0/1 Knapsack problem using Dynamic programming.

→ Consider the knapsack instance $n=3$, $(w_1, w_2, w_3) = (2, 3, 4)$ and $(P_1, P_2, P_3) = (1, 2, 5)$

$m = 6$.

Let $f_n(m)$ represents the profit obtainable with a knapsack of capacity 'm' and 'n' objects.

To get optimal solution:-

Set $x_n = 0$ if $(P_1, w_1) \in S^{n-1}$

If $(P_1, w_1) \notin S^{n-1}$, then $(P_1 - P_n, w_1 - w_n) \in S^{n-1}$

we can set $x_n = 1$.

If the pair is present in S^n & S^{n-1} then set $x_n = 0$ else $x_n = 1$.

Solution:-

$S^0 = \{(\overset{P}{0}, \overset{w}{0})\}$ - Initially no objects in the knapsack.

$S_1^0 = \{(\overset{P_1}{1}, \overset{w_1}{2})\}$

$S^1 = \text{Merge } S^0 \text{ \& } S_1^0 = \{(0,0), (1,2)\}$

$\therefore S^1 = \{(\underset{+2}{0}, \underset{+3}{0}), (\underset{+2}{1}, \underset{+3}{2})\}$

Add (P_2, w_2) to S^1 then u will get S_1^1

$S_1^1 = \{(2,3), (3,5)\}$

$S^2 = S^1 + S_1^1$

$\therefore S^2 = \{(0,0), (1,2), (2,3), (3,5)\}$.

Add (P_3, W_3) to all the tuples of S^2 to get S^2_1
$(5, 4)$.

$S^2_1 = \{(5,4), (6,6), (7,7), (8,9)\}$.

$S^3 = S^2 + S^2_1$

$\therefore S^3 = \{(0,0), (1,2), (2,3), (3,5), (5,4),$
$(6,6), (7,7), (8,9)\}$.

Write the tuples in ~~their~~ the increasing order of their weights.

$S^3 = \{(0,0), (1,2), (2,3), (5,4), \cancel{(3,5)}, (6,6),$
$(7,7), (8,9)\}$.

Tuples $(3,5)$ is discarded, since as weight increases, profit should also increase.

$(5,4), (3,5)$

weight is increased from 4 to 5 but profit did not it is decreased.

Therefore $(3,5)$ should be discarded.

i.e we can get profit 5 by putting only 4 objects where as in case of $(3,5)$ we are getting only profit as 3 by putting 5 objects.

$(3,5)\ (5,4)$
$p_i\ w_i\ \ p_j\ w_j$

if $(p_i < p_j)$ & $(w_i > w_j)$
Then (p_i, w_i) is eliminated
This is called "Purging Rule"

$\therefore S^3 = \{(0,0), (1,2), (2,3), (5,4), (6,6), (7,7), (8,9)\}$.

determine the value of $f_n(m)$ in S^n.
$n=3$, $m=6$ so $f_3(6)$ in $S^3 = (6,6) \in S^3$.

and $\underset{6,\ 6}{(p_1, w_1)} \notin S^2$ so set $\underline{\underline{x_3 = 1}}$.

the pair $(6,6)$ came from the pair $(1,2)$
since $(p_1 - \underset{p_3}{p_n}, w_1 - \underset{w_3}{w_n})$
$= (6-5, 6-4) = (1,2)$.

$x_2 =$

$(1,2) \in S^2$ and $(1,2) \in S^1$
$\therefore \underline{\underline{x_2 = 0}}$.

$x_1 =$ $(1,2) \in S^1$ and $(1,2) \notin S^0$

$\therefore x_1 = 1$
Hence optimal solution is $\underset{x_1\ x_2\ x_3}{(1,0,1)}$.

