MATH 366 - PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 3 SOLUTIONS

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1. Solve the following first-order equations:

(a) $\begin{cases} u_t + u_x - u = 0 \\ u(x,0) = \cos(x) \end{cases}$

Step 1: Introduce characteristic coordinates.
This is a linear homogeneous equation with constant coefficients. We use the method of characteristic coordinates. The equation is of the form $u_t + cu_x + \lambda u = f(x,t)$, where $c = 1$, $\lambda = -1$, and $f(x,t) = 0$. We introduce new variables s and τ:

$$s = x - t$$
$$\tau = t$$

Step 2: Transform the PDE into an ODE.
Using the chain rule, we find the derivatives of u with respect to t and x:

$$u_t = \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = u_s(-1) + u_\tau(1) = -u_s + u_\tau$$
$$u_x = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial x} = u_s(1) + u_\tau(0) = u_s$$

Substitute these into the original PDE:

$$(-u_s + u_\tau) + (u_s) - u = 0$$
$$u_\tau - u = 0$$

Step 3: Solve the ODE.
This is a first-order linear ODE with respect to τ:

$$\frac{du}{d\tau} = u \implies \frac{du}{u} = d\tau$$
$$\ln|u| = \tau + g(s)$$

where $g(s)$ is an arbitrary function of s. The solution is:

$$u(s,\tau) = e^{\tau+g(s)} = G(s)e^{\tau}$$

where $G(s) = e^{g(s)}$ is another arbitrary function.
Step 4: Return to original variables and apply the initial condition.
Substitute back $s = x - t$ and $\tau = t$:

$$u(x,t) = G(x-t)e^{t}$$

Apply the initial condition $u(x,0) = \cos(x)$:

$$u(x,0) = G(x-0)e^{0} = G(x)$$

So, we have $G(x) = \cos(x)$.
Step 5: Write the final solution.
Replacing $G(x-t)$ with $\cos(x-t)$, the final solution is:

$$\boxed{u(x,t) = \cos(x-t)e^{t}}$$

(b) $\begin{cases} 2u_x + u_t + 3u = 0 \\ u(x,0) = \frac{1}{1+x^2} \end{cases}$

Step 1: Determine the characteristic curves.
The characteristic curves for $2u_x + u_t + 3u = 0$ satisfy the ODE:

$$\frac{dt}{dx} = \frac{1}{2}$$

Integrating gives the characteristic lines: $t = \frac{1}{2}x + k$, which can be written as $x - 2t =$ constant.
Step 2: Solve the PDE along the characteristics.
We parameterize the characteristic curves by a variable s. The characteristic equations are:

$$\frac{dx}{ds} = 2, \quad \frac{dt}{ds} = 1$$

Along these curves, the derivative of u is:

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = 2u_x + u_t$$

Substituting this into the PDE gives an ODE in s:

$$\frac{du}{ds} + 3u = 0$$

Step 3: Solve the ODE.
The solution to this ODE is $u = Ce^{-3s}$. The constant C is constant along a characteristic, so it can be an arbitrary function of the characteristic variable, $k = x - 2t$. Let's call it

$f(x-2t)$. Since $\frac{dt}{ds} = 1$, we can choose the parameterization such that $s = t$. The general solution is therefore:

$$u(x,t) = f(x-2t)e^{-3t}$$

Step 4: Apply the initial condition.
We are given $u(x,0) = \frac{1}{1+x^2}$.

$$u(x,0) = f(x-2(0))e^{-3(0)} = f(x)$$

Thus, $f(x) = \frac{1}{1+x^2}$.
Step 5: Write the final solution.
Substitute the function f into the general solution:

$$\boxed{u(x,t) = \frac{1}{1+(x-2t)^2}e^{-3t}}$$

(c) $\begin{cases} u_x + 3u_y - 5u = 2x^2 \\ u(x,0) = 2x^2 + 10x + 2 \end{cases}$

Step 1: Set up the characteristic ODEs.
The characteristic curves are defined by $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 3$. Along these curves, the PDE becomes an ODE for u: $\frac{du}{dt} = 5u + 2x^2$.
Step 2: Solve for the characteristic curves.
Let a characteristic curve start at $(s,0)$ at $t = 0$.

$$x(t) = t + s$$
$$y(t) = 3t$$

Step 3: Solve the ODE for u.
The ODE for u along a characteristic is $\frac{du}{dt} - 5u = 2x(t)^2 = 2(t+s)^2$. The integrating factor is $I(t) = e^{-5t}$.

$$\frac{d}{dt}(ue^{-5t}) = 2(t+s)^2e^{-5t}$$

Integrating both sides (using integration by parts for the right side):

$$\begin{aligned} u(t,s)e^{-5t} &= 2\int (t^2 + 2st + s^2)e^{-5t}dt + F(s) \\ &= -2e^{-5t}\left[\frac{t^2}{5} + \frac{2st}{5} + \frac{s^2}{5} + \frac{2t}{25} + \frac{2s}{25} + \frac{2}{125}\right] + F(s) \end{aligned}$$

So, the general solution along characteristics is:

$$u(t,s) = -2\left[\frac{t^2}{5} + \frac{2st}{5} + \frac{s^2}{5} + \frac{2t}{25} + \frac{2s}{25} + \frac{2}{125}\right] + F(s)e^{5t}$$

Step 4: Apply the initial condition.
The initial condition is at $y = 0$, which corresponds to $t = 0$. At $t = 0$, we have $x = s$.

$$u(s,0) = -2\left[\frac{s^2}{5} + \frac{2s}{25} + \frac{2}{125}\right] + F(s)$$

We are given $u(x,0) = 2x^2 + 10x + 2$. So, $u(s,0) = 2s^2 + 10s + 2$.

$$2s^2 + 10s + 2 = -\frac{2s^2}{5} - \frac{4s}{25} - \frac{4}{125} + F(s)$$

$$F(s) = \left(2 + \frac{2}{5}\right)s^2 + \left(10 + \frac{4}{25}\right)s + \left(2 + \frac{4}{125}\right) = \frac{12}{5}s^2 + \frac{254}{25}s + \frac{254}{125}$$

Step 5: Write the final solution in terms of x and y.

