Affine spaces, Vector spaces
Transformation Matricies

Affine Space Basics

→ Vector spaces are simply AFFINE spaces but with an origin

→ let V be some vector space, A be set with 'point' elements

∴ A is affine space associated with V if $\forall$ following rules are met:

① $\forall$ points $\in A \wedge \forall$ vectors $\in V$, (point + vector) $\mapsto Q \in A$

② $\forall (x, y) \in V \& \forall P \in A$,

$(P+x)+y = P+(x+y)$ ← additive assoc.

③ $\forall P \in A$, $P + \underline{0} = P$ (unit of vector addition)

④ $\forall (P, Q) \in A$, $\exists$ unique $m \in V \mid P+m = Q$

→ $m = Q-P$ ∴ $m = \overrightarrow{PQ}$ & $\overrightarrow{QP} = -m$

∵ if $\exists$ random $(P, Q, R) \in A$, $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

→ if A = affine space assoc. with vector space V, let O be arbitrary point $\in A$ && consider vectors starting at O

∴ vector space $V = \overrightarrow{OX}$, where X = random point in A

∴ We have defined Vector space in terms of affine space

→ but what if we need affine space in terms of vector space??

→ consider V but $\exists$ 2 copies: V_p, V_v

elements of V are points || vectors

→ if (A (which has value a) is random point within V_p) $\wedge$ (x is random vector within V_v)

→ $B = A + x = a + x$ $\quad \therefore x = \vec{AB}$

B belongs to $\underline{V_p}$

→ by saying "$x = \vec{AB}$", we assign vector $x = b - a$ to 2 points $A(=a)$ and $B(=b)$

e.g. $\mathbb{R}^n$, real number vector space, can ALSO be points

→ consider $M = (m_1, m_2, \dots, m_n)$ and $T = (t_1, t_2, \dots, t_n)$ } points, not vectors

→ They can FORM vector $\vec{MT} = T - M = (t_1 - m_1, t_2 - m_2, \dots, t_n - m_n)$

1.2.1 - Euclidean affine spaces

→ let E^n = n-dimensional Euclidean vector space (i.e. vector space which has SCALAR PRODUCT)

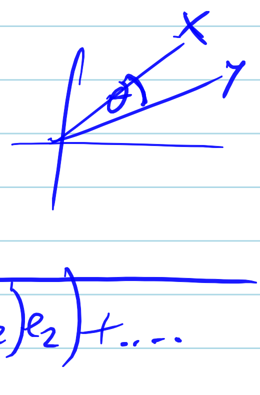
→ let $\{e_i\}$ = some orthonormal basis in E^n

→ if E^n has 2 copies → E_p^n : set of points; → E_v^n : set of vectors

→ $\forall (A, B) \in E_p^n \mid A = \sum_{i=1}^{n} a_i e_i$ & $B = \sum_{i=1}^{n} b_i e_i$,

these points define vector $\vec{AB} \in E_v^n$, as:

$$\vec{AB} = B - A = \sum_{i=1}^{n} (b_i - a_i) e_i$$



$\therefore$ distance$(A, B) = |\vec{AB}| = \sqrt{(\vec{AB}, \vec{AB})}$

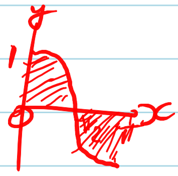
$= \sqrt{((b_1 - a_1)e_1 \cdot (b_1 - a_1)e_1) + ((b_2 - a_2)e_2 \cdot (b_2 - a_2)e_2) + \dots}$

$= \sqrt{\sum_{i=1}^{n} (b_i - a_i)^2}$

→ also if ∃ angle θ between vectors X and Y, then:

→ $(x,y) = x_1y_1 + x_2y_2 + \dots + x_ny_n = |x||y|\cos\theta$

∴ θ is acute / obtuse / perpendicular if (x,y) is $>$ / $<$ / $=$ 0



#FamousCBSinequality

→ from $0 \le \theta \le 180^\circ$, $0 \le \cos\theta \le 1$

∴ when $\theta = 0$, X and Y are COLLINEAR

∴ $(x,y)^2 = (x_1y_1 + x_2y_2 + \dots + x_ny_n)^2 \le ((x_1)^2 + (x_2)^2 + \dots + (x_n)^2)((y_1)^2 + (y_2)^2 + \dots + (y_n)^2)$
$= (x,x)(y,y)$

∴ if $X = \lambda Y \mid \lambda \in \mathbb{R}$, $(x,y)^2 = (x,x)(y,y)$ and (X,Y) are collinear.