Max profit 6 since $(6,6)$

Eg 2:-

Generate the sets S^i, $0 \le i \le 4$ when $(w_1, w_2, w_3, w_4) = (10, 15, 6, 9)$ and $(p_1, p_2, p_3, p_4) = (2, 5, 8, 1)$, $m = 24$

$$S^0 = \{\overset{p}{(0}, \overset{w}{0)}\}$$

$$S^0_1 = (p_1, w_1) = \{(2, 10)\}$$

$$S^1 = S^0 + S^0_1$$

$$\therefore S^1 = \{(0,0), (2,10)\}$$

add $(p_2, w_2) = (5, 15)$ to S^1

$$S^1_1 = \{(5, 15), (7, 25)\}$$

$$S^2 = S^1 + S^1_1$$

$$\therefore S^2 = \{(0,0), (2,10), (5,15), (7,25)\}$$

add $(p_3, w_3) = (8, 6)$ to S^2

$$S^2_1 = \{(8,6), (10,16), (13,21), (15,31)\}$$

$$S^3 = S^2 + S^2_1$$

$$S^3 = \{(0,0), (2,10), (5,15), (7,25), (8,6), (10,16), (13,21), (15,31)\}$$

add $(p_4, w_4) = (1, 9)$ to S^3

$$S^3_1 = \{(1,9), (3,19), (6,24), (8,34), (9,15), (11,25), (14,30), (16,40)\}$$

$$S^4 = S^3 + S^3_1$$

$S^4 = \{(0,0), (2,10), (5,15), (7,25), (8,6), (10,16),$
$(13,21), (15,31), (1,9), (3,19), (6,24), (8,34),$
$(9,15), (11,25), (14,30), (16,40)\}$

Write the tuples in the increasing order of their ~~tuples~~ weights.

$S^4 = \{(0,0), (8,6), \cancel{(1,9)}, \cancel{(2,10)}, \cancel{(5,15)}, (9,15),$ $\cancel{(11,25)}$
$(10,16), \cancel{(3,19)}, (13,21), \cancel{(6,24)}, \cancel{(7,25)}, (14,30),$
$(15,31), \cancel{(8,34)}, (16,40)\}.$

$\therefore S^4 = \{(0,0), (8,6), (9,15), (10,16), (13,21),$
$(14,30), (15,31), (16,40)\}.$

determine the value of $f_n(m)$ in S^n

$n = 4, \quad m = 24$

$P_i \; W_i$

So $f_4(24)$ in $S^4 = (6,24) \in S^4$.

$(6,24) \notin S^3$.

$\therefore x_4 = 1$.

$(6,24)$ came from the pair $(5,15)$

since $(P_i - P_n, W_i - W_n)$

$(6-1, 24-9) = (5,15)$

$(5,15) \in S^3$, $(5,15) \in S^2$

$\therefore x_3 = 0$

~~$(5,15) \in S^2$, $(5,15) \notin S^1$~~

~~$\therefore x_2 = 1$.~~

~~$(5,15) \notin S^1$~~

~~$\therefore x_1 = 1$.~~

~~$\therefore$ The optimal solution is $(1\ 1\ 0\ 1)$~~ ~~$x_1\ x_2\ x_3\ x_4$~~

here $(P1, W1) = (13,21)$

x_4:

$(13,21) \in S^4$ & $\in S^3$

$\therefore x_4 = 0$.

Since $x_4 = 0$, we need not consider $(P1-Pn, W1-Wn)$ for x_3.

x_3:

$(13,21) \in S^3$ & $\notin S^2$

$\therefore x_3 = 1$.

Since $x_3 = 1$ we have to conside $(P1-Pn, W1-Wn)$

x_2:

$(13-P_3, 21-W_3) = (13-8, 21-6) = (5,15)$

$(5,15) \in S^2$ & $\notin S^1$

$\therefore x_2 = 1$.

Since $x_2 = 1$ we ~~need not~~ have to consider $(P1-Pn, W1-Wn)$

x_1:

$\therefore (P1-P2, W1-W2) = (\cancel{13}\,5-5, \cancel{21}\,15-15)$

$= \cancel{(8,6)}\ (0,0)$

$(0,0) \in S^1$ & $\in S^0$

$\therefore x_1 = 0$

$\therefore$ optimal set $x_i = (0110)$