COMP27112

Computer Graphics and Image Processing

3: Transformations

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What we'll cover

1. Types of geometrical transformation
2. Vector and matrix representations
3. Homogeneous coordinates
4. Composite transformations

1-3 are probably recap for most people

4: possibly new for some people

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5. *Using transformations in OpenGL*

5: reference material to help with lab

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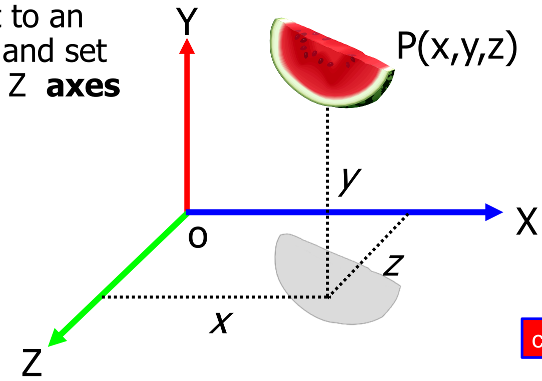
6. *Essential vector geometry*

6: reference material for now and later

3D Cartesian coordinates

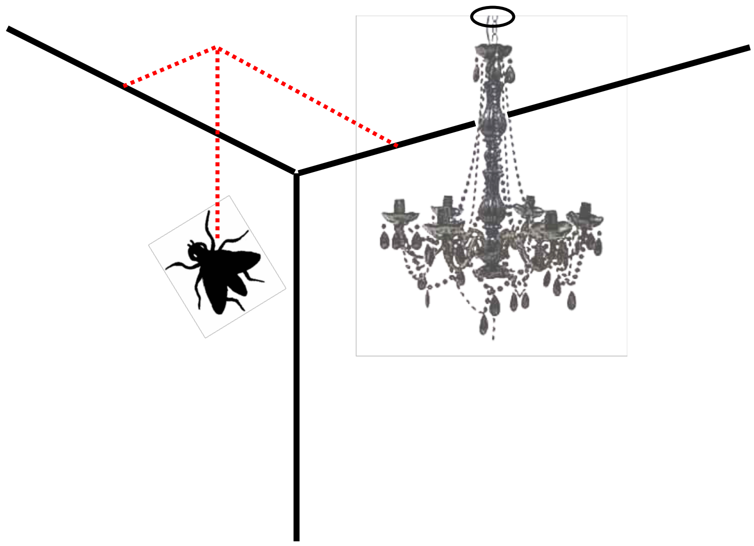
- A coordinate represents a point in space, measured with respect to an **origin** and set of X, Y, Z **axes**

René Descartes, 1596-1650, inventor of Cartesian coordinates



crucial concept

3



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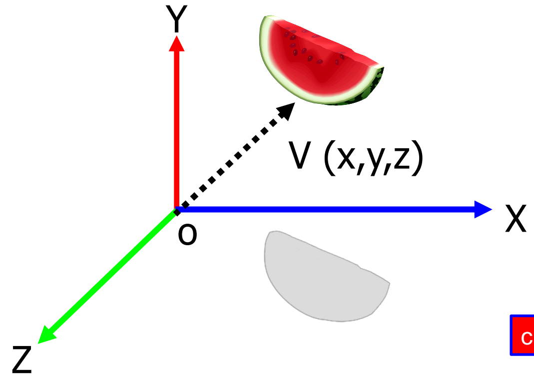
Vectors

- A vector represents a **direction** in space, with respect to a set of X, Y, Z axes. It has a characteristic length. A vector of length 1 is known as a “unit vector”



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Coordinates and Vectors

- Both coordinates and vectors can be represented by a triple of x,y,z values. In computer graphics we usually write these in column format:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- In the previous slides
  - The melon is **at** Point **P**
  - The **spatial relationship** between the origin and the melon is described by the vector **V**

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Danger: 2 different representations

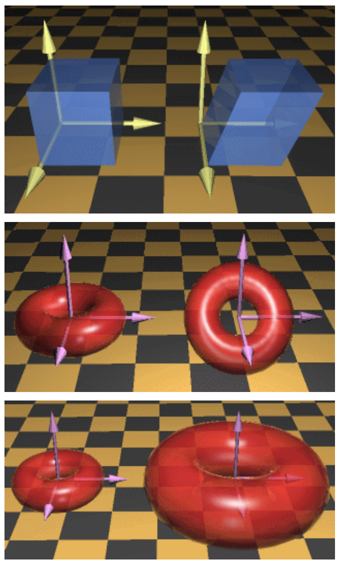
- We can write a vector as either a column or a row:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x & y & z \end{bmatrix}$$

- OpenGL uses **column** vectors
- Some other systems, like MATLAB, use **row** vectors
- Yes, this is confusing!
- The two representations are equivalent, but a transformation matrix used with column vectors is the **transpose** of the equivalent matrix used with row vectors
- Choose column or row and stick with that

Geometrical Transformations

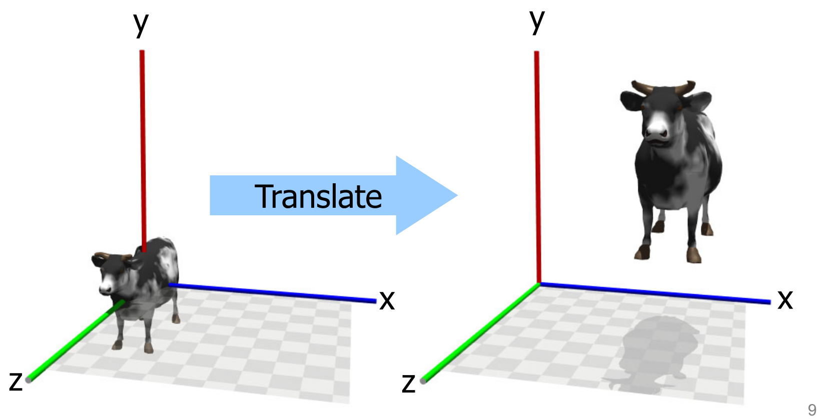
- Define geometry as sets of vertices
- Apply transformations to vertices to change them
- Examples: translation, scaling, rotation
- To transform a whole shape, we transform all its vertices



