Lecture 6

How to know if the estimator is good or bad?
If you got different estimators by different methods which one will you prefer?

In order to decide which point estimator of a particular parameter is the best one to use we need to examine their statistical properties and develope some criteria for comparing estimators
(we don't know population parameter)

Criterias →
- unbiasedness
- Min variance unbiased estimator MVUE
- efficiency
- Consistency

[I] unbiased estimators

if $\hat{\theta}$ is an estimator of θ

$\hat{\theta} - \theta$ = error diff between sample value and estimated value / expected of parameter

We call $bias(\hat{\theta}) = E(\hat{\theta}) - \theta$ الحيود عن القيمة الأصلية

$\hat{\theta}$ is said to be unbiased when $\boxed{E(\hat{\theta}) = \theta}$

(if on the average it's value are equal to θ)

$\hat{\theta}$ is called Asymptotically unbiased when $E(\hat{\theta}) \xrightarrow[n \to \infty]{} \theta$

e.g. In normal distr

$\hat{\mu} = \bar{X}$ (by MME, by MLE)

$E(\hat{\mu}) = E(\bar{X}) = \mu$

$\therefore \hat{\mu}$ is unbiased estimator of μ

The mean squared error MSE of estimator

$$MSE(\hat\theta) = E(\hat\theta-\theta)^2$$

$$= E(\hat\theta - E(\hat\theta) + E(\hat\theta) - \theta)^2$$

$$= E\left((\hat\theta - E(\hat\theta)) + (E(\hat\theta)-\theta)\right)^2$$

$$= E\left((\hat\theta - E(\hat\theta))^2 + (\underbrace{E(\hat\theta)-\theta}_{bias})^2 + \cancel{2(\hat\theta - E(\hat\theta))(\underbrace{E(\hat\theta)-\theta}_{bias})}^{0}\right)$$

$$= E(\hat\theta - E(\hat\theta))^2 + (E(\hat\theta)-\theta)^2$$

$$MSE(\hat\theta) = Var(\hat\theta) + (bias(\hat\theta))^2$$

$$E((\hat\theta - E(\hat\theta))(\underbrace{E(\hat\theta)-\theta}_{bias}))$$
$$= \underbrace{E(\hat\theta-\theta)}_{Constant}\underbrace{(E(\hat\theta) - E(\hat\theta))}_{0}$$
$$= Zero$$

$$Var(x) = E(x-\mu)^2$$
$$= E(x - E(x))^2$$
$$bias(\hat\theta) = E(\hat\theta) - \theta$$

If $\hat\theta$ is unbiased bias $(\hat\theta) = 0$

$$\boxed{MSE(\hat\theta) = Var(\hat\theta)}$$

$$MSE(\hat\theta) = E(\hat\theta - E(\hat\theta))^2$$

and also $\boxed{E(\hat\theta) = \theta}$

Ex(2.8) If X has binomial distr with Parameters n, p

$$X \sim Bin(n, p).$$

Show that $\hat p = \frac{X}{n}$ is unbiased estimator of p

In Binomial distr $E(x) = np$, $V(x) = np(1-p)$

$$E(\hat p) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(x) = \frac{1}{n}np = p$$

$\therefore$ $\hat p$ is unbiased estimator of p

Ex (2.9) Suppose that X is a r.v. with mean μ. and variance σ^2. let $X_1, X_2, \dots X_n$ be a random sample from X. Show that the sample mean $\bar{X}$ and the sample variance S^2 are unbiased estimators of μ, σ^2

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

X_i identically distr.

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n} E(\Sigma X_i)$$

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$$E(\bar{X}) = \frac{1}{n} n\mu = \mu$$

$\therefore \bar{X}$ is unbiased estimator for μ

$$S^2 = \frac{1}{n-1}\Sigma(X_i - \bar{X})^2$$

$$E(S^2) = \sigma^2$$

from chi-squar. look

$\therefore S^2$ is unbiased estimator of σ^2

Recall that

if $U \sim \chi^2_\nu$

$E(U) = \nu, \quad V(U) = 2\nu$

if $U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$E(U) = n-1$$

$$E\left(\frac{(n-1)S^2}{\sigma^2}\right) = n-1$$

$$\frac{(n-1)}{\sigma^2} E(S^2) = n-1$$

$$\boxed{E(S^2) = \sigma^2}$$

(easier proof than the one in the book)

To get $V(S^2)$

$$V(U) = 2(n-1)$$

$$V\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

$$\frac{(n-1)^2}{\sigma^4} V(S^2) = 2(n-1)$$

$$V(S^2) = \frac{2\sigma^4}{n-1}$$ (will need it later)

