MATH 366 - PARTIAL DIFFERENTIAL EQUATIONS

ASSIGNMENT 4 SOLUTIONS

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Question 1: General Solutions of the Wave Equation

(a) Find the general solution for the wave equation

$$u_{tt} = 64u_{xx}$$

Solution:

The equation is in the standard form for a one-dimensional wave equation, $u_{tt} = c^2 u_{xx}$. By comparison, we see that $c^2 = 64$, which gives a wave speed of $c = 8$.

The general solution is given by d'Alembert's formula:

$$u(x,t) = f(x + ct) + g(x - ct)$$

where f and g are arbitrary functions. Substituting the value of c, we get:

$$\boxed{u(x,t) = f(x + 8t) + g(x - 8t)}$$

(b) Find the general solution for the wave equation

$$4u_{tt} - 25u_{xx} = 0$$

Solution:

We first rearrange the equation to the standard form, $u_{tt} = c^2 u_{xx}$:

$$4u_{tt} = 25u_{xx} \implies u_{tt} = \frac{25}{4}u_{xx}$$

By comparison, we see that $c^2 = \frac{25}{4}$, which gives a wave speed of $c = \frac{5}{2}$.

The general solution is given by d'Alembert's formula:

$$u(x,t) = f(x + ct) + g(x - ct)$$

where f and g are arbitrary functions. Substituting the value of c, we get:

$$\boxed{u(x,t) = f\left(x + \frac{5}{2}t\right) + g\left(x - \frac{5}{2}t\right)}$$

(c) Find the general solution for the wave equation

$$u_{xx} = 16u_{tt}$$

Solution:

We first rearrange the equation to the standard form, $u_{tt} = c^2 u_{xx}$:

$$16u_{tt} = u_{xx} \implies u_{tt} = \frac{1}{16}u_{xx}$$

By comparison, we see that $c^2 = \frac{1}{16}$, which gives a wave speed of $c = \frac{1}{4}$.

The general solution is given by d'Alembert's formula:

$$u(x,t) = f(x + ct) + g(x - ct)$$

where f and g are arbitrary functions. Substituting the value of c, we get:

$$\boxed{u(x,t) = f\left(x + \frac{1}{4}t\right) + g\left(x - \frac{1}{4}t\right)}$$

Question 2: Initial Value Problems

(a) Solve the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^{-x^2} \\ u_t(x,0) = \cos^2(x) \end{cases}$$

Note: Corrected typos from the assignment sheet for clarity ($z^2 \to c^2$ and $e^{-2} \to e^{-x^2}$).

Solution:

We use d'Alembert's formula for an initial value problem:

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Here, the initial position is $\phi(x) = e^{-x^2}$ and the initial velocity is $\psi(x) = \cos^2(x)$.

The first part of the solution, from the initial position, is:

$$\frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right]$$

For the second part, we evaluate the integral of the initial velocity:

$$\begin{aligned} \int_{x-ct}^{x+ct} \cos^2(s)ds &= \int_{x-ct}^{x+ct} \frac{1}{2}(1+\cos(2s))ds \\ &= \frac{1}{2}\left[s + \frac{1}{2}\sin(2s)\right]_{x-ct}^{x+ct} \\ &= \frac{1}{2}\left[(x+ct-(x-ct)) + \frac{1}{2}(\sin(2(x+ct)) - \sin(2(x-ct)))\right] \\ &= \frac{1}{2}\left[2ct + \cos(2x)\sin(2ct)\right] \\ &= ct + \frac{1}{2}\cos(2x)\sin(2ct) \end{aligned}$$

Combining the parts according to d'Alembert's formula, the final solution is:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(e^{-(x+ct)^2} + e^{-(x-ct)^2}\right) + \frac{1}{2c}\left(ct + \frac{1}{2}\cos(2x)\sin(2ct)\right) \\ &= \boxed{\frac{1}{2}\left(e^{-(x+ct)^2} + e^{-(x-ct)^2}\right) + \frac{t}{2} + \frac{1}{4c}\cos(2x)\sin(2ct)} \end{aligned}$$

(b) Solve the initial value problem

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = \sin(5x) \\ u_t(x,0) = \frac{1}{5}\cos(x) \end{cases}$$

Solution:

We use d'Alembert's formula for an initial value problem with wave speed $c = 1$:

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds$$

Here, the initial position is $\phi(x) = \sin(5x)$ and the initial velocity is $\psi(x) = \frac{1}{5}\cos(x)$.

