MATH 366 - PARTIAL DIFFERENTIAL EQUATIONS

Assignment 5 Solutions

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1 Solve the wave equations

1.1

Problem: Solve the initial value problem for the wave equation on an infinite string:

$$\begin{cases} u_{tt} = 49u_{xx} \\ u(x,0) = 0 \\ u_t(x,0) = \sin(6x) \end{cases}$$

Solution: This is the wave equation $u_{tt} = c^2 u_{xx}$ with $c^2 = 49$, so $c = 7$. The problem is set on an infinite domain, so we use d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Here, the initial position is $f(x) = 0$ and the initial velocity is $g(x) = \sin(6x)$.

Step 1: Apply the initial conditions to d'Alembert's formula. Since $f(x) = 0$, the first part of the formula is zero.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(6s)\,ds$$

Step 2: Evaluate the integral. With $c = 7$, we have:

$$\begin{aligned} u(x,t) &= \frac{1}{2(7)}\int_{x-7t}^{x+7t} \sin(6s)\,ds \\ &= \frac{1}{14}\left[-\frac{1}{6}\cos(6s)\right]_{x-7t}^{x+7t} \\ &= -\frac{1}{84}\left[\cos(6(x+7t)) - \cos(6(x-7t))\right] \\ &= -\frac{1}{84}\left[\cos(6x+42t) - \cos(6x-42t)\right] \end{aligned}$$

Step 3: Simplify using the trigonometric identity. We use the identity $\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$. Let $A = 6x + 42t$ and $B = 6x - 42t$.

$$\frac{A+B}{2} = \frac{12x}{2} = 6x$$

$$\frac{A-B}{2} = \frac{84t}{2} = 42t$$

Substituting these into the solution:

$$\begin{aligned} u(x,t) &= -\frac{1}{84}\left[-2\sin(6x)\sin(42t)\right] \\ &= \frac{2}{84}\sin(6x)\sin(42t) \end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{42}\sin(6x)\sin(42t)}$$

1.2

Problem: Solve the initial-boundary value problem for the wave equation on a finite string:

$$\begin{cases} u_{tt} = 4u_{xx}, \quad 0 < x < 2, \quad t > 0 \\ u(0,t) = 0, \quad u(2,t) = 0 \\ u(x,0) = \sin(\pi x) \\ u_t(x,0) = 2\sin(2\pi x) \end{cases}$$

Solution: This is the wave equation on a finite domain with length $L = 2$ and wave speed $c = 2$. We use the method of separation of variables. The general solution has the form:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

Substituting $L = 2$ and $c = 2$:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(n\pi t) + B_n \sin(n\pi t) \right] \sin\left(\frac{n\pi x}{2}\right)$$

Step 1: Apply the initial position condition $u(x,0) = \sin(\pi x)$. At $t = 0$:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \sin(\pi x)$$

By comparing the terms in the series with the given function, we can see that this is a Fourier sine series where only one coefficient is non-zero. The term $\sin(\pi x)$ corresponds to the case where $\frac{n\pi}{2} = \pi$, which means $n = 2$. Therefore, $A_2 = 1$ and all other $A_n = 0$ for $n \neq 2$.

Step 2: Apply the initial velocity condition $u_t(x,0) = 2\sin(2\pi x)$. First, we find the derivative of the general solution with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-n\pi A_n \sin(n\pi t) + n\pi B_n \cos(n\pi t) \right] \sin\left(\frac{n\pi x}{2}\right)$$

At $t = 0$:

$$u_t(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin\left(\frac{n\pi x}{2}\right) = 2\sin(2\pi x)$$

By comparing terms, we see that the term $\sin(2\pi x)$ corresponds to the case where $\frac{n\pi}{2} = 2\pi$, which means $n = 4$. For $n = 4$, the coefficient must be 2:

$$4\pi B_4 = 2 \implies B_4 = \frac{2}{4\pi} = \frac{1}{2\pi}$$

All other $B_n = 0$ for $n \neq 4$.

Step 3: Combine the results to form the final solution. The solution is the sum of the non-zero terms we found: the $n = 2$ term from the cosine part and the $n = 4$ term from the sine part.

$$u(x,t) = A_2 \cos(2\pi t) \sin\left(\frac{2\pi x}{2}\right) + B_4 \sin(4\pi t) \sin\left(\frac{4\pi x}{2}\right)$$

$$\boxed{u(x,t) = \cos(2\pi t)\sin(\pi x) + \frac{1}{2\pi}\sin(4\pi t)\sin(2\pi x)}$$

2 Solve the heat equations

2.1

Problem: Solve the initial-boundary value problem for the heat equation:

$$\begin{cases} u_t = 3u_{xx}, \quad t > 0, \quad x \in [0, L] \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = 20 \end{cases}$$

Solution: This is the heat equation with thermal diffusivity $k = 3$ on a finite domain of length L. The general solution from separation of variables is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

Substituting $k = 3$:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-3(n\pi/L)^2 t}$$

Step 1: Apply the initial condition $u(x,0) = 20$. At $t = 0$:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 20$$

The coefficients B_n are the Fourier sine coefficients of the function $f(x) = 20$.

