Lecture 9

Chapter 4

Tests of Hypotheses

A statistical hypothesis is assumption or statement, which may or may not be true, concerning one or more populations

ex. A factory has a machine that dispenses 80 mL of fluid in a bottle. An employee believes that the average amount of fluid is not 80 mL

Hyp mean μ_0

$H_0 : \mu = 80$ mL

null hypothesis

claim

$H_a : \mu \neq 80$

alternative Hyp.

the employee has to prove his claim -- How ??

ex. Using 40 samples, he measures the average amount dispensed by the machine to be 78 mL with standard deviation of 2.5 mL

At 95% Confidence level, is there enough evidence to support his idea ??

Sample : $n = 40$, $\overline{X} = 78$, $S = 2.5$

Confidence level $= 1 - \alpha$

Significance Level $= \alpha$

Result → reject H_0 with $100(1-\alpha)\%$ confi level
(If we have evidence) (accept H_a)

→ do not reject H_0 with $100(1-\alpha)\%$ conf level
(If we have no evidence)

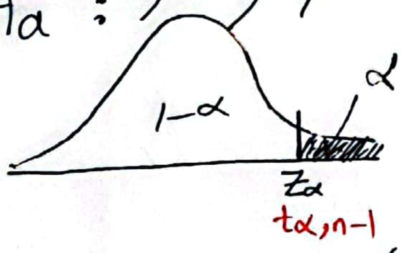
If we reject H_0 we say that the test is <u>significant</u> معتبر/مهم محقق للهدف

Two kinds of tests

<u>One sided test</u>
one (tailed) test

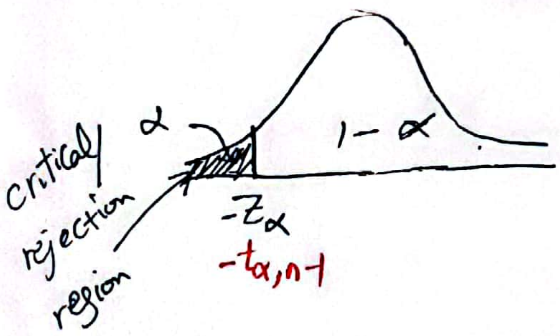
$H_0: \mu = \mu_0$

1- $H_a: \mu > \mu_0$ (right tail test)



if $Z_c > Z_\alpha$ reject H_0

2- $H_a: \mu < \mu_0$ (left tail test)

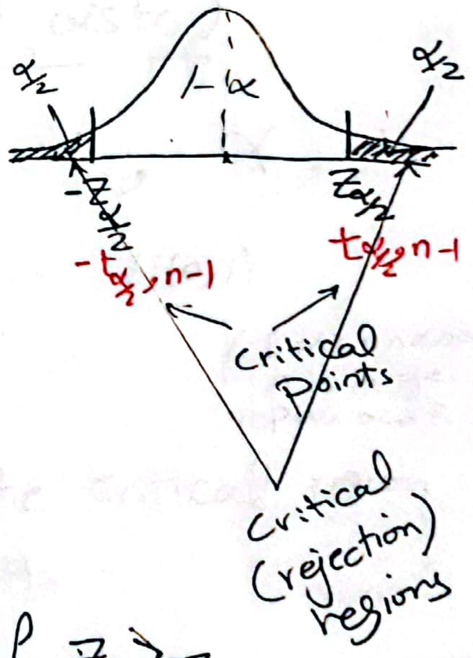


if $Z_c < -Z_\alpha$ reject H_0

<u>two sided test</u>
two (tailed) test

$H_0: \mu = \mu_0$

$H_a: \mu \neq \mu_0$



if $Z_c > Z_{\alpha/2}$ → reject H_0
or $Z_c < -Z_{\alpha/2}$

Type I and Type II error

	H_0 is true	H_0 is false
Do not reject H_0	✓	Type II error
Reject H_0	Type I error	✓

Type I error : we reject H_0 when it is true

Type II error : we don't reject H_0 when it is false

Prob of type I error = α (level of significance)

Tests Concerning The population Mean μ

Case ① We know σ^2 or n is large

(use normal distr)