2 Minimum variance unbiased estimator

$\because MSE(\hat{\theta}) = Var(\hat{\theta}) + \underbrace{(bias(\hat{\theta}))^2}_{0}$

if we have two unbiased estimators $\hat{\theta}_1, \hat{\theta}_2$ of θ

$Var(\hat{\theta}_1) < Var(\hat{\theta}_2)$

$\hat{\theta}_1$ is better estimator than $\hat{\theta}_2$

3 Efficiency

if $\hat{\theta}_1, \hat{\theta}_2$ are two estimators of θ

$$eff(\hat{\theta}_2, \hat{\theta}_1) = \frac{MSE(\hat{\theta}_1)}{MSE(\hat{\theta}_2)}$$

relative eff. of $\hat{\theta}_2$ to $\hat{\theta}_1$

If relative eff. < 1 $\hat{\theta}_1$ is more efficient estimator of θ than $\hat{\theta}_2$

e.g. suppose that we wish to estimate the mean μ of population, we have a random sample of n observations $X_1, X_2, \ldots, X_n$.

we wish to compare two possible estimators for μ the sample mean $\bar{X}$ and a single observation say X_1

$E(\bar{X}) = \mu$ $\qquad E(X_1) = \mu$

both $\bar{X}, X_1$ are unbiased estimators of μ

But

$$MSE(\bar{X}) = Var(\bar{x}) = \frac{\sigma^2}{n} \quad \text{less}$$

$$MSE(X_1) = Var(X_1) = \sigma^2$$

Since $MSE(\bar{x}) < MSE(X_1)$

$\bar{X}$ is <u>more efficient estimator</u> of μ than X_1

[4] Consistency

The estimator $\hat{\theta}$ is called a consistent estimator of the parameter θ iff

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2 \to 0 \quad \text{as } n \to \infty$$

(that is $\overset{\text{unbiased}}{E(\hat{\theta}) \to \theta}$ as $n \to \infty$)

$MSE(\hat{\theta}) = V(\hat{\theta}) \to 0$ as $n \to \infty$

e.g. let $\hat{\mu} = \bar{x}$

$E(\hat{\mu}) = E(\bar{x}) = \mu$ $\quad$ $\hat{\mu}$ is unbiased

$$MSE(\hat{\mu}) = Var(\hat{\mu}) = var(\bar{x}) = \frac{\sigma^2}{n}$$

$$MSE(\hat{\mu})\Big|_{as\, n \to \infty} = 0$$

$\therefore \bar{X}$ is consistent estimator

means that $\bar{X} \to \mu$ when $n \to \infty$ (without knowing value of μ!!)

e.g. let $\hat{\sigma}^2 = S^2$ Show that sample variance is unbiased consistent estimator

$E(\hat{\sigma}^2) = E(S^2) = \sigma^2$ S^2 is unbiased estimator of σ^2

$Var(\hat{\sigma}^2) = var(S^2) = \frac{2\sigma^4}{n-1}$ (page 3)

As $n \to \infty$ $MSE(S^2) \to 0$

S^2 is Consistent estimator of σ^2

as $n \to \infty$ $S^2 \to \sigma^2$

ex in Binomial distr.

$\hat{P} = \frac{X}{n}$

$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{nP}{n} = P$ unbiased

$Var(\hat{p}) = Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X)$

$= \frac{1}{n^2} nP(1-P)$

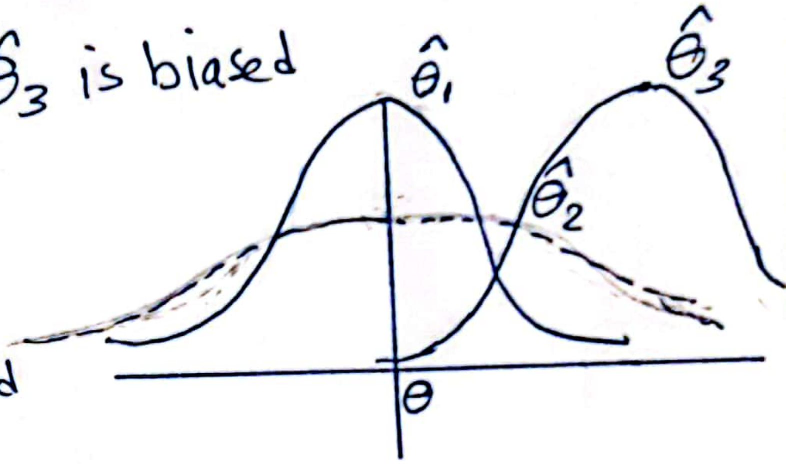
$Var(\hat{p}) = \frac{P(1-P)}{n}$

as $n \to \infty$ $var(\hat{p}) = MSE(\hat{p}) \to 0$

$\therefore \hat{p}$ is Consistent estimator

EX ① Given three estimators $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ for population parameter θ as shown in the figure

which estimator will you choose? why?



