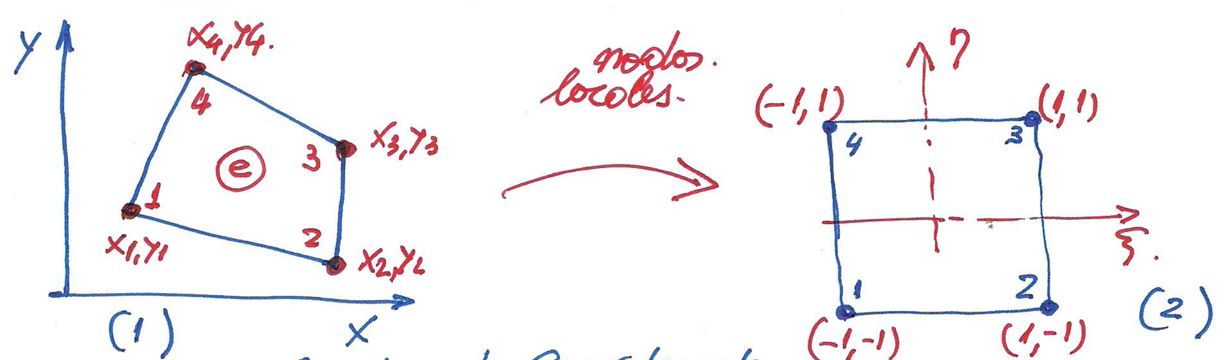
ELEMENTO RECTANGULAR GENÉRICO



Cambio de Coordenadas.

Vamos a obtener la matriz de rigidez de un elemento cuadrilátero genérico de 4 nodos

Cuando el elemento tiene la forma de la figura, obtener la expresión analítica de su matriz de rigidez es bastante complicado.

Para la expresión de su matriz de rigidez se opta por la integración numérica. Para facilitar este proceso de integración se procede al cambio de coordenadas, es decir las coordenadas cartesianas iniciales (x,y) se transforman en las naturales (ξ,η) de la figura (2). La integración se realiza en el dominio regular de las coordenadas naturales

En primer lugar recurrimos a las mismas funciones de forma con que aproximamos los desplazamientos

$$\Rightarrow \boxed{x=\sum_{i=1}^{n} N_i(\xi,\eta)\,x_i \quad ; \quad y=\sum_{i=1}^{n} N_i(\xi,\eta)\,y_i} \qquad (1)$$

Vamos a integrar la matriz de rigidez:

$$\boxed{\underline{\underline{K}}^e=\int_A \underline{\underline{B}}^T\,\underline{\underline{D}}\,\underline{\underline{B}}\,dA.} \qquad (2)$$

Como nos interesan las derivadas (gradientes) para calcular $\underline{\underline{B}}$ $\Rightarrow$ aplicando la regla de la cadena:

$$\boxed{\begin{aligned}\frac{\partial N_i}{\partial \xi}&=\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}\\ \frac{\partial N_i}{\partial \eta}&=\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}\end{aligned}} \qquad (3)$$

En forma matricial:

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\[2ex] \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{\underline{J}^e \text{ Matriz Jacobiana}} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\[2ex] \dfrac{\partial N_i}{\partial y} \end{bmatrix} \qquad (4)$$

$\underline{J}^e$: Matriz Jacobiana de transformación de coordenadas naturales a cartesianas.

La relación inversa:

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\[2ex] \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \underbrace{\frac{1}{\underbrace{|J_0^e|}_{\text{determinante Jacobiano}}} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\[2ex] -\dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}}_{\underline{J}^{e-1}} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\[2ex] \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} \qquad (5)$$

Al integrar:

$$\boxed{\int_y \int_x dx\,dy = \int_\eta \int_\xi |J^{(e)}|\, d\xi\, d\eta} \qquad (6)$$

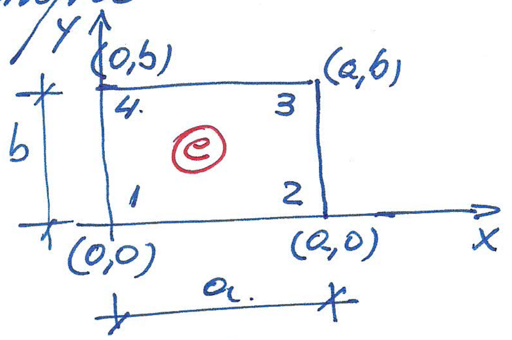
Para calcular el Jacobiano partimos de la expresión (4) donde los elementos de $\underline{J}^e$ se obtienen:

$$\boxed{\begin{aligned} \frac{\partial x}{\partial \xi} &= \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i \;; & \frac{\partial y}{\partial \xi} &= \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i \\ \frac{\partial x}{\partial \eta} &= \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} x_i \;; & \frac{\partial y}{\partial \eta} &= \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} y_i \end{aligned}} \qquad (7)$$

de esta expresión determinamos el JACOBIANO

$$\underline{\underline{J}}^e = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \quad (8)$$

Como ejemplo tomemos un elemento regular de lados a y b. como se ve en la figura.



