Manifolds

So far we have seen topological manifold of dimension d $(M,\mathcal{O})$. Question is that is it enough to define notions like differentiability. Based on discussions in Lecture 3; it is a resounding No.



So we wish to define the notion of differentiable

curve : $\mathbb{R} \longrightarrow M$

function : $M \longrightarrow \mathbb{R}$

maps : $M \longrightarrow N$

1. Strategy : Choose a chart (U,x) and consider the portion of the curve in chart domain.

$$\begin{tikzcd} \gamma : \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & U \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$

Idea : To lift the undergraduate notion of differentiability of a curve in $\mathbb{R}^d$ to a notion of differentiability of a curve in M.

Problem : Can this be well defined under change of chart?

$$\begin{tikzcd} & y(U\cap V) \subseteq \mathbb{R}^d \\ \gamma : \mathbb{R} \arrow[ur, "y\circ\gamma"] \arrow[r] \arrow[ddr, "x\circ\gamma"'] & U\cap V \neq \phi \arrow[u, "y"] \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \\ & x(U\cap V) \subseteq \mathbb{R}^d \arrow[uuu, bend right, "y\circ x^{-1}"'] \end{tikzcd}$$

There could be another chart with another chart domain & it could have another chart map at the same time. If the other chart domain is V and intersection between U & V is empty then ok. But if $U \cap V \neq \phi$ then in the overlap region there is problem becoz one chart map might tell; yes it is differentiable & other might tell it isn't. If that happens it is catastrophic.

$$\underbrace{y\circ\gamma}_{\text{may be only continuous and not differentiable}} = \underbrace{\overset{\mathbb{R}^d\to\mathbb{R}^d}{(y\circ x^{-1})}}_{\text{continuous}} \circ \underbrace{\overset{\mathbb{R}\to\mathbb{R}^d}{(x\circ\gamma)}}_{\text{undergrad diff'ble}} = y\circ(x^{-1}\circ x)\circ\gamma$$

So at first sight, strategy does not work out.

2. Compatible Charts

In section 1, we used any imaginable charts on top. manifold $(M,\mathcal{O})$.

To emphasize this, we may say that we took U and V from the Maximal Atlas $\mathcal{A}$ of $(M,\mathcal{O})$

Defn: Two charts (U,x) and (V,y) of a topological manifold are called $\circledast$ - compatible if either;

(a) $U \cap V = \phi$

or

(b) $U \cap V \neq \phi$, then the chart transition maps,

$$y \circ x^{-1} : \underbrace{x(U \cap V)}_{\subseteq \mathbb{R}^d} \longrightarrow \underbrace{y(U \cap V)}_{\subseteq \mathbb{R}^d}$$

$$x \circ y^{-1} : \underbrace{y(U \cap V)}_{\subseteq \mathbb{R}^d} \longrightarrow \underbrace{x(U \cap V)}_{\subseteq \mathbb{R}^d}$$

have undergraduate $\circledast$ - property.

$\circledast$ - could be any property we want ☺

Philosophy

Defn: An Atlas $\mathcal{A}_{\circledast}$ is a $\circledast$- compatible atlas if any two charts in $\mathcal{A}_{\circledast}$ are $\circledast$- compatible.

Defn: A $\circledast$- manifold is a triple $(\underbrace{M, \mathcal{O}}_{\text{top. man.}}, \mathcal{A}_{\circledast})$ $\mathcal{A}_{\circledast} \subseteq \mathcal{A}_{maximal}$.

$\mathcal{A}_{\circledast} \subseteq \mathcal{A}_{maximal}$

$\circledast$	
C^0	$C^0(\mathbb{R}^d \to \mathbb{R}^d)$ $\equiv$ continuous maps wrt $\mathcal{O}_{st}$
C^1	$C^1(\mathbb{R}^d \to \mathbb{R}^d)$ $\equiv$ diff'ble (once) and result is continuous.
C^k	k times continuously diff'ble.
D^k	k times differentiable
$\vdots$	
C^∞	$C^\infty(\mathbb{R}^d \to \mathbb{R}^d)$
$\cup\!\!\mid$	
C^ω	$\exists$ Taylor expansion (multi dim).

So now our defn. of differentiability of curve - can be called k-times differentiable on a manifold when we manage to find an Atlas that is a C^k Atlas.