$\hat{\theta}_1, \hat{\theta}_2$ are unbiased, $\hat{\theta}_3$ is biased

$E(\hat{\theta}_1) = \theta$

$E(\hat{\theta}_2) = \theta$

their distribution are centered at θ

but $\hat{\theta}_1$ has smaller variance than $\hat{\theta}_2$

$\Rightarrow \hat{\theta}_1$ is more efficient (less MSE)

$\Rightarrow$ Choose estimator $\hat{\theta}_1$

Ex②

Consider $\hat{\theta}_1, \hat{\theta}_2$ are two estimators for Population parameter θ such that

$E(\hat{\theta}_1) = \theta \quad , \quad var(\hat{\theta}_1) = \frac{\theta^2}{3}$

$E(\hat{\theta}_2) = \frac{\theta}{3} \quad , \quad var(\hat{\theta}_2) = \frac{\theta^2}{9}$

Find
1] bias $(\hat{\theta}_2)$
2] efficiency $(\hat{\theta}_2 / \hat{\theta}_1)$ which estimator is more efficient

$E(\hat{\theta}_1) = \theta \Rightarrow \hat{\theta}_1$ is unbiased $\Rightarrow$ bias $(\hat{\theta}_1) = 0$

$$\text{bias}(\hat{\theta}_2) = E(\hat{\theta}_2) - \theta = \frac{\theta}{3} - \theta = \frac{-2\theta}{3}$$

$$\therefore MSE(\hat{\theta}_1) = var(\hat{\theta}_1) + (\underbrace{\text{bias}(\hat{\theta}_1)}_{0})^2$$

$$MSE(\hat{\theta}_1) = \frac{\theta^2}{3}$$

$$MSE(\hat{\theta}_2) = \frac{\theta^2}{9} + \left(\frac{-2\theta}{3}\right)^2 = \frac{5\theta^2}{9}$$

$$\text{efficiency}(\hat{\theta}_2/\hat{\theta}_1) = \frac{MSE(\hat{\theta}_1)}{MSE(\hat{\theta}_2)} = \frac{\frac{\theta^2}{3}}{\frac{5\theta^2}{9}} = \frac{3}{5} < 1$$

$\therefore \hat{\theta}_1$ is more efficient

Ex(3) let us define $\acute{S}^2 = \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n}$, estimator for σ^2

Show that $E(\acute{S}^2) = \left[\frac{n-1}{n}\right]\sigma^2$

and hence $\acute{S}^2$ is a biased estimator for σ^2

and Find bias($\acute{S}^2$), MSE($\acute{S}^2$), Is $\acute{S}^2$ Consistent ?

$$\because S^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}, \quad \acute{S}^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}$$

$$\acute{S}^2 = \frac{(n-1)\, S^2}{n}$$

$$E(\acute{S}^2) = E\left(\frac{n-1}{n} S^2\right)$$

$$E(\acute{S}^2) = \frac{n-1}{n} E(S^2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

$\therefore$ $\acute{S}^2$ is biased estimator for σ^2

$$\boxed{bias(\hat{\theta}) = E(\hat{\theta}) - \theta}$$

$$MSE(\hat{\theta}) = V(\hat{\theta}) + (bias(\hat{\theta}))^2$$

$$E(S^2) = \sigma^2$$

$$V(S^2) = \frac{2\sigma^4}{n-1}$$

$$bias(\acute{S}^2) = E(\acute{S}^2) - \sigma^2$$

$$= \frac{(n-1)}{n}\sigma^2 - \sigma^2$$

$$= \left[-\frac{1}{n}\right]\sigma^2$$

$$MSE(\acute{S}^2) = Var(\acute{S}^2) + (bias(\acute{S}^2))^2$$

$$Var(\acute{S}^2) = Var\left(\frac{(n-1)}{n} S^2\right)$$

$$= \frac{(n-1)^2}{n^2} V(S^2) = \frac{(n-1)^2}{n^2}\,\frac{2\sigma^4}{n-1}$$

$$Var(\acute{S}^2) = \frac{2(n-1)}{n^2} \sigma^4$$

$$\therefore MSE(\acute{S}^2) = var(\acute{S}^2) + (bias(\acute{S}^2))^2$$

$$= \frac{2(n-1)\sigma^4}{n^2} + \frac{1}{n^2}\sigma^4$$

$$MSE(\acute{S}^2) = \frac{2n-1}{n^2} \sigma^4$$

$$\boxed{\begin{aligned} MSE(S^2) &= \\ var(S^2) &= \frac{2\sigma^4}{n-1} \end{aligned}}$$

As $n \to \infty$ $= \lim_{n\to\infty} MSE(\acute{S}^2)$

$$= \lim_{n\to\infty} \frac{2n-1}{n^2} \sigma^4$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n} - \frac{1}{n^2}}{1} \sigma^4$$

$= 0$ $\therefore S'^2$ is Consistent estimato

Regards

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