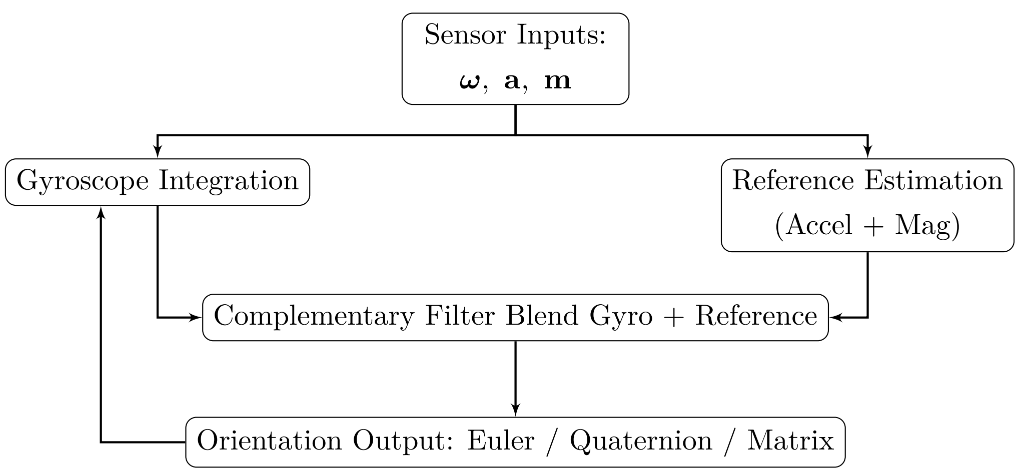
Complementary Filter for 3D Attitude Estimation in AHRS Systems

Introduction

Attitude and Heading Reference Systems (AHRS) estimate the orientation of a body in 3D space by combining data from inertial sensors. The **complementary filter** is one of the simplest and most efficient algorithms for estimating roll, pitch, and yaw by fusing gyroscope, accelerometer, and magnetometer data.

1 Complementary Filter Architecture

The following diagram illustrates the data flow in a complementary filter for orientation estimation:



This flow diagram represents how the complementary filter uses both high-frequency gyroscope data and low-frequency corrections from the accelerometer and magnetometer to estimate robust orientation.

2 Basic Principle of the Complementary Filter

A complementary filter takes advantage of the frequency characteristics of different sensors:

- Gyroscopes are accurate for short-term angular velocity but drift over time.
- Accelerometers provide a long-term reference for roll and pitch via gravity.
- Magnetometers provide a long-term reference for yaw via the Earth's magnetic field.

The idea is to combine these sources using a high-pass filter (for the gyroscope) and a low-pass filter (for the accelerometer and magnetometer), resulting in a stable and drift-compensated orientation estimate.

All measurements from the gyroscope, accelerometer, and magnetometer are assumed to be expressed in the **body-fixed frame**, and orientation is computed relative to the **NED (North–East–Down)** frame.

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4 Mathematical Formulation

4.1 Estimation of Roll, Pitch, and Yaw

Let ϕ, θ, and ψ be roll, pitch, and yaw, respectively. Let ϕ_g, θ_g, and ψ_g be the angles obtained by integrating gyroscope data:

$$\phi_g(t) = \phi(t - \Delta t) + \omega_x(t) \cdot \Delta t$$

$$\theta_g(t) = \theta(t - \Delta t) + \omega_y(t) \cdot \Delta t$$

$$\psi_g(t) = \psi(t - \Delta t) + \omega_z(t) \cdot \Delta t$$

Let ϕ_a, θ_a be the roll and pitch derived from the accelerometer (gravity vector):

$$\phi_a = \arctan 2(a_y, a_z), \quad \theta_a = \arctan 2(-a_x, \sqrt{a_y^2 + a_z^2})$$

Let ψ_m be the yaw angle derived from the magnetometer, using **tilt compensation** to correct for current roll and pitch. First, the magnetometer vector $\mathbf{m} = [m_x, m_y, m_z]^T$ is transformed as:

$$\begin{aligned} m_x^c &= m_x \cos\theta + m_z \sin\theta \\ m_y^c &= m_x \sin\phi \sin\theta + m_y \cos\phi - m_z \sin\phi \cos\theta \\ \psi_m &= \arctan 2(-m_y^c, m_x^c) \end{aligned}$$

Final complementary filter:

$$\phi = \alpha \cdot \phi_g + (1 - \alpha) \cdot \phi_a$$

$$\theta = \alpha \cdot \theta_g + (1 - \alpha) \cdot \theta_a$$

$$\psi = \alpha \cdot \psi_g + (1 - \alpha) \cdot \psi_m$$

Where $\alpha \in [0, 1]$ is the filter coefficient.

4.2 Direct Estimation of the Orientation Quaternion

The orientation quaternion $q = [q_0, q_1, q_2, q_3]^T$ can be directly estimated by combining the gyroscope-integrated quaternion q_g with a reference quaternion q_a constructed from the accelerometer and magnetometer readings.