→ PROOF of CBS inequality ¶¶¶:

→ consider roots of polynomial $Ax^2 + 2Bx + C$, where:
$A = (x,x)$, $B = (x,y)$, $C = (y,y)$

RTP: this has 0 or 1 real roots

→ $\Delta = 4B^2 - 4AC$, which ≤ 0 for 0 or 1 real roots

∴ $B^2 - AC = (x,y)^2 - (x,x)(y,y) \le 0$

∴ $B^2 \le AC$, i.e. $(x,y)^2 \le (x,x)(y,y)$ #CBS

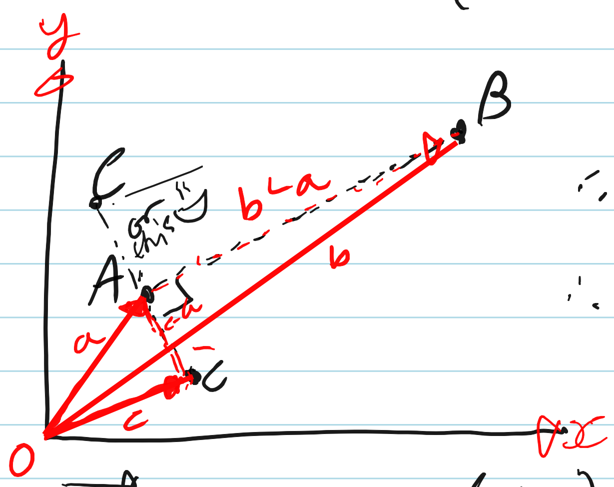
Homework 1 – Affine spaces problems (Q1-2)

Q1(a) $\mathbb{R}^2$ = affine space of points, $A=(2,3)$, $B=(6,6)$

$$|\vec{AB}| = \sqrt{(6-2)^2+(6-3)^2} = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25}$$
$$= 5 \text{ units.}$$

Q1(b) "if $|\vec{AC}|=1$, and $\vec{AC}$ is orthogonal to $\vec{AB}$" $\to$ C?

$$(c-a, c-a) = 1, \quad (c-a, b-a) = 0$$



Let $C=(x,y)$

$\therefore \vec{AB} \perp \vec{AC}$

$\therefore (\vec{AB}, \vec{AC}) = 0$

$\vec{AB} = (4,3)$

$$\therefore \vec{AC} = c - a = (x,y)-(2,3) = ((x-2),(y-3))$$

$$\therefore (\vec{AB},\vec{AC}) = ((4,3),((x-2),(y-3)))$$
$$= 4(x-2)+3(y-3) = 0$$

(i) $4(x-2) = -3(y-3)$

(ii) For equality, $x-2 = -3s$, $y-3 = 4s$, $\exists$ some

$\because |\vec{AC}| = 1 \quad \therefore (x-2)^2+(y-3)^2 = 1$

$$\therefore 9s^2+16s^2 = 1 \quad \therefore 25s^2 = 1$$

$$\therefore s^2 = \frac{1}{25} \quad \therefore s = \pm\frac{1}{5}$$

(iii) Consider $s = +\frac{1}{5} \to x - \frac{10}{5} = \frac{-3}{5} \quad \therefore x = \frac{10-3}{5} = \frac{7}{5} = 1\frac{2}{5}$

$y - \frac{15}{5} = \frac{4}{5} \qquad y = \frac{15+4}{5} = \frac{19}{5} = 3\frac{4}{5}$

$s = -\frac{1}{5} \rightarrow x - \frac{10}{5} = \frac{3}{5} \quad \therefore x = \frac{13}{5} = 2\frac{3}{5}$

$y - \frac{15}{5} = -\frac{4}{5} \quad \therefore y = \frac{11}{5} = 2\frac{1}{5}$

Q2) R^2: $A = (2,1)$, $B = (2+a, 1+b)$, $C = (2+p, 1+q)$

area of $\triangle ABC = ?$ let $\theta = \angle BAC$

let $M = \begin{pmatrix} a & b \\ p & q \end{pmatrix} \therefore \det(M) = aq - bp$

$$\text{area} = \frac{1}{2}|\vec{AB}||\vec{AC}|\sin\angle BAC \qquad (x,y) = |x||y|\cos\theta$$

$$= \frac{1}{2}\cdot\sqrt{(a^2+b^2)}\cdot\sqrt{(p^2+q^2)}\cdot\sqrt{1-\cos^2\theta}$$

$$= \frac{1}{2}\cdot\sqrt{(a^2+b^2)\cdot(p^2+q^2)\cdot\left(1-\left(\frac{(\vec{AB}, \vec{AC})}{|\vec{AB}|\cdot|\vec{AC}|}\right)^2\right)}$$

$$= \frac{1}{2}\cdot\sqrt{(a^2+b^2)\cdot(p^2+q^2)\cdot\left(1-\frac{((a,b),(p,q))^2}{(a^2+b^2)\cdot(p^2+q^2)}\right)}$$

$$= \frac{1}{2}\cdot\sqrt{(a^2+b^2)\cdot(p^2+q^2)-(ap+bq)^2}$$

$$= \frac{1}{2}\cdot\sqrt{\cancel{a^2p^2}+a^2q^2+b^2p^2+\cancel{b^2q^2}-(\cancel{a^2p^2}+2apbq+\cancel{b^2q^2})}$$

$$= \frac{1}{2}\cdot\sqrt{a^2q^2-2aqbp+b^2p^2} = \frac{1}{2}\cdot\sqrt{(aq-bp)^2} = \cancel{\frac{|aq-bp|}{2}}$$

$$= \boxed{\frac{|\det(M)|}{2}} \quad \therefore \triangle ABC\ \text{area} = \frac{1}{2}|\vec{AC}\times\vec{AB}|$$