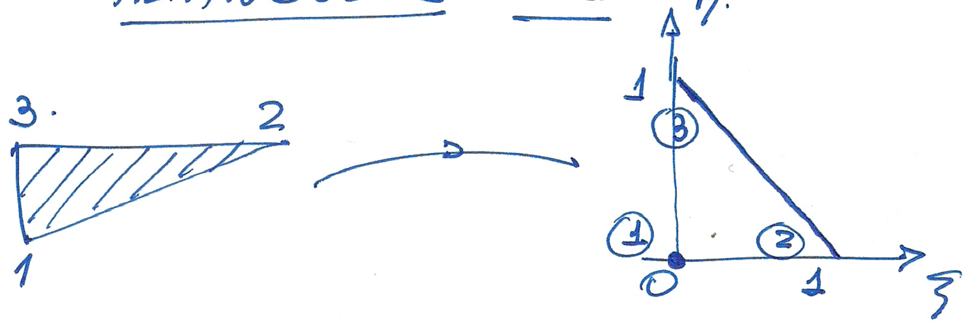
ELEMENTOS ISOPARAMETRICOS.
TRIANGULOS 2Δ



$$N_1(\xi,\eta) = \xi \qquad N_3(\xi,\eta) = 1-\xi-\eta$$
$$N_2(\xi,\eta) = \eta$$

Transformación de coordenadas locales a globales

$$\begin{pmatrix} x \\ y \end{pmatrix}(\xi,\eta) = N_1(\xi,\eta)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + N_2(\xi,\eta)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + N_3(\xi,\eta)\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix}}_{\text{Coordenadas globales.}} = \underbrace{\begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) \end{bmatrix}}_{[\underline{N}_e]} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}}_{\text{DATO}}$$

Matriz de Rigidez.

$$K_{ij} = \int_{\Omega_{ele}} \begin{pmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} d\Omega \qquad i,j \in [1,2,3].$$

En coordenadas naturales

$$K_{ij} = \int_0^1 \int_0^{1-\eta} J_e^{-T} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot J_e^{-T} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \overbrace{\underbrace{|J|}_{\text{determinante.}}\, d\xi\, d\eta}^{d\Omega} \qquad i,j \in [1,2,3].$$

Ojo! El determinante NO DEBE SER NEGATIVO o CERO!!

$$\underline{J_e} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Sabemos que:

$$x = \sum_{i=1}^{3} N_i(\xi,\eta)\, x_i$$

$$y = \sum_{i=1}^{3} N_i(\xi,\eta)\, y_i$$

Reemplazando.

$$\underline{J_e} = \begin{bmatrix} \displaystyle\sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \xi} x_i & \displaystyle\sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \eta} x_i \\ \displaystyle\sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \xi} y_i & \displaystyle\sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \eta} y_i \end{bmatrix}$$

NOTA:

$$\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial N}{\partial \eta} = \frac{\partial N}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N}{\partial y}\frac{\partial y}{\partial \eta}$$

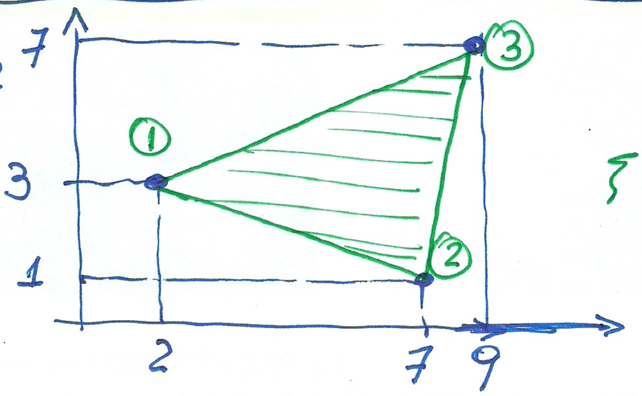
$$\Rightarrow \begin{pmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{pmatrix} = \underline{J_e}^T \begin{pmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{pmatrix}$$

