→ Equivalence Relations:

→ reflexive, symmetric, transitive

→ 2 bases are ~ related IF they have SAME orientation

① $\{\vec{e_i}\} \sim \{\vec{e_j}\}$: the fact above

② THIS is EQUIVALENCE relation

→ reflexive: $\{e_i\} = \{e_i\} \cdot I$

→ $\det(I) = 1, > 0$

→ symmetric: consider $[e_i]$, $[\bar{e}_i]$

① if $[e_i] \sim [\bar{e}_i]$, $[e_i] = [\bar{e}_i]T$

$\therefore \det(T) > 0$

→ if $[\bar{e}_i] = [e_i]T^{-1}$ $\therefore \det(T^{-1}) = \frac{1}{\det(T)}$

$\therefore \det(T^{-1}) > 0$

→ Transitivity:

Consider $\underset{①}{[e_i]}$, $\underset{②}{[e_i']}$, $\underset{③}{[\bar{e}_i]}$

→ $[e_i] \sim [e_i']$, $[e_i'] \sim [\bar{e}_i]$

→ $\underbrace{\det(① \cdot ③)}_{>0} = \underbrace{\det(① \cdot ②)}_{>0} \cdot \underbrace{\det(② \cdot ③)}_{>0}$

→ Consider two bases: $[e_1, e_2, e_3, \dots, e_n]$

∴ different orientation $[e_2, e_1, e_3, \dots, e_n]$

swap!

→ Proposition: let $[e_i]$, $[\bar{e}_i]$ be two bases with opposite orientation

∴ For any third basis $[e_i']$
$[e_i'] \sim [e_i]$ OR
$[e_i'] \sim [\bar{e}_i]$

→ $\det(T^{13}) = \det(T^{12}) \cdot \det(T^{23})$

∴ if $\det(T^{13}) < 0$, opposite orientation

∴ logically, one of the RHS components must be +ve

→ Orientation of vector space:

→ equivalence class of bases in the vector space

→ enough to choose one basis vector to DEFINE orientation of vector space

→ Example 0: consider $\mathbb{R} = \mathbb{R}^1$

→ $e = 1$, $\bar{e} = 2$, $e' = 8$

$T = 2$ $\quad T = -8$

Ex 1: consider $\mathbb{R} = \mathbb{R}^2$, $\{\vec{e}, \vec{f}\}$

$\rightarrow \{\vec{f}^\theta, \vec{e}^\theta\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, swap of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$