Chapter 5

ANALYSIS OF VARIANCE
ANOVA

5.1 One Way (One Factor) ANOVA

Suppose that we have k independent random samples from k populations (or treatments or a factor with k levels). The jth value from the ith population is denoted by $\mathbf{x_{ij}}$, that is,

Population (treatment)	Observations				Totals	Means
1	x_{11}	x_{12}	$\cdots$	x_{1n}	$x_{1.}$	$\bar{x}_{1.}$
2	x_{21}	x_{22}	$\cdots$	x_{2n}	$x_{2.}$	$\bar{x}_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	x_{k1}	x_{k2}	$\cdots$	x_{kn}	$x_{k.}$	$\bar{x}_{k.}$

Table 5.1. Table of Sample Values for the Completely Randomized Design

and we shall assume that the corresponding random variables $\mathbf{X_{ij}}$, which are all **independent**, have **normal** distributions with respective means $\boldsymbol{\mu_i}$ and the **common variance σ^2**

We may describe these observations by the linear model

$$x_{ij} = \mu_i + \varepsilon_{ij}$$

for i=1, 2,..., k, and j=1, 2,...,n, where ε_{ij} are values of **N=nk** independent random variables having normal distributions with zero means and the common variance $\boldsymbol{\sigma^2}$. Given k populations, we may refer to the ***grand mean*** of all the observations as μ, that is,

$$\mu = \frac{1}{k}\sum_{i=1}^{k}\mu_i$$

The amount by which a group mean differs from the grand mean is called the ***treatment effect***. The **i^{th} treatment effect** is given by

$$\alpha_i = \mu_i - \mu \quad \text{where} \quad \sum_{i=1}^{k}\alpha_i = \sum_{i=1}^{k}(\mu_i - \mu) = 0$$

If we substitute for $\boldsymbol{\mu_i}$ in the above model, we obtain the more generalized model,

$$x_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad \text{with} \qquad \sum_{i=1}^{k} \alpha_i = 0$$

The null hypothesis we shall want to test is that the population means are all equal, namely,

$$H_o : \mu_1 = \mu_2 = \ldots = \mu_k = \mu$$

or equivalently that

$$H_o : \alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$$

Correspondingly, the alternative hypothesis is that the populations means are not all equal, namely, that

$$H_a : \alpha_i \neq 0 \text{ for at least one } i.$$

Let

$$\bar{x}_{i.} = \frac{1}{n}\sum_{j=1}^{n} x_{ij} \quad , \quad \text{and} \quad \bar{x}_{..} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}$$

where **N = nk** is the total number of observations.

The test procedure is called the analysis of variance. The name "analysis of variance" results from partitioning total variability in the data into its component parts. The total variability in the data is measured by the ***total (corrected) sum of squares***, abbreviated by **SST** and is given by

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..})^2 = n.\sum_{i=1}^{k}(\bar{x}_{i.} - \bar{x}_{..})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})^2$$

or symbolically as

$$SST = SS(tr) + SSE$$

where

$$SS(tr) = n.\sum_{i=1}^{k}(\bar{x}_{i.} - \bar{x}_{..})^2 = n\sum_{i=1}^{k}\bar{x}_{i.}^2 - N\bar{x}_{..}^2$$

is called the ***treatment sum of squares*** and measures the differences **between** (or **among**) treatments (sometimes denoted by **SSA**) and

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})^2 = SST - SS(tr)$$

is called the ***error sum of squares*** and measures the chance variation (namely the variation **within** the samples and sometimes denoted by **SSW**).

Since we have assumed that ε_{ij} are independent $N(0, \sigma^2)$, the observations x_{ij} are also independent $N(\mu+\alpha_i, \sigma^2)$. Thus

$$SST/\sigma^2 \sim \chi^2_{N-1}$$

We may also show that if H_o is true,

$$SS(tr)/\sigma^2 \sim \chi^2_{k-1}$$

and

$$SSE/\sigma^2 \sim \chi^2_{N-k}$$

Therefore, under H_o, the statistic

$$F_c = \frac{SS(tr)/(k-1)}{SSE/(N-k)} = \frac{MS(tr)}{MSE}$$

follows the $F_{k-1,\ N-k}$ distribution. The quantities **MS(tr)** and **MSE** are called **mean squares**.

