3)

a) Given, matrix A of size $m\times n$ $(m \leq n)$ and

$P = A^t A$, $Q = AA^t$.

<u>To Prove</u>: $y^t P y \geq 0$; $z^t Q z \geq 0$ for vectors y, z with corresponding sizes so that the products $y^t P y$ and $z^t Q z$ are defined. Also prove that eigen values of P and Q are non-negative.

<u>Proof</u>:-

$$y^t P y = y^t . (A^t A) . y$$

$$= (y^t A^t) . (Ay)$$

$$= (Ay)^t . (Ay).$$

$\because$ y is a vector, Ay is also vector and Let.

$V_1 = Ay$.

$$\boxed{\therefore\ y^t P y = V_1^t . V_1 = \|V_1\|^2 \geq 0.}$$

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Similarly,

$$z^t Q z = z^t (AA^t) z$$

$$= (z^t A)(A^t z)$$

$$= (A^t z)^t \cdot (A^t z).$$

Again, Since z is vector, $A^t z$ is also a Vector and,

Let $V_2 = A^t z$.

$$\boxed{\therefore z^t Q z = V_2^t \cdot V_2 = \|V_2\|^2 \geq 0.}$$

$\boxed{\therefore \text{Hence Proved}}$

Let 'λ' be a eigenvalue of P. Then, we have:-

$Pv = \lambda v$, where v is Corresponding eigen vector.

~~multip~~ Left multiply by v^t on both sides:-

$\Rightarrow v^t P v = v^t \lambda v \Rightarrow v^t P v = \lambda (v^t v) = \lambda (\|v\|^2)$.

$$\boxed{\therefore \lambda = \frac{v^t P v}{\|v\|^2}}$$

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Since we proved that, $v^t P v \geq 0 \quad \forall v.$

$$\boxed{\therefore \lambda = \frac{v^t P v}{\|v\|^2} \geq 0.}$$

$\boxed{\therefore \text{ All eigen Values of } P \text{ are non-negative}}$

Similarly, Let λ_1 be ~~the~~ an eigenvalue of Q, Then,

we have

$Q v_1 = \lambda_1 v_1$ where v_1 is the corresponding eigen vector.

Left Multiply by v_1^t on both sides:

$$\Rightarrow v_1^t Q v_1 = \lambda_1 (v_1^t v_1) = \lambda_1 (\|v_1\|^2).$$

$$\boxed{\therefore \lambda_1 = \frac{v_1^t Q v_1}{\|v_1\|^2}}$$

We also proved that, $v_1^t Q v_1 \geq 0 \quad \forall v_1$

$$\boxed{\therefore \lambda_1 = \frac{v_1^t Q v_1}{\|v_1\|^2} \geq 0.}$$

$\boxed{\therefore \text{ All eigenvalues of } Q \text{ are non-negative.}}$

$\boxed{\text{Hence Proved}}$

3)

b) Problem Statement: If 'u' is an eigenvector of P with eigenvalue λ, show that 'Au' is an eigenvector of Q with eigenvalue λ. If 'v' is an eigenvector of Q with eigenvalue μ, show that $A^T v$ is an eigenvector of P with eigenvalue μ. Also find the number of elements in u, v.

Answer:

Since, u is an eigenvector of P with eigenvalue λ, we have:

$$Pu = \lambda u$$

also $P = A^T A$.

$$\boxed{\therefore A^T A u = \lambda u} \quad - (1)$$

Now consider,

$$Q(Au) = (AA^T)(Au) \quad (\because Q = AA^T)$$

$$= A(A^T A u)$$

$$= A(\lambda u)$$

$$= \lambda (Au).$$

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$\therefore Q(Au) = \lambda(Au)$

$\therefore Au$ is an eigen vector of Q with eigen value λ

Again, Since, v is an eigen ~~value~~ vector of Q with eigen value μ, we have :-

$$Qv = \mu v.$$

$$\Rightarrow AA^t v = \mu v \quad (\because Q = AA^t) \quad - (2).$$

Now Consider,

$$P(A^t v) = (A^t A)(A^t v) \quad (\because P = A^t A)$$

$$= A^t (AA^t v)$$

$$= A^t (\mu v)$$

$$= \mu (A^t v).$$

$$\therefore P(A^t v) = \mu (A^t v)$$

$\therefore A^t v$ is an eigenvector of P with eigen value μ.

$\therefore$ Hence Proved

We know that,

$P = A^t A$, $Q = A A^t$, and Size of $A = m \times n$.

$\therefore$ Size of $P = n \times n$, Size of $Q = m \times m$.

Since, u is eigenvector of P, Pu should be valid.

So, Size of u should be $n \times 1$.

Since, v is eigenvector of Q, Qv should be valid.

So, Size of v should be $m \times 1$.

$\boxed{\therefore \text{Number of elements in } u = n}$

$\boxed{\therefore \text{Number of elements in } v = m}$

3)

c) Problem statement:- If v_i is an eigenvector of Q and we define $u_i = \frac{A^T v_i}{\|A^T v_i\|}$. Then prove that there exists some real, non-negative γ_i such that $Au_i = \gamma_i v_i$

Answer:-

Let λ_i be the eigenvalue corresponding to eigenvector v_i.

$$\therefore \quad Q v_i = \lambda_i v_i$$

$$\boxed{\Rightarrow \; A A^t v_i = \lambda_i v_i} \quad (\because Q = AA^t) \quad -①$$

Now, consider:-

$$Au_i = A\left(\frac{A^T v_i}{\|A^T v_i\|}\right) = \frac{A A^T v_i}{\|A^T v_i\|}$$

$$= \frac{\lambda_i v_i}{\|A^T v_i\|} \quad (\text{from } ①)$$

$$\therefore Au_i = \left(\frac{\lambda_i}{\|A^T v_i\|}\right) v_i$$

Let $r_i = \dfrac{\lambda_i}{\|A^t v_i\|}$.