Translation

- Applies a 3D shift (tx, ty, tz) to all coordinates

$$x' = x + tx$$
$$y' = y + ty$$
$$z' = z + tz$$



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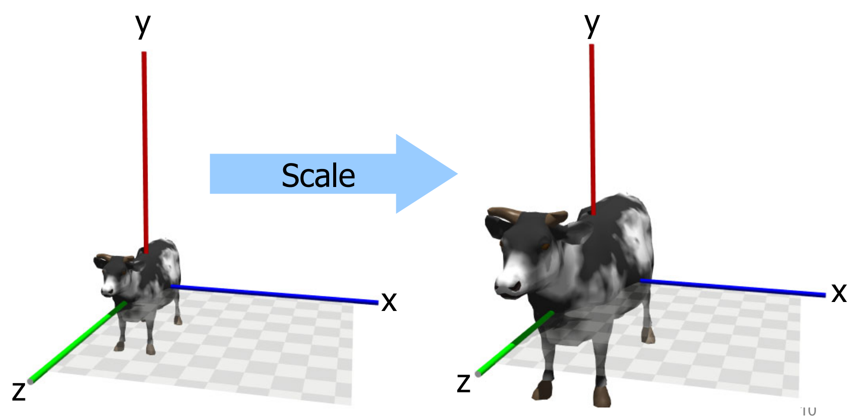
Scaling

- Applies a 3D scale (sx, sy, sz) to all coordinates (with respect to the origin)

$$x' = x \,.\, sx$$
$$y' = y \,.\, sy$$
$$z' = z \,.\, sz$$



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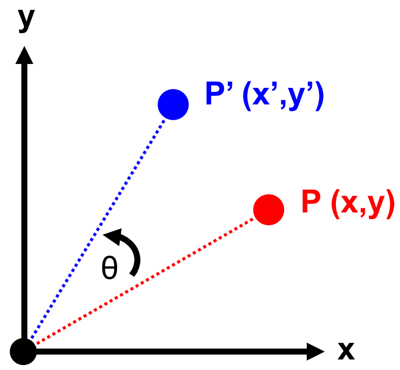
Rotation (2D)

- Before we look at 3D, let's look at the 2D case.
- We want to rotate point **P** about the origin, by angle θ



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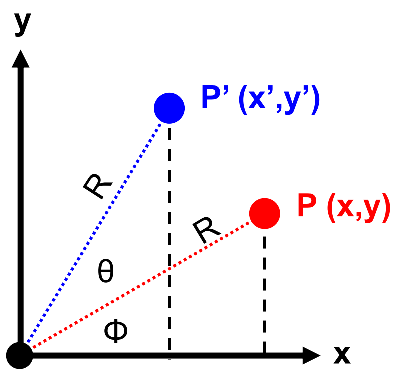
Rotation (2D)

- $x = R\cos\Phi$
- $y = R\sin\Phi$
- $x' = R\cos(\theta+\Phi)$
  - $x' = R\cos\Phi\cos\theta - R\sin\Phi\sin\theta$
- $y' = R\sin(\theta+\Phi)$
  - $y' = R\cos\Phi\sin\theta + R\sin\Phi\cos\theta$
- Substituting for $R\cos\Phi$ and $R\sin\Phi$ gives:
- $x' = x\cos\theta - y\sin\theta$
- $y' = x\sin\theta + y\cos\theta$



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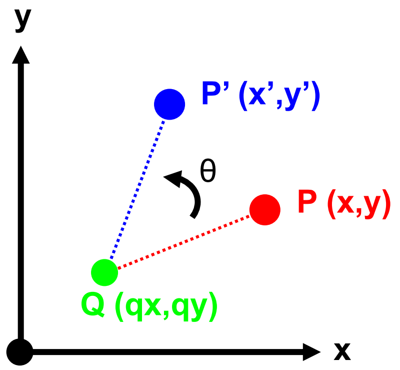
Rotation (2D) about a point

- What if we want to rotate about **some other point** than the origin?
- How do we rotate point **P** about point **Q** ?
- We break the problem into 3 simpler steps:
  - Translate by (-qx,-qy) to place Q at the origin
  - Do the rotation by θ
  - Translate back again by (+qx,+qy)
- This is **always the way** we do complex transformations… see later



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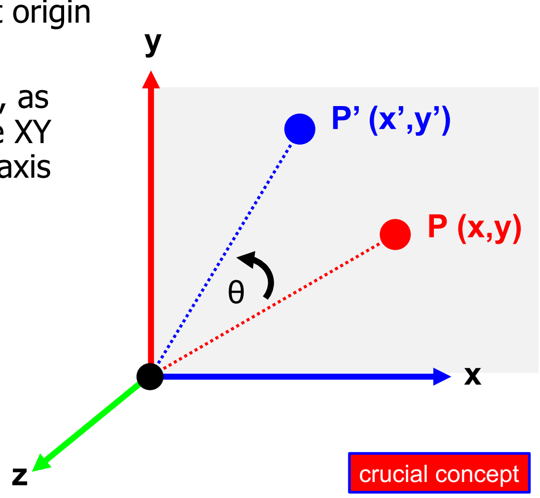
Rotation (3D)

- Rotation of **P** about origin in 2D…
- …is the same in 3D, as rotation of **P**, in the XY plane, about the Z-axis
- So:
- $x' = x\cos\theta - y\sin\theta$
- $y' = x\sin\theta + y\cos\theta$
- $z' = z$ (no change)