It can be shown that, under the alternative hypothesis, the expected value of the numerator of the F_c is greater than the expected value of the denominator. Consequently, we should reject H_o if F_c is large. This implies an upper-tail, one-tail critical region. Therefore, we would reject H_o if

$$F_c > F_{\alpha,\ k-1,N-k}$$

The details of the test procedure are usually presented in the following kind of **analysis of variance table.**

Source of variation	Sum of Squares	Degrees of Freedom	Mean Squares	F_c
Treatments	**SS(tr)**	**k - 1**	**MS(tr)**	**MS(tr)/MSE**
Error	**SSE**	**N - k**	**MSE**	
Total	**SST**	**N - 1**		

Table 5.2. The One-Way ANOVA table

Note

An estimate for the grand mean μ is the overall mean of the observations, i.e.

$$\hat{\mu} = \bar{x}_{..}$$

and the estimate of any treatment effect is just the difference between the treatment average and the grand mean average, i.e.

$$\hat{\alpha}_i = \bar{x}_{i.} - \bar{x}_{..} \quad , \quad i = 1,\ 2, \ldots,\ k$$

An estimate for the experimental variance σ^2 is

$$\hat{\sigma}^2 = \frac{SSE}{N-k} = MSE$$

Example 5.1

Researchers studied the association between birth mothers' smoking habits and the birth weights of their babies. Group 1 consisted of nonsmokers. Group 2 comprised smokers who smoked less than one pack of cigarettes per day. Group 3 smoked more than one but fewer than two packs per day. Group 4 smoked more than two packs per day.

Table 5.3 **Birth Weights in grams of Infants (n = 11 in Each Group) by Mother's Smoking Status**

	Group 1	Group 2	Group 3	Group 4
1	3510	3444	2608	2232
2	3174	3111	2555	2331
3	3580	2890	3100	2200
4	3232	3002	1775	2121
5	3884	2995	2985	2001
6	3982	3101	2479	1566
7	4055	3400	2901	1676
8	3459	3764	2778	1783
9	3998	2997	2099	2002
10	3852	3031	2500	2118
11	3421	3120	2322	1882
Totals	**40147**	**34855**	**28102**	**21912**
Means	**3649.7**	**3168.6**	**2553.7**	**1992.0**

Use the above table to construct an ANOVA table for the test of no mean differences in birth weight among the groups. What is the p-value for this test? What do you conclude about the effect of smoking on birth weight? Estimate the experimental variance, the overall mean and the treatment effects.

Solution

We assume that the birth weights are normally distributed with common variance $\boldsymbol{\sigma}^2$. We have to test;

H_o : The mean birth weights of the four groups are equal

H_a : The mean birth weights of the four groups are not all equal

Let $\alpha = 0.05$

The required means are

$\bar{\mathbf{x}}_{1.} = \mathbf{3649.7}$, $\bar{\mathbf{x}}_{2.} = \mathbf{3168.6}$, $\bar{\mathbf{x}}_{3.} = \mathbf{2553.7}$ $\bar{\mathbf{x}}_{4.} = \mathbf{1992.0}$ and $\bar{\mathbf{x}}_{..} = \mathbf{2841}$

and the remaining calculations are shown in the following ANOVA table

Source of variation	Sum of Squares	Degrees of Freedom	Mean Sum of Squares	F_c	p
Treatments	**17212407**	**3**	**5737469**	**4.01**	**< 0.001**
Error	**3812309**	**40**	**95308**		
Total	**21024716**	**43**			

Since p-value is very small (< 0.001), the null hypothesis must be rejected, and we conclude that the mean birth weights of the four groups are not equal.

