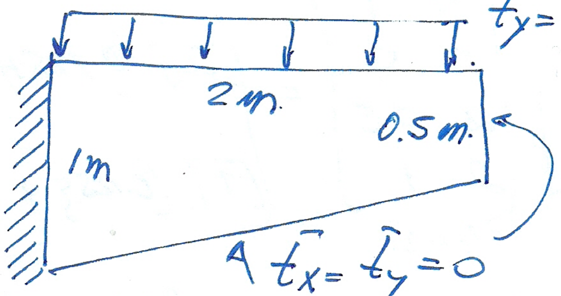
EJEMPLO

Consideremos el siguiente problema de elasticidad lineal. La arista vertical izquierda está fija como si tuviera un empotramiento. El fondo y la arista vertical derecha es de libre tracción ($\bar{t}_x=\bar{t}_y=0$)

La arista superior tiene una carga de compresión de $\bar{t}_y=-20$ N/m. El modulo de elasticidad del material es $E=3\cdot 10^7$ [Pa] (Pa = N/m^2) y el coef. de Poisson $\nu=0.3$. Se consideran condiciones de tensión plana

$$D=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$$

Discretizar el dominio con elementos cuadriláteros.



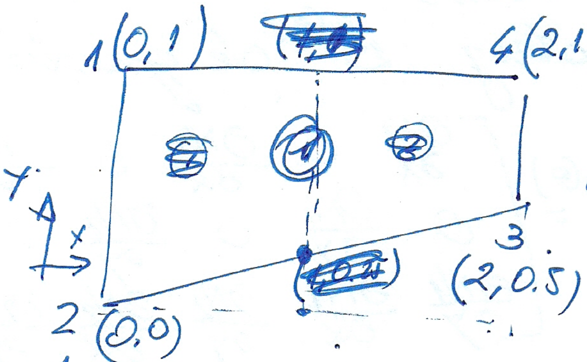
La matriz de elasticidad es:

$$D=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}=\frac{3\cdot 10^7}{1-0.3^2}\begin{bmatrix}1. & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & \frac{1-0.3}{2}\end{bmatrix}=\frac{3\cdot 10^7}{0.91}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}$$

$$D=3.2967\times 10^7\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}$$

LA PLANTEO COMO UN SOLO ELEMENTO

Con 1 elemento



Un solo elemento

$$[x^e \;\; y^e]=\begin{bmatrix}x_1^e & y_1^e\\ x_2^e & y_2^e\\ x_3^e & y_3^e\\ x_4^e & y_4^e\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\\ 2 & 0.5\\ 2 & 1\end{bmatrix}.$$

Matriz coordenadas

$y=a+bx$
$a=0$
$0.5=a+b\cdot 2$
$\frac{0.5}{2}=b$

Las funciones de forma son:

$$\hat{N}_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$\hat{N}_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$\hat{N}_3 = \frac{1}{4}(1+\xi)(1+\xi)$$
$$\hat{N}_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

$$\underset{=}{J}^{(e)} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} =$$

$$\underset{=}{J}^{(e)} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}.$$

$$\frac{1}{4} \underset{2\times 4}{\begin{bmatrix} (\eta-1) & (1-\eta) & (1+\eta) & (-\eta-1) \\ (\xi-1) & (-\xi-1) & (1+\xi) & (1-\xi) \end{bmatrix}} \underset{4\times 2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0.5 \\ 2 & 1 \end{bmatrix}} = \begin{bmatrix} [0] & [0.125\eta - 0.375] \\ [1] & [0.125\xi + 0.125] \end{bmatrix}$$

$$\underset{=}{J}^{(e)-1} \begin{bmatrix} \frac{1+\xi}{3-\eta} & 1 \\ \frac{8}{\eta-3} & 0 \end{bmatrix}$$

la inversa del Jacobiano

DETERMINANTE del JACOBIANO

$$\boxed{|J^e| = -0.125\eta + 0.375}$$

La matriz $B^{(e)}$ de gradientes será:

$$\underset{=}{B}^{(e)} \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

Para calcular la matriz de rigidez se usa 4 puntos de Gauss (2 en c/dirección)

$$(\xi, \eta) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right); \quad W_i = W_j = 1. \quad \text{Pesos.}$$