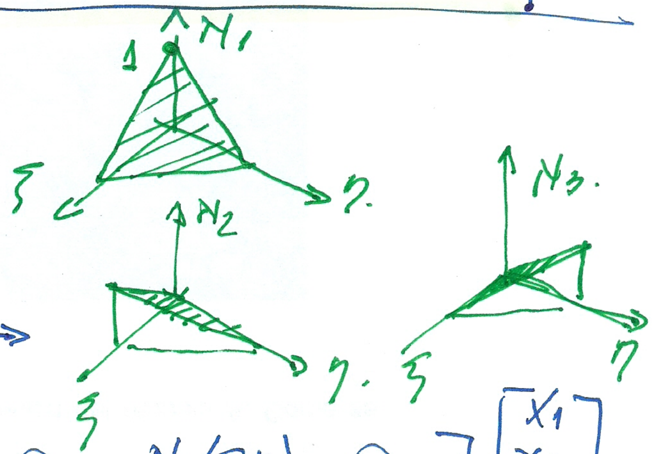
donde.

$$\boxed{J_e^{-T} \begin{pmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{pmatrix}}$$

por eso usamos J_e^{-T} en ⊗

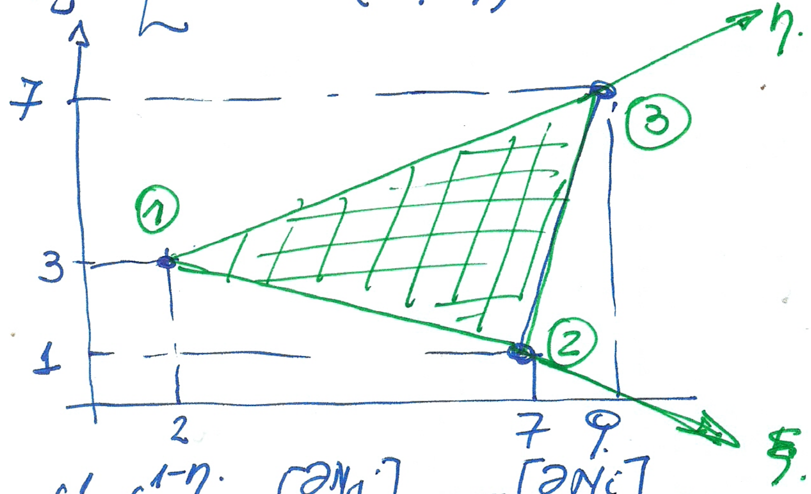
Ejemplo





$$\begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix} = \begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} X(\xi,\eta) \\ Y(\xi,\eta)\end{bmatrix} = \begin{bmatrix} (1-\xi-\eta) & 0 & \xi & 0 & \eta & 0 \\ 0 & (1-\xi-\eta) & 0 & \xi & 0 & \eta \end{bmatrix} \begin{bmatrix}2\\3\\7\\1\\9\\7\end{bmatrix}$$



$$K_{ij} = \int_0^1 \int_0^{1-\eta} J_e^{-T} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} J_e^{-T} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} |J_e| \, d\xi \, d\eta \qquad i,j \in [1,2,3].$$

$$J_e = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \Rightarrow J^T = \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \underbrace{\begin{bmatrix} 1-\xi-\eta & \xi & \eta \end{bmatrix}}_{[N_e](\xi,\eta)} \underbrace{\begin{bmatrix} 2 & 3 \\ 7 & 1 \\ 9 & 7 \end{bmatrix}}_{\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}}$$

$$\Rightarrow J_e = \begin{bmatrix} 5 & 7 \\ -2 & 4 \end{bmatrix}$$

Su determinante $|J_e| = 34$.

$$J_e^{-T} = \frac{1}{|J_e^T|} \begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix}.$$

Tomamos la Inversa del Jacobiano

Armamos la matriz de rigidez.