An estimate for the experimental (error) variance σ^2 is

$$\hat{\sigma}^2 = \frac{\mathbf{SSE}}{\mathbf{N - k}} = \mathbf{MSE = 95308}$$

c- The estimate of the grand mean is $\hat{\mu} = \bar{\mathbf{x}}_{..} = \mathbf{2841}$

The estimates of the treatment effects are:

$$\hat{\alpha}_1 = \bar{\mathbf{x}}_{1.} - \bar{\mathbf{x}}_{..} = \mathbf{808.7},$$

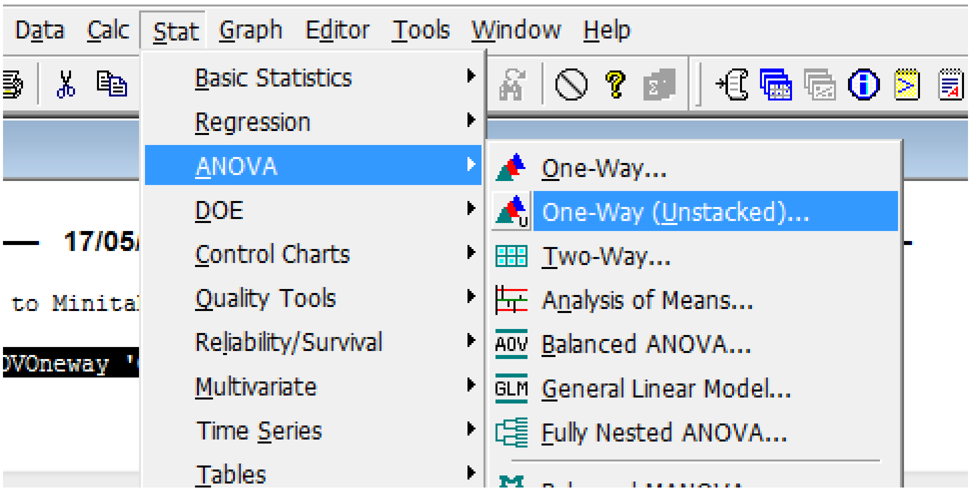
$$\hat{\alpha}_2 = \mathbf{x}_{2.} - \bar{\mathbf{x}}_{..} = \mathbf{327.6},$$

$$\hat{\alpha}_3 = \mathbf{x}_{3.} - \bar{\mathbf{x}}_{..} = \mathbf{-287.3} \text{ and}$$

$$\hat{\alpha}_4 = \mathbf{x}_{4.} - \bar{\mathbf{x}}_{..} = \mathbf{-849.0} \qquad \text{(Note that } \sum_{i=1}^{4} \hat{\alpha}_i = \mathbf{0}\text{)}$$

MINITAB Solution

Put the data of "weights" in 4 columns (variables) name "Gp1, Gp2, Gp3 and Gp4"; then click on **Stat** → **ANOVA** → **One-Way (unstacked).** we obtain the following output:




```
MTB > AOVOneway 'Gp1'-'Gp4'.
```

One-way ANOVA: Gp1, Gp2, Gp3, Gp4

```
Source   DF        SS       MS      F      P
Factor    3  17205520  5735173  60.32  0.000
Error    40   3802867    95072
Total    43  21008387

S = 308.3   R-Sq = 81.90%   R-Sq(adj) = 80.54%
```

```
                                  Individual 95% CIs For Mean Based
on Pooled
                                  StDev
Level    N    Mean   StDev     +---------+---------+---------+-
--------
Gp1     11  3649.7  317.1                                (--
*--)
Gp2     11  3168.6  260.4                        (--*--)
Gp3     11  2554.7  391.6               (---*--)
Gp4     11  1992.0  242.0      (--*--)
                               +---------+---------+---------+-
--------
                                  1800      2400      3000      3600

Pooled StDev = 308.3
```

5.2 Two Way (Factor) ANOVA

In many experimental situations, there are two or more factors that are of simultaneous interest. Suppose that we have k levels of factor A (or treatments) and n levels of factor B (or Blocks). Then there are nk possible combinations consisting of one level of factor A and one of factor B; each such combination is called a "**cell**" or a "**treatment**". In this section we consider the case in which there is a single observation in each cell. An important special case of this type is a **randomized block design**, in which a single factor A is of primary interest but another factor, '**blocks**" is created in order to control for extraneous variability in experimental units or subjects. Thus, if y_{ij} for i=1, 2, ..., k and j=1,2, ..., n are values of independent random variables having normal distributions with the respective means μ_{ij} and the common variance σ^2 , we shall consider the array,

	Block 1	**Block 2**	**...**	**Block n**
Treatment 1	x_{11}	x_{12}	...	x_{1n}
Treatment 2	x_{21}	x_{22}	...	x_{2n}
...	...	...	...	...
Treatment k	x_{k1}	x_{k2}	...	x_{kn}

