# THE TRAVELING SALESPERSON PROBLEM

Let $G = (V, E)$ be a directed graph with edge costs $c_{ij}$. The variable $c_{ij}$ is defined such that $c_{ij} > 0$ for all $i$ and $j$ and $c_{ij} = \infty$ if $\langle i, j \rangle \notin E$. Let $|V| = n$ and assume $n > 1$. A tour of $G$ is a directed simple cycle that include every vertex in $V$. The cost of a tour is the sum of the cost of the edges on the tour. The travelling salesperson problem is to find a tour of minimum cost.

Application :- Suppose we have to route a postal van to pickup mail from mail boxes at $n$ different sites. An $n+1$ vertex graph can be used to represent the situation. One vertex represents the post office from which the postal van starts and to which it must return. Edge $\langle i, j \rangle$ is assigned a cost equal to the distance from $i$ to site $j$. The route taken by the postal van is a tour, and we are interested in finding a tour of minimum length.

The tour to be a simple path that starts and ends at vertex 1. Every tour consists of an edge $\langle 1, k\rangle$ for some $k \in V-\{1\}$ and a path from vertex $k$ to vertex 1. The path from vertex $k$ to vertex 1 goes through each vertex in $V-\{1,k\}$ exactly once. It is easy to see that if the tour is optimal, then the path from $k$ to 1 must be a shortest $k$ to 1 path going through all vertices in $V-\{1,k\}$. Hence, the principle of optimality holds. Let $g(i,S)$ be the length of a shortest path starting at vertex $i$, going through all vertices in $S$, and terminating at vertex 1. The function $g(1, V-\{1\})$ is the length of an optimal salesperson tour. From the principle of optimality it follows that

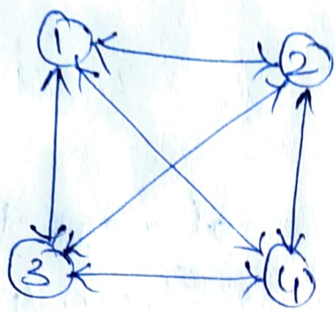
$$g(1, V-\{1\}) = \min_{2\le k\le n}\{c_{1k} + g(k, V-\{1,k\})\} \quad \text{—(1)}$$

Generalizing, we obtain (for $i \notin S$)

$$g(i,S) = \min_{j\in S}\{c_{ij} + g(j, S-\{j\})\} \quad \text{—(2)}$$

Clearly $g(i,\emptyset) = c_{i1}$ for $1 \le i \le n$. Hence we can use (2) to obtain $g(i,S)$ for all $S$ of size 1. Then we can obtain $g(i,S)$ for $S$ with $|S| = 2$, and so on. When $|S| < n-1$ the values of $i$ and $S$ for which $g(i,S)$ is needed are such that $i \ne 1$, $1 \notin S$, and $i \notin S$.

Example 1 :– Consider the directed graph. The edge lengths are given by matrix c.




G.

$$C = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 10 & 15 & 20 \\ 2 & 5 & 0 & 9 & 10 \\ 3 & 6 & 13 & 0 & 12 \\ 4 & 8 & 8 & 9 & 0 \end{array}$$

$g(i, \phi) = c_{i1}, \quad 1 \le i \le n.$

$g(1, \phi) = c_{11} = 0$

$g(2, \phi) = c_{21} = 10$

$g(3, \phi) = c_{31} = 6$

$g(4, \phi) = c_{41} = 8$

Consider $|S| = 1$.

It means that set contains only one element. Before solving this problem we make an assumption that the sales person starts at vertex 1, from that he can move to vertex 2. If he visits vertex 2, from that vertex, he can visit either vertex 3 or 4.

$\therefore g(2, 3)$ 2 to 3
or
$g(2, 4)$ 2 to 4.

From vertex 1, ~~next~~ he can also visit 3 instead of 2; in this case from 3, next he can visit 2 or 4.

$\therefore$ $g(3,2)$

$g(3,4)$

From vertex 1, he can also visit 4 instead of 3, in this case from 4, next he can visit 2 or 3.