We have n, $\bar{X}$, σ, α, μ_0

$\boxed{1}$ $$Z_c = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

Calculated Z-value

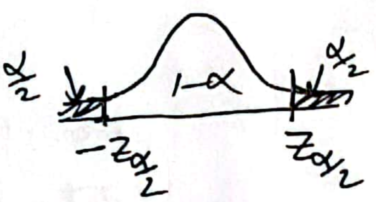
(if σ is unknown n is large replace $\sigma \to S$)

$\boxed{2}$ If Z_c falls in the critical region reject H_0

else do not reject H_0

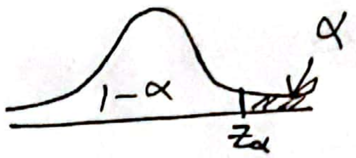
Recall that

two sided | **one side**



$P(-Z_{\alpha/2} < Z < Z_{\frac{\alpha}{2}}) = 1-\alpha$

$\Phi(Z_{\alpha/2}) = 1 - \frac{\alpha}{2}$



$P(Z > Z_\alpha) = \alpha$

$P(Z < Z_\alpha) = 1-\alpha$

$\Phi(Z_\alpha) = 1-\alpha$

$P(Z < -Z_\alpha) = \alpha$

$P(Z > -Z_\alpha) = 1-\alpha$

$\Phi(-Z_\alpha) = \alpha$

EX (4.1) It has been found from experience that the mean breaking strength of a particular brand of thread is 9.5 ounces with a standard deviation of 1.4 ounces. Recently a sample of 36 pieces of thread showed a mean breaking strength of 8.94 ounces. Can one conclude at a significance level 0.05, that the thread has become inferior? اقل / اضعف

$H_0 : \mu = 9.5$ $\quad \mu_0 = 9.5, \ \sigma = 1.4$

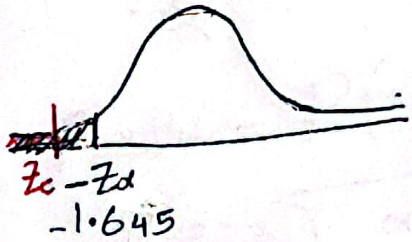
$n = 36 > 30$

$H_a : \mu < 9.5$ اضعف

$\bar{X} = 8.94$

$\alpha = 0.05$

$$Z_c = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{8.94 - 9.5}{\frac{1.4}{\sqrt{36}}} = -2.4$$



From Minitab $Z_{0.05} = 1.645$

$-Z_\alpha = -Z_{0.05} = -1.645$

Compare Z_c by $-Z_\alpha$

we found $Z_c < -Z_\alpha$

Z_c falls in rejection region

Conclusion :-

reject H_0 at %5 level of significance

(%95 level of confidence)

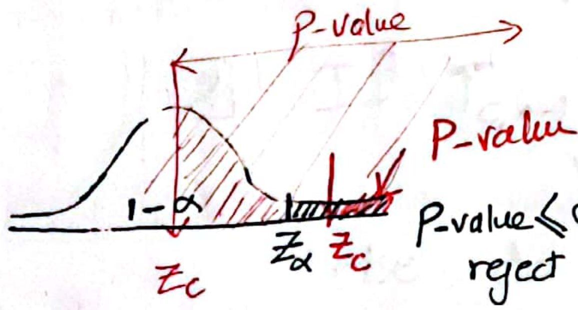
we conclude that the thread has become inferior

The previous example uses the Z-value test

Another way is to use the P-value test

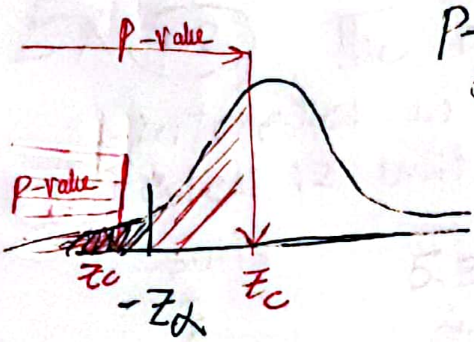
P- value test (used by Minitab)