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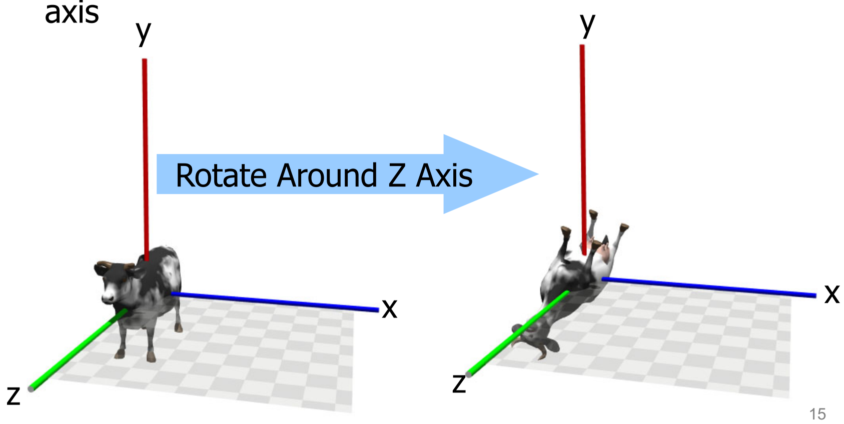
Rotation (3D)

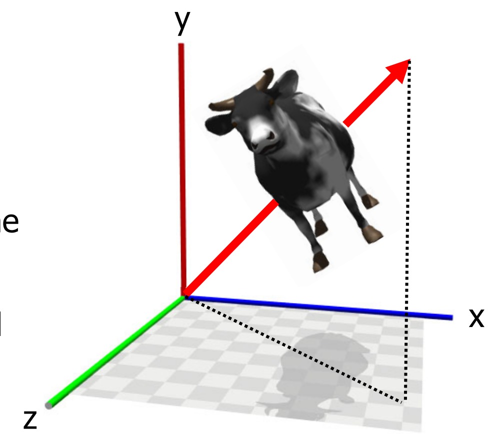
- 3D rotations are relative to an **axis**, e.g. a rotation by angle θ about the Z axis

$$x' = x \,.\, \cos\theta \; - y \,.\, \sin\theta$$
$$y' = x \,.\, \sin\theta \; + y \,.\, \cos\theta$$
$$z' = z$$



Rotation (3D) about a vector

- In 3D we often want to rotate (or scale) about an **arbitrary** axis vector
- This is analogous to the 2D case of rotating (or scaling) about an arbitrary point
- And we approach the same way: as a sequence of steps (which we will describe later)



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Representing transformations

- We've seen the following equations for representing some transformations:

Translation	Scale	Rotation (about Z)
$x' = x + tx$	$x' = x \,.\, sx$	$x' = x \,.\, \cos\theta \; - y \,.\, \sin\theta$
$y' = y + ty$	$y' = y \,.\, sy$	$y' = x \,.\, \sin\theta \; + y \,.\, \cos\theta$
$z' = z + tz$	$z' = z \,.\, sz$	$z' = z$

- They're all different!
- It would be very convenient if we could use a single **homogeneous** (the same) representation
- We use **vectors** and **matrices**

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Using matrices: scaling

- A transformation changes a vector into another vector
- We can represent this change using a **matrix**
- Example, scale (x,y,z) by (2,3,5):

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Multiply elements row by column:
  - **x'** = 2*x + 0*y + 0*z = **2x**
  - **y'** = 0*x + 3*y + 0*z = **3y**
  - **z'** = 0*x + 0*y + 5*z = **5z**

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Using matrices: rotation (about Z)

- Example, rotate (x,y,z) about Z axis by θ:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Multiply elements row by column:
  - **x'** = cosθ*x - sinθ*y + 0*z = **xcosθ - ysinθ**
  - **y'** = sinθ*x + cosθ*y + 0*z = **xsinθ + ycosθ**
  - **z'** = 0*x + 0*y + 1*z = **z**

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Using matrices: translation

- Example, translate (x,y,z) by (tx,ty,tz) :

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Q: What should we put in the matrix?
- A: We can't do it ! (try it)
- What's the solution then?

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Using matrices: translation

- To incorporate translation, we have to add an **extra row and column** to the matrix, and an **extra term** to our coordinates:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

This may seem like an arbitrary "fix", but it goes deeper.

And later we will see a use for the new bottom row of the matrix, for doing projections.

- Multiply elements row by column:
  - **x'** = 1*x + 0*y + 0*z + tx*1= **x + tx**
  - **y'** = 0*x + 1*y + 0*z + ty*1= **y + ty**
  - **z'** = 0*x + 0*y + 1*z + tz*1= **z + tz**
  - **w'** = 0*x + 0*y + 0*z + 1*1= **1**

crucial concept

Homogeneous coordinates

- In order to use a consistent matrix representation for all kinds of linear transformations, we've had to add an extra coordinate, **w**, to our usual 3D coordinate (x,y,z)
- This (x,y,z,w) form is called **homogeneous coordinates**
- But where is this **w**? Is it in a 4th spatial dimension?
- **Yes it is!**
- The mathematical details are beyond the scope of this course
- Usually, w=1, and we just ignore it
- When it is not, we need to "normalise"… see later, when we cover perspective

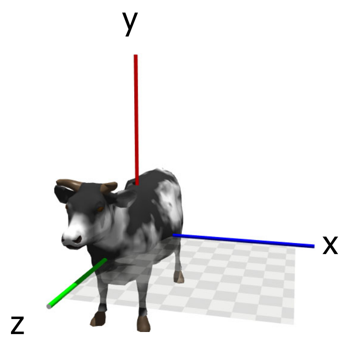
August Möbius, 1790-1868, inventor of homogeneous coordinates

crucial concept

3D Scale matrix

$$P' = T \cdot P$$

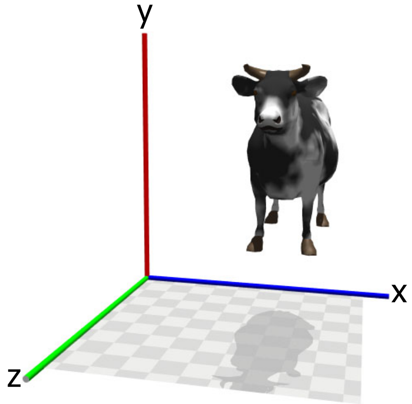
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Translation matrix