First, integrate angular velocity from the gyroscope using:

$$\dot{q}_g = \frac{1}{2} q_g \otimes \begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad q_g(t) = q_g(t - \Delta t) + \dot{q}_g(t) \cdot \Delta t$$

To estimate q_a, we use the accelerometer and magnetometer as reference vectors. The gravity vector in the NED frame is $\mathbf{g}_n = [0, 0, 1]^T$ and the magnetic field vector is $\mathbf{m}_n = [1, 0, 0]^T$ (assuming north-aligned field).

Normalize the accelerometer and magnetometer measurements:

$$\mathbf{g}_b = \frac{\mathbf{a}}{\|\mathbf{a}\|}, \quad \mathbf{m}_b = \frac{\mathbf{m}}{\|\mathbf{m}\|}$$

Apply the TRIAD algorithm to find the orientation quaternion q_a that aligns $\mathbf{g}_b \to \mathbf{g}_n$ and $\mathbf{m}_b \to \mathbf{m}_n$:

1. Define orthonormal bases:

$$\begin{aligned} \mathbf{v}_1^b &= \mathbf{g}_b, & \mathbf{v}_2^b &= \frac{\mathbf{g}_b \times \mathbf{m}_b}{\|\mathbf{g}_b \times \mathbf{m}_b\|}, & \mathbf{v}_3^b &= \mathbf{v}_1^b \times \mathbf{v}_2^b \\ \mathbf{v}_1^n &= \mathbf{g}_n, & \mathbf{v}_2^n &= \frac{\mathbf{g}_n \times \mathbf{m}_n}{\|\mathbf{g}_n \times \mathbf{m}_n\|}, & \mathbf{v}_3^n &= \mathbf{v}_1^n \times \mathbf{v}_2^n \end{aligned}$$

2. Construct direction cosine matrices:

$$R_b = [\mathbf{v}_1^b \ \mathbf{v}_2^b \ \mathbf{v}_3^b], \quad R_n = [\mathbf{v}_1^n \ \mathbf{v}_2^n \ \mathbf{v}_3^n]$$

3. Compute the rotation matrix:

$$R = R_n R_b^T$$

4. Extract the quaternion q_a from R using the standard DCM-to-quaternion conversion:

$$\begin{aligned} q_0 &= \frac{1}{2}\sqrt{1 + R_{11} + R_{22} + R_{33}} \\ q_1 &= \frac{R_{32} - R_{23}}{4q_0}, \quad q_2 &= \frac{R_{13} - R_{31}}{4q_0}, \quad q_3 = \frac{R_{21} - R_{12}}{4q_0} \end{aligned}$$

The complementary filter then blends the gyroscope and reference quaternions:

$$q(t) = \text{normalize}(\alpha q_g(t) + (1-\alpha) q_a(t))$$

4.3 Direct Estimation of the Rotation Matrix

The orientation matrix $R(t) \in SO(3)$ can be computed by integrating the angular velocity and correcting using the accelerometer and magnetometer.

From gyroscope integration:

$$\dot{R}_g = R_g[\boldsymbol{\omega}]_\times, \quad R_g(t) = R_g(t-\Delta t) + \dot{R}_g(t) \cdot \Delta t$$

where $[\boldsymbol{\omega}]_\times$ is the skew-symmetric matrix:

$$[\boldsymbol{\omega}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Construct a reference rotation matrix R_a from body-frame measurements of gravity $\mathbf{g}_b$ and magnetic field $\mathbf{m}_b$ using the TRIAD basis:

$$\begin{aligned} \mathbf{v}_1^b &= \mathbf{g}_b, \\ \mathbf{v}_2^b &= \frac{\mathbf{g}_b \times \mathbf{m}_b}{\|\mathbf{g}_b \times \mathbf{m}_b\|}, \\ \mathbf{v}_3^b &= \mathbf{v}_1^b \times \mathbf{v}_2^b \end{aligned}$$

Then:

$$R_a = [\mathbf{v}_1^b \ \mathbf{v}_2^b \ \mathbf{v}_3^b]$$

The complementary filter updates the rotation matrix as:

$$R(t) = \text{orthonormalize}(\alpha R_g(t) + (1-\alpha) R_a(t))$$

where `orthonormalize` ensures that $R(t)$ remains a valid rotation matrix.

5 Filter Tuning

The coefficient α controls the balance between fast gyroscope response and slow correction from the accelerometer and magnetometer:

- A typical value is $\alpha = 0.98$.
- Higher values trust the gyroscope more (less correction).
- Lower values increase sensitivity to drift correction.

6 Advantages and Limitations

Advantages

- Simple and computationally efficient.
- Real-time performance suitable for embedded systems.
- Full 3D orientation coverage with all three sensors.

Limitations

- Sensitive to magnetometer disturbances (e.g., motors, metal).
- Accuracy degrades in highly dynamic conditions.
- Cannot fully represent orientation using quaternions or rotation matrices without correction.