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Solving the Heat Transport Equation

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ABSTRACT

The heat transport equation is a fundamental partial differential equation that models the distribution of temperature over time. It is present in many fields such as thermodynamics, material sciences, and environmental modeling. This paper seeks to solve the equation using the Explicit Finite Difference Method and Crank Nicholson method. The two methods were compared, and time intervals and space intervals were varied so as to see what the differences are between the two. Crank Nicholson was shown to provide more accurate solutions for coarser grids, but when certain conditions were met, the differences between the two methods were shown to be minimal. Future work may extend these methods to higher dimensions or incorporate adaptive mesh refinement techniques for improved performance in complex scenarios.

Keywords: Heat Transport Equation; Partial Differential Equations; Explicit Finite Difference; Crank-Nicholson

INTRODUCTION

The heat transport equation is a fundamental partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time Launder (2005). Its mathematical form, $u_t = u_{xx}$, expresses the temporal change in temperature (u_t) as proportional to the spatial curvature (u_{xx}) of the temperature distribution. This equation appears in thermodynamics, material science, and environmental modeling, making it essential to mathematical physics.

In thermodynamics, the heat equation governs the distribution of temperature within solids, liquids, and gases. It is essential for designing systems that require thermal management, such as engines, cooling systems, and thermal insulation. In material science, it helps predict thermal stresses and phase transitions, which are critical for developing materials with specific thermal properties. In environmental science, it is used to model heat transfer in oceans and the atmosphere, which contributes to climate studies and the prediction of weather patterns Gaskell & Krane (2024); Rau et al. (2012).

In this paper, we investigate the heat transport equation in a one-dimensional domain ($0 \leq x \leq 1$) with time ($t \geq 0$). The problem is defined with an initial condition:

$$f(x) = \sin\left(\frac{1}{2}\pi x\right) + \frac{1}{2}\sin(2\pi x),$$

and boundary conditions:

$$g_0 t = 0 \quad \text{and} \quad g_1 t = \exp\left(-\frac{\pi^2 t}{4}\right).$$

The exact solution to this problem is known, which allows us to make direct comparisons between numerical and analytical results.

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To solve the heat equation numerically, we employ two methods: the Explicit Finite Difference Method and the Crank-Nicholson Method. The Explicit Finite Difference Method approximates the equation using discrete time and space intervals, providing a straightforward implementation but requiring us to be careful of grid sizes to maintain stability and avoid divergence. On the other hand, the Crank-Nicholson Method, a semi-implicit scheme, is very stable and usually offers higher accuracy but is much more complex.

In addition to implementing these methods, we will explore how grid resolution, specifically coarser and finer values of the time step (Δt) and space interval (Δx), affects the accuracy and computational efficiency of each method by varying the grid parameters incrementally.

INITIAL AND BOUNDARY CONDITIONS

As stated in the introduction, the initial condition for the equation at t = 0 for 0 ≤ x ≤ 1 is

$$f(x) = \sin\left(\frac{1}{2}\pi x\right) + \frac{1}{2}\sin(2\pi x),$$

and the boundary conditions are

$$g_0 t = 0 \quad \text{and} \quad g_1 t = \exp\left(-\frac{\pi^2 t}{4}\right).$$

for x = 0 and x = 1 for all t ≥ 0. These conditions are important as they prevent the heat equation from having infinite or undefined solutions by limiting the behavior to x = 0 or x = 1. The initial condition is a combination of two sine waves. This means that the system starts as a combination of two waves but changes according to the heat equation. At t = 0, the system is static and will only move when t 1. The boundary conditions ensures that the solution models real world processes like a fixed temperature region. These conditions also serve to help with our calculations as they need to be used when applying the Explicit Finite Difference Method and Crank-Nicholson method. The boundary conditions in this problem are both Dirichlet Conditions because they specify the value of the solution at the boundaries.