Theorem (w.o. proof) ~~Any $C^{k\geq 1}$ manifold $(M, \mathcal{O}, \mathcal{A}_{C^{k\geq 1}})$~~

Any $C^{k\geq 1}$ - atlas denoted as $\mathcal{A}_{C^{k\geq 1}}$ of a topological manifold contains a C^∞ - atlas.
meaning that we have a topological manifold where we simply cannot remove charts in such a way that all that remains is still an Atlas and has continuously differentiable transition functions. Some topological manifolds cannot be given such a structure but to those where we can achieve that transition maps are at least once continuously differentiable; in such an atlas we can remove more and more of such charts until we even have a C^∞ atlas.

so the difficult step is from C^0 atlas to C^1 atlas; this might or might not work to choose such an Atlas for a given topological manifold. But once we arrive here C^1-atlas, ~~we trickle down right upto C^∞-atlas.~~

Smooth Manifold:

Thus we may w.l.o.g always consider C^∞-manifolds. These are also called as "Smooth Manifolds", unless we wish to define Taylor expandibility / complex differentiability etc.

Defn: A smooth manifold $(M, \theta, \mathcal{A})$; (M, θ) topological space
$\mathcal{A}$ - C^∞ atlas.

Lecture - Topological Spaces

A topology is the weakest structure that can be established on a set in order to have two very important notions of convergence and continuity.

Defn: Let M be some set. Then a choice of $\theta \subseteq \mathcal{P}(M)$ is called a topology on M if

(i) $\phi \in \theta$ and $M \in \theta$

(ii) $\mathcal{U}, \mathcal{V} \in \theta \Rightarrow \cap\{\mathcal{U}, \mathcal{V}\} \in \theta$ ($\mathcal{U} \cap \mathcal{V}$)

(iii) $\mathcal{C} \subseteq \theta \Rightarrow \cup \mathcal{C} \in \theta$

Remark: Unless $|M| = 1$ there are different topologies θ one can choose one and the same set.

An interesting table;

$\lvert M \rvert$	# Topologies
1	1
2	4
3	29
4	355
5	6942
6	209527
7	9,535,241

Terminology: ⊛ The pair (M, θ) is then called a Topological space.

⊛ Elements of a topology are called open sets.

Examples (a) M any set, $\mathcal{O} = \{\phi, M\}$ is a topology on M. Also called "Chaotic Topology".

(b) M any set. $\mathcal{O} = \mathcal{P}(M)$, is a topology on M. Also called as "Discrete Topology".

(c) $M = \{1, 2, 3\}$; $\mathcal{O} = \{\phi, \{1\}, \{2\}, \{1,2\}, \underbrace{\{1,2,3\}}_{M}\}$

- can be easily checked that it is a topology — one of 29 topologies

(d) Important and heavily used example :

$M = \mathbb{R}^d := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{d \text{ times}}$

$\mathcal{O}_{\text{standard } \mathbb{R}^d}$ constructed in 3-steps

(i) Defn: $\forall x \in \mathbb{R}^d$; $\forall r \in \mathbb{R}^+$

define $B_r(x) = \left\{ y \in \mathbb{R}^d \;\middle|\; \underbrace{\sqrt{\sum_{i=1}^{d} (y^i - x^i)^{2n}}}_{\|x-y\|_{2n}} < r \right\}$

Open ball around x in $\mathbb{R}^d$

We shall mainly deal with $n=1$

(ii) $\mathcal{U} \in \mathcal{O}_{\text{standard } \mathbb{R}^d}$

$\Updownarrow$

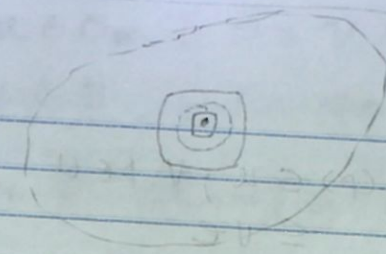
$\forall p \in \mathcal{U} : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$

Intuitively : $n=1$

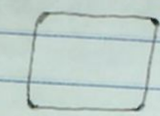
lies outside if boundary of $\mathcal{U}$ is included

How would a "ball" in $n=2$ case look like

— would look like a TV set



$(n=2)$

$n = 10{,}000$ $B_r(p)$  $n \to \infty$ $\|\ \|_\infty$  Infinity norm.

