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1 Introduction

In aquaponic systems, it is commonly reported that certain plant nutrients are lacking. Furthermore, it is oftentimes observed that some compounds reach a concentration plateau even though they are continuously supplied to the system. Thus, there has to be a constraint that is controlling the concentration of these compounds in the water. If microbial uptake is neglected, two potential constraints are

- the chemical satiation concentration
- nutrient depletion via water exchange

. It was found that the satiation concentrations of most of the plant nutrients are about three orders of magnitude higher than the concentrations observed in aquaponic systems. Thus, the water exchange might be what the right thing to look for in some cases.

2 Equations

2.1 General considerations

In general, the mass balance of an aquaculture system can be described considering the mass inputs m_{in} and outputs m_{out}.

$$m_{acc} = m_{in} - m_{out} \tag{1}$$

Depending on the values of M_{in} and m_{out} the resulting scenarios in terms of the mass m_{acc} of a compound in the system are

- $m_{in} > m_{out}$: Accumulation
- $m_{in} = m_{out}$: Balance
- $m_{in} < m_{out}$: Depletion

The accumulation rate over time can be described as

$$\Delta m_{acc} = \frac{dV_{tot}c}{dt} = V_{tot}\frac{dc}{dt} \tag{2}$$

, due to the constant volume V_{tot} of the system. We assume that a substance (for instance a nutrient) is added continuously to the system at a constant mass. Thus, we assume a constant percentage water exchange of the total system volume, a constant mass input of feed and other substances. We can then write

$$V_{tot}\frac{dc}{dt} = Q_{in}c_{in} - Q_{out}c_{out} \tag{3}$$

, with Q_{in} and Q_{out} describing the exchange volume per day and thus a volume flow per time unit and c_{in} and c_{out} as the concentration of the substance in the liquid entering or leaving the system, respectively.

2.2 Scenario 1: Accumulation

Considering a system with a water discharge of 0%, which would be the case if only evaporation water was replaced, Equation 3 would be simplified to a constant mass input:

$$V_{tot}\frac{dc}{dt} = Q_{in}c_{in} \tag{4}$$

The constant input would eventually lead to a linear increase in the concentration of the targeted substance.

2.3 Scenario 2: Balance

The starting point of the inflow-outflow mass balance is Equation 3. Because we are replacing the amount of water that we remove with the same amount of exchange water, we can write

$$Q_{in} = Q_{out} = Q \tag{5}$$

. We can then substitute equation 5 into equation 3, divide by Q and obtain

$$\frac{V_{tot}}{Q}\frac{dc}{dt} = c_{in} - c_{out} \tag{6}$$

. Eventually, this equation can be integrated and rearranged with the following steps (see also Howe et al., Principles of Water Treatment).

$$\frac{Q}{V_{tot}}\int_0^t dt = \int_0^c \frac{dc}{c_{in}-c_{out}} \tag{7}$$

$$\frac{1}{-1}\ln|c_{in}-c_{out}|_0^c = \frac{Q}{V}t \tag{8}$$

$$-\ln|c_{in}-c_{out}| - (-\ln|c_{in}-c_{out}|) = \frac{Q}{V}t \tag{9}$$

$$-\ln(c_{in}-c_{out}) + \ln(c_{in}) = -\frac{Q}{V}t \tag{10}$$

$$\ln(\frac{c_{in}-c_{out}}{c_{in}}) = -\frac{Q}{V}t \tag{11}$$

$$c_{in}-c_{out} = c_{in}\cdot \exp^{-\frac{Q}{V}t} \tag{12}$$

$$1-\frac{c_{out}}{c_{in}} = \exp^{-\frac{Q}{V}t} \tag{13}$$

$$\frac{c_{out}}{c_{in}} = 1-\exp^{-\frac{Q}{V}t} \tag{14}$$

The expression $\frac{Q}{V}$ in the equation is denoting for the **hydraulic retention time (HRT)**

$$\frac{Q}{V} = \tau \tag{15}$$

, so that equation 15 can be substituted into 14, resulting in

$$\frac{c_{out}}{c_{in}} = 1-\exp^{-\frac{t}{\tau}} \tag{16}$$

. Equation 16 can now be rearranged and gives

$$c_{out} = c_{in}(1-\exp^{-\frac{t}{\tau}}) \tag{17}$$

. This model is generally known as the **Continuous Flow Stirred Tank Reactor (CFSTR)** model and is widely used in the field of chemical engineering. As shown in Figure 1, the model approaches 1 for $\lim_{t\to\infty} f(t)$.