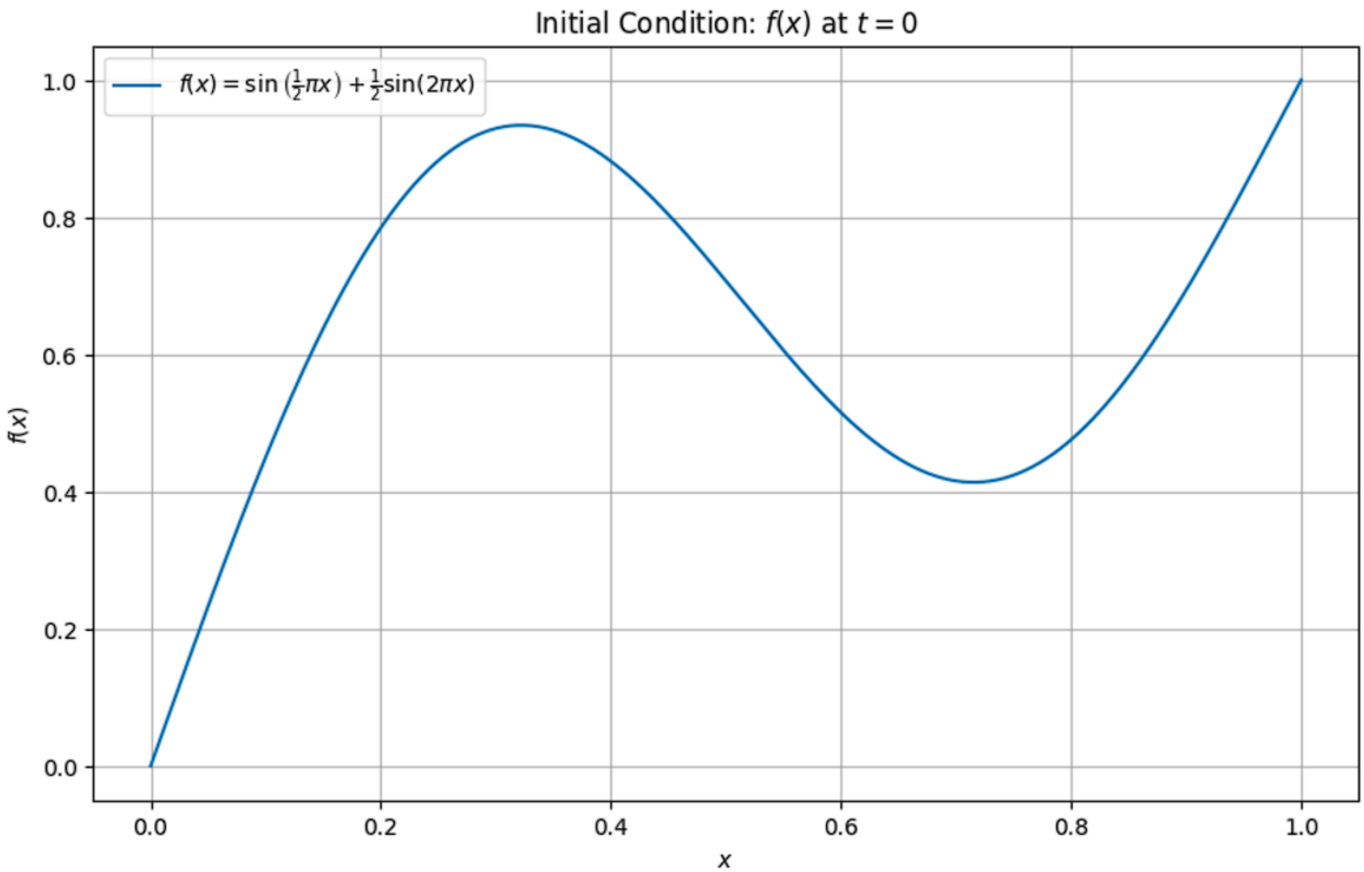

Figure 1. The initial condition at t = 0 for 0 ≤ x ≤ 1

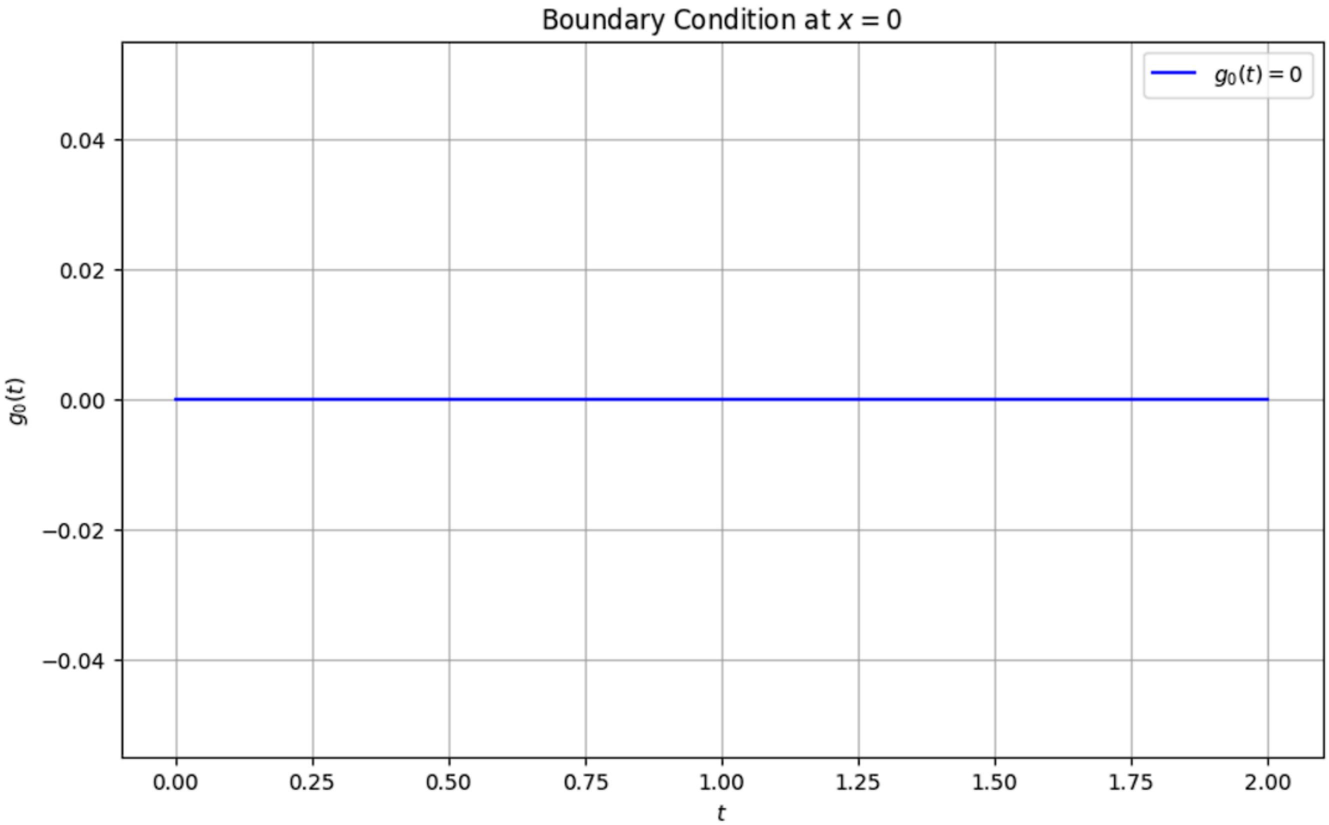

Figure 2. The boundary condition at x = 0

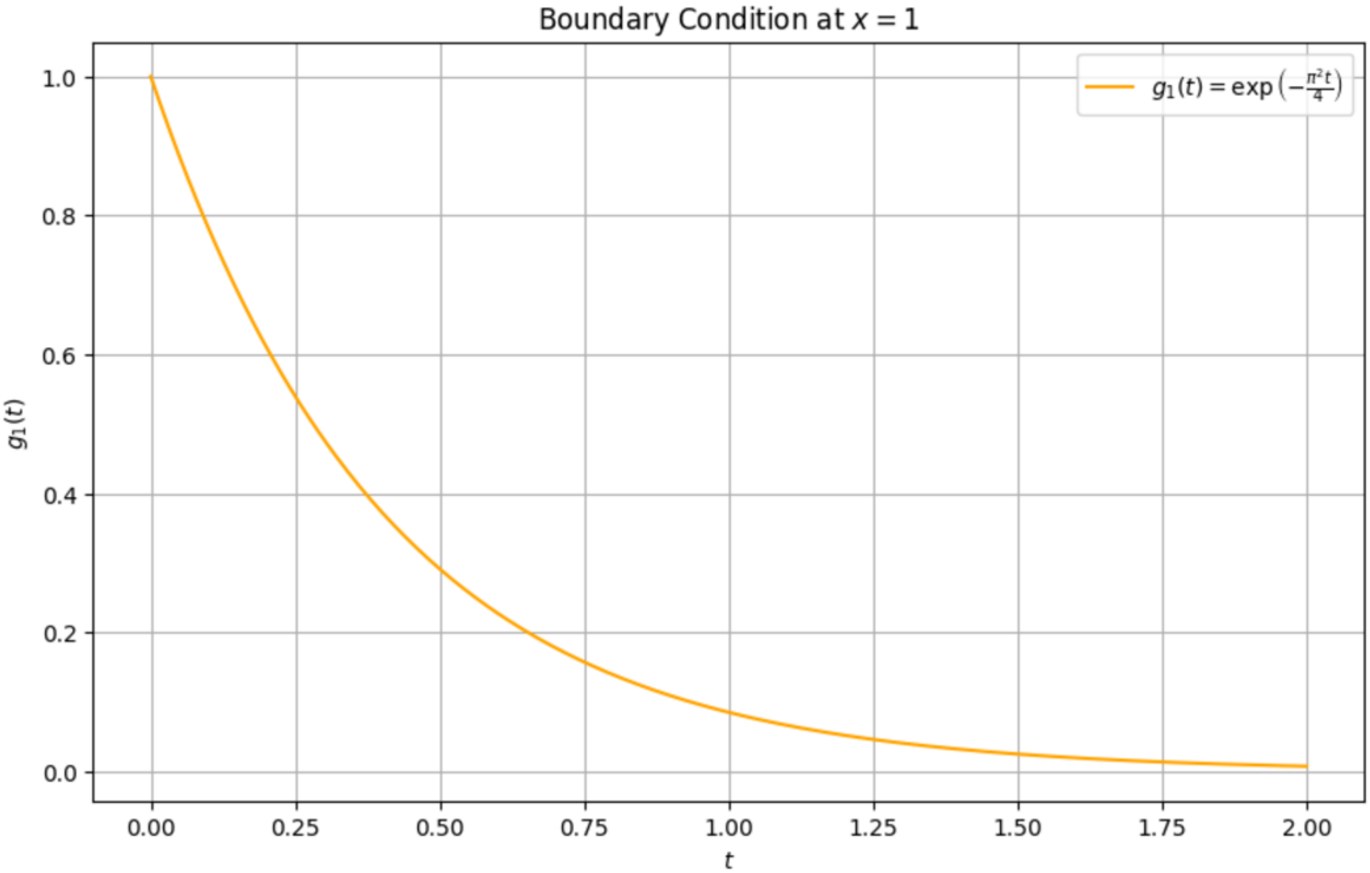

Figure 3. The boundary condition at x = 1

As can be seen from the plots, when x = 1 the initial condition has a value of 1 when t = 0, and the boundary condition is decreasing exponentially. This means that at x = 1, and at t = 0 before time has passed, the amount of heat at that position in space in time is one. Moreover, by observing the graph that models the boundary condition at x = 1, it can be seen that heat dissipates exponentially as time passes.

EXPLICIT FINITE METHOD

The Explicit Finite Method is a numerical method which involves calculating values at future time steps based only on quantities from previous time steps, using iterative methods to approximate points and solutions. This is one of the methods to solve for one dimensional boundary problems. The specific method we are using is called the Forward Time and Center Space (FTCS) Method.

$$T_i^{n+1} = \lambda(1-2)T_i^n + \lambda(T_{i-1}^n + T_{i+1}^n)$$

where λ is equal to $\frac{\triangle t}{\triangle x^2}$ is called mesh ratio parameter. Where the mesh ratio must stay below .5 or else it destabilizes. This highlights the problem with many numerical methods in which it is technically impossible to get the exact solution, even if the mesh ratio is minuscule and the step sizes are as well. Smaller sizes often get better approximation though they are often off by 10^{-16} to 10^{-24} which to us might be negligible but could matter a ton more in certain applications of numerical methods in real life.Adak (2020)