one sided test



P-value $\leq \alpha$

reject H_0



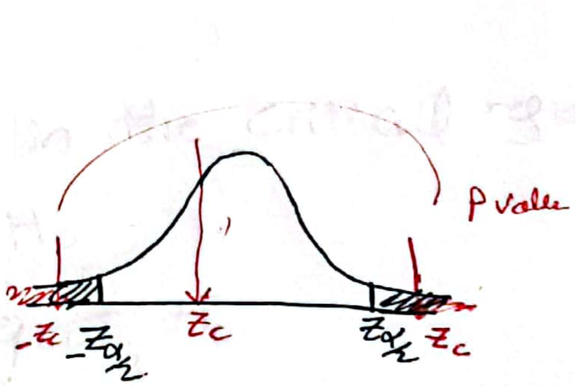
P-value $> \alpha$

do not reject H_0

$$P\text{-value} = P(Z > |Z_c|)$$

$P(Z > Z_c)$ or $P(Z < -Z_c)$

two-sided test



$$P\text{-value} = 2P(Z > |Z_c|)$$

if P-value $< \alpha$

reject H_0

else P-value $> \alpha$

Do not reject H_0

Ex (4.1) by p-value

$Z_c = -2.4$

$$\begin{aligned}P\text{value} &= P(Z > |2.4|)\\ &= 1 - P(Z \leq 2.4)\\ &= 1 - \phi(2.4) \rightarrow \text{Cumulative by Minitab}\\ &= 1 - 0.992\\ &= 0.008\end{aligned}$$

Since $\alpha = 0.05$ P-value $\leq \alpha$ reject H_0

Case (2) (σ is unknown, $n < 30$)
(use t-distribution)

We have $n, \bar{x}, s, \alpha, \mu_0$

[1] $T_{c,n-1} = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$

[2] If $T_{c,n-1}$ falls in the critical region
reject H_0
else do not reject H_0

Ex (4.2) The manufacturer of a power supply is interested in the mean of output Voltage. He has tested 12 units, chosen at random with the following results

5.34, 4.90, 5.07, 5.25, 5.65, 5.45, 5.25, 5.35
4.86, 5.54, 5.44, 4.90

Test the hypothesis that the true mean Voltage does not equal 5. Use $\alpha = 0.05$

$\mu_0 = 5$, $\alpha = 0.05$

two sided

$H_0: \mu = 5$ vs $H_a: \mu \neq 5$

$n = 12 < 30$, σ is unknown $\Rightarrow$ use t distr.

by Minitab (or Calculator)

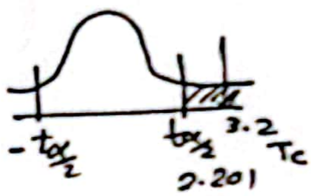
$\bar{X} = \frac{1}{n} \Sigma x_i = 5.25$

$S = 0.2642$

$$T_c = \frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}} = \frac{5.25 - 5}{\frac{0.2642}{\sqrt{12}}} = 3.28$$

from Minitab get $t_{\frac{\alpha}{2}, n-1} = t_{0.025, 11} = 2.201$

Since $T_c > t_{\frac{\alpha}{2}, n-1} \Rightarrow$ reject H_0



or

Using p-value test

$$\text{p-value} = 2P(T > |T_c|)$$

$$= 2\,P(T > 3.28)$$

$$= 2\,(1 - \overset{0.9965}{\Phi}(3.28))$$

Cumulative, d.o.f = 11

$= 0.007$

p-value $< \alpha$ reject H_0

Tests Concerning Two Populations

Population 1

μ_1

σ_1^2

$\Downarrow$

Sample n_1

$\overline{X}_1$

Population 2

μ_2

σ_2^2

$\Downarrow$

Sample n_2

$\overline{X}_2$

$H_0 : \mu_1 - \mu_2 = d_0$ Vs. $H_a : \mu_1 - \mu_2 < d_0$ one sided

or $\mu_1 - \mu_2 > d_0$

or $\mu_1 - \mu_2 \neq d_0$ two sided

Case ① σ_1^2, σ_2^2 are known (or n_1 & n_2 are large)

(use normal distr)

1] Calculate $$Z_c = \frac{\overline{X}_1 - \overline{X}_2 - d_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