$$P' = T \cdot P$$

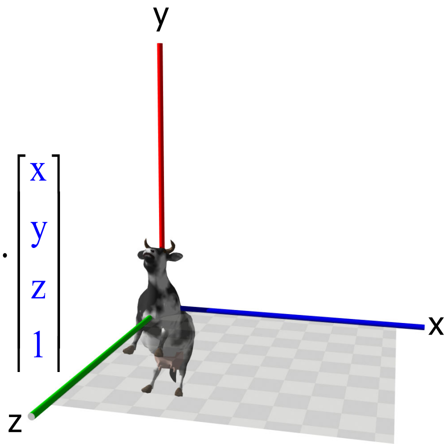
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Rotation matrix (around X axis)

$$P' = T \cdot P$$

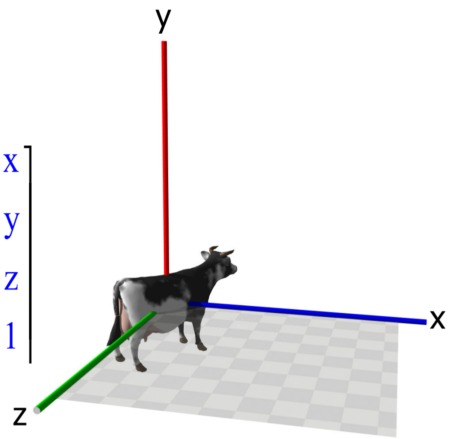
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Rotation matrix (around Y axis)

$$P' = T \cdot P$$

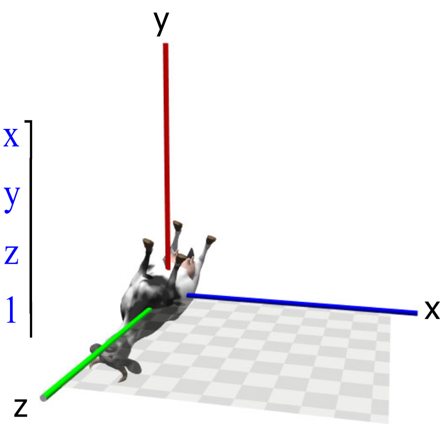
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Rotation matrix (around Z axis)

$$P' = T \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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Matrix transformations recap

- The transformation $\mathbf{T_1}$ changes point **P** to **P′**

$$P' = T_1 \cdot P \qquad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Where the 16 values **a … p** in the matrix determine what kind of transformation it is

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Composing transformations

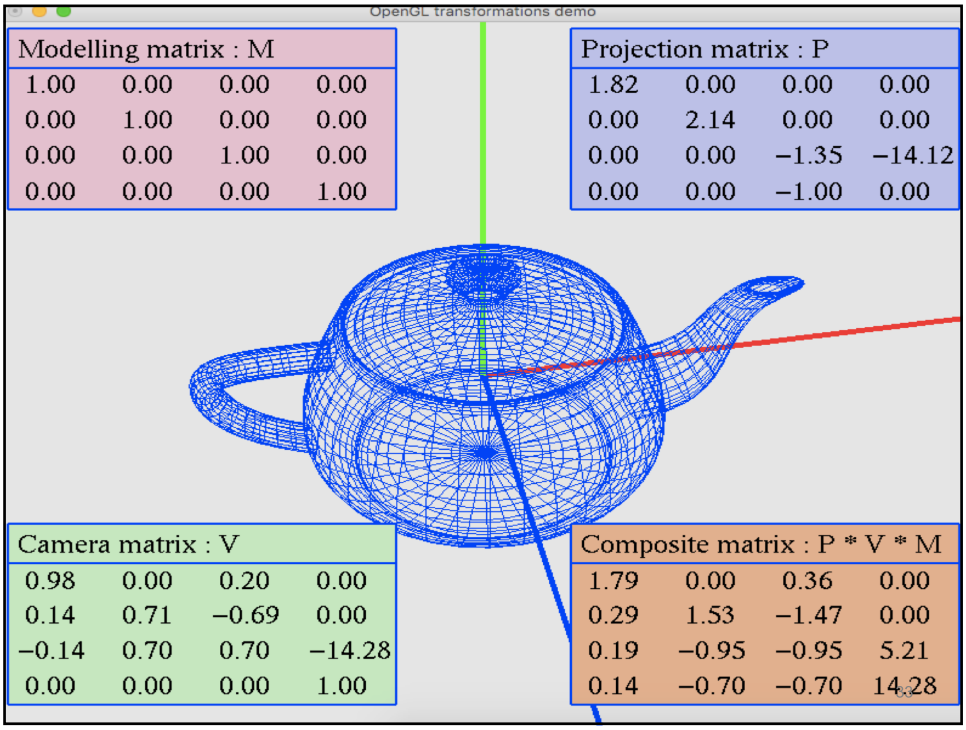
- What if we now apply a transformation $\mathbf{T_2}$ to **P′** ?

$$P'' = T_2 \cdot P'$$
$$P'' = T_2 \cdot T_1 \cdot P$$

- We can apply this double transformation to **P** in one go, if we multiply the matrices $\mathbf{T_1}$ and $\mathbf{T_2}$ together to obtain the **composite** transformation $\mathbf{T_C}$

$$T_C = T_2 \cdot T_1$$
$$P'' = T_C \cdot P$$

crucial concept



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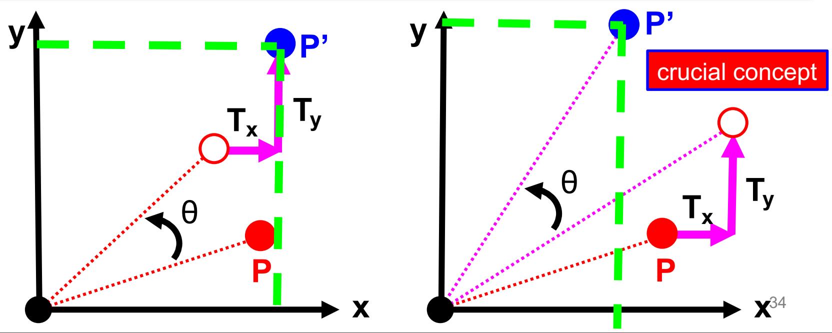
Non-commutativity, or "order matters"

- Matrix multiplications (transformations) are in general non-commutative
- Given two matrices **M_1** and **M_2**, **M_1** * **M_2** ≠ **M_2** * **M_1**

In the left example, **P** is first rotated by θ, and then shifted by (Tx,Ty). In the right example, **P** is first shifted by (Tx,Ty) and then rotated by θ. The results are different. Performing transformations in the correct order is crucial.



