Lecture 5 - more on orientation

① → $V = \mathbb{R}^2$

→ let $\{\vec{e}_x, \vec{e}_y\}$ be a basis

Let $\{\vec{a}, \vec{b}\}$ be aribitrary <u>ordered</u> set of vectors

$\vec{a} = a_x \vec{e}_x + a_y \vec{e}_y$

$\vec{b} = b_x \vec{e}_x + b_y \vec{e}_y$

$\therefore \{\vec{a}, \vec{b}\} = \{\vec{e}_x, \vec{e}_y\} \boxed{\begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}}$ ← Trans. matrix

☺ $\boxed{\det\begin{pmatrix}\text{Transition} \\ \text{matrix}\end{pmatrix} \neq 0} \longleftrightarrow \{\vec{a}, \vec{b}\}$ is a basis

→ $\det(T_m) > 0$ iff $\{\vec{a}, \vec{b}\}$ is basis and this basis has same orientation on $\{\vec{e}_x, \vec{e}_y\}$

Denote: $\{\vec{e}_x, \vec{e}_y\}$ - Left

☺ $\{\vec{a}, \vec{b}\}$ is $\left\langle \begin{matrix}\text{Left} \\ \text{Right}\end{matrix} \right\rangle$ basis $\longleftrightarrow \det(T) \left\langle \begin{matrix} +\text{ve} \\ -\text{ve} \end{matrix}\right.$

② → $V = \mathbb{R}^3$: $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$, a RIGHT basis

→ Let $\{\vec{a}, \vec{b}, \vec{c}\}$ be arbitrary ORDERED set of vectors

$\therefore \{\vec{a}, \vec{b}, \vec{c}\} = \{\vec{e}_x, \vec{e}_y, \vec{e}_z\} \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$

T →

GIVEN: → $\{\vec{a}, \vec{b}, \vec{c}\}$ is a basis $\longleftrightarrow \det(T) \neq 0$

$\therefore \{\vec{a}, \vec{b}, \vec{c}\} \left\langle \begin{matrix} \text{Left} \\ \text{Right} \end{matrix} \right\rangle$ basis $\longleftrightarrow \det(T) \left\langle \begin{matrix} -ve \\ +ve \end{matrix} \right.$

→ Orientation of linear operations

→ let P be linear operator in Vector space V

→ let $\{\vec{e}_i\}$ be an arbitrary basis

$\therefore P\{\vec{e}_i\} = \sum_{k=1}^{n} \vec{e}_k \cdot P_{ki}$

if $\det P = 0$, $\{\vec{e}_i\}$ isn't basis (obviously)

→ NON degenerate : $\det(P) \neq 0$

$\therefore$ if $\det(P) > 0$, $\{\vec{e}_i\} \sim \{\vec{e}_j\}$

$\qquad\qquad < 0$, $\{\vec{e}_i\} \nsim \{\vec{e}_j\}$

→ $P \left\langle \begin{matrix} \text{preserves} \\ \text{changes} \end{matrix} \right\rangle$ vector orientation if $\det(P) \left\langle \begin{matrix} > 0 \\ < 0 \end{matrix} \right.$

Rotations and orthogonal operators

- $\mathbb{E}^n$ $(n=2,3)$
- Let A be orthogonal in $\mathbb{E}^2$
  - Consider $\{\vec{e}, \vec{f}\}$ arbitrary orthonormal basis
    - This is where $(\vec{e}, \vec{e}) = (\vec{f}, \vec{f}) = 1$, $(\vec{e}, \vec{f}) = 0$.
  - orthogonal matrix A: $A^T A = I$

    transpose
  - let $A^T \cdot A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \begin{cases} a^2 + b^2 = 1 \\ ac + bd = 0 \quad (2) \\ c^2 + d^2 = 1 \end{cases}$ let $a = \cos\varphi$, $b = \sin\varphi$, $c = \cos\theta$, $d = \sin\theta$

$\therefore$ using (2): $\cos\varphi\cos\theta + \sin\varphi\sin\theta = 0$.

$\therefore \cos(\varphi - \theta) = 0$

$\therefore \varphi - \theta = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$

$(k=0)$

case 1: $\theta = \varphi - \pi/2$

$\therefore A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$\therefore$ now remembering $\{\vec{e}, \vec{f}\}$: