Travelling Sales Person Problem (Using LCBB)

Time complexity is $O(n^2 2^n)$ in case of dynamic programming.
In Branch & Bound, the time complexity is same $O(n^2 2^n)$ in worst case, but using good bounding fns, will enable these Branch & Bound algs to solve some problem instances in much less time than required by the dynamic programming.

- let $G = (V, E)$ be a directed graph let c_{ij} be the cost of edge $\langle i, j \rangle$ $c_{ij} = \infty$ if $\langle i, j \rangle \notin E$ & let $|V| = n$. Every tour starts & ends at vertex 1.

- To use LCBB to search the travelling salesperson state space tree, we need to define a cost fn $c(\cdot)$ & two other fns $\hat{c}(\cdot)$ & $u(\cdot)$ such that $\hat{c}(r) \leq c(r) \leq u(r)$ for all nodes r. The cost $c(\cdot)$ is such that the solution node with least $c(\cdot)$ corresponds to a shortest tour in G

- * A better $\hat{c}(\cdot)$ can be obtained by using the reduced cost matrix corresponding to G.
A simple $\hat{c}(\cdot)$ such that $\hat{c}(A) \leq c(A)$ $\forall A$ is obtained by defining $\hat{c}(A)$ to be the length of the path defined at node A

Procedure for solving TSP Problem :-

Step 1 :-

Find out the reduced cost matrix from a given cost matrix (A matrix is to be reduced iff every row & column is reduced)
This can be obtained as follows.

Find out (i) row-reduction
(ii) column-reduction.

(i) Row-Reduction :- A row (column) is said to be reduced iff it contains atleast one zero and all remaining entries are non-negative.

Take the min element from 1st row, subtract that element from 1st row (including that element), next take min element from 2nd row, subtract that element from 2nd row (including that element). Similarly apply same procedure for all rows. After applying row-reduction, we will get one resultant matrix. For this apply column reduction.

(ii) Column Reduction :-

Take min element from 1st column, subtract that element from 1st column. Similarly apply same process for remaining columns.

Now we should find row-wise reduction sum & column wise reduction sum

Row wise reduction sum:-

The sum of the elements which subtracted from rows.

Column wise reduction sum:-

The sum of the elements which subtracted from columns.

Cumulative reduction = row wise reduction sum + column wise reduction sum.

- Subtracting a constant t from every entry in one column or row of the cost matrix reduces the length of every tour by exactly t. A min cost tour remains min-cost tour following this subtraction operation. If t is chosen to be the min entry in row i (col j), then subtracting it from all entries in row i (col j) introduces a zero into row i (col j). Repeating this as often as needed, the cost matrix can be reduced. ~~The total amount~~

- The total amount (cumulative reduction) subtracted from the cols and rows is a lower bound on the length of the min-cost tour, and can be used as $\hat{c}$ value for the root of the state space tree.

* Hence all tours in the original graph have a length atleast value of cumulative reduction.

Step 2 :- For root node, we take cumulative reduction as lower bound & ∞ as upper bound.

- We can associate a reduced cost matrix with every node in the TSP state space tree. Let A be the reduced cost matrix for node R. Let S be the child of R such that the tree edge (R, S) corresponds to including edge $\langle i, j \rangle$ in the tour. If S is not a leaf, then the reduced cost matrix for S may be obtained as follows.

(i) ~~change~~ If path $\langle i, j \rangle$ is considered, change all entries in row i, col j of A to ∞. This prevents the use of any more edges leaving vertex i or entering vertex j.

(ii) Set $A(j,1)$ to ∞. This prevents the use of edge $\langle j, 1 \rangle$.

(iii) Apply row reduction & col reduction except for rows & cols containing ∞. Also find cumulative reduction sum (r).

(iv) Let the resulting matrix be B. Calculate

$$\hat{c}(S) = \hat{c}(R) + A(i,j) + r$$

$\hat{c}(S)$ is lower bound for node S.

$\hat{c}(R)$ is lower bound value of S's parent.

$A(i,j)$ is entry in reduced cost matrix A at node R.

r = cumulative reduction sum.

For upper bound fn, u we can use $U(R) = \infty$ for all nodes R.

$\hat{c}(S)$ is called ranking fn. repeat step 2 until all nodes are visited.

Example :-

Given Cost matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ \\ \end{array}\begin{array}{c}\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix} \\ \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array}\end{array}$$

Step1 :-

(i) Perform row reduction $R_1 - 10$, $R_2 - 2$, $R_3 - 2$, $R_4 - 3$, $R_5 - 4$.

Then the resultant matrix is

$$\begin{array}{c}\begin{bmatrix} \infty & 10 & 20 & 0 & 1 \\ 13 & \infty & 14 & 2 & 0 \\ 1 & 3 & \infty & 0 & 2 \\ 16 & 3 & 15 & \infty & 0 \\ 12 & 0 & 3 & 12 & \infty \end{bmatrix} \\ \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array}\end{array}$$

(ii) Perform col reduction $C_1 - 1$, $C_3 - 3$.

$$\begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

Row wise reduction sum $= 10+2+2+3+4 = 21$

Col wise reduction sum $= 1+3 = 4$

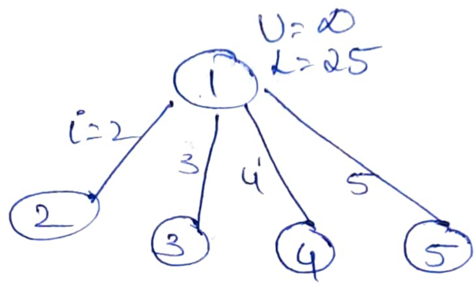
Cumulative reduction sum $= 21+4 = 25$

Hence Reduced cost matrix indicated by:

$$A = \begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

→ Reduced cost matrix at node 1.

Starting vertex node is 1, from node 1 he can visit next; 2, 3, 4 or 5 vertices. This can be shown as



The variable 'i' indicates next node that sales person will visit.

For starting node, i.e node 1 has lower bound as cumulative sum i.e 25.
Upper bound is ∞.

i=2 means, next he visits node 2
i=3 means, next he visits node 3. & so on

Step 2:- Apply step 2, for reduced cost matrix A.

Consider path $(\underset{i}{1}, \underset{j}{2})$

(i) change all entries of 1st row & 2nd col of A to ∞. Then matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{bmatrix}$$

Set A(2,1) to ∞, resultant matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ \\ \end{array}\begin{array}{c}\left[\begin{array}{ccccc} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{array}\right] \\ \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array}\end{array} \rightarrow \text{reduced cost matrix at node 2}$$

(iii) Apply row & col reduction, except for rows & cols containing ∞.

Row reduction: R_2-0, R_3-0, R_4-0, R_5-0.

row reduction sum $= 0+0+0+0 = 0$.

Col reduction:- C_1-0, C_3-0, C_4-0, C_5-0.

Col reduction sum $= 0+0+0+0 = 0$.

Cumulative reduction $(\gamma) = 0+0 = 0$.

(iv) $\hat{c}(S) = \hat{c}(R) + A(1,2) + \gamma$

$\hat{c}(S) = 25 + 10 + 0 = 35$

Consider the path $(\underset{i}{1},\underset{j}{3})$:-

change all entries of 1^{st} row, 3^{rd} col of A to ∞. set (3,1) to ∞. Then resultant matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 15 & 3 & \infty & \infty & 0 \\ 11 & 0 & \infty & 12 & \infty \end{bmatrix}$$

Apply row wise reduction i.e. $R_2 - 0$, $R_3 - 0$,
$R_4 - 0$, $R_5 - 0$

$\therefore$ row reduction sum is $= 0$.

Apply col wise reduction $C_1 - 11$, $C_2 - 0$,
$C_4 - 0$, $C_5 - 0$. resultant matrix

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 4 & 3 & \infty & \infty & 0 \\ 0 & 0 & \infty & 12 & \infty \end{bmatrix}$$

$\rightarrow$ reduced cost matrix at node 3.

Cumulative reduction $(r) = 0 + 11 = 11$.

$$\hat{c}(S) = \hat{c}(R) + A(1,3) + r$$
$$= 25 + 17 + 11$$
$$= \underline{\underline{53}}$$

Consider path $(1,4)$:-

change all entries of 1^{st} row, 4^{th} col of A to ∞. Set $A(4,1)$ to ∞. Then the resultant matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$

Apply row wise reduction, $R_2 - 0$, $R_3 - 0$, $R_4 - 0$, $R_5 - 0$.

$\therefore$ row reduction sum $= 0$.

Apply col wise reduction $C_1 - 0$, $C_2 - 0$, $C_3 - 0$, $C_5 - 0$.

$\therefore$ col reduction sum $= 0$.

Cumulative reduction sum = 0+0 = 0.

∴ Reduced matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$ — Reduced cost matrix at node 4.

$$\hat{c}(S) = \hat{c}(R) + A(1,4) + r$$
$$= 25 + 0 + 0 = 25$$

Consider the path (1,5) :—

Change all entries of 1st row, 5th col of A to ∞. Set $A(5,1)$ to ∞.

The resultant matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & 2 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 15 & 3 & 12 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix}$$

Apply row wise reduction : $R_2 - 2$, $R_3 - 0$, $R_4 - 3$, $R_5 - 0$.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix}$$

row reduction sum = 2+0+3+0 = 5

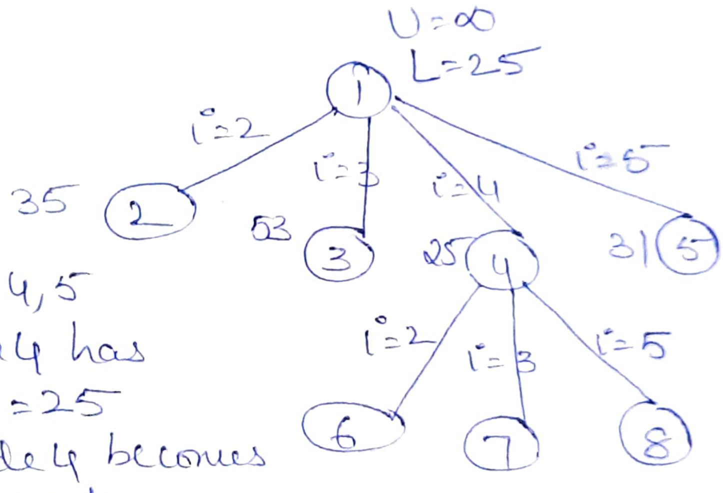
Apply col reduction : $c_1 - 0, c_2 - 0, c_3 -$

Col reduction sum = 0+0+0+0 = 0

Cumulative reduction sum = 5+0 = 5

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix}$$ - Reduced matrix at node 5

$$\hat{c}(S) = \hat{c}(R) + A(1,5) + r$$
$$= 25 + 1 + 5$$
$$= \underline{\underline{31}}$$



Among 2,3,4,5 nodes, node 4 has min $\hat{c}(S) = 25$

Hence, node 4 becomes next E-node.

So, we get nodes 6,7,8 live nodes as vertices 1,4 are already considered, the options for node 4 are i=2, i=3, i=5. Now compute the cost of the nodes 6,7,8. whichever gets min cost that will become next E-node.

Now cost matrix A is changed ~~as~~ to reduced cost matrix at node 4.

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$

– Reduced cost matrix at node 4. as we as are exploring node 4.

Consider the path (4,2):–

Change all entries of 4th row, 2nd col of A to ∞. Set $A(2, \cancel{4}\,1)$ to ∞.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

Apply row reduction

row wise reduction sum = 0.

Apply col wise reduction

col wise reduction sum = 0.

$r = 0 + 0 = \underline{\underline{0}}.$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

– Reduced cost matrix at node 6.

$$\hat{c}(S) = \hat{c}(R) + A(4,2) + r$$
$$= 25 + 3 + 0 = \underline{\underline{28}}$$

Consider the path (4,3) :-

change all entries of 4^{th} row, 3^{rd} col A to ∞, Set A(3,1) to ∞. Then the resultant matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & \infty & 0 \\ \infty & 3 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & 0 & \infty & \infty & \infty \end{bmatrix}$$

Apply row wise reduction: $R_3 - 2$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & 0 & \infty & \infty & \infty \end{bmatrix}$$

Apply col wise reduction: $C_1 - 11$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & \infty & \infty & \infty \end{bmatrix}$$ → Reduced cost matrix at node 7

$r = 2 + 11 = \underline{13}$

$\hat{c}(S) = \hat{c}(R) + A(4,3) + r$

$= 25 + 12 + 13$

$= \underline{50}$

Consider the path (4,5):-

Change all entries of 4th row, 5th col of A to ∞. Set A(5,1) to ∞. The resultant matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix}$$

Apply row wise reduction: $R_2 - 11$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix}$$

Apply col wise reduction:

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix}$$ → Reduced cost matrix at node 8

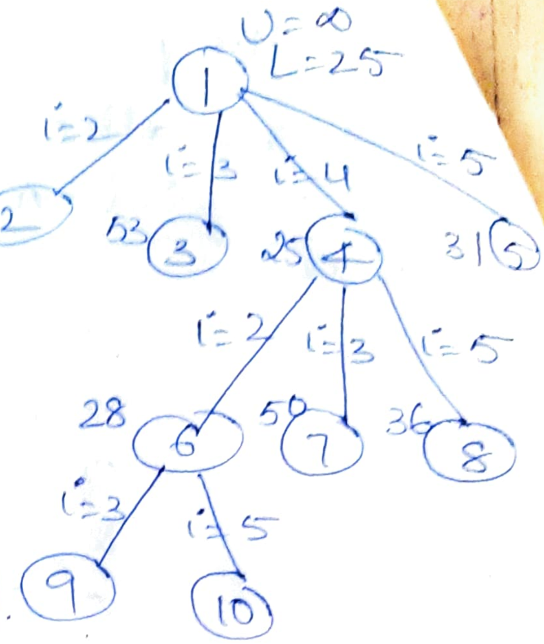
$r = 11 + 0 = \underline{\underline{11}}$

$\hat{c}(S) = \hat{c}(R) + A(4,5) + r$

$= 25 + 0 + 11$

$= \underline{\underline{36}}$

At this point node 6 has got min cost, hence it becomes next E-node & nodes 9 & 10 are generated with edge value i=3 & i=5 since the vertices i=1, i=4, i=2 are already considered.