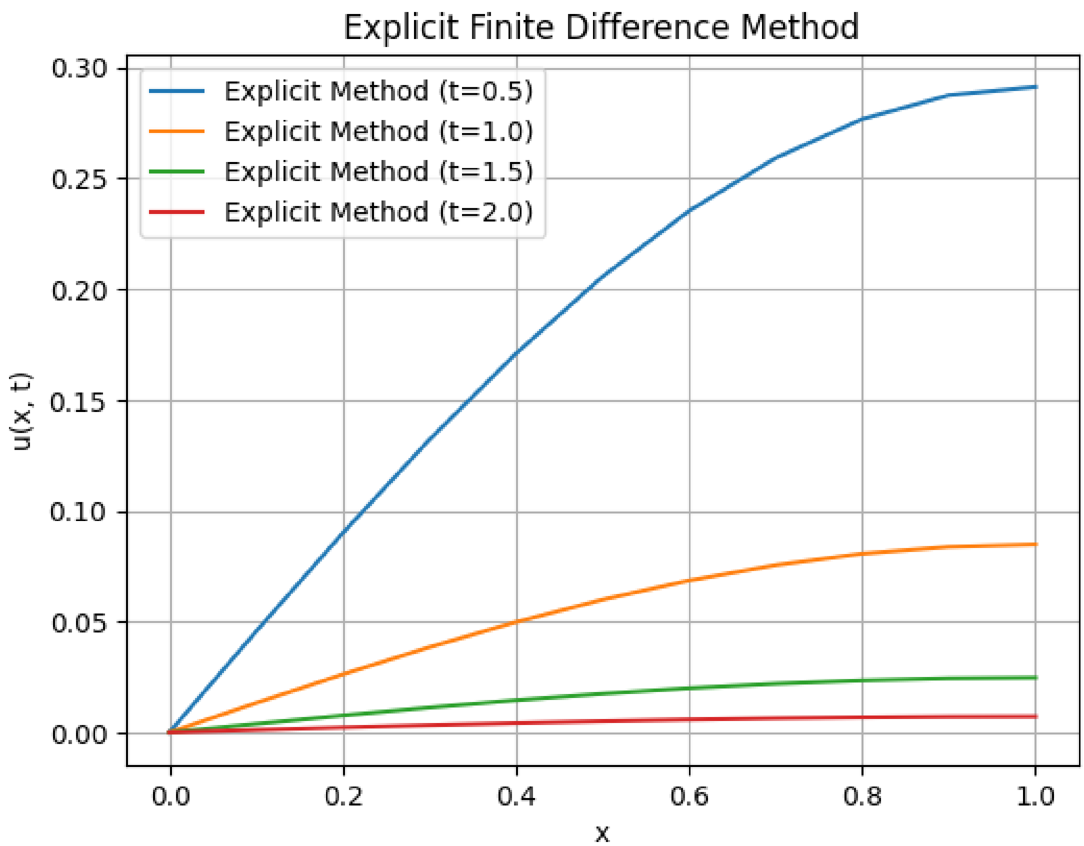

Figure 4. Heat equation at t= 0.5, 1.0, 1.5, 2.0 using the Explicit Finite Difference Method

