dec 7

Chapter 3

Interval estimation

when point est. is not accurate

estimating measurement

let $X_1, X_2, \dots, X_n$ be a R.S from a distribution with unknow Parameter θ

$$P(L \leq \theta \leq U) = 0.95 = 1-\alpha$$

L: lower Confidence limit

θ: parameter

U: upper Conf. limit

0.95: Confidence Coeff.

%95 Confidence level (Coeff)

(L, U) is Called %95 Confidence interval (Limits) for θ

$\downarrow$

$100(1-\alpha)\%$ C.I.

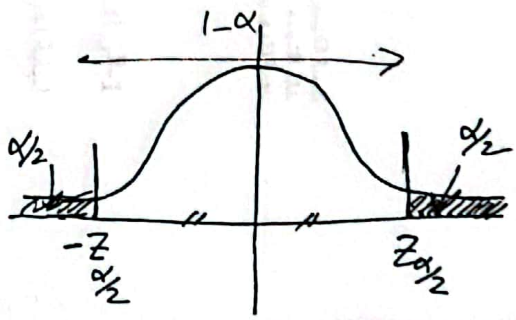
$$P_r(L \leq \theta \leq U) = 1-\alpha$$

A) Confidence Interval for μ (population mean) when σ^2 is Known

let $X_1, X_2, \dots, X_n$ be a R.s. from $N(\mu, \sigma^2)$, σ^2 is Known

then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

then $Z = \dfrac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$



$$P(-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}) = 1-\alpha$$

$$P\left(-Z_{\alpha/2} \leq \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq Z_{\alpha/2}\right) = 1-\alpha$$

$$\boxed{\Phi(Z_{\alpha/2}) = 1-\frac{\alpha}{2}}$$

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$$P\left(-Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X}-\mu \leq Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(-\bar{X}-Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\bar{X}+Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(\bar{X}+Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \geq \mu \geq \bar{X}-Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$\boxed{P\left(\underbrace{\bar{X}-Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}}_{L} \leq \underbrace{\mu}_{\text{population Parameter}} \leq \underbrace{\bar{X}+Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}}_{U}\right) = 1-\alpha}$$

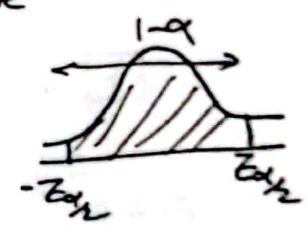
where $\bar{X}$ sample mean of size n

$Z_{\alpha/2}$ Value of standard normal distr. Leaving an area $\frac{\alpha}{2}$ to the right

$$\boxed{\phi(Z_{\alpha/2}) = 1-\frac{\alpha}{2}}$$

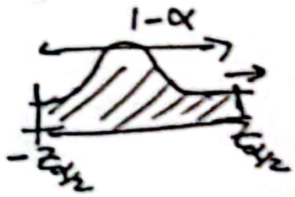
as $(1-\alpha)$ increases $\longrightarrow$ $Z_{\alpha/2}$ increase

C.I will be more wide



as $(1-\alpha)$ decrease $\longrightarrow$ $Z_{\alpha/2}$ decrease

C.I will be more tight



as n increase $\longrightarrow$ C.I more tight

$$\text{Range} = U-L = \left(\bar{X}+Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)-\left(\bar{X}-Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$\text{Range} = 2\,Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$= 2\,E$ error

Ex (3.1)

A Sample size of $n=25$ is drawn from a normal Population with Unknown mean μ and Variance 16, have $\bar{X}=15$

Find a 95% and 99% Confidence interval of μ

$n=25$, $\sigma^2=16$, $\bar{X}=15$
$\sigma=4$

95% C.I

$1-\alpha=0.95 \Rightarrow \alpha=0.05 \Rightarrow \frac{\alpha}{2}=0.025$

to get $Z_{\alpha/2}$

$$\phi(Z_{\alpha/2}) = 1-\alpha/2$$

$$\phi(Z_{\alpha/2}) = 0.975$$

(I have the area CDF) want Point

Using Minitab/ Calc/ Prob distr / normal

- Inverse Cumulative
  Mean $\boxed{0}$, S.deviation $\boxed{1}$
  input const 0.975 ↵

$$\phi(1.96) = 0.975$$

$$\therefore Z_{\frac{\alpha}{2}} = 1.96$$

Thus %95 C.I. is given by

$$\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + Z_{\alpha/2}\frac{\sigma}{n}$$

$$\boxed{13.43 \leq \mu \leq 16.57}$$

$$P(13.43 \leq \mu \leq 16.57) = 0.95$$

For 99% C.I

$1-\alpha = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$

$\Phi(Z_{\alpha/2}) = 1-\frac{\alpha}{2}$

$\Phi(Z_{\alpha/2}) = 0.995$

Similarly, by minitab $Z_{\alpha/2} = 2.58$

thus 99% C.I is given by

$\boxed{13 < \mu < 17}$

Note: 99% C.I. is more wide than 95% C.I.

In Medical applications we choose 99% C.I.

