Sheet 2

1. Find method of moments estimators (MME's) of Θ based on a random sample X_1, ..., X_n from each of the following pdf's:

   a- f(x; θ) = $\theta x^{\theta-1}$; 0 < x < 1, zero otherwise; θ > 0.

   b- f(x; θ) = (θ + 1)$x^{-\theta-2}$; 1 < x, zero otherwise; θ > 0.

2. Find maximum likelihood estimators (MLE's) for θ based on a random sample of size n for each of the pdfs in problem [1].

3. Let X_1, X_2, ..., X_n be random sample from a geometric distribution
   f($x; \theta$) = $\theta(1-\theta)^{x-1}$ for x = 1,2,3,....

   Find a formula for estimating θ by using,
   a- the method of moments b- the method of maximum likelihood.

4. Let X_1, X_2, ..., X_n be a random sample from a geometric distribution, X ~ GEO(p). Find the MLE's of the following quantities:

   a- E(X) = 1/p. b- Var(X) = (1-p)/p^2.

   b- P[X > k] = $(1-p)^k$ for arbitrary k = 1, 2, ...

   (Hint: Use the invariance property of MLE's)

5. If X_1, X_2, ..., X_n constitute a random sample from a population given by the p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta^2} x e^{-x/\theta} & x > 0;\ \theta > 0 \\ 0 & otherwise \end{cases}$$

   a - Find the maximum likelihood estimator $\hat{\theta}$ for the parameter θ.
   b - Show that the method of moments gives the same estimator $\hat{\theta}$ for θ.
   c - Prove that $\hat{\theta}$ is unbiased and consistent estimator for θ.
   (Hint: $\int_0^\infty x^m e^{-x/\theta}\, dx$ = m! θ^{m+1} for any +ve integer m)

6. If $X_1, X_2, \ldots, X_n$ is a random sample from the Poisson distribution

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!} \quad , \qquad x = 0, 1, 2, \ldots$$

a- Find the maximum likelihood estimator for the parameter θ.

b- Prove that $\hat{\theta}$ is an unbiased consistent estimator for θ

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from EXP(θ) and define $\hat{\theta}_1 = \bar{X}$ and $\hat{\theta}_2 = n\bar{X}/(n+1)$

   a) Find the bias$(\hat{\theta}_1)$ and bias$(\hat{\theta}_2)$
   b) Find the variances of $\hat{\theta}_1$ and $\hat{\theta}_2$
   c) Find the MSE's of $\hat{\theta}_1$ and $\hat{\theta}_2$
   d) Compare the variances of $\hat{\theta}_1$ and $\hat{\theta}_2$ for n = 2
   e) Compare the MSE's of $\hat{\theta}_1$ and $\hat{\theta}_2$ for n = 2

8. Let X_1, X_2 and X_3 be a random sample from a population having mean μ and variance σ^2. Consider the following estimators:

$$\hat{\mu}_1 = \frac{2X_1 + X_2 - X_3}{2} \qquad \& \qquad \hat{\mu}_2 = \frac{3X_1 + 2X_2 - X_3}{4}$$

compare these two estimators. Which do you prefer? Why?

9. Suppose that $\hat{\theta}_1$ and $\hat{\theta}_2$ are estimators of the parameter θ.We know that E($\hat{\theta}_1$) = θ, var($\hat{\theta}_1$) = 10, and E($\hat{\theta}_2$) = $\theta/2$, var($\hat{\theta}_2$) = 4. Which estimator is "best"? In what sense it is best?

10. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators of θ. The estimator $\hat{\theta}_2$ is said to be more efficient than $\hat{\theta}_1$ if

a – var($\hat{\theta}_1$)> var($\hat{\theta}_2$)

b- MSE($\hat{\theta}_1$)> MSE($\hat{\theta}_2$)

c - E($\hat{\theta}_1$)> E($\hat{\theta}_2$)

d- None of the above.

11. Let $\hat{\theta}_1$ *and* $\hat{\theta}_2$ be two unbiased estimators of θ. The estimator $\hat{\theta}_1$ is said to be more efficient than $\hat{\theta}_2$ if

a - E($\hat{\theta}_1^2$)> E($\hat{\theta}_2^2$) b- E($\hat{\theta}_1^2$)< E($\hat{\theta}_2^2$)

c - E($\hat{\theta}_1$)> E($\hat{\theta}_2$) d- E($\hat{\theta}_1$)< E($\hat{\theta}_2$)

12. Suppose that $\hat{\theta}_1$, $\hat{\theta}_2$ *and* $\hat{\theta}_3$ are three estimators of the parameter θ. We know that E($\hat{\theta}_1$) = E($\hat{\theta}_2$) =θ, E($\hat{\theta}_3$) $\neq \theta$, var($\hat{\theta}_1$) = 12, var($\hat{\theta}_2$) = 10 and E($\hat{\theta}_3 - \theta)^2 = 6.$

Then the most efficient estimator between them is:

a - $\hat{\theta}_1$ b- $\hat{\theta}_2$ c- $\hat{\theta}_3$ d- None of the above

13. Let X be a random variable with mean μ and variance σ^2. Given two independent random samples of size 30 and 50 with sample means $\bar{X}_1$ and $\bar{X}_2$, respectively. Show that

$$\bar{X} = \alpha\bar{X}_1 + (1 - \alpha)\bar{X}_2$$

is an unbiased estimator of μ. Find the value of α that minimizes var($\bar{X}$) . Let $\mu = \frac{\bar{X}_1 + \bar{X}_2}{2}$ be another estimator for μ, compare these two estimators. Which do you prefer? Why?