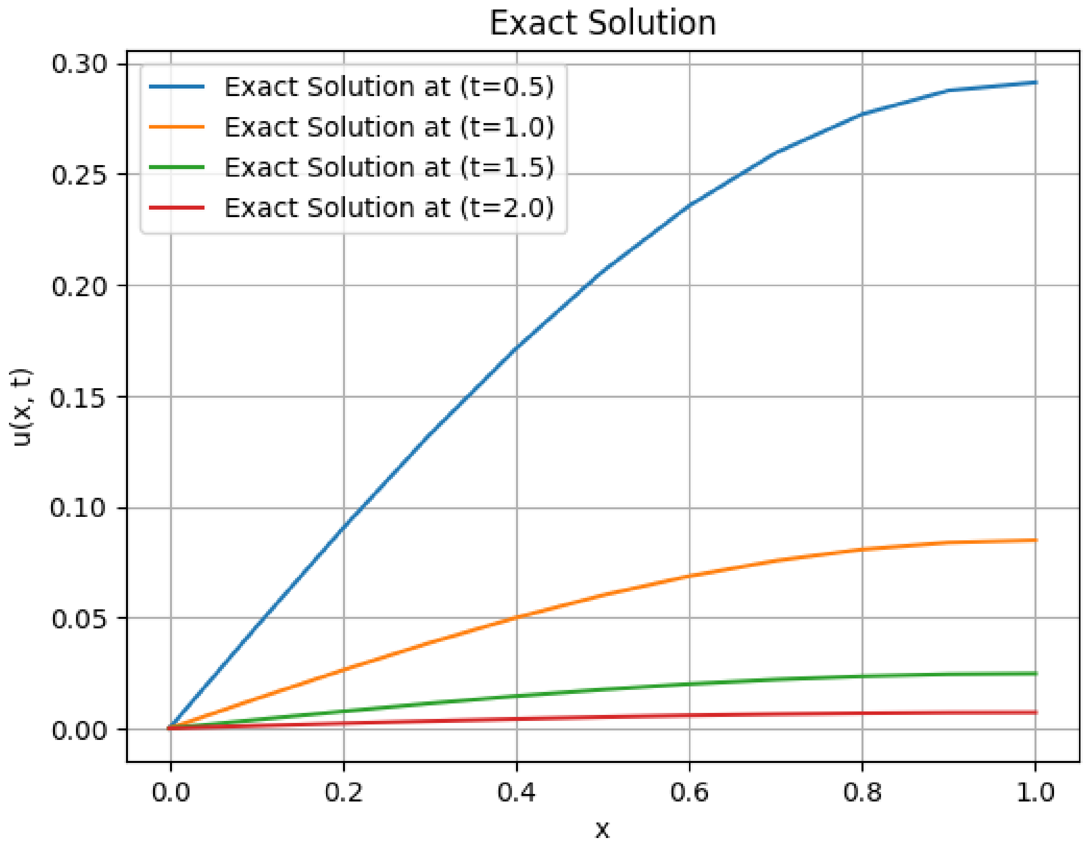

Figure 5. Heat equation at t= 0.5, 1.0, 1.5, 2.0 using the exact solution

Differences

We can see from the graph that there are minimal differences and even the python console output are the same, due to the differences being so minuscule that the console output which goes to 18 decimals has no difference between the exact solution and the explicit solution. Which then highlights how numerical methods using small step sizes results in great results. The error in the explicit method is called Local truncation error in which each iteration due to the jump in computation between the steps, it will be missing all the numbers that possibly could have an effect on the next iteration. This error becomes smaller as you have smaller step sizes and becomes unstable if the mesh ratio parameter is greater than .5. Jammy et al. (2019)

Crank-Nickelson Method

Crank-Nickelson method is a numerical method is based on the trapezoidal rule, giving second-order convergence in time. It is the average of the forward and backward Euler's method and it has the significant advantage of being unconditionally stable unlike the explicit method which is conditionally stable based on the mesh ratio parameter.

$$-\lambda T_{i-1,j-1} + \lambda(1+2)T_{i,j} - \lambda T_{i+1,j+1} = \lambda T_{i-1,j} + 2(1-\lambda)T_{i,j} + \lambda T_{i+1,j}$$

where λ is equal to $\frac{k}{h^2}$ this ratio does not need to less than .5 though it would be more beneficial if the ratio was low, as it indicates that the step sizes are smaller and shows that it provides a more accurate approximation. Liu & Hao (2022)