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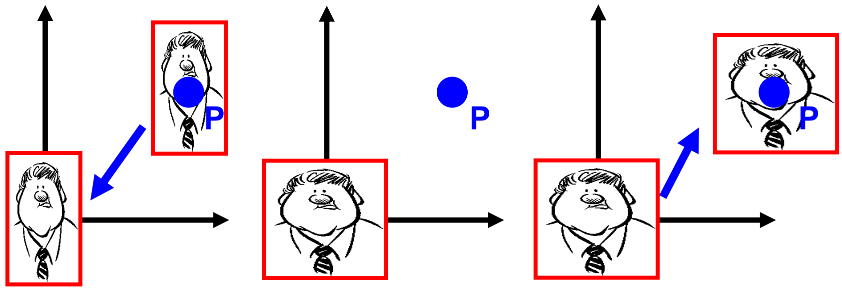
Composite transformations, 2D example

- How to scale an object about an arbitrary point **P**, in 2D?
- We split the operation into three simpler transformations:



M_1: Translate to origin

M_2: Scale about the origin

M_3: Translate back

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Composite transformations, 2D example

- We want to scale an object by (sx, sy) about an arbitrary 2D point **P** (px, py).
- We split this into simpler steps:
  - Step 1: Construct the translation matrix **M_1** which shifts the object to the origin, by (–px, –py)
  - Step 2: Construct the matrix **M_2** which scales the object by (sx, sy) with respect to the origin
  - Step 3: Construct the translation matrix **M_3** which shifts the object back by (px, py)
- The composite transformation is **$M_3 \cdot M_2 \cdot M_1$**
- Note the ORDER: **M_1** first, then **M_2** then **M_3**
- The **key** to this process is in Step 3, where matrix **M_3** **undoes** the effect of matrix **M_1**

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Undoing a transformation

- Matrix **A**: shift by (Tx, Ty, Tz):

$$\begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Matrix **B**: shift by (-Tx, -Ty, -Tz):

$$\begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & -T_y \\ 0 & 0 & 1 & -T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Matrix product of **A** and **B**:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

identity matrix, **I**

- **A** and **B** are called **inverses**
  - So multiplying a point **P** by **A**, then **B**, has no effect on **P**

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Matrix inverses

- Two matrices **A** and **B** are said to be inverses of each other if: **A** x **B** = **I**, where **I** is the identity matrix
- For a matrix **M**, we write its inverse as $\mathbf{M}^{-1}$
- So, **M** x $\mathbf{M}^{-1}$ = **I**
- In other words, if a matrix **M** does some transformation on a point **P**, $\mathbf{M}^{-1}$ undoes it, restoring **P**
- Given **M**, there are algorithms for computing $\mathbf{M}^{-1}$
- **BUT**, not all matrices actually have an inverse!
  - Example: how can you undo a transformation that makes all y-coordinates 0? The original information has been destroyed…

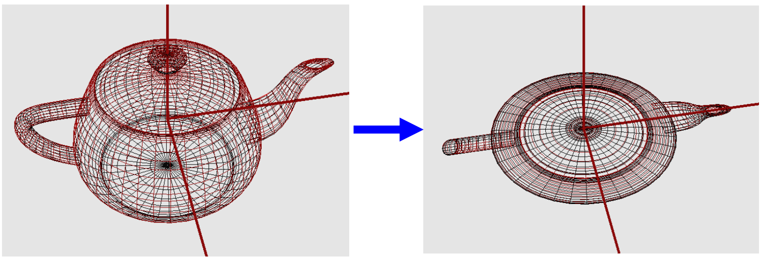
crucial concept

Non-invertible transformations

- Example: this scale transformation will set all y-coordinates to 0:

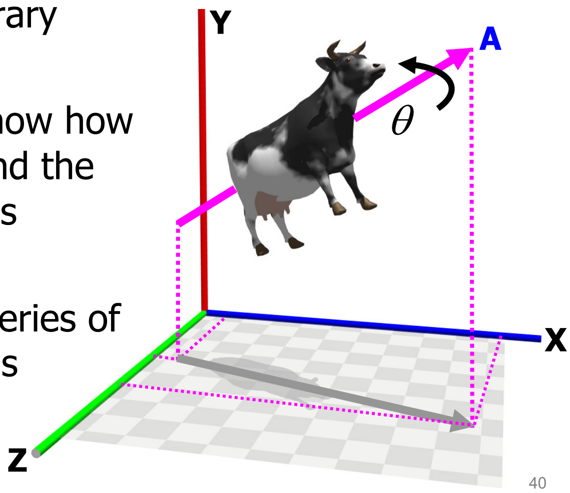
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has no inverse – it is "singular". In general, a transformation is singular if it throws away information



Composite transformations, 3D example

- We want to rotate by θ about an arbitrary 3D vector **A**
- But we only know how to rotate around the X, Y and Z axes
- We reduce the problem to a series of transformations



crucial concept

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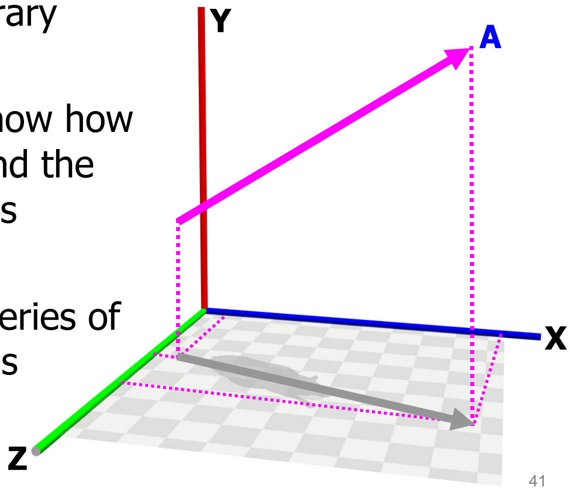
Composite transformations, 3D example