The model for a two-way ANOVA (without interaction) is

$$x_{ij} = \mu + \alpha_i + \beta_j + e_{ij}$$

for i=1, 2,..., k, and j=1, 2,...,n, where e_{ij} are values of nk independent random variables having normal distributions with zero means and the common variance σ^2. Here μ is to the ***grand mean***, the ***treatment effects*** α_i are such that

$$\sum_{i=1}^{k} \alpha_i = 0$$

the ***block effects*** $\boldsymbol{\beta_j}$ are such that

$$\sum_{j=1}^{n} \beta_j = 0$$

Note that:

$$\mu = \frac{1}{nk} \sum_{i=1}^{k}\sum_{j=1}^{n} \mu_{ij} \qquad \text{and} \qquad \mu_{ij} = \mu + \alpha_i + \beta_j$$

The two null hypothesis we shall want to test are that the treatment effects are all equal to zero and the block effects are all equal to zero, namely,

$\mathbf{H_o : \alpha_i = 0}$ for i=1, 2, ..., k (i.e. factor A has no effect or row means are equal)

and

$\mathbf{H_o' : \beta_j = 0}$ for j=1, 2, ..., n (i.e. factor B has no effect or column means are equal)

The alternative hypotheses are

$$\mathbf{H_a : \alpha_i \neq 0} \quad \text{for at least one i}$$

and

$$\mathbf{H_a : \beta_j \neq 0} \quad \text{for at least one i}$$

The two-way analysis is based on the following generalization of partitioning the total sum of squares as

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..})^2 = n\sum_{i=1}^{k}(\bar{x}_{i.} - \bar{x}_{..})^2 + k\sum_{j=1}^{n}(\bar{x}_{.j} - \bar{x}_{..})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2$$

or symbolically as

$$SST = SS(tr) + SSB + SSE$$

where

$$SSB = k \cdot \sum_{j=1}^{n} (\bar{x}_{.j} - \bar{x}_{..})^2 = k\sum_{j=1}^{n} \bar{x}_{.j}^2 - N\bar{x}_{..}^2$$

is called the ***block sum of squares*** and measures the differences ***between*** (or ***among***) blocks (or columns). The treatments sum of squares SS(tr), the error sum of squares SSE and the total sum of squares SST are given as before.

Therefore, under the null hypotheses, we have

$$SST/\sigma^2 \sim \chi^2_{N-1} \;,\; SS(tr)/\sigma^2 \sim \chi^2_{k-1} \text{ and } SSB/\sigma^2 \sim \chi^2_{n-1}$$

and

$$SSE/\sigma^2 \sim \chi^2_{N-k-n+1}$$

Source of variation	Sum of Squares	D.F.	Mean Square	F_c
Treatments	SS(tr)	k - 1	MS(tr)	F_c = MS(tr)/MSE
Blocks	SSB	n - 1	MSB	F_c' = MSB/MSE
Error	SSE	(n-1)(k-1)	MSE	
Total	SST	N - 1		

Table 5.2. The two-Way ANOVA table

Note that the d.f. corresponding to SSE is (N-1) -(k-1)-(n-1) = (k-1)(n-1).

Therefore, we would reject $\mathbf{H_o}$ if

$$F_c = \frac{SS(tr)/(k-1)}{SSE/(n-1)(k-1)} = \frac{MS(tr)}{MSE} > F_{\alpha, k-1, (k-1)(n-1)}$$

and we would reject $\mathbf{H_o'}$ if

$$F'_c = \frac{SSB/(n-1)}{SSE/(k-1)(n-1)} = \frac{MSB}{MSE} > F_{\alpha, n-1, (k-1)(n-1)}$$

The two-way analysis of variance table is shown in table 5.2

Example 5.2

A laboratory technician measures the breaking strength of each of 4 kinds of linen threads by using 3 different measuring instruments $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ and obtains the following results, in ounces.

	$\mathbf{I_1}$	$\mathbf{I_2}$	$\mathbf{I_3}$
Thread 1	21	20	19
Thread 2	25	26	27
Thread 3	19	23	21
Thread 4	21	23	22

Perform the appropriate analysis of variance to test the null hypothesis concerning the kinds of linen threads and the measuring instruments using the 0.05 level of significance. Estimate the experimental variance, the overall mean, the treatment effects and the block effects. Let $\alpha = 0.05$

Solution

We assume that the breaking strength measurements are normally distributed with common variance σ^2.

