0/1 Knapsack Problem (LCBB Solution)

To use the branch-and-bound technique to solve any, problem, it is first necessary to conceive of a state space tree for the problem. Branch-and-Bound technique is applicable for only minimization problems, where as Knapsack problem is a maximization problem. This difficulty is easily overcome by replacing the objective fn $\sum p_i x_i$ by the fn $-\sum p_i x_i$. Clearly $\sum p_i x_i$ is maximized iff $-\sum p_i x_i$ is minimized. This modified Knapsack problem is stated as

$$\text{minimize } -\sum_{i=1}^{n} p_i x_i$$

$$\text{Subject to } \sum_{i=1}^{n} w_i x_i \leq m$$

$$x_i = 0 \text{ or } 1, \quad 1 \leq i \leq n.$$

Every leaf node in the state space tree representing an assignment for which $\sum_{1\leq i\leq n} w_i x_i \leq m$ is an answer (or solution) node. All other leaf nodes are infeasible. For a min-cost-answer node to correspond to any optimal solution, we need to define

$$c(x) = -\sum_{1\leq i\leq n} p_i x_i \text{ for every answer node } x.$$

The cost $c(x) = \infty$ for infeasible leaf nodes.

For non leaf nodes, $c(x)$ is recursively def. to be $\min\{c(\text{lchild}(x)), c(\text{rchild}(x))\}$

We now need two fns $\hat{c}(x)$ and $u(x)$ such that $\hat{c}(x) \leq c(x) \leq u(x)$ for every node x

$\hat{c}(\cdot)$ - lower bound
$u(\cdot)$ - upper bound.

Compute $\hat{c}$ & u as follows :-

1. go on including the objects into the Knapsack completely till it is not possible to accomodate a ~~portion~~ particular object completely into the Knapsack.

2. The total profit of objects included in the ~~Ka~~ Knapsack forms upperbound i.e $u(\cdot)$. with -ve sign

3. For getting lower bound $\hat{c}(\cdot)$:- take the fraction of the object which was not being accomodated in the Knapsack with -ve sign. This is done using

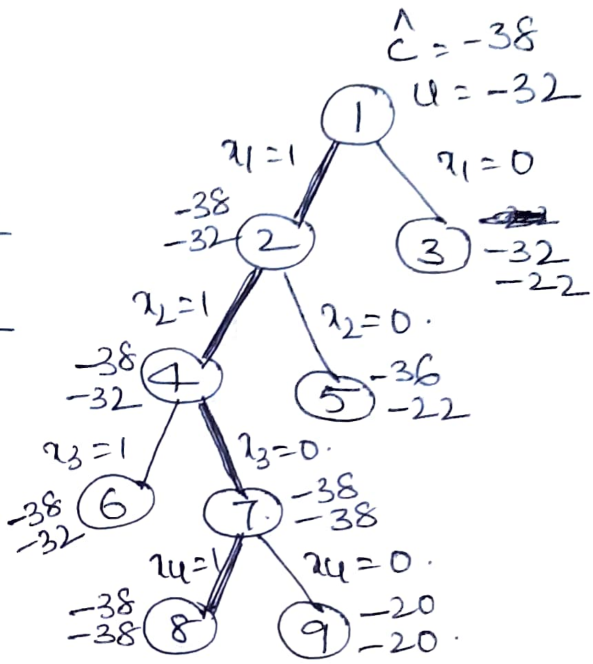
$$\hat{c}(\cdot) = u + \left[\text{remaining wt of the bag} * \frac{\text{Profit of next item}}{\text{wt of next item}}\right]$$

Example :-

$n=4, (P_1, P_2, P_3, P_4) = (10, 10, 12, 18)$

$m = 15, (w_1, w_2, w_3, w_4) = (2, 4, 6, 9)$.

Using LCBB



At Node 1:

$u(1) = 2+4+6 = 12 < 15$

cannot accomodate w_4

$\therefore u(1) = 10+10+12 = -32$

sum of corresponding profits.

$\hat{c}(1) = -32 + (-3 \times \frac{18}{9})$

$= -32 + (-6) = -38$

node 1 is present E-node and hence nodes 2 & 3 are generated.

left child always connected with edge value $x_i = 1$, right child $x_i = 0$.

Al node 2 :-

$x_1 = 1$

hence 1st object should be considered

$u(1) = \cancel{10+10+12}\ 2+4+6 < 15$

cannot accomodate w_4

$\therefore u(1) = 10+10+12 = -32$

$\hat{c}(2) = -32 + (-3 \times \frac{18}{9})$

$= -32 + (-6) = -38$

At node 3 :-

$x_1 = 0$.

1st object should not be considered.

$u(3) = 4+6 = 10 < 15$

w_4 cannot be accomodated

$\therefore u(3) = 10+12 = -22$.