In Part ⓐ, we proved that all eigen value of Q are non-negative.

$\boxed{\therefore \lambda_i \geq 0.}$

we also know that, $\|A^t v_i\| \geq 0$.

$$\therefore r_i = \frac{\lambda_i}{\|A^t v_i\|} \geq 0.$$

$$\boxed{\therefore A u_i = r_i v_i \quad \text{where} \quad r_i = \frac{\lambda_i}{\|A^t v_i\|} \geq 0.}$$

$\boxed{\therefore \text{There exists non-negative } r_i \text{ such that } A u_i = r_i v_i}$

$\boxed{\therefore \text{Hence Proved}}$

3)

d)

Problem Statement :- Define $U = [v_1 | v_2 | \cdots v_m]$ and $V = [u_1 | u_2 \cdots | u_m]$. Now show that $A = U \Gamma V^T$ where Γ is a diagnol matrix Containing the non-negative Values $\gamma_1, \gamma_2, \cdots \gamma_m$.

Answer :-

We have,
$$u_i^t u_j = 0, \quad i \neq j \qquad v_i^t v_j = 0, \quad i \neq j$$
$$= 1, \quad i = j \qquad = 1, \quad i = j$$

where u_i, u_j are eigenvectors of P and v_i, v_j are eigenvectors of Q.

$$\therefore U^T U = \begin{bmatrix} v_1^t \\ v_2^t \\ | \\ v_m^t \end{bmatrix} [v_1 | v_2 | \cdots v_m]$$

$$= [e_{ij}]_{m \times m}, \quad \text{where } e_{ij} = v_i^t v_j$$

$$\boxed{\therefore e_{ij} = 0, \quad i \neq j \\ = 1, \quad i = j}$$

~~where, $e_{ij} = V$.~~

$$\boxed{\Rightarrow U^T U = I_{m\times m}}$$

$\boxed{\therefore U \text{ is an Orthonormal matrix.}}$

$$\boxed{\therefore U^T U = I_{m\times m} = U U^T.} \quad - (1)$$

Similarly we have,

$$V^T V = \begin{bmatrix} u_1^t \\ u_2^t \\ \vdots \\ u_m^t \end{bmatrix} [u_1 | u_2 | \cdots u_m]$$

$$= [f_{ij}]_{m\times m} \quad \text{where} \quad f_{ij} = u_i^t u_j$$

$$\therefore f_{ij} = 0, \quad i \neq j$$
$$= 1, \quad i = j$$

$\boxed{\Rightarrow V^T V = I_{m\times m}}$ $\boxed{\Rightarrow V \text{ is an Orthonormal matrix.}}$

$$\boxed{\therefore V^T V = I_{m\times m}, \quad V V^T = I_{n\times n}} \quad - (2)$$

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Now Consider,

$$U^T A V = \begin{bmatrix} v_1^t \\ v_2^t \\ \vdots \\ v_m^t \end{bmatrix} \cdot A \cdot \left[u_1 \mid u_2 \mid \cdots \mid u_m \right]$$

we Know that, $A \cdot \left[u_1 \mid u_2 \mid \cdots \mid u_m \right] = \left[Au_1 \mid Au_2 \mid \cdots \mid Au_m \right]$.

$$\therefore \; U^T A V = \begin{bmatrix} v_1^t \\ v_2^t \\ \vdots \\ v_m^t \end{bmatrix} \left[Au_1 \mid Au_2 \mid \cdots \mid Au_m \right]$$

In Part (c), we Proved that, $\forall \; u_i, v_i$, there exists a non-negative r_i Such that $Au_i = r_i v_i$.

$$\therefore \; U^T A V = \begin{bmatrix} v_1^t \\ v_2^t \\ \vdots \\ v_m^t \end{bmatrix} \left[r_1 v_1 \mid r_2 v_2 \mid \cdots \; r_m v_m \right]$$

$$= \left[x_{ij} \right]_{m \times m}$$

where $x_{ij} = v_i^t \, r_j v_j = r_j \left(v_i^t v_j \right)$.

$\therefore \; x_{ij} = 0, \quad i \neq j$

$= r_j, \quad i = j.$

$\therefore$ $U^T A V$ is a diagnol matrix with diagnol entries as $r_1, r_2, \ldots r_m$.

So, Let $\Gamma = U^T A V$.

Now Consider,

$$A = \underset{m\times m}{(UU^T)} \cdot \underset{m\times n}{A} \cdot \underset{n\times n}{(VV^T)}$$ $\left(\because UU^T = I_{m\times m}, \; VV^T = I_{n\times n}, \text{ because they are orthonormal matrices} \right)$

$$= U(U^T A V) \cdot V^T$$

$\therefore A = U \Gamma V^T$ where Γ is a diagnol matrix with non-Zero $r_1, r_2, \ldots, r_m$ as diagnol entries.

$\therefore$ Hence Proved.