(If σ_1^2, σ_2^2 unknown n_1, n_2 large you can replace $\sigma_1^2 \to S_1^2, \sigma_2^2 \to S_2^2$)

2] If Z_c falls in the critical region reject H_0

else donot reject H_0

Ex 4.3 Two types of chemical Solutions A, B were tested for their PH values. Analysis of 40 samples of A showed a mean PH of 7.55 with a st. deviation of 0.06. Analysis of 50 samples of B showed a mean PH of 7.48 with st. dev. 0.08. Test the hypothesis that the two types of solutions have different PH values using $\alpha = 0.01$

$H_0 : \mu_1 \underset{(d_0=0)}{=} \mu_2$ Vs $H_a : \mu_1 \neq \mu_2 \ (d_0 \neq 0)$ two sided

$n_1 = 40 \quad \bar{X}_1 = 7.55, \quad S_1 = 0.06$

$n_2 = 50 \quad \bar{X}_2 = 7.48, \quad S_2 = 0.08$

n_1 & $n_2 > 30 \Longrightarrow$ Use normal distr.

Calculate
$$Z_c = \frac{(\bar{X}_1 - \bar{X}_2) - d_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{7.55 - 7.48}{\sqrt{\frac{(0.06)^2}{40} + \frac{(0.08)^2}{50}}}$$
$$= 4.74$$

from Minitab $Z_{\frac{\alpha}{2}} = Z_{0.005} = 2.58$

$Z_c > Z_{\alpha/2} \Longrightarrow$ reject H_0

acept $H_a \Rightarrow$ accept that the two types have different PH values

by Minitab p-value $0.000 < \alpha$
reject H_0

Case 2 σ_1, σ_2 are unknown and n_1 & n_2 are small
(use - t - distr.)

Assume $\sigma_1^2 = \sigma_2^2 = \sigma^2$. (If not equals $\sigma_1 \neq \sigma_2$ solve it by Minitab)

$$T_c = \frac{(\bar{X}_1 - \bar{X}_2) - d_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}, \quad d.o.f = n_1 + n_2 - 2$$

where

$$S_p^2 = \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}$$

Pooled Variance

if T_c falls in the rejection region $\Rightarrow$ reject H_0

Ex (4.4) A sample of 9 plants of the same Variety was grown in one type of soil in a greenhouse and after a fixed time they were removed and dried. Their dry weights were 27.5, 22.3, 24.7, 26.1, 26.5, 20.0, 31.0, 25.3, 28.6 gm

A further Sample of all similar Plants was grown in identical Conditions but in another type of soil, their weights were 31.8, 30.3, 26.4, 24.2, 27.8, 29.1, 25.5, 28.9, 30.0, 24.9, 31.7

Do the two soil types have different effects on the Plants

$H_0: \mu_1 = \mu_2$ vs. $H_a: \mu_1 \neq \mu_2$ two sided test

$\alpha = 0.05$

Calc/Col statistics/ Mean, st dev

$n_1 = 9$, $\bar{X}_1 = 25.78$, $S_1 = 3.27$

$n_2 = 11$, $\bar{X}_2 = 28.24$, $S_2 = 2.68$

$$S_p^2 = \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2} = \frac{8(3.27)^2 + (10)(2.68)^2}{9 + 11 - 2}$$

$$S_p^2 = 8.74$$

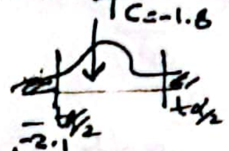
$$S_p = 2.957$$

$$T_c = \frac{(\overline{X}_1 - \overline{X}_2) - d_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{25.78 - 28.24}{2.957 \sqrt{\frac{1}{9} + \frac{1}{11}}}$$

$$T_c = -1.85$$

get $t_{\frac{\alpha}{2}, n_1+n_2-2} = t_{0.025, 18} \simeq 2.1$

Critical value $\left(\phi(t_{\alpha/2, 18}) = 1 - \frac{\alpha}{2}\right)$



Since $T_c > -t_{\frac{\alpha}{2}, n_1+n_2-2}$

then do not reject H_0 at 0.05 level of significance.

we conclude that two type of soil have the same effects on the plants

by Minitab p-value $= 0.081 > \underset{0.05}{\alpha}$