We have to test;

H_o : The mean breaking strength measurements of the 4 kinds of threads are all equal

H_a : The mean breaking strength measurements of the 4 kinds of threads are not all equal

and

H_o': The mean breaking strength measurements using the instruments $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ are all equal

H_a' : The mean breaking strength measurements using the instruments $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ are all equal

The required means and sums of squares are

Row means : $\bar{\mathbf{x}}_{1.} = \mathbf{20}$, $\bar{\mathbf{x}}_{2.} = \mathbf{26}$, $\bar{\mathbf{x}}_{3.} = \mathbf{21}$ and $\bar{\mathbf{x}}_{4.} = \mathbf{22}$.

Column means: $\bar{\mathbf{x}}_{.1} = \mathbf{21.5}$, $\bar{\mathbf{x}}_{.2} = \mathbf{23}$ and $\bar{\mathbf{x}}_{.3} = \mathbf{22.25}$.

$$\mathbf{SST} = \sum_{i=1}^{k}\sum_{j=1}^{n}(\mathbf{x}_{ij} - \bar{\mathbf{x}}_{..})^2 = \mathbf{76.25}$$

$$\mathbf{SS\,(tr)} = \mathbf{n}\,.\sum_{i=1}^{k}(\bar{\mathbf{x}}_{1.} - \bar{\mathbf{x}}_{..})^2 = \mathbf{62.25}$$

$$\mathbf{SSB} = \mathbf{k}\,.\sum_{j=1}^{n}(\bar{\mathbf{x}}_{.j} - \bar{\mathbf{x}}_{..})^2 = \mathbf{4.5}$$

Then, by subtraction,

$$\mathbf{SSE = SST - SSB - SS(tr) = 9.5}$$

and the remaining calculations are shown in the following ANOVA table

Source	S.S.	d.f.	M.S.	F_c	P
Treatments	62.25	3	20.75	$F_c = 13.133$	0.0048
Blocks	4.5	2	2.25	$F_c' = 1.424$	0.3118
Error	9.5	6	1.58		
Total	76.25	11			

From the **p-va**lues given in the table, H_o must be rejected, and we conclude that the mean breaking strength measurements of the 4 kinds of threads are not all equal, i.e. there is a difference in mean breaking strength of the kinds of threads, while H_o' cannot be rejected, and we conclude that the instruments $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ are all equally in measuring the breaking strength of threads

MINITAB Solution

Put all data of " Breaking Strength " in one column (variable) name " Breaking Strength" say and create another 2 indicator variables " threads " (treatments) and another variable "instruments " (Blocks) and name it Age. The teaching method variable should contains 1's, 2's and 3's in front of the corresponding observation and age variable should contains 1's, 2's, 3's, 4's and 5's in front of the corresponding observation. Then click on

Stat → ANOVA → Two-Way

we obtain the following output:

↓	C1	C2	C3
	Break-Str	Threads	Instruments
1	21	1	1
2	25	2	1
3	19	3	1
4	21	4	1
5	20	1	2
6	26	2	2
7	23	3	2
8	23	4	2
9	19	1	3
10	27	2	3
11	21	3	3
12	22	4	3