(iii) Proof that $\mathcal{O}_{\text{standard } \mathbb{R}^d}$ is indeed a topology ;

(1) $\phi \in \mathcal{O}_{st}$

$\forall p \in \mathcal{O} : \ldots$

$\underbrace{\forall p : \underbrace{(\underbrace{p \in \phi}_{F} \Rightarrow \ldots)}_{T}}_{T}$

(2) $M \in \mathcal{O}_{st}$ because any $B_r(x) \in M$

$M = \mathbb{R}^d$

(3) $\mathcal{U}, V \in \mathcal{O}_{st} \overset{?}{\Rightarrow} \mathcal{U} \cap V \in \mathcal{O}_{st}$

Consider $p \in \mathcal{U} \cap V \Rightarrow p \in \mathcal{U}$ and $p \in V$

But $\exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$

Also $\exists s \in \mathbb{R}^+ : B_s(p) \subseteq V$

Now take $\min(r, s)$.

$\Rightarrow B_{\min(r,p)}(p) \subseteq \mathcal{U}$ $\subseteq V$ $\Rightarrow \boxed{B_{\min(r,p)} \subseteq \mathcal{U} \cap V}$

③ $\mathcal{C} \subseteq \mathcal{O} \overset{?}{\Rightarrow} \bigcup \mathcal{C} \in \mathcal{O}$

Let $\mathcal{U} \in \mathcal{C} \Rightarrow \exists\, r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U} ;\ \forall\, p \in U$ $\subseteq \bigcup \mathcal{C}$

$\Rightarrow \boxed{\bigcup \mathcal{C} \in \mathcal{O}}$

$\blacksquare$

2. Construction of New Topologies From Given Topologies

Defn/Thm.: Let $(M, \mathcal{O})$ be a topological space. Let $N \subseteq M$. Then

$$\underbrace{\mathcal{O}|_N}_{\subseteq \mathcal{P}(N)} = \{ \mathcal{U} \cap N \mid \mathcal{U} \in \mathcal{O} \}$$ is a topology on N,

called an induced topology. So $\mathcal{O}_N$ has been contructed from the topology on the bigger set by intersecting any open set in the bigger set with N.

Proof: (1) $\emptyset \in \mathcal{O}|_N$?

Yes ; Since $\emptyset = \underbrace{\underset{\mathcal{O}}{\emptyset} \cap N}_{\in \mathcal{O}|_N}$

$N \in \mathcal{O}|_N$?

Yes ; Since $N = N \cap M = \underbrace{\underset{\mathcal{O}}{M} \cap N}_{\in \mathcal{O}|_N}$

~~(2) $U, V \in \mathcal{O}|_N \overset{?}{\Rightarrow} U \cap V \in \mathcal{O}|_N$~~

(2) $\underbrace{S, T \in \mathcal{O}|_N}$ $\overset{?}{\Rightarrow} S \cap T \in \mathcal{O}|_N$

$\Rightarrow \exists\, \mathcal{U} \in \mathcal{O} : S = \mathcal{U} \cap N$

$\exists\, V \in \mathcal{O} : T = V \cap N$

Then $S \cap T = (\mathcal{U} \cap N) \cap (V \cap N)$

$= \underbrace{\underset{\mathcal{O}}{(\mathcal{U} \cap V)} \cap N}_{\in \mathcal{O}|_N}$

(3) can be similarly checked to be true.

Example of Induced Topology :

Ex. $(\mathbb{R}, \mathcal{O}_{st})$

$N = [-1, 1] := \{ x \in \mathbb{R} \mid -1 \leq x \leq 1 \}$

$(N, \mathcal{O}_{st}|_N)$

Claim: $(0, 1] \notin \mathcal{O}_{st}$ $\quad$ $(0,1] \notin \mathcal{O}_{st}$ means it is not an open set.

but $(0, 1] = \underbrace{\underset{\mathcal{O}_{st}}{(0,2)} \cap \underset{N}{[-1,1]}}_{\in \mathcal{O}_{st}|_N}$ – So a set can be non open in "big" space but can be open in the induced topology on some subset if it happens to be a subset of the subset.

So one must judge "openness" of a set w.r.t the Topology under consideration.