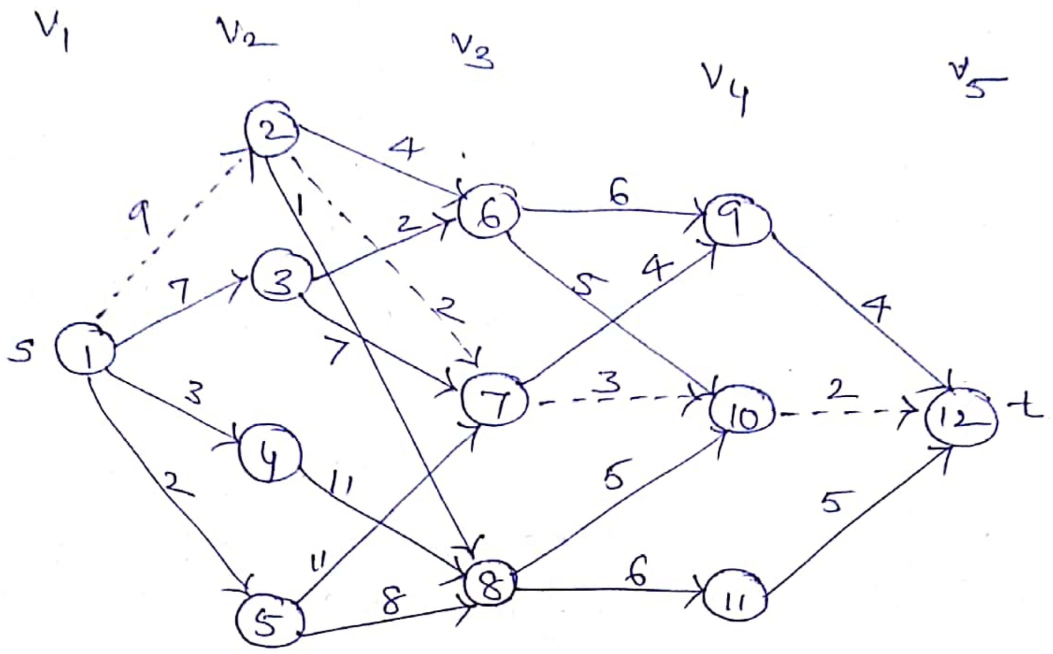
MULTISTAGE GRAPHS

A multistage graph $G = (V, E)$ is a directed graph in which the vertices are partitioned into $k \geq 2$ disjoint sets V_i, $1 \leq i \leq k$. In addition, if $\langle u, v \rangle$ is an edge in E, then $u \in V_i$ and $v \in V_{i+1}$ for some i, $1 \leq i < k$. The sets V_1 and V_k are such that $|V_1| = |V_k| = 1$. Let s and t, respectively, be the vertices in V_1 and V_k. The vertex s is the source, and t is the sink. Let $c(i, j)$ be the cost of edge $\langle i, j \rangle$. The cost of a path from s to t is the sum of the costs of the edges on the path. The multistage graph problem is to find a minimum-cost path from s to t. Each set V_i defines a stage in the graph. Because of the constraints on E, every path from s to t starts in stage 1, goes to stage 2, then to stage 3, then to stage 4 and so on, and eventually terminates in stage k. Following figure shows a Five-stage graph. A minimum cost s to t path is indicated by the broken edges.



Five-stage graph.

First stage and last stage should always contain only one vertex.

$G(V,E)$. $|V| = n$.

k = no. of stages.

$\therefore$ $(n-2)$ vertices should be distributed in $k-2$ stages.

Edges should be connected as $V_i \rightarrow V_{i+1}$ only i.e stage 1 cannot be connected to stage 3.

<u>Using Greedy Method</u>: The minimum-cost path is 1-5-8-10-12 and cost is 17.

i.e lesser costs are considered at every stage.

but consider the path

1-2-7-10-12

Cost is = 16.

∴ greedy method is failed to find the optimal path.