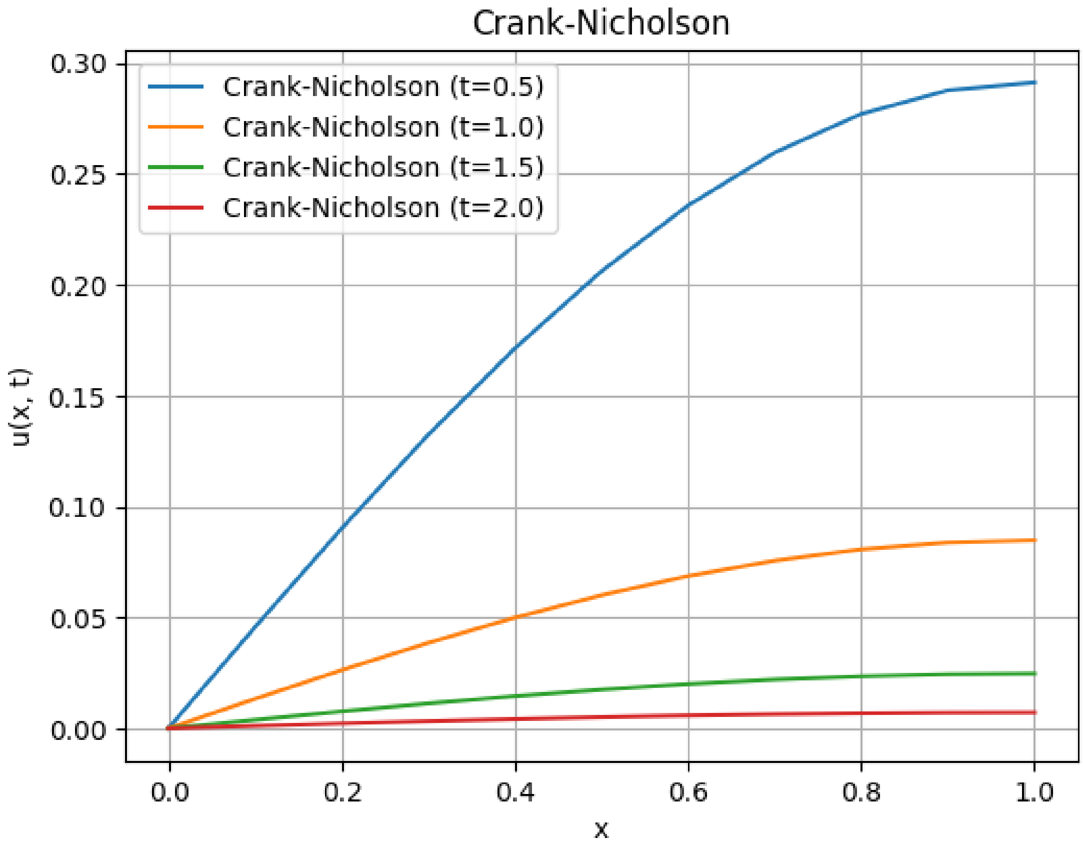

Figure 6. Example Figure

DIFFERENCES USING VARYING LEVELS OF STEP SIZE

When using coarse time steps which are rather big in computation, the Crank-Nickelson Method still works while the Explicit method no longer gives us a stable answer. This is due to the ratio which is conditional to the Explicit method and not conditional to the Crank-Nickelson Method. Which is the main difference, varying levels of step sizes that fit the condition have no differences at least in the 10^{-18} to 10^{-24} range as python does not give us every decimal. This just concludes that the Crank-Nickelson Method is far better due to the lack of restrictions and accurate solutions. Bieniasz et al. (1995)

SUMMARY / CONCLUSION

In this paper, we investigate numerical solutions of the differential equation of heat transport using two finite difference methods: Explicit Finite Difference Method (EFDM) and Crank-Nicolson Method (CNM). Our implementation of EFDM shows that with a small time and space step to satisfy the stability condition ($\lambda = \Delta t/\Delta x^2 < 0.5$), the method yields accurate approximation of the temperature distribution over time. The CNM, being unconditionally stable because of its implicit nature, allowed for larger time steps without sacrificing stability or accuracy. This advantage translates into reduced computational cost while maintaining high fidelity in the numerical solution, as confirmed by the minimal differences observed when comparing the CNM results with the exact solution. Our comparative analysis revealed that while both methods can produce accurate results, the CNM offers superior stability and efficiency, especially when coarser grids are used. Furthermore, we observed that varying the grid resolution had minimal impact on the accuracy of the CNM, whereas the EFDM required finer grids to maintain stability and accuracy. This underscores the suitability of the CNM for problems where computational efficiency is critical. Future work may involve extending these numerical methods to higher-dimensional heat equations or incorporating nonlinear terms to model more complex physical phenomena. Furthermore, exploring adaptive mesh refinement techniques could further enhance the efficiency and accuracy of numerical solutions, particularly in regions with steep temperature gradients. In conclusion, our study highlights the importance of selecting appropriate numerical methods based on the specific requirements of the problem at hand. The advantages of the Crank-Nicolson method in stability and accuracy make it a preferable choice for simulating heat transport in various scientific and engineering applications. Using the strengths of implicit methods such as the CNM, we can achieve reliable and efficient solutions to partial differential equations that are fundamental to understanding and predicting thermal behaviors in complex systems.

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