2)

To Prove :- The direction f perpendicular to e for which $f^t cf$ is maximized, is the eigenvector of C with the second highest eigenvalue. (where e is eigenvector corresponding to highest eigenvalue)

Proof:-

We'll use Lagrange multipliers to prove the above statement by imposing the constraints $f^t f = 1$ and $f^t e = 0$.

$\therefore$ Consider the function:-

$$J(f) = f^t Cf - \lambda(f^t f - 1) - \alpha f^t e$$

$$\frac{\partial J}{\partial f} = \frac{\partial (f^t C f)}{\partial f} - \lambda \frac{\partial (f^t f)}{\partial f} - \alpha \frac{\partial (f^t e)}{\partial f}$$

In class we proved that:-

$$\frac{\partial (f^t c f)}{\partial f} = 2cf, \quad \frac{\partial (f^t f)}{\partial f} = 2f,$$

$$\frac{\partial (f^t e)}{\partial f} = e.$$

$$\therefore \frac{\partial J}{\partial f} = 2Cf - 2\lambda f - \alpha e.$$

For maximum Value, we have $\frac{\partial J}{\partial f} = 0.$

$$\boxed{\therefore 2Cf - 2\lambda f - \alpha e = 0} \quad - (i)$$

Multiply the above equation with e^t..

$$\therefore 2e^tCf - 2\lambda e^tf - \alpha e^te = 0.$$

$$\therefore 2e^tCf - 0 - \alpha = 0 \quad (\because e^tf = 0,\ e^te = 1)$$

$$\boxed{\therefore \alpha = 2e^tCf.} \quad - (1)$$

Since, **e** is the eigenvector Corresponding to highest eigenvalue λ_o, we have:-

$$Ce = \lambda_o e$$

$$\therefore (Ce)^t = \lambda_o e^t \Rightarrow e^tC^t = \lambda_o e^t$$

$\because$ C is Covariance matrix, it's Symmetric.

$$\therefore C = C^t. \boxed{\Rightarrow e^tC = \lambda_o e^t} \quad - (2)$$

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Substituting ② in ①, we get :-

$$\alpha = 2(\lambda_0 e^t) \cdot f = 2\lambda_0 (e^t f) = 0.$$

$$\boxed{\therefore \alpha = 0.} - ③$$

Substituting ③ in ① we get :-

$$2Cf - 2\lambda f = 0$$

$$\boxed{\Rightarrow Cf = \lambda f}.$$

$\therefore$ f is one of the eigen vectors of 'C' and 'λ' is Corresponding eigen Value.

we also have, $f^t C f = f^t (\lambda f) = \lambda$.

And, we need to maximise $f^t C f$. So, we need to maximise 'λ'.

Since, f is Perpendicular to e, λ cannot be equal to highest eigen Value. So, $\boxed{\lambda = \text{Second highest eigen value}}$

and $\boxed{f = \text{eigenvector Corresponding to Second highest eigen Value}}$

$\boxed{\therefore \text{Hence Proved}}$

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