Methods of Point estimation

2 The Method of Maximum Likelihood

First, lets Recall that

Lecture (5)

Summation

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} c\, f(x_i) = c f(x_1) + c f(x_2) \dots c f(x_n)$$
$$= c \sum f(x_i)$$

$$\sum_{i=1}^{n} \left(f(x_i) \pm g(x_i)\right) = \sum_{i=1}^{n} f(x_i) \pm \sum_{i=1}^{n} g(x_i)$$

Product

$$\prod_{i=1}^{n} f(x_i) = f(x_1) \cdot f(x_2) \cdot f(x_3) \dots f(x_n)$$

$$\prod_{i=1}^{n} c = c^n$$

$$\prod_{i=1}^{n} c\, g(x_i) = c g(x_1) \cdot c g(x_2) \cdot c g(x_3) \dots c g(x_n)$$
$$= c^n \prod_{i=1}^{n} g(x_i)$$

$$\prod_{i=1}^{n} g_1(x_i)\, g_2(x_i) = \prod_{i=1}^{n} g_1(x_i) \cdot \prod_{i=1}^{n} g_2(x_i)$$

Ln

$\ln e = 1$

$\ln(ab) = \ln a + \ln b$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln(1) = 0 \quad \ln(\infty) = \infty \quad \ln(0) = -\infty$

$\ln a^n = n \ln a$

$\ln e^m = m$

$\ln(a+b) \neq$ Can't be Simplified

Differentiation

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} (f(x))^n = n (f(x))^{n-1} \cdot f'(x)$$

$$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Method of MaxLikelihood MLE

Let $X_1, X_2, \dots, X_n$ random sample of size n

$X_1, X_2, \dots, X_n$ are i.i.d

the likelihood function is the joint pdf (or pmf of random sample

$$L(\theta) = f_{joint}(x_1, x_2, \dots, x_n; \theta)$$

$$L(\theta) = f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta)$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i ; \theta)$$

f^n of unknow Parameter θ only

MLE (maximum likelihood estimator) of $\hat{\theta}$ is the value of θ that maximize the likelihood f^n $L(\theta)$

MLE $\left.\frac{dL(\theta)}{d\theta}\right|_{\theta=\hat{\theta}} = 0$

Recall

Max point of $f(x)$ at $f'(x)=0$

Search for $\hat{\theta}$ that maximize the value of $L(\theta)$

Note that any value of θ that maximize $L(\theta)$ will also maximize the $\ln L(\theta)$ log-likelihood

Loglikelihood $L^*(\theta) = \ln \overset{\text{product}}{L(\theta)}$

MLE $\boxed{\frac{dL^*}{d\theta} = \left.\frac{d}{d\theta}\left(\ln L(\theta)\right)\right|_{\theta=\hat{\theta}} = 0}$

Steps

1- write $L(\theta) = f(x_1;\theta) \cdot f(x_2;\theta) \cdots f(x_n;\theta)$
joint pdf, use properties of Π

2- take ln of both sides

$$L^*(\theta) = \ln f(x_1;\theta) + \ln f(x_2;\theta) \cdots + \ln f(x_n;\theta)$$

3- differentiate w.r.t θ
equate the equation to zero to get $\hat{\theta}$

Ex① If $x_1, x_2 .., x_n$ are the value of a random sample from a Bernoulli Population.

Find the MLE of its parameter θ

if $X \sim$ Bernoulli (θ)

$$f(x;\theta) = \theta^{x} (1-\theta)^{1-x} \qquad x=0,1$$

pdf

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \prod_{i=1}^{n} \theta^{x_i} \cdot \prod_{i=1}^{n} (1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n-\sum x_i}$$

Recall

In Bernoulli trial one trial with Prob of success $= P$

, Prob of fail $= 1-P$

X: No of success

$$f(x) = P^{x} (1-P)^{1-x}$$
$$x = 0, 1$$

$X \sim$ Bernoulli (P)

Bernoulli is binomial $(n=1, P)$

take ln for both sides

$$L^{*}(\theta) = \ln\left(\theta^{\sum x_i} (1-\theta)^{n-\sum x_i}\right)$$

$$L^{*}(\theta) = \sum x_i \ln\theta + (n-\sum x_i)\ln(1-\theta)$$

Differentiate w.r.t θ to get MLE

Note

$$\theta^{x_1} \cdot \theta^{x_2} \cdot \theta^{x_3} \ldots = \theta^{x_1+x_2+\ldots}$$

$$(1-\theta)^{1-x_1}(1-\theta)^{1-x_2}(1-\theta)^{1-x_3}\cdots = (1-\theta)^{n-\sum x_i}$$