∴ we have to use Dynamic programming.

Principle of Optimality:-

The decision of going from i to v_i is optimal only when v_i to j is also optimal. This is called Principle of Optimality. This states that whatever the initial state and decision 's' the remaining sequence of decisions must be optimal with regard to the state resulting from first decision.

* let cost(i,j) represents the cost of reaching destination 't' from vertex 'j' present in stage 'i'. Let cost(i,j) be the cost of this path. Then using forward approach, we obtain.

$$\underset{\substack{\text{stage}}}{\text{Cost}}\,(i,j) = \min_{\substack{l \in V_{i+1} \\ (j,l) \in E}} \{ c(j,l) + \text{Cost}(i+1, l) \}$$

stage vertex

c(j,l) is direct cost edge <j,l>. ——— (1)

Let $D(i, j) = l$ that minimizes the equation ①.

First consider the only ~~stage~~ vertex 1 which is in stage 1.

$$\therefore \; Cost(\underset{stage}{1}, \underset{vertex}{1}) = \min_{l \in 2,3,4,5} \{ c(1,2) + cost(2,2),$$
$$c(1,3) + cost(2,3),$$
$$c(1,4) + cost(2,4),$$
$$c(1,5) + cost(2,5) \}.$$

Here we need to find out

$Cost(2,2)$, $cost(2,3)$, $cost(2,4)$, $cost(2,5)$.

$$\therefore \; Cost(\underset{i}{2}, \underset{j}{2}) = \min_{l \in 6,7,8} \{ c(2,6) + cost(3,6),$$
$$c(2,7) + cost(3,7),$$
$$c(2,8) + cost(3,8) \}.$$

$$cost(\underset{i}{2}, \underset{j}{3}) = \min_{l \in 6,7} \{ c(3,6) + cost(3,6),$$
$$c(3,7) + cost(3,7) \}$$

$$cost(\underset{i}{2}, \underset{j}{4}) = \min_{l \in 8} \{ \cancel{cost}\, c(4,8) + cost(3,8) \}.$$

$$cost(2,5) = \min_{l \in 7,8} \{ c(5,7) + cost(3,7),$$
$$c(5,8) + cost(3,8) \}.$$

Step1 $cost(k-1, j) = \cancel{cost}\; c(j,t), j \in V_{k-1}$

Step2 $cost(k-2, j)$. for $j \in V_{k-2}$

Step3 $cost(k-3, j)$ $\forall j \in V_{k-3}$.

Step4 $cost(k-4, j)$ $\forall$ $j \in V_{k-4}$ so on. $cost(1, s)$.

Here we need to find out
$cost(3,6)$, $cost(3,7)$, $cost(3,8)$

$\therefore cost(\underset{i}{3},\underset{j}{6}) = \min_{l \in 9,10} \{ c(6,9) + cost(4,9),$
$c(6,10) + cost(4,10) \}.$

$cost(\underset{i}{3},\underset{j}{7}) = \min_{l \in 9,10} \{ c(7,9) + cost(4,9),$
$c(7,10) + cost(4,10) \}.$

$cost(\underset{i}{3},\underset{j}{8}) = \min_{l \in 10,11} \{ c(8,10) + cost(4,10),$
$c(8,11) + cost(4,11) \}$

Here we need to find out
$cost(4,9)$, $cost(4,10)$, $cost(4,11)$.

$\therefore cost(\underset{i}{4},\underset{j}{9})$ here $= k=5$ and $i = k-1$
$4 = 5-1$
$4 = 4$

whenever 'i' becomes i.e $i = k-1$
Then $cost(i,j) = c_{jt} \rightarrow c(j,t)$.