$\hat{c}(3) = -22 + \left(-5 \times \frac{18}{9}\right)$

$-22 + (-10) = -32$.

Perform $u - \hat{c}$ to find which node becomes next E-node. Select the min value node.

at node 2 :- $\cancel{-38-(-32)=}$

$-32-(-38) = 6$

[or compare $\hat{c}$ take least cost].

at node 3

$-22-(-32) = 10$.

hence node (3) is killed.

Therefore Next E-node is (2). Nodes 4 & 5 are generated.

At Node 4 :-

$x_1 = 1, x_2 = 1$. Consider 1st & 2nd objects.

$u(4) = 2+4+6 = 12 < 15$

$\therefore u(4) = 10+10+12 = -32$.

$\hat{c}(4) = -32 + \left(-3 \times \frac{18}{9}\right)$

$= -32 + (-6) = -38$.

At- Node 5 :—

$x_1 = 1,\ x_2 = 0$ 2nd object should not be considered.

$u(5) = 2 + 6 = 8 < 15$

w_4 cannot be accomodated.

$\therefore u(5) = 10 + 12 = -22.$

$\hat{c}(5) = -22 + \left(-7 \times \frac{18^2}{7}\right)$

$= -22 + (-14) = -36$

Next E-node is 4 because it has least $\hat{c} = -38$, compared with node 5 its $\hat{c} = -36$. hence node 5 is killed.

Therefore node 6 & 7 are generated.

At- Node 6 :—

$x_1 = 1,\ x_2 = 1,\ x_3 = 1$

objects 1, 2, & 3 should be considered.

$u(6) = 2 + 4 + 6 = 12 \leq 15$ w_4 cannot be accomodated.

$\therefore u(6) = 10 + 10 + 12 = -32$

$\hat{c}(6) = -32 + \left(-3 \times \frac{18^2}{7}\right)$

$\hat{c}(6) = -32 + (-6) = -38.$

At Node 7:

$x_1 = 1, x_2 = 1, x_3 = 0.$ obj 3 should not be considered.

$u(7) = 2 + 4 + 9 = 15 \leq 15$ w_4 can be fully accomodated.

$\therefore u(7) = 10 + 10 + 18 = -38.$

$\hat{c}(7) = -38 + (-0 \times 0) = -38.$

At this point

at node 6 $\hat{c} = -38$

at node 7 $\hat{c} = -38.$

So take the difference of $u - \hat{c}$ to decide Next E-node.

at node 6 $-32 - (-38) = 6$

at node 7 $-38 - (-38) = 0.$

node 7 has min difference value

hence Next E-node is node 7. Nodes 8 & 9 are generated. Node 6 is killed.

At this point the 'u' is updated to new upper bound. -38 (which upper bound of node 7).

Though we new upper bounds at node 3 and node 5 they should not be considered since they are killed.

At Node 8

$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$

$\therefore u(8) = 10 + 10 + 18 = -38$

$\hat{c}(8) = -38 + (-0 \times 0) = -38.$

At Node 9:

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 0.$$

$\therefore\ u(9) = 10+10 = -20.$

$\hat{c}(9) = -20 + (-0 \times 0) = -20.$

~~Compare~~ All the objects are considered at diff levels of state space tree.

Now compare $\hat{c}$ values of Nodes 8 & 9.
discard the max value node
hence node 9 is discarded.

$\therefore$ The solution vector is
$(x_1 x_2 x_3 x_4) = (1, 1, 0, 1)$ and
profit earned is -38