$$K^1 = \int_\Omega \underset{=}{B}^{eT} \underset{=}{D}\, \underset{=}{B}^e \, d\Omega = \int_{-1}^{1}\int_{-1}^{1} B^{eT} D\, B^e |J^e|\, d\xi\, d\eta.$$

$$K^1 = \int_{\Omega^1} B_e^T D B_e |J^e| d\xi d\eta = \sum_{i=1}^{2} \sum_{j=1}^{2} W_i \cdot W_j \cdot \{B_e^T D^e B_e |J^e|\}_{(\xi_i, \eta_j)}$$ ②

pesos de la integración 1D en las direcciones

$i \to \xi$, $\eta \to j$

Calculamos la contribución de $B^{(e)}$ desde los puntos de Gauss $(\xi, \eta) = \left(-\frac{1}{\sqrt{3}}; \frac{-1}{\sqrt{3}}\right)$ y se obtiene lo siguiente:
Primero saco esto y luego formo B

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix} = J^{(e)^{-1}}_{(\xi_i; \eta_j)} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}_{(\xi_i; \eta_j)}$$

por ejemplo $\cancel{\frac{\partial N_1}{\partial x}}$ $\frac{\partial N_1}{\partial \xi} = \eta - 1 = -\frac{1}{\sqrt{3}} - 1 = \frac{-1-\sqrt{3}}{\sqrt{3}}$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix} = J^{e^{-1}}_{(\xi_i;\eta_i)} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \eta} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}_{(\xi_i;\eta_i)}$$

$$= \begin{bmatrix} -0.44 & -0.06 & 0.12 & 0.38 \\ 0.88 & -0.88 & -0.24 & 0.24 \end{bmatrix}.$$

$$B^e_{(\xi_i;\eta_i)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

$(-1/\sqrt{3}, -1/\sqrt{3})$

$$B^{e^T} = \begin{bmatrix} -0.44 & 0 & -0.06 & 0 & 0.12 & 0 & 0.38 & 0 \\ 0 & 0.88 & 0 & -0.88 & 0 & -0.24 & 0 & 0.24 \\ 0.88 & -0.44 & -0.88 & -0.06 & -0.24 & 0.12 & 0.24 & 0.38 \end{bmatrix}$$

$$K^1 = \underbrace{W_1 W_1}_{=1\ \ =1} \left\{ B^{e^T} D\, B^e\, |J^e| \right\}_{\xi_i;\eta_i}$$

son los pesos.

$$|J^e| = -0.125\left(\frac{-1}{\sqrt{3}}\right) + 0.375$$

$$|J^e| = \underline{0.46962}.$$

$$D = 3.2967 \times 10^7 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

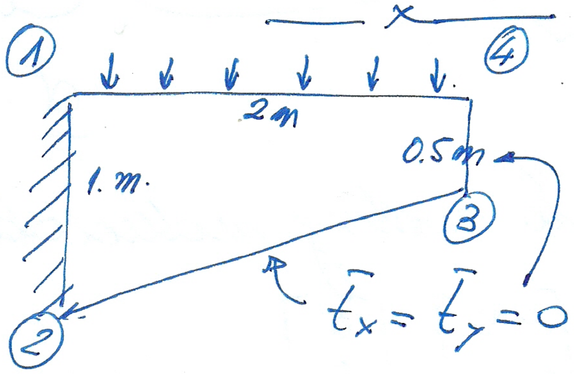
Hacer el calculo

$$\underbrace{\underset{8\times3}{B^{e^T}}\ \underset{3\times3}{D}\ \underset{3\times8}{B^e}}_{8\times8}$$

Obtengo la contribución para mi único elemento.

K^1 = Sumamos las contribuciones de c/punto ⇒

$$K^1 = \begin{bmatrix} 1.49 & -0.74 & -0.66 & 0.16 & -0.98 & 0.65 & 0.15 & -0.07 \\ & 2.75 & 0.24 & -2.46 & 0.66 & -1.68 & -0.16 & 1.39 \\ & & 1.08 & 0.33 & 0.15 & -0.16 & -0.56 & -0.41 \\ & \text{SIMETRICA} & & 2.6 & -0.08 & 1.39 & -0.41 & -1.53 \\ & & & & 2 & -0.82 & -1.18 & 0.25 \\ & & & & & 3.82 & 0.33 & -3.53 \\ & & & & & & 1.59 & 0.25 \\ & & & & & & & 3.67 \end{bmatrix}$$