$$\frac{dL^{*}(\theta)}{d\theta} = 0$$

$$\sum x_i \left(\frac{1}{\hat\theta}\right) + (n-\sum x_i)\frac{-1}{1-\hat\theta} = 0$$

$$\frac{\sum x_i}{\hat\theta} = \frac{n-\sum x_i}{1-\hat\theta}$$

$$(1-\hat{\theta}) \sum X_i = \hat{\theta}(n-\sum X_i)$$

$\div n$

$$(1-\hat{\theta}) \frac{\sum X_i}{n} = \hat{\theta}\left(\frac{n}{n} - \frac{\sum X_i}{n}\right)$$

$$(1-\hat{\theta}) \bar{X} = \hat{\theta}(1-\bar{X})$$

$$\bar{X} - \cancel{\bar{X}\hat{\theta}} = \hat{\theta} - \cancel{\hat{\theta}\bar{X}}$$

$$\boxed{\hat{\theta} = \bar{X}}$$

estimator of θ (Probability of success)

Ex(2) let $X_1, X_2, \dots, X_n$ be a random sample from exponential population

Find MLE for the Parameter θ

$X \sim \exp(\lambda)$

$$\underset{\text{pdf}}{f(x)} = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \qquad \lambda > 0, x > 0$$

$$f(x_i;\theta) = \frac{1}{\theta} e^{-\frac{x_i}{\theta}}$$

$$\underset{\text{joint}}{L(\theta)} = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \underset{\text{Const}}{\frac{1}{\theta}} e^{-\frac{x_i}{\theta}}$$

$$e^{-\frac{x_1}{\theta}} + e^{-\frac{x_2}{\theta}} + e^{-\frac{x_3}{\theta}} \dots = e^{-\frac{\sum x_i}{\theta}}$$

$$= \frac{1}{\theta^n} \prod_{i=1}^{n} e^{-\frac{x_i}{\theta}}$$

$$L(\theta) = \frac{1}{\theta^n} e^{-\frac{\sum x_i}{\theta}} = \theta^{-n} e^{-\frac{\sum x_i}{\theta}}$$

$$L^*(\theta) = \ln L(\theta) = \ln \theta^{-n} e^{-\frac{\sum x_i}{\theta}}$$

$$L^*(\theta) = -n \ln\theta - \frac{\sum x_i}{\theta}$$

$$\frac{d}{d\theta} L^*(\theta) = 0 \Rightarrow -n\left(\frac{1}{\hat\theta}\right) - \sum x_i \left(\frac{-1}{\hat\theta}\right)^2 = 0$$

$$\sum x_i \left(\frac{1}{\hat\theta^2}\right) = \frac{n}{\hat\theta}$$

$$\boxed{\hat\theta = \frac{\sum x_i}{n} = \bar{x}}$$

If several unknown Parameters

$$\theta_1, \theta_2, --- \theta_k$$

$$\frac{\partial}{\partial\theta_1} L(\theta_1, \theta_2, \dots, \theta_k) = 0$$

$$\frac{\partial}{\partial\theta_2} L(\theta_1, \theta_2, \cdots, \theta_k) = 0$$

$$\vdots$$

$$\frac{\partial}{\partial\theta_k} L(\theta_1, \theta_2, --, \theta_k) = 0$$

then solve k equation in k Unknowns

Ex ③ If $X_1, X_2, \dots, X_n$ Constitute a random Sample from normal population with the mean μ, variance σ^2

Find joint MLE of μ, σ^2

if $X \sim N(\mu, \sigma^2)$

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{(given)}$$

$$L(\mu,\sigma^2) = \prod_{i=1}^{n} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}$$

$$L(\mu,\sigma^2) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\Sigma(x_i-\mu)^2}{2\sigma^2}}$$

take ln of both sides $\boxed{\ln\frac{1}{\sigma^2} = \underbrace{\ln 1}_{0} - \ln\sigma^2}$

$$L^*(\mu,\sigma^2) = \text{Const} + \frac{n}{2}\ln\left(\frac{1}{\sigma^2}\right) - \frac{\Sigma(x_i-\mu)^2}{2\sigma^2}$$

$$= \text{Const.} - \frac{n}{2}\ln\sigma^2 - \frac{\Sigma(x_i-\mu)^2}{2\sigma^2}$$

$$\frac{\partial L^*(\mu,\sigma^2)}{\partial\mu} = \cancel{\frac{-1}{2\sigma^2}}\, \cancel{2}\Sigma(x_i-\hat{\mu})\cancel{(-1)} = 0$$