Two-way ANOVA: Break-Str versus Threads, Instruments

```
Source        DF     SS       MS      F      P
Threads        3  62.25  20.7500  13.11  0.005
Instruments    2   4.50   2.2500   1.42  0.312
Error          6   9.50   1.5833
Total         11  76.25

S = 1.258   R-Sq = 87.54%   R-Sq(adj) = 77.16%

               Individual 95% CIs For Mean Based on
               Pooled StDev
Threads  Mean  -------+---------+---------+---------+--
1          20  (------*------)
2          26                        (------*------)
3          21      (------*------)
4          22          (------*------)
               -------+---------+---------+---------+--
                    20.0      22.5      25.0      27.5

                    Individual 95% CIs For Mean Based on
                    Pooled StDev
Instruments   Mean  ----+---------+---------+---------+-----
1            21.50  (------------*------------)
2            23.00               (------------*------------)
3            22.25         (------------*------------)
                    ----+---------+---------+---------+-----
                     20.4      21.6      22.8      24.0
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EXERCISES

[1] Researchers examined bone strength. They collected 10 cadaveric femurs from subjects in three age groups: young (19–49 years), middle-aged (50–69 years), and elderly (70 years or older) [Note: one value was missing in the middle-aged group]. One of the outcome measures (W) was the force in Newtons required to fracture the bone. The following table shows the data for the three age groups.

Young (Y)	Middle-aged (MA)	Elderly (E)
193.6	125.4	59.0
137.5	126.5	87.2
122.0	115.9	84.4
145.4	98.8	78.1
117.0	94.3	51.9
105.4	99.9	57.1
99.9	83.3	54.7
74.0	72.8	78.6
74.4	83.5	53.7
112.8		96.0

Carry out the appropriate analysis of variance to see whether the bone strength depends on age. State the null hypothesis and construct the ANOVA table. Use $\alpha = 0.5$. Estimate the experimental (error) variance, the overall mean and the coating type effects.

[2] The following table shows the arterial plasma epinephrine concentrations (nanograms per milliliter) found in 10 laboratory animals during three types of anesthesia:

	Animal									
Anesthesia	1	2	3	4	5	6	7	8	9	10
A	0.28	0.50	0.68	0.27	0.31	0.99	0.26	0.35	0.38	0.34
B	0.20	0.38	0.50	0.29	0.38	0.62	0.42	0.87	0.37	0.43
C	1.23	1.34	0.55	1.06	0.48	0.68	1.12	1.52	0.27	0.35

Can we conclude from these data that the three types of anesthesia, on the average, have different effects? Let $\alpha = .05$.

[3] **(a)** Complete the following ANOVA table. .05

(b) How many treatments were compared?

(c) How many observations were analyzed?

(d) At the $\alpha = 0.05$ level of significance, can one conclude that the treatments have different effects? Why?

Source	SS	d.f.	MS	F	P
Treatments	154.999	4	*	*	*
Error	*	*	*		
Total	200.4773	39			

[4] Consider the following ANOVA table.

Source	SS	d.f.	MS	F
Treatments	231.505	2	115.753	2.824
Blocks	98.500	7	14.071	
Error	573.750	14	40.982	

(a) How many treatments were compared?
(b) How many observations were analyzed?
(c) At the 0.05 level of significance, can one conclude that the treatments have different effects? Why?

[5] Four different coatings are being considered for corrosion protection of metal pipe. The pipe will be buried in three different types of soil. To investigate whether the amount of corrosion depends either on the coating or on the type of soil, 12 pieces of pipe are selected. Each piece is coated with one of the four coatings and buried in one of the three types of soil for a fixed time, after which the amount of corrosion (depth of maximum pits, in .0001 in.) is determined. The data appears in the following table

	Coating (A)			
Soil Type (B)	1	2	3	4
1	63	54	48	51
2	49	51	43	41
3	50	48	50	52

Carry out the appropriate analysis of variance to see whether the amount of corrosion depends on either the type of coating used or the type of soil. Estimate the experimental (error) variance, the overall mean and the coating type effects.

[6] Complete the following ANOVA table and state which design was used.

Source	SS	d.f.	MS	F	p
Treatments		3			
Blocks	183.5	3			
Error	26.0				
Total	709.0	15			

[7] The following are the cholesterol contents in milligrams per package that three laboratories obtained for 6-ounce packages of four very similar diet foods:

	Diet food				
Laboratory	**A**	**B**	**C**	**D**	**Mean**
1	**2.6**	**3.0**	**3.2**	**2.8**	**2.9**
2	**3.1**	**3.9**	**3.3**	**3.7**	**3.5**
3	**3.0**	**3.3**	**3.4**	**3.1**	**3.2**
Mean	**2.9**	**3.4**	**3.3**	**3.2**	

Perform the appropriate analysis of variance and test the null hypothesis concerning the diet foods and the laboratories at the 0.05 level of significance. Write down the corresponding model, H_o and H_1. State the assumptions about the populations, your decision and conclusion. Estimate the common error variance, the overall mean and the laboratories type effects.

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