In Business " " " 95% C.I.

default value 95% C.I. $\Rightarrow \alpha = 0.05$

Note: If $X_1, X_2, \dots, X_n$ are R.s. from $N(\mu, \sigma^2)$, n is small $(n<30)$, σ^2 is unknown

then

$$\left(\bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right)$$

replace $\sigma \to S$, $Z_{\alpha/2} \to t_{\alpha/2}$

Maximum Error and Sample size

Used early to get the sample size in terms of error

$$\because P\left(\bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

(reliability Coeff)

$$P\left(-z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \bar{X} - \mu \leq z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(\underbrace{|\bar{X} - \mu|}_{\text{error}} \leq \underbrace{z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}}_{\text{Max error}}\right) = 1-\alpha$$

(margin of error)

What are sample size n

$$\underset{\text{error}}{E} = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

Recall
$|x| < a$
$-a < x < a$

$$\sqrt{n} = \frac{z_{\frac{\alpha}{2}} \sigma}{E}$$

$$\boxed{n = \left(\frac{z_{\frac{\alpha}{2}} \sigma}{\underset{\text{Max error}}{E}}\right)^2}$$

, n is integer (take the Ceil)
e.g. $n = 35.2$
Choose $n = 36$

$$\text{Range} = U - L = 2E$$

if σ is known ✓

if σ is unknown you will have max, min value of observation

$$\sigma \simeq \frac{R}{4} \simeq \frac{|Max - Min|}{4}$$

Ex (3.2) The life, in hours, of a 150-watt light bulb is known to be approx. normally distr. with, $\sigma = 25$ hrs. What sample size should be taken in order to be 95% Confident that the error in estimating the mean life is less than 5 hrs?

$\sigma = 25$, $\alpha = 0.05 \Rightarrow Z_{\alpha/2} = Z_{0.025} = 1.96$

as before

$$n = \left[\frac{Z_{\alpha/2}\,\sigma}{E}\right]^2$$

$\Phi(Z_{0.025}) = 1 - 0.025 = 0.975$

$$n = \left(\frac{(1.96)(25)}{5}\right)^2 = 96$$

Ex (3.3) A medical research worker intends to use the mean of a random Sample of size 120 to estimate the mean blood pressure of women in their fifties. If, based on experience, he knows that $\sigma = 10.5$ mm, what can he assert with Prob. 0.99 about the max error?

$\sigma = 10.5$, $\alpha = 0.01$, $Z_{\alpha/2} = Z_{0.005} = 2.58$

(as before)

$n = 120$

$$\text{Max error } E = \frac{Z_{\alpha/2}\,\sigma}{\sqrt{n}} = \frac{(2.58)(0.01)}{\sqrt{120}} \simeq 2.47 \text{ mm}$$

Recall that (**Case 1**: C.I for μ when σ^2 is known, sample of size n taken from normal (or non-normal, $n \geq 30$))

$$P\left(\bar{X} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

estimator — Population mean — Confidence level

$[L, U]$ $100(1-\alpha)\%$ Confidence interval

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$\bar{X}$: sample mean (estimator)

$Z_{\alpha/2}$: reliability coeff

$\frac{\sigma}{\sqrt{n}}$: standard error of the mean

$Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ → Max error, margin of error

القانون مش حفظ لانه هيكون في f.s. , الشطارة انك تعرف تستخدم القانون صح

Case 2: Confidence interval for μ when σ^2 is unknown and $n < 30$

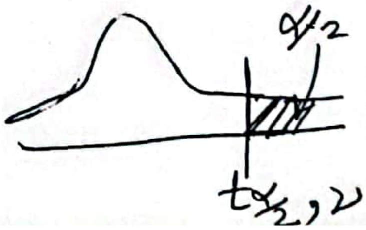
let $X_1, X_2, \ldots X_n$ be a R.S of size $n < 30$ taken from normal Population $N(\mu, \sigma^2)$

then $100(1-\alpha)\%$ conf. interval for μ

$$\bar{X} - t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}$$

$Z_{\frac{\alpha}{2}} \longrightarrow t_{\alpha/2, n-1}$

$\sigma \longrightarrow S$



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Where $\overline{X}$: Sample mean

S : " st. deviation

$t_{\frac{\alpha}{2},\nu}$: Value of t distr. with $\nu = n-1$ degrees of freedom leaving an area of $\frac{\alpha}{2}$ to the right

Ex 3.4 Nine measurements of reaction time of an individual to certain stimuli were recorded as 0.28, 0.33, 0.30, 0.32, 0.27, 0.29, 0.27, 0.31, 0.33 seconds. Find 95% confidence interval for the actual mean reaction time

$n = 9$

$$\overline{X} = \frac{1}{9}\sum_{i=1}^{9} X_i = 0.3 \quad \text{by Calculator}$$

or by Minitab

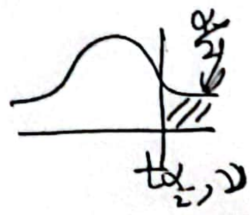
$$S^2 = \frac{1}{9-1}\sum_{i=1}^{9}(X_i - \overline{X})^2 = 0.000575$$

$$S = \sqrt{S^2} = 0.024 \quad \text{(Calculator or Minitab)}$$

$$1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$$

$$\boxed{\Phi(t_{\frac{\alpha}{2},8}) = 1 - \frac{\alpha}{2}}$$

$$\Phi(t_{\frac{\alpha}{2},8}) = 0.975$$



Using Minitab

Calc / prob dist. / t / • inverse Cumulative

Degree of freedom 8

input Const 0.975

$t_{\frac{\alpha}{2},8} = 2.306$

$$0.3 - 2.306\left(\frac{0.024}{3}\right) \leq \mu \leq 0.3 + 2.306\left(\frac{0.024}{3}\right)$$

$$0.282 \leq \mu \leq 0.318$$

Regards

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