$$f_\Gamma^{(e)} = \int_\Gamma N^T \bar{t}\, d\Gamma = \int_{-1}^{1} N^T \bar{t}\Big|_{\xi=-1,\eta} d\eta = \begin{bmatrix} \int_{-1}^{1} N_1 d\eta & 0 \\ 0 & \int_{-1}^{1} N_1 d\eta \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \int_{-1}^{1} N_4 d\eta & 0 \\ 0 & \int_{-1}^{1} N_4 d\eta \end{bmatrix} \begin{bmatrix} 0 \\ -20 \end{bmatrix} =$$

↳ valor de $\underline{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

las fuerzas superficiales están en los nodos 1 y 4 dirección η.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -20 \end{bmatrix} = \begin{bmatrix} 0 \\ -20 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20 \end{bmatrix} \Rightarrow f^e = \begin{bmatrix} r_{x1} \\ r_{y1} - 20 \\ r_{x2} \\ r_{y2} \\ 0 \\ 0 \\ 0 \\ -20 \end{bmatrix}$$

$$\underset{\text{vector desplaz.}}{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{se calculan}$$

$\left.\begin{matrix} r_{x1} & r_{y1} \\ r_{x2} & r_{y2} \end{matrix}\right\}$ reacciones en los nodos 1 y 2 ya que allí está la sujeción

El Sistema Quedaría así

$$10^{7}\underbrace{\begin{bmatrix} 1.49 & -0.74 & -0.66 & 0.16 & -0.98 & 0.65 & 0.15 & -0.07 \\ & 2.75 & 0.24 & -2.46 & 0.66 & -1.68 & -0.16 & 1.39 \\ & & 1.08 & 0.33 & 0.15 & -0.16 & -0.56 & -0.41 \\ & & & 2.6 & -0.07 & 1.39 & -0.41 & -1.53 \\ & & & & 2. & -0.82 & -1.18 & 0.25 \\ & & & & & 3.82 & 0.33 & -3.53 \\ & & & & & & 1.59 & 0.25 \\ & & & & & & & 3.67 \end{bmatrix}}_{\underline{\underline{K}}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}}_{\underline{d}} = \underbrace{\begin{bmatrix} R_{x1} \\ R_{y1}-20 \\ R_{x2} \\ R_{y2} \\ 0 \\ 0 \\ 0 \\ -20 \end{bmatrix}}_{\underline{f}}$$

K_e^R

Solamente calculamos la matriz reducida.

$$\begin{bmatrix} 2 & -0.82 & -1.18 & 0.25 \\ -0.82 & 3.82 & 0.33 & -3.53 \\ -1.18 & 0.33 & 1.59 & 0.25 \\ 0.25 & -3.53 & 0.25 & 3.67 \end{bmatrix} \begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -20 \end{bmatrix}$$

$$\underline{d}_R = 10^{-6}\begin{bmatrix} -1.17 \\ -9.67 \\ 2.67 \\ -9.94 \end{bmatrix} \rightarrow \text{la } \underline{d} \text{ total sería } = 10^{-6}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.17 \\ -9.67 \\ 2.67 \\ -9.94 \end{bmatrix}$$

nodos restring.

Si quiero calcular las tensiones:

$$\sigma^e(\xi_1,\eta_1) = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_{\xi_1,\eta_1} = \underline{\underline{D}}^e \underline{\varepsilon}^e(\xi_1,\eta_1) = \underline{\underline{D}}^e \underline{\underline{B}}^e(\xi_1,\eta_1) d^e$$

en el punto de Gauss

$$\xi_1,\eta_1 = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$3.3\times10^{7}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} -0.44 & 0 & -0.06 & 0 & 0.12 & 0 & 0.38 & 0 \\ 0 & 0.88 & 0 & -0.88 & 0 & -0.24 & 0 & 0.24 \\ 0.88 & -0.44 & -0.88 & -0.06 & -0.24 & 0.12 & 0.24 & 0.38 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.17 \\ -9.67 \\ 2.67 \\ -9.94 \end{bmatrix}$$

me da $\begin{bmatrix} -12.5 \\ -5.64 \\ -45.5 \end{bmatrix}$