$\therefore$ $g(4,2)$

$g(4,3)$.

All these 6 possibilities can be determined using eq (2) as follows.

$$g(2,3) = \min_{j \in S} \{c_{23} + g(3,\phi)\}$$

$$= 9 + 6 = \underline{\underline{15}}$$

$$g(2,4) = \min_{j \in S} \{c_{24} + g(4,\phi)\}$$

$$= 10 + 8 = \underline{\underline{18}}$$

$$g(3,2) = \min_{j \in S} \{c_{32} + g(2,\phi)\}$$

$$= 13 + 5 = \underline{\underline{18}}$$

$$g(3,4) = \min_{j \in S} \{c_{34} + g(4,\phi)\}$$

$$= 12 + 8 = \underline{\underline{20}}$$

$$g(4,2) = \min_{j \in S} \{ c_{42} + g(2,\emptyset) \}$$
$$= 8 + 5 = \underline{\underline{13}}$$

$$g(4,3) = \min_{j \in S} \{ c_{43} + g(3,\phi) \}$$
$$= 9 + 6 = \underline{\underline{15}}$$

Consider $|S| = 2$

then set contains two values, so we can place two vertices in S. From starting vertex 1, next he can visit either 2 or 3 or 4.

If he visits 2, then from 2 he can visit either 3 or 4. So determine

$\therefore g(2, \{3,4\})$

If he visits 3 instead of 2, then from 3 he can visit either 2 or 4.

$\therefore$ determine $g(3, \{2,4\})$

If he visits 4 instead of 2 & 3, then from 4 he can visit either 2 or 3.

$\therefore$ determine $g(4, \{2,3\})$

$$g(2, \{3,4\}) = \min_{j \in S}^{j=3,4} \{ c_{23} + g(3,4), c_{24} + g(4,3) \}$$
$$= \min\{9 + 20, 10 + 15\}$$
$$= \min\{29, 25\}$$
$$= \underline{\underline{25}}$$

$$g(3,\{2,4\}) = \min_{\substack{j=3,4 \\ j\in S}} \{c_{32} + g(2,4),\ c_{34} + g(4,2)\}$$

$$= \min\{13+18,\ 12+13\}$$

$$= \min\{31, 25\}$$

$$= \underline{\underline{25}}$$

$$g(4,\{2,3\}) = \min_{\substack{j=2,3 \\ j\in S}} \{c_{42} + g(2,3),\ c_{43} + g(3,2)\}$$

$$= \min\{8+15,\ 9+18\}$$

$$= \min\{23, 27\}$$

$$= \underline{\underline{23}}$$

Now consider $|S| = 3$.

Here S contains 3 elements. After starting from vertex 1, next he can visit 2 or 3 or 4.

$\therefore$ determine $g(1,\{2,3,4\})$.

$$g(1,\{2,3,4\}) = \min_{j\in S} \{c_{12} + g(2,\{3,4\}),\ c_{13} + g(3,\{2,4\}),\ c_{14} + g(4,\{2,3\})\}$$

$$= \min\{10+25,\ 15+25,\ 20+23\}$$

$$= \min\{35, 40, 43\}$$

$$= \underline{\underline{35}}$$

$\therefore$ An optimal tour of the graph has length $\underline{35}$.

A tour of this length can be constructed if we retain with each $g(i, S)$ the value of $j$ that minimizes the right hand side of eq (2).

Let $J(i, S)$ be this value, Then

$J(1, \{2,3,4\}) = 2$. $\because$ for $j=2$ it is 35

Means $1 \to 2$

After reaching 2, from 2 for which $j$ value it is giving minimum value.

i.e $J(2, \{3,4\}) = 4$ $\because$ for $j=4$ it is 25

$\therefore$ $1 \to 2 \to 4$.

After reaching 4, from 4 for which $j$ value it is giving minimum value

i.e $J(4, 3) = 3$ $\because$ only one value in S.

$\therefore$ The path is

i.e $1 \to 2 \to 4 \to 3 \to 1$.

$10 + 10 + 9 + 6 = \underline{\underline{35}}$

$\therefore$ Min cost $= \underline{\underline{35}}$