The first part of the solution, from the initial position, simplifies using a sum-to-product identity:

$$\begin{aligned} \frac{1}{2}[\sin(5(x+t)) + \sin(5(x-t))] &= \frac{1}{2}[2\sin(5x)\cos(5t)] \\ &= \sin(5x)\cos(5t) \end{aligned}$$

For the second part, we evaluate the integral of the initial velocity:

$$\begin{aligned}
\frac{1}{2}\int_{x-t}^{x+t} \frac{1}{5}\cos(s)ds &= \frac{1}{10}[\sin(s)]_{x-t}^{x+t} \\
&= \frac{1}{10}[\sin(x+t) - \sin(x-t)] \\
&= \frac{1}{10}[2\cos(x)\sin(t)] \\
&= \frac{1}{5}\cos(x)\sin(t)
\end{aligned}$$

Combining the two parts gives the final solution:

$$\boxed{u(x,t) = \sin(5x)\cos(5t) + \frac{1}{5}\cos(x)\sin(t)}$$

Question 3: Boundary Value Problems

(a) Solve the boundary value problem

$$y'' + 4y = 0, \quad y(0) = -2, \quad y\left(\frac{\pi}{4}\right) = 10$$

Solution:

The characteristic equation for the ordinary differential equation is $r^2 + 4 = 0$, which has complex roots $r = \pm 2i$. The general solution is therefore of the form:

$$y(x) = c_1\cos(2x) + c_2\sin(2x)$$

We now apply the two boundary conditions to find the constants c_1 and c_2.

Applying the first condition, $y(0) = -2$:

$$\begin{aligned}
y(0) = c_1\cos(0) + c_2\sin(0) &= -2 \\
c_1(1) + c_2(0) &= -2 \\
c_1 &= -2
\end{aligned}$$

Applying the second condition, $y(\frac{\pi}{4}) = 10$:

$$\begin{aligned}
y\left(\frac{\pi}{4}\right) = c_1\cos\left(2\cdot\frac{\pi}{4}\right) + c_2\sin\left(2\cdot\frac{\pi}{4}\right) &= 10 \\
c_1\cos\left(\frac{\pi}{2}\right) + c_2\sin\left(\frac{\pi}{2}\right) &= 10 \\
c_1(0) + c_2(1) &= 10 \\
c_2 &= 10
\end{aligned}$$

Substituting the determined constants back into the general solution gives the unique solution to the boundary value problem:

$$\boxed{y(x) = -2\cos(2x) + 10\sin(2x)}$$

(b) Solve the boundary value problem

$$y'' + 4y = 0, \quad y(0) = -2, \quad y(2\pi) = -2$$

Solution:

The ordinary differential equation is the same as in the previous problem, so the general solution is:

$$y(x) = c_1\cos(2x) + c_2\sin(2x)$$

We apply the boundary conditions to find the constants c_1 and c_2.

Applying the first condition, $y(0) = -2$:

$$\begin{aligned} y(0) = c_1 \cos(0) + c_2 \sin(0) &= -2 \\ c_1 &= -2 \end{aligned}$$

Applying the second condition, $y(2\pi) = -2$:

$$\begin{aligned} y(2\pi) = c_1 \cos(4\pi) + c_2 \sin(4\pi) &= -2 \\ c_1(1) + c_2(0) &= -2 \\ c_1 &= -2 \end{aligned}$$

Both conditions impose the same constraint, $c_1 = -2$, but neither condition places any restriction on the constant c_2. Therefore, c_2 is an arbitrary constant, and the boundary value problem has infinitely many solutions. The solution is the family of functions:

$$\boxed{y(x) = -2\cos(2x) + c_2 \sin(2x), \quad \text{for any constant } c_2}$$

(c) Solve the boundary value problem

$$y'' + 9y = \cos(x), \quad y'(0) = 5, \quad y\left(\frac{\pi}{2}\right) = -\frac{5}{3}$$

Solution:

This is a non-homogeneous boundary value problem. The general solution is the sum of the complementary solution (y_c) and a particular solution (y_p).

First, we solve the homogeneous equation $y'' + 9y = 0$. The characteristic equation $r^2 + 9 = 0$ gives roots $r = \pm 3i$. The complementary solution is:

$$y_c(x) = c_1 \cos(3x) + c_2 \sin(3x)$$

Next, using the method of undetermined coefficients for the particular solution, we guess $y_p(x) = A\cos(x) + B\sin(x)$. Substituting this into the ODE yields $A = 1/8$ and $B = 0$. Thus, the particular solution is:

$$y_p(x) = \frac{1}{8}\cos(x)$$

The general solution to the ODE is $y(x) = y_c(x) + y_p(x)$:

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{8}\cos(x)$$

We need its derivative to apply the boundary conditions:

$$y'(x) = -3c_1 \sin(3x) + 3c_2 \cos(3x) - \frac{1}{8}\sin(x)$$

Now we apply the boundary conditions. From $y'(0) = 5$:

$$\begin{aligned} y'(0) = -3c_1(0) + 3c_2(1) - \frac{1}{8}(0) &= 5 \\ 3c_2 &= 5 \implies c_2 = \frac{5}{3} \end{aligned}$$

From $y(\frac{\pi}{2}) = -\frac{5}{3}$:

$$\begin{aligned} y\left(\frac{\pi}{2}\right) = c_1(0) + c_2(-1) + \frac{1}{8}(0) &= -\frac{5}{3} \\ -c_2 &= -\frac{5}{3} \implies c_2 = \frac{5}{3} \end{aligned}$$

Both conditions determine that $c_2 = 5/3$, but they leave c_1 as an arbitrary constant. The BVP has infinitely many solutions:

$$\boxed{y(x) = c_1 \cos(3x) + \frac{5}{3}\sin(3x) + \frac{1}{8}\cos(x), \quad \text{for any constant } c_1}$$