$$K_{ij}^e = \int_0^1 \int_0^{1-\eta} J_e^{-T} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot J_e^{-T} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \cdot |J_e| \, d\xi \, d\eta.$$

$$K_{ij}^e = \int_0^1 \int_0^{1-\eta} \left[\frac{1}{34} \begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix}\right] \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot \left[\frac{1}{34} \begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix}\right] \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} 34 \, d\xi \, d\eta.$$

o sea $\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$

Ejemplo

$$K_{21}^e = \int_0^1 \int_0^{1-\eta} \frac{1}{34}\begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{\begin{bmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{bmatrix}} \cdot \frac{1}{34}\begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\begin{bmatrix} \frac{\partial N_2}{\partial \xi} \\ \frac{\partial N_2}{\partial \eta} \end{bmatrix}} 34 \, d\xi \, d\eta$$

$$K_{21}^e = \int_0^1 \int_0^{1-\eta} \frac{1}{34}\begin{bmatrix} -6 \\ 2 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \end{bmatrix} d\xi \, d\eta = -1.118 \int_0^1 \int_0 d\xi \, d\eta =$$

$$\boxed{K_{21}^e = -1.118 \cdot \frac{1}{2} = -0.559}$$

y así con el resto de los componentes.

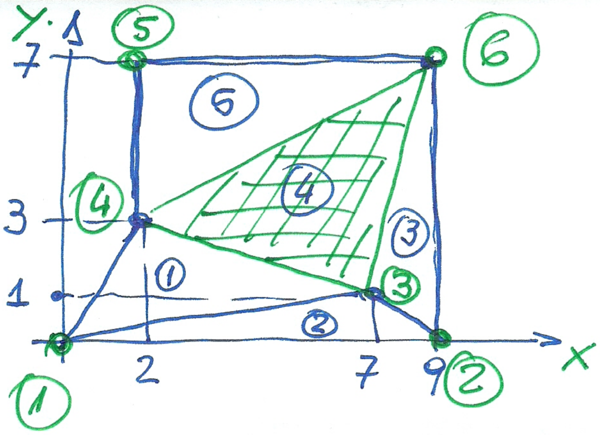
Si tuviera una carga distribuida en todo el elemento $p(x)$ en la dirección x.

$$f^e = \begin{bmatrix} \int_{\Omega e} p(x) N_1(x,y)\, dx \\ \int_{\Omega e} p(x) N_2(x,y)\, dx \\ \int_{\Omega e} p(x) N_3(x,y)\, dx \end{bmatrix} = \begin{bmatrix} \int_0^1 \int_0^{1-\eta} p(\xi) N_1(\xi,\eta) |J_e| \, d\xi \, d\eta \\ \int_0^1 \int_0^{1-\eta} p(\xi) N_2(\xi,\eta) |J_e| \, d\xi \, d\eta \\ \int_0^1 \int_0^{1-\eta} p(\xi) N_3(\xi,\eta) |J_e| \, d\xi \, d\eta \end{bmatrix}$$

$|J_e|$ = determinante en el ejemplo $= 34$

En caso que tengamos que ensamblar.

Dado el sig. problema:



V

El elemento anterior está inserto en un sistema global. cuyo dominio. se ve en la figura.

$$
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & & & & & & \\
2 & & & & & & \\
3 & & & K_{22}^{(4)} & K_{21}^{(4)} & & K_{23}^{(4)} \\
4 & & & K_{12}^{(4)} & K_{11}^{(4)} & & K_{13}^{(4)} \\
5 & & & & & & \\
6 & & & K_{32}^{(4)} & K_{31}^{(4)} & & K_{33}^{(4)} \\
\hline
\end{array}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}
=
\begin{bmatrix} \\ \\ f_2^{(4)} \\ f_1^{(4)} \\ \\ f_3^{(4)} \end{bmatrix}
$$

$$\underline{K}\,\underline{u} = \underline{f}$$

local. $\rightarrow$ 1 2 3

global. $\rightarrow$ 4 3 6

$$
\text{local}\left\{
\begin{array}{cc}
1 & 4 \\
2 & 3 \\
3 & 6
\end{array}
\right.
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}
\begin{array}{c} 4 \\ 3 \\ 6 \end{array}
$$

global.

——— x ———