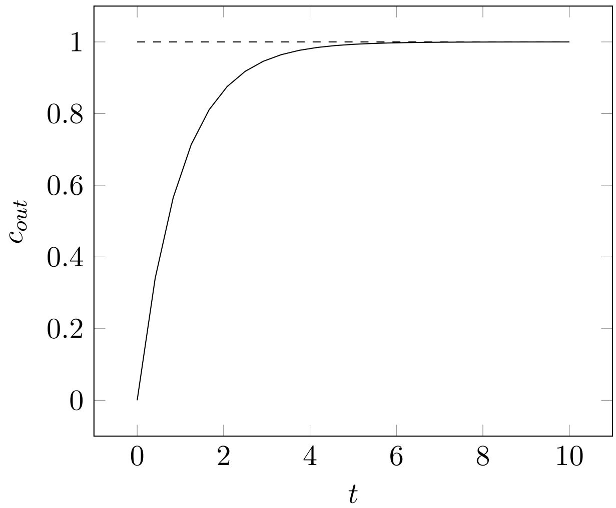

Figure 1: Asymptotic behaviour of the CFSTR model for $\lim_{t\to\infty} f(t)$.

2.3.1 Time until steady state

The time until a steady state in terms of the concentration is reached can be calculated as follows

$$\frac{c_{out}}{c_{in}} = 0.95 = 1 - \exp^{-\frac{t}{\tau}} \tag{18}$$

$$\exp^{-\frac{t}{\tau}} = 1 - 0.95 \tag{19}$$

$$-\frac{t}{\tau} = \ln 0.05 = -3 \tag{20}$$

$$t_{95\%} = 3\tau \tag{21}$$

, with 95% of the maximum concentration possible reached. The result shows, that it takes three times the retention time to reach this state. To come closer to the actual steady state where no change in concentration is expected to take place any more, the value for 99% of the maximum concentration is calculated.

$$\frac{c_{out}}{c_{in}} = 0.99 = 1 - \exp^{-\frac{t}{\tau}} \tag{22}$$

$$\exp^{-\frac{t}{\tau}} = 1 - 0.99 \tag{23}$$

$$-\frac{t}{\tau} = \ln 0.01 = -4.6 \tag{24}$$

$$t_{99\%} = 4.6\tau \tag{25}$$

As can be seen, the maximum concentration is reached after 4.6 times the retention time.

2.3.2 Substance losses over time

To calculate the total loss of a substance over time in terms of mass, Equation 17 can be integrated.

$$c_{out} = \int_0^t c_{in}(1 - \exp^{-\frac{t}{\tau}}) \tag{26}$$

$$c_{out} = c_{in} \int_0^t 1 - \exp^{-\frac{t}{\tau}} \tag{27}$$

$$m_{out} = c_{in} V_{out} \int_0^t 1 - \exp^{-\frac{t}{\tau}} \tag{28}$$

$$m_{out} = c_{in} V_{out} + \tau \cdot \exp^{-\frac{t}{\tau}} + c \tag{29}$$

3 Input routes of substances

The input routes of substances into aquaculture systems are

1. Source water
2. Fish feed
3. "Auxiliary substances" for water management

. The input concentration c_{in} can therefore be rewritten as

$$c_{in} = \frac{m_{in}}{V_{in}} = \frac{m_w + m_f + m_a}{V_{\text{in}}} \tag{30}$$

, with m_w, m_f and m_a denoting for the nutrient mass supplied by water, feed and auxiliary substances, respectively.