$$\Sigma(x_i - \underset{\text{value}}{\hat{\mu}}) = 0$$

$$\Sigma x_i - n\hat{\mu} = 0$$

$$\boxed{\hat{\mu} = \frac{\Sigma x_i}{n} = \bar{x}}$$

$$L^*(\mu, \sigma^2) = \text{Const} - \frac{n}{2}\ln\sigma^2 - \frac{\sum(x_i-\mu)^2}{2}\frac{1}{\sigma^2}$$

$$\frac{\partial L^*(\mu,\sigma^2)}{\partial\sigma^2} = 0 \Longrightarrow -\frac{n}{2}\frac{1}{\sigma^2} - \frac{\sum(x_i-\mu)^2}{2}\frac{-1}{(\sigma^2)^2}$$

take care treat σ^2 as one symbol θ

$$\frac{-n}{\cancel{2}\hat{\sigma}^{\cancel{2}}} - \frac{1}{\cancel{2}}\sum(x_i-\mu)^2\frac{-1}{(\hat{\sigma}^2)^2} = 0$$

$\frac{1}{\theta} \to \frac{-1}{\theta^2}$

$$\frac{\sum(x_i-\mu)^2}{\hat{\sigma}^2} = n$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\mu)^2$$

$$\hat{\mu} = \overline{x}$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\overline{x})^2}$$

Problem 4 : sheet (2)

[4] let $X_1, X_2, \dots X_n$ be a random sample from a geometric distribution

$$f(x;\theta) = \theta(1-\theta)^{x-1}$$

$x = 1, 2, 3, \dots$

θ: prob of success

x: No of trials

Find a formula for estimating θ by using
(a) Method of Moments
(b) Method of ML

(a) Method of Moments

Population

$$\mu_1 = E(x) = \sum_{x=1}^{\infty} x\, f(x) = \frac{1}{\theta}$$

Sample

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\mu_1 = m_1$$

$$\frac{1}{\theta} = \bar{x} \Rightarrow \boxed{\hat{\theta} = \frac{1}{\bar{x}}}$$

(b) Method of MLE

$$L(\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1}$$

نستخدم خواص $\prod$

$$= \theta^n \prod_{i=1}^{n} (1-\theta)^{x_i-1}$$

$$= \theta^n (1-\theta)^{\sum x_i - n}$$

$(1-\theta)^{x_1-1}(1-\theta)^{x_2-1}(1-\theta)^{x_3-1}\dots$

$= (1-\theta)^{\sum x_i - n}$

$$L(\theta) = \theta^{n} (1-\theta)^{\sum x_i - n}$$

$$L^*(\theta) = \ln \theta^{n} (1-\theta)^{\sum x_i - n}$$

$$L^*(\theta) = n \ln \theta + (\sum x_i - n) \ln (1-\theta)$$

$$\frac{d L^*(\theta)}{d\theta} = 0$$

$$n \frac{1}{\hat{\theta}} + (\sum x_i - n) \frac{-1}{1-\hat{\theta}} = 0$$

$$\frac{\sum x_i - n}{1-\hat{\theta}} = \frac{n}{\hat{\theta}}$$

$$\hat{\theta} \sum x_i - \cancel{n\hat{\theta}} = n - \cancel{n\hat{\theta}}$$

$$\hat{\theta} \sum x_i = n$$

$$\boxed{\hat{\theta} = \frac{n}{\sum x_i} = \frac{1}{X}}$$

Invariance Property of MLE's

let $\hat{\theta}$ be the MLE of θ

then $g(\hat{\theta})$ is a MLE for $g(\theta)$

where g is one-to one fn

e.g if $\hat{\theta}$ is MLE for variance θ

then $\sqrt{\hat{\theta}}$ is the MLE for standard deviation $\sqrt{\theta}$

Ex Problem 4, 5 Ch. 2

for Geometric(P)

4)

MLE of p $= \hat{P} = \frac{1}{\bar{X}}$

5) $\because \hat{P} = \frac{1}{\bar{X}}$, Find the MLE of the following quantities

a) $E(x) = \frac{1}{P}$ MLE of $E(x) = \frac{1}{\hat{P}} = \frac{1}{\frac{1}{\bar{X}}} = \bar{X}$

$\hat{E}(x)$

b) $V(x) = \frac{1-P}{P^2}$ MLE $V(x) = \frac{1-\hat{P}}{\hat{P}^2} = \frac{1-\frac{1}{\bar{X}}}{\frac{1}{\bar{X}^2}} = \bar{X}^2 - \bar{X}$

$\hat{V}(x)$

c) $P(x>K) = (1-P)^K$ MLE of $P(x>K) = (1-\frac{1}{\bar{X}})^K$

$K=1,2,\ldots$ $\hat{P}(x>K)$