Here $t=12$

$\therefore cost(4,9) = c_{9t} = \underline{\underline{4}}.$

$cost(4,10) = c_{10t} = \underline{\underline{2}}$

$cost(4,11) = c_{11t} = \underline{\underline{5}}$

$\therefore \text{Cost}(3,6) = \min\{6+4, 5+2\}$

$= \min\{10, 7\}$

$= \underline{\underline{7}}$

$\text{Cost}(3,7) = \min\{4+4, 3+2\}$

$= \min\{8, 5\}$

$= \underline{\underline{5}}$

$\text{Cost}(3,8) = \min\{5+2, 6+5\}$

$= \min\{7, 11\}$

$= \underline{\underline{7}}$

$\text{cost}(2,2) = \min\{4+7, 2+5, 1+7\}$

$= \min\{11, 7, 8\}$

$= \underline{\underline{7}}$

$\text{cost}(2,3) = \min\{2+7, 7+5\}$

$= \min\{9, 12\}$

$= \underline{\underline{9}}$

Scanned with CamScanner

$cost(2,4) = \min\{11+7\}$

$= \underline{18}$

$cost(2,5) = \min\{11+5, 8+7\}$

$= \min\{16, 15\}$

$= \underline{\underline{15}}$

$\therefore cost(1,1) = \min\{9+7, 7+9, 3+18, 2+15\}$

$= \min\{16, 16, 21, 17\}$

$= \underline{\underline{16}}$

$\therefore$ Cost of minimum-length path $= \underline{\underline{16}}$.

$D(1,1) = 2$ $\because$ for $l=2$ it is giving min.

$\therefore 1 \rightarrow 2$

After reaching ②, from ② which 'l' value is giving ~~...~~ min cost

i.e ~~D=2~~, cost(2,2) here $l \in 6,7,8$.

for $l=7$ it is min

$\therefore 1 \rightarrow 2 \rightarrow 7$.

After reaching ⑦, from ⑦ which 'l' value is giving min cost

~~i.e~~

∴ Cost (3, T)

here $l = 9, 10$.

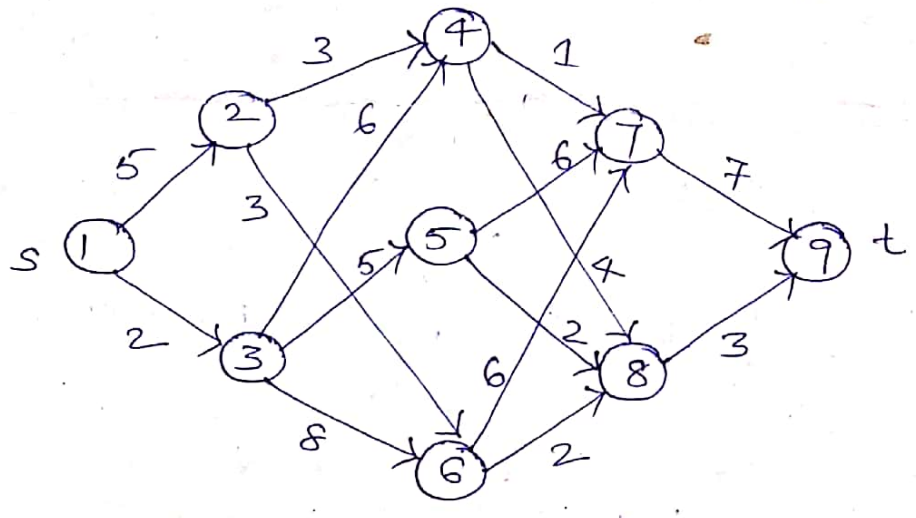
for $l = 10$ it is giving min cost

∴ $1 \to 2 \to 7 \to 10 \to 12$.

$9 + 2 + 3 + 2 = 16$ is matching with the value which is got from formula.

Example 1:-

Find the minimum cost path in the following multi stage graph using backward approach.



$$Bcost(i,j) = \min_{\substack{l \in V_{i-1} \\ (l,j) \in E}} \{ Bcost(i-1, l) + c(l,j) \}$$

Step 1 :- Bcost (2, j) = c (1, j).

Step 2 :- Bcost (3, j) i.e i=3

Step 3 :- i=4 and so on.