Step 2: Calculate the Fourier coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L 20\sin\left(\frac{n\pi x}{L}\right) dx = \frac{40}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$B_n = -\frac{40}{n\pi}\left[\cos(n\pi) - \cos(0)\right] = -\frac{40}{n\pi}((-1)^n - 1) = \frac{40(1-(-1)^n)}{n\pi}$$

This expression is non-zero only for odd values of n.

- If n is even, $B_n = 0$.
- If n is odd, $B_n = \frac{40(1-(-1))}{n\pi} = \frac{80}{n\pi}$.

Step 3: Write the final solution. We sum only over the odd values of n.

$$\boxed{u(x,t) = \sum_{n=1,\text{ odd}}^{\infty} \frac{80}{n\pi}\sin\left(\frac{n\pi x}{L}\right) e^{-3(n\pi/L)^2 t}}$$

2.2

Problem: Solve the initial-boundary value problem for the heat equation:

$$\begin{cases} u_t = 2u_{xx}, \quad t > 0, \quad x \in [0, L] \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = x^2 \end{cases}$$

Solution: This is the heat equation with $k = 2$. The general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-2(n\pi/L)^2 t}$$

Step 1: Apply the initial condition $u(x,0) = x^2$. At $t = 0$:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = x^2$$

The coefficients B_n are the Fourier sine coefficients of $f(x) = x^2$.

Step 2: Calculate the Fourier coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right) dx$$

Using integration by parts twice, the indefinite integral is:

$$\int x^2 \sin(ax)dx = \frac{2x}{a^2}\sin(ax) - \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\cos(ax) \quad \text{where } a = \frac{n\pi}{L}$$

Evaluating the definite integral:

$$\begin{aligned}
B_n &= \frac{2}{L}\left[\frac{2xL^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right) - \left(\frac{x^2 L}{n\pi} - \frac{2L^3}{n^3\pi^3}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L \\
&= \frac{2}{L}\left[\left(0 - \left(\frac{L^3}{n\pi} - \frac{2L^3}{n^3\pi^3}\right)\cos(n\pi)\right) - \left(0 - \left(-\frac{2L^3}{n^3\pi^3}\right)\cos(0)\right)\right] \\
&= \frac{2}{L}\left[-\frac{L^3}{n\pi}(-1)^n + \frac{2L^3}{n^3\pi^3}(-1)^n - \frac{2L^3}{n^3\pi^3}\right] \\
&= 2L^2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2((-1)^n - 1)}{n^3\pi^3}\right]
\end{aligned}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} 2L^2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2((-1)^n - 1)}{n^3\pi^3}\right]\sin\left(\frac{n\pi x}{L}\right) e^{-2(n\pi/L)^2 t}}$$

3 Solve the Laplace equation

Problem: Solve the boundary value problem for Laplace's equation on a rectangle:

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < 1 \\ u(0, y) = 0, \quad u(\pi, y) = 0 \\ u(x, 0) = 0, \quad u(x, 1) = \sin(4x) \end{cases}$$

Solution: This is Laplace's equation on a rectangle with $L = \pi$ and $H = 1$. Three boundary conditions are homogeneous. We use separation of variables, assuming a solution of the form $u(x, y) = h(x)\phi(y)$.

Step 1: Separate the variables. This leads to two ODEs with a separation constant $-\lambda$:

- $h''(x) + \lambda h(x) = 0$, with $h(0) = 0$ and $h(\pi) = 0$.
- $\phi''(y) - \lambda\phi(y) = 0$, with $\phi(0) = 0$.

Step 2: Solve the eigenvalue problem for $h(x)$. The boundary value problem for $h(x)$ has eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2, \quad h_n(x) = \sin(nx), \quad \text{for } n = 1, 2, 3, \ldots$$

Step 3: Solve the ODE for $\phi(y)$. For each $\lambda_n = n^2$, the ODE for $\phi(y)$ is:

$$\phi_n''(y) - n^2\phi_n(y) = 0$$

The general solution is $\phi_n(y) = C_n \cosh(ny) + D_n \sinh(ny)$. Applying the boundary condition $\phi(0) = 0$:

$$\phi(0) = C_n \cosh(0) + D_n \sinh(0) = C_n = 0$$

So the solution for $\phi(y)$ is of the form $\phi_n(y) = D_n \sinh(ny)$.

Step 4: Form the general solution. The general solution is the superposition of all product solutions:

$$u(x, y) = \sum_{n=1}^{\infty} D_n \sin(nx) \sinh(ny)$$

Step 5: Apply the non-homogeneous boundary condition. We use the final condition $u(x, 1) = \sin(4x)$:

$$u(x, 1) = \sum_{n=1}^{\infty} D_n \sin(nx) \sinh(n) = \sin(4x)$$

By comparing the series to the function, we see that this is a Fourier series where only the coefficient for $n = 4$ is non-zero.

$$D_4 \sinh(4) = 1 \implies D_4 = \frac{1}{\sinh(4)}$$

All other coefficients $D_n = 0$ for $n \neq 4$.

Step 6: Write the final solution. The solution consists of only the $n = 4$ term.

$$\boxed{u(x, y) = \frac{\sinh(4y)}{\sinh(4)} \sin(4x)}$$