- We want to rotate by θ about an arbitrary 3D vector **A**
- But we only know how to rotate around the X, Y and Z axes
- We reduce the problem to a series of transformations
- (let's remove the cow)



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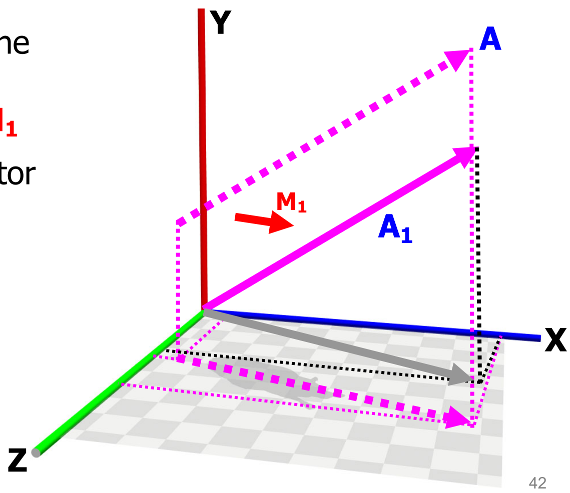
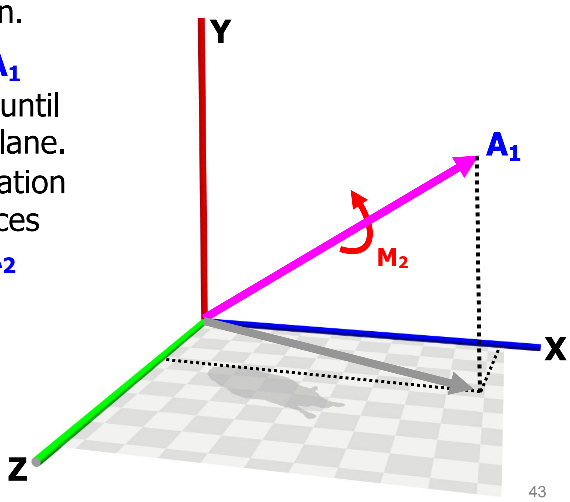
Composite transformations, 3D example

- **Step 1:** We shift vector **A** until it passes through the origin. This is transformation $\mathbf{M_1}$
- Call this new vector $\mathbf{A_1}$



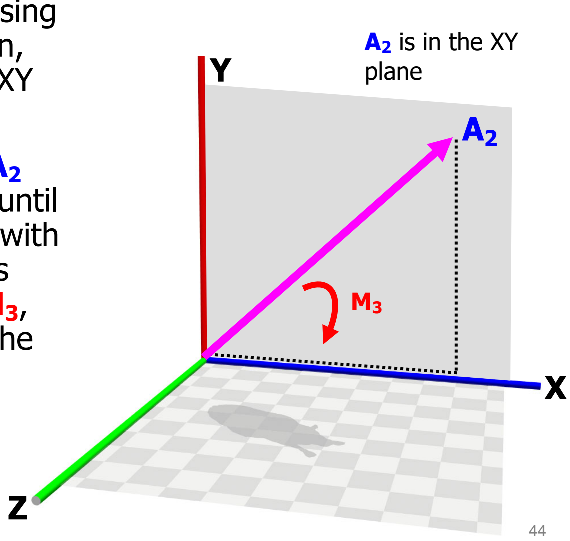
Composite transformations, 3D example

- So now $\mathbf{A_1}$ is passing through the origin.
- **Step 2:** Rotate $\mathbf{A_1}$ about the X-axis until it lies in the XY plane. This is transformation $\mathbf{M_2}$, and it produces the new vector $\mathbf{A_2}$



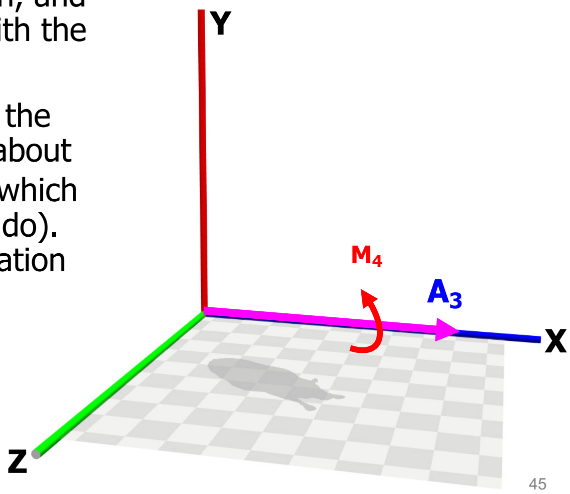
Composite transformations, 3D example

- So now $\mathbf{A_2}$ is passing through the origin, and it lies in the XY plane.
- **Step 3:** Rotate $\mathbf{A_2}$ about the Z axis until it it is coincident with the X-axis. This is transformation $\mathbf{M_3}$, and it produces the new vector $\mathbf{A_3}$



Composite transformations, 3D example

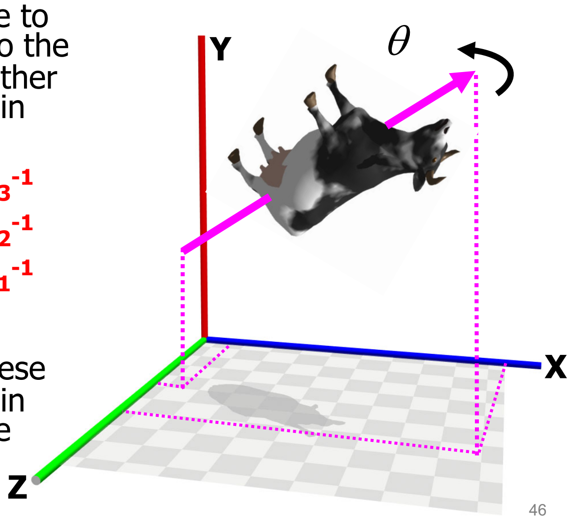
- So now $\mathbf{A_3}$ is passing through the origin, and it is coincident with the X-axis.
- **Step 4:** Perform the desired rotation about the X-axis by θ (which we know how to do). This is transformation $\mathbf{M_4}$