Now reduced cost matrix A is changed to reduced cost matrix at node 6.

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

<u>Consider the path (2,3)</u> :-

change all entries of 2nd row, 3rd col of A to ∞. Set A(3,1) to ∞.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & \infty & \infty & \infty \end{bmatrix}$$

Apply row wise reduction:- $R_3 - 2$, $R_5 - 11$

rowwise reduction sum = 2+11 = 13.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \end{bmatrix}$$

Apply col wise reduction, ~~Col to~~

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \end{bmatrix}$$ — Reduced cost matrix at node 9.

Col wise reduction sum = 0.

$r = 13 + 0 = \underline{\underline{13}}$

$\hat{c}(S) = \hat{c}(R) + A(2,3) + r$

$= 28 + 11 + 13$

$= \underline{\underline{52}}$

Consider the path (2, 5) :-

Change all entries of 2nd row, 5th col of A to ∞. Set A(5, 1) to ∞. Then the resultant matrix is.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix}$$

Row wise reduction sum = 0.

Col wise reduction sum = 0.

$r = 0 + 0 = \underline{\underline{0}}$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix}$$ – Reduced cost matrix at node

$$\hat{C}(S) = \hat{C}(R) + A(2,5) + r$$
$$= 28 + 0 + 0$$
$$= \underline{\underline{28}}$$

Next Enode to be explored is node 10. Since it has least cost

Now reduced cost matrix A is change to reduced cost matrix at node 10.

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix}$$

Consider the path (5,3) :-

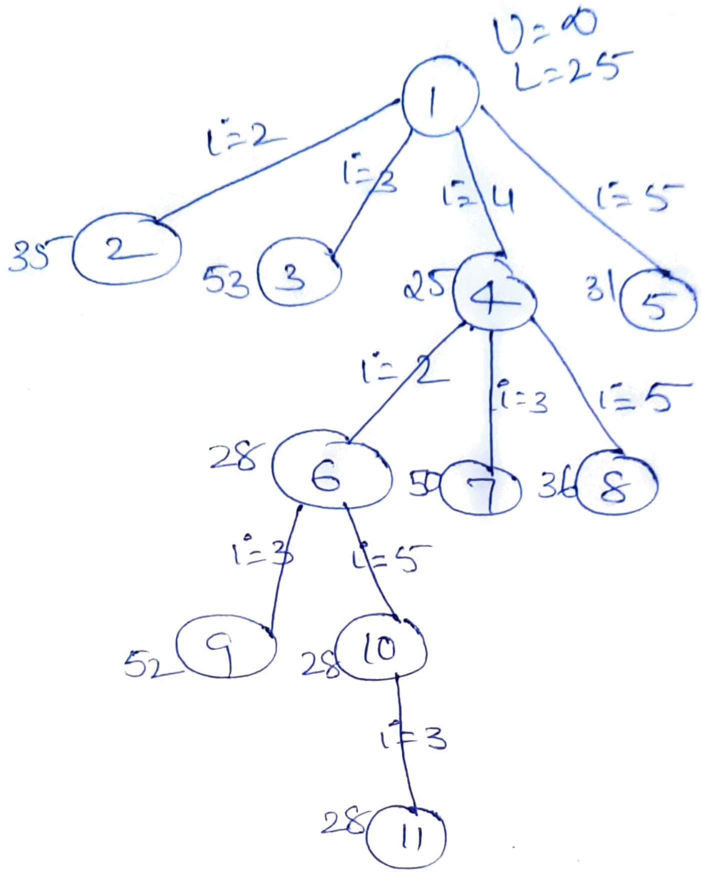
Change all entries of 5^{th} row, 3^{rd} col of A to ∞. Set A(3,1) to ∞.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Apply row & col wise reduction

$$r = 0 + 0 = 0.$$

$$\hat{C}(S) = \hat{C}(R) + A(5,3) + r$$
$$= 28 + 0 + 0 = \underline{28}$$



Hence the path for travelling salesperson problem is the path from root to answer node i.e $1 \to 4 \to 2 \to 5 \to 3 - 1$

From this path we get min-cost as : $10 + 6 + 2 + 7 + 3 = \underline{28}$