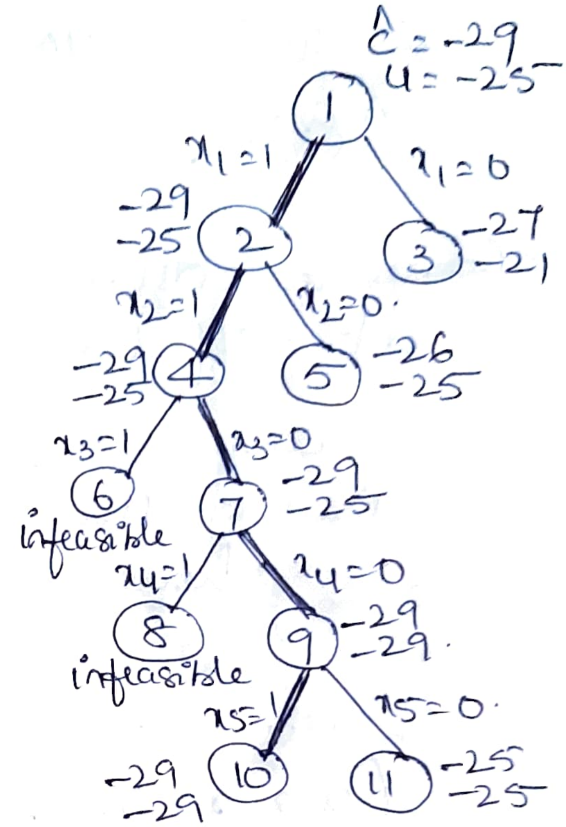
Answer node is 8 bcoz

$$\sum_{1 \le i \le n} w_i x_i = 2 + 4 + 0 + 9 = 15 \le 15.$$

Example 2

$n=5$, $(P_1, P_2, P_3, P_4, P_5) = (10, 15, 6, 8, 4)$

$m=12$, $(w_1, w_2, w_3, w_4, w_5) = (4, 6, 3, 4, 2)$ Using LCBB



At Node 1

$u(1) = 4+6 = 10 \leq 12$

$\therefore u(1) = 10+15 = -25$

$\hat{c}(1) = -25 + (-2 * \frac{\cancel{6}^{2}}{\cancel{3}})$

$= -25 + (-4) = -29$

At Node 2 :-

$x_1 = 1$

$u(2) = 10+15 = -25$

$\hat{c}(2) = -25 + (-2 * \frac{\cancel{6}^{2}}{\cancel{3}})$

$= -25 + (-4) = -29$

At Node 3 :-

$x_1 = 0$

$u(3) = 6+3 = 9 \leq 12$

$\therefore u(3) = 15+6 = -21$

$\hat{c}(3) = -21 + (-3 * \frac{\cancel{8}^{2}}{\cancel{4}})$

$= -21 + (-6) = -27$

Node 2 has min $\hat{c} = -29$ when campared with node 3

hence Next E-node is node 2, nodes 4 & 5 are generated. Node 3 is killed.

At Node 4 :-

$x_1 = 1, x_2 = 1$

$u(4) = 4 + 6 = 10 \leq 12$

$\therefore u(4) = 10 + 15 = -25$

$\hat{c}(5) = -25 + (-2 \times \frac{6^2}{3})$

$= -25 + (-4) = -29$

At Node 5 :-

$x_1 = 1, x_2 = 0.$

$u(5) = 4 + 3 + 4 = 11 \leq 12$

$\therefore u(5) = 10 + 6 + 8 = -24$

$\hat{c}(5) = -24 + (-1 \times \frac{4}{2})$

$= -24 + (-2) = -26.$

Node (4) has min $\hat{c} = 29$ when compared with node 5.
hence (4) becomes Next E-node, Nodes 6 & 7 are generated. Node 5 is killed.

At Node 6 :- $x_1 = 1, x_2 = 1, x_3 = 1$

$4 + 6 + 3 = 13 > 12$

$\therefore$ it is leading to infeasible solution and it is immediately killed.

~~Hence~~

At-Node 7 :-

$x_1 = 1,\ x_2 = 1,\ x_3 = 0.$

$u(7) = 4 + 6 = 10 \le 12$

$\therefore u(7) = 10 + 15 = -25$

$\hat{c}(7) = -25 + (-2 \times \frac{\cancel{8}^{2}}{\cancel{4}})$

$= -25 + (-4) = -29.$

Next E-node is node 7 and nodes 8 & 9 are generated.

At-Node 8 :-

$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1.$

$4 + 6 + 4 = 14 > 12$

Therefor Node 8 is infeasible solution.

At-Node 9 :-

$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 0.$

$u(9) = 4 + 6 + 2 = 12 \le 12$ $\quad$ x_5 can be fully accomodated.

$\therefore u(9) = 10 + 15 + 4 = 29$

$\hat{c}(9) = -29 + (0 \times 0) = -29.$

Next E-node is node 9., Nodes 10 & 11 are generated.

At-Node 10:-

$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 1.$

$u(10) = 4+6+2 = 12 \leq 12$

$\therefore u(10) = 10+15+4 = -29.$

$\hat{c}(10) = -29 + (-0 \times 0) = -29$

At-Node 11:-

$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0$

$u(11) = 4+6 = 10 \leq 12$

$\therefore u(11) = 10+15 = -25$

$\hat{c}(11) = -25 + (-0 \times 0) = -25$

All the objects are considered.

Now compare $\hat{c}$ values of Nodes 10 & 11, discard the max value node. hence Node 11 is discarded.

$\therefore$ The solution vector is

$(x_1\, x_2\, x_3\, x_4\, x_5) = (1, 1, 0, 0, 1)$ and

profit earned is $\underline{-29}$

Answer node is 10 becoz

$$\sum_{1 \leq i \leq n} w_i x_i = 12 \leq 12$$

Exercise:- $n=5$, $m=15$

$$(P_1, P_2, P_3, P_4, P_5) = (w_1, w_2, w_3, w_4, w_5) = (4, 4, 5, 8, 9)$$

Solve using LCBB.