Composite transformations, 3D example

- So that's the rotation done. All we have to do now is to undo the effect of all the other transformations, in reverse order
- **Step 5:** apply $\mathbf{M_3^{-1}}$
- **Step 6:** apply $\mathbf{M_2^{-1}}$
- **Step 7:** apply $\mathbf{M_1^{-1}}$

- If we apply all these transformations, in order, we achieve what we want



Rotation about an arbitrary 3D axis

- Summary of the steps we've gone through:
  1. Construct the matrix $\mathbf{M_1}$ which translates $\mathbf{A}$ so it passes through the origin. The new vector is $\mathbf{A_1}$.
  2. Construct $\mathbf{M_2}$ which rotates $\mathbf{A_1}$ about the X-axis (although we could use a different axis), mapping it into the XY plane. The new vector is $\mathbf{A_2}$.
  3. Construct $\mathbf{M_3}$, which rotates $\mathbf{A_2}$ about the Z-axis, mapping it onto the X-axis. The new vector is $\mathbf{A_3}$.
  4. Construct $\mathbf{M_4}$, which applies the required rotation by θ about the X-axis.
  5. Construct the inverse matrices, to undo the effects of $\mathbf{M_3}$, $\mathbf{M_2}$ and $\mathbf{M_1}$.
- The entire transformation is thus:
  - $\mathbf{P'} = \mathbf{M_1^{-1}} \cdot \mathbf{M_2^{-1}} \cdot \mathbf{M_3^{-1}} \cdot \mathbf{M_4} \cdot \mathbf{M_3} \cdot \mathbf{M_2} \cdot \mathbf{M_1} \cdot \mathbf{P}$

Reference

- The following slides are reference material, to assist your lab work.

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Transformations in OpenGL (1)

- OpenGL maintains two transformation matrices internally:
  - the **"modelview"** matrix, used for transforming the geometry you draw, and specifying the camera
  - the **"projection matrix"**, used for controlling the way the camera image is projected onto the screen (see later)
- Every 3D point you ask OpenGL to draw is automatically transformed by these two matrices before it is drawn (and you cannot prevent this happening)
  - $\mathbf{P}_{\mathbf{drawn}}$ = **ProjectionMatrix** x **ModelviewMatrix** x $\mathbf{P}_{\mathbf{specifed}}$
- For full details, see Chapter 5 of the OpenGL manual.

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Transformations in OpenGL (2)

- OpenGL provides functions for easily dealing with transformations. Here are some:

OpenGL

```
glTranslatef(tx, ty, tz)
glScalef(sx, sy, sz)
glRotatef(theta, rx, ry, rz)
```

- When we call one of these functions, OpenGL creates a corresponding temporary matrix **TMP**, and then multiples the **modelview** matrix by **TMP**, and then throws away **TMP**

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Transformations in OpenGL (3)

- An example. We want to **first** rotate and **then** shift the teapot.

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity(); // M= identity matrix (I)
glTranslatef(tx, ty, tz);
// OpenGL computes temp translation matrix T,
// then sets M= M x T, so now M is T
glRotatef(theta, 0.0, 1.0, 0.0);
// OpenGL computes temp rotation matrix R,
// then sets M= M x R, so M is now T x R
glutWireTeapot(1.0);
```

- Notice the **order** we call the functions in… it's the **reverse** of how we would write it down logically.

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Transformations in OpenGL (4)

- What if we want a series of steps, as we saw earlier?
- Sometimes there are OpenGL functions which come to our rescue. For example, `glRotatef()` will conveniently compute a matrix for rotation of angle θ about the vector (x, y, z) which passes through the origin.