From Step 2, $t = y/3$ and $s = x - t = x - y/3$. The term in brackets in the general solution simplifies to $\frac{1}{5}(t+s)^2 + \frac{2}{25}(t+s) + \frac{2}{125} = \frac{x^2}{5} + \frac{2x}{25} + \frac{2}{125}$.

$$u(x,y) = -2\left(\frac{x^2}{5} + \frac{2x}{25} + \frac{2}{125}\right) + F(x - y/3)e^{5y/3}$$

$$\boxed{u(x,y) = -\frac{2x^2}{5} - \frac{4x}{25} - \frac{4}{125} + \left(\frac{12}{5}\left(x - \frac{y}{3}\right)^2 + \frac{254}{25}\left(x - \frac{y}{3}\right) + \frac{254}{125}\right)e^{5y/3}}$$

2. Derive the general solution for the following PDE:

(a) $u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$

Step 1: Introduce new coordinate system.
Let the new coordinates be defined by the characteristic directions:

$$x' = x + 2y$$
$$y' = 2x - y$$

Step 2: Transform the PDE using the chain rule.
The partial derivatives in the new system are:

$$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'} + 2u_{y'}$$
$$u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 2u_{x'} - u_{y'}$$

Substitute these into the PDE:

$$(u_{x'} + 2u_{y'}) + 2(2u_{x'} - u_{y'}) + (2x - y)u = 2x^2 + 3xy - 2y^2$$
$$5u_{x'} + y'u = 2x^2 + 3xy - 2y^2$$

Step 3: Express the right-hand side in new coordinates.
We note that $x'y' = (x + 2y)(2x - y) = 2x^2 + 3xy - 2y^2$. The transformed PDE is:

$$5u_{x'} + y'u = x'y'$$

Step 4: Solve the transformed ODE.
This is a linear first-order ODE in x', treating y' as a parameter. The homogeneous solution for $5u_{x'} + y'u = 0$ is $u_H = F(y')e^{-\frac{1}{5}x'y'}$. For a particular solution, we try $u_P = A(y')x' + B(y')$. $5(A) + y'(Ax' + B) = x'y' \implies (5A + By') + Ay'x' = x'y'$. Comparing coefficients of x', we get $Ay' = y' \implies A = 1$. Comparing constant terms, $5A + By' = 0 \implies 5 + By' = 0 \implies B = -5/y'$. So, $u_P(x', y') = x' - \frac{5}{y'}$. The general solution is $u = u_H + u_P$:

$$u(x', y') = x' - \frac{5}{y'} + F(y')e^{-\frac{1}{5}x'y'}, \quad y' \neq 0$$

Step 5: Return to original variables.
Substitute back $x' = x + 2y$ and $y' = 2x - y$:

$$u(x, y) = (x + 2y) - \frac{5}{2x - y} + F(2x - y)e^{-\frac{1}{5}(x+2y)(2x-y)}$$

$$\boxed{u(x, y) = x + 2y - \frac{5}{2x - y} + F(2x - y)e^{-\frac{1}{5}(2x^2+3xy-2y^2)}, \quad y \neq 2x}$$

(b) $2u_x + 3u_y - 5u = 2x^2$

Step 1: Use the coordinate method based on characteristics.
The characteristic curves satisfy $\frac{dy}{dx} = \frac{3}{2}$, so $3x - 2y =$ constant. We choose a new coordinate system where one coordinate is constant along the characteristics. Let:

$$x' = 3x - 2y$$
$$y' = x$$

Step 2: Transform the PDE.
The partial derivatives are:

$$u_x = 3u_{x'} + u_{y'}$$
$$u_y = -2u_{x'}$$

Substitute into the PDE:

$$2(3u_{x'} + u_{y'}) + 3(-2u_{x'}) - 5u = 2(y')^2$$
$$6u_{x'} + 2u_{y'} - 6u_{x'} - 5u = 2(y')^2$$
$$2u_{y'} - 5u = 2(y')^2 \implies u_{y'} - \frac{5}{2}u = (y')^2$$

Step 3: Solve the transformed ODE.
This is a linear ODE in y'. The integrating factor is $I(y') = e^{\int -5/2dy'} = e^{-5y'/2}$.

$$\frac{d}{dy'}(ue^{-5y'/2}) = (y')^2e^{-5y'/2}$$

Integrate with respect to y' (using integration by parts):

$$\int (y')^2e^{-ay'}dy' = -e^{-ay'}\left(\frac{(y')^2}{a} + \frac{2y'}{a^2} + \frac{2}{a^3}\right)$$

With $a = 5/2$:

$$u(x', y')e^{-5y'/2} = -e^{-5y'/2}\left(\frac{2(y')^2}{5} + \frac{8y'}{25} + \frac{16}{125}\right) + F(x')$$
$$u(x', y') = -\left(\frac{2(y')^2}{5} + \frac{8y'}{25} + \frac{16}{125}\right) + F(x')e^{5y'/2}$$

where $F(x')$ is an arbitrary function.
Step 4: Return to original variables.
Substitute back $x' = 3x - 2y$ and $y' = x$:

$$\boxed{u(x, y) = -\left(\frac{2x^2}{5} + \frac{8x}{25} + \frac{16}{125}\right) + F(3x - 2y)e^{5x/2}}$$