```
glRotatef(GLfloat theta,
          GLfloat x, GLfloat y,
          GLfloat z);
```
OpenGL

- Therefore, if we call this transformation **R**, we can express our previous rotation-about-arbitrary-vector example as $\mathbf{M_1^{-1}} \cdot \mathbf{R} \cdot \mathbf{M_1}$
- There are OpenGL functions for loading your own matrix from the modelview or projection matrices, and for multiplying them together. But in practice, it's not necessary to use these much.

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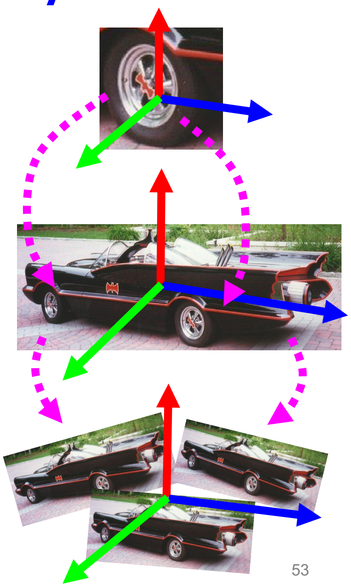
Model and world coordinate systems

- Often an object is defined in a local **modelling coordinate system**. E.g. modelling a car wheel with an origin at the wheel centre.
- **Modelling transformations** are used to **instance** multiple copies of an object in the scene, e.g. translate and rotate the wheels onto a car body
- The entire car may then have further transformations applied, like translation to simulate its movement.
- A global **world coordinate system** is used to specify the position of objects in the entire scene.



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Essential vector geometry

- Vectors provide a very convenient way of thinking about many of the manipulations we might want to perform on an object in 3D space
- In fact, it's the only sensible way to work (and essential for rendering, as we shall see later)
- Understanding a small amount of vector maths goes a long way in 3D graphics…
  - Addition and Subtraction
  - Scalar multiplication
  - Vector normalization
  - Dot product
  - Cross product

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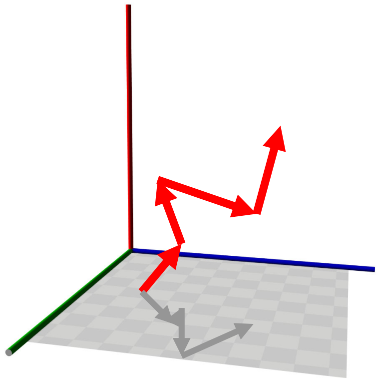
Vector addition

- To add two vectors of the same order, add the components…

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ 1 \end{bmatrix}$$

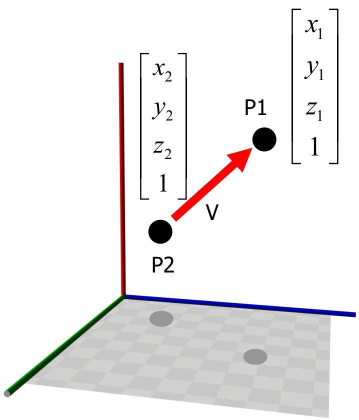


- Why is this useful…?
- … moves a point through space in a known direction

Vector subtraction

- To subtract two vectors of the same order, subtract the components…

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} - \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ 1 \end{bmatrix}$$



V= P1 - P2

- Why is this useful…?
- … represents 'a line' between two points

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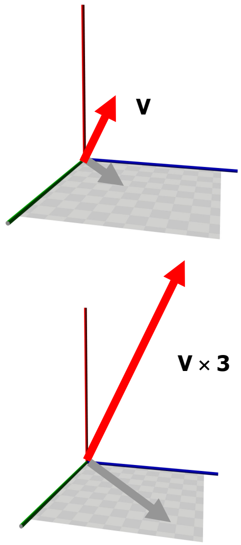
Multiplication by a scalar

- Multiply the individual components by a scalar ***C***

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} \times C = \begin{bmatrix} x_1 \times C \\ y_1 \times C \\ z_1 \times C \\ 1 \end{bmatrix}$$

- Why is this useful…?
- … moves a point along a vector by a given amount



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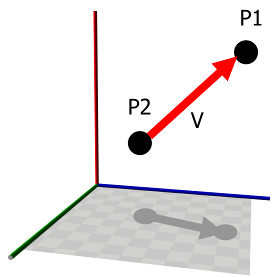
Vector magnitude

- Gives the 'length' or size of a vector

$$V = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad |V| = \sqrt{x^2 + y^2 + z^2}$$

- If we have 'a line' joining two points, the magnitude of the vector between them represents their distance in 3D space



V= P1 - P2

Distance from P1 to P2 is |V|

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Vector normalization

- **Normalization** is the process of taking an arbitrary (but non-zero) vector **V**, and converting it into a vector $\hat{\mathbf{V}}$ (V-hat) **of length 1,** which points in the same direction
- Calculate the length L of **V**, and divide its *x*, *y* and *z* components by this value
- Essential operation in rendering

$$\mathbf{V} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$L = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{\mathbf{V}} = \begin{bmatrix} x/L \\ y/L \\ z/L \\ 1 \end{bmatrix}$$

Vector multiplication

- There are two ways of multiplying vectors
- One results in a **scalar value**, and is called the **dot product** (aka "inner product")
- The other results in a **vector**, and is called the **cross product** (aka "outer product")
- Both are essential operations in 3D graphics

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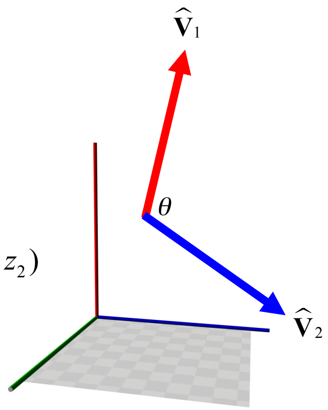
The Dot Product

- is the scalar product of the individual components

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix} = (x_1 \times x_2) + (y_1 \times y_2) + (z_1 \times z_2)$$

- For normalized vectors, their dot product is the cosine of the angle between them
- Essential for rendering



$$\cos\theta = \hat{\mathbf{V}}_1 \bullet \hat{\mathbf{V}}_2$$

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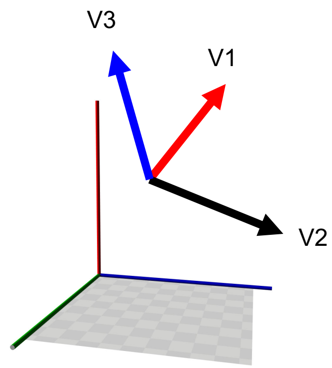
The Cross Product

- is a vector, defined as follows:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 \times z_2 - z_1 \times y_2 \\ z_1 \times x_2 - x_1 \times z_2 \\ x_1 \times y_2 - y_1 \times x_2 \\ 1 \end{bmatrix}$$

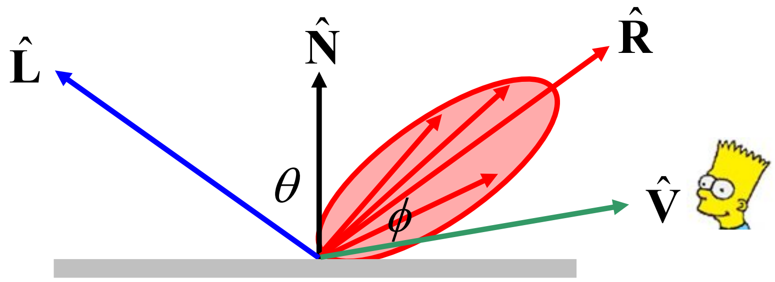
- For two vectors, their cross product is a third vector **perpendicular** to them both (forming a right handed system)



V3 = V1 x V2

Vector geometry is essential

- All these properties of vectors are essential in 3D graphics:
  - for defining and manipulating geometry
  - for specifying and evaluating rendering



- There are many vector manipulation libraries available that hide the underlying maths and make vector manipulation easy