(3)

Las funciones de forma en coordenadas naturales para este elemento son:

$$\boxed{\begin{aligned} N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \; ; \; N_2 = \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \; ; \; N_4 = \tfrac{1}{4}(1-\xi)(1+\eta). \end{aligned}} \quad (9)$$

con una expresión única: $N_i = \frac{1}{4}(1+\xi_i \xi)(1+\eta_i \eta)$

con $i = 1, 2, 3, 4$.

(Ver figura (2)).

Las coordenadas son:

$$\boxed{\begin{aligned} &x_1 = 0 \; y_1 = 0; \; x_2 = a; \; y_2 = 0 \\ &x_3 = a \quad y_3 = b. ; \; x_4 = 0 \quad y_4 = b. \end{aligned}} \quad (10)$$

$$\Rightarrow x = \sum_{i=1}^{n=4} N_i x_i = \tfrac{1}{4}(1+\xi)(1-\eta)\, a + \tfrac{1}{4}(1+\xi)(1+\eta) \cdot a$$

$$\boxed{x = \tfrac{1}{2}(1+\xi)\, a.} \quad (11)$$

$$y = \sum_{i=1}^{n=4} N_i y_i = \tfrac{1}{4}(1+\xi)(1+\eta)\, b + \tfrac{1}{4}(1-\xi)(1+\eta)\, b = \tfrac{1}{2}(1+\eta)\, b.$$

$$\boxed{y = \tfrac{1}{2}(1+\eta)\, b.} \quad (12)$$

Para calcular la matriz $\underline{\underline{J}}^e$ (jacobiana).

$$\underline{\underline{J}}^e = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} a/2 & 0 \\ 0 & b/2 \end{bmatrix}$$

luego el determinante es: $\boxed{|J^e| = \frac{ab}{4}}$ (13)

La inversa de $\underline{\underline{J}}^e$ para poder transformar de (ξ,η) a (x,y) será:

$$\underline{\underline{J}}^{e^{-1}} = \frac{1}{|J^e|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} = \frac{1}{\frac{ab}{4}} \begin{bmatrix} b/2 & 0 \\ 0 & a/2 \end{bmatrix} = \frac{4}{ab} \begin{bmatrix} b/2 & 0 \\ 0 & a/2 \end{bmatrix}.$$

$$\Rightarrow \boxed{\underline{\underline{J}}^{e^{-1}} = \frac{4}{ab} \begin{bmatrix} b/2 & 0 \\ 0 & a/2 \end{bmatrix}}. \quad (14)$$

Luego con esto determinamos la matriz de rigidez del elemento cuadrilátero genérico.-

En el caso estructural tenemos dos grados de libertad por nodo $\Rightarrow$ (u_i, v_i) desplazamientos.-

Llamamos $\boxed{\delta e = [u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4]^T}$ son nuestras incógnitas. VECTOR DESPLAZAMIENTOS.

$\underline{\underline{J}}^e$?

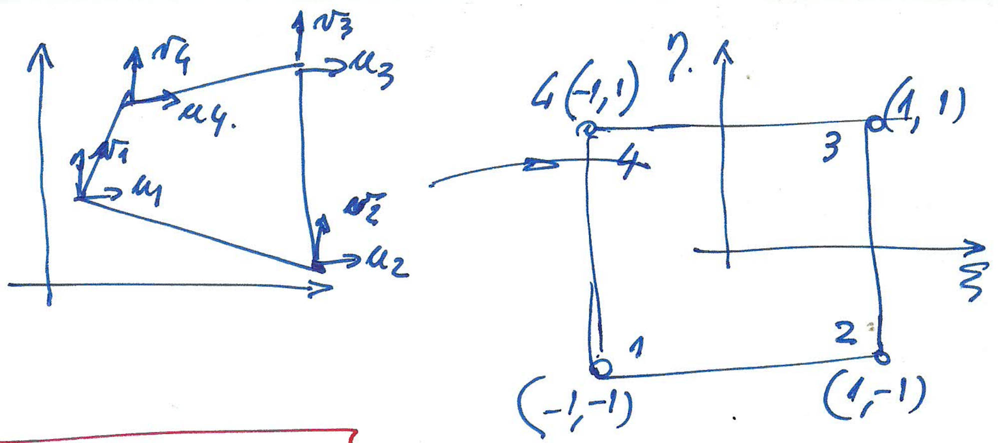
sabemos que:

$$\boxed{\begin{aligned} u(x,y) &= \sum_{i=1}^{n=4} N_i(\xi,\eta)\, u_i \\ v(x,y) &= \sum_{i=1}^{n} N_i(\xi,\eta)\, v_i \end{aligned}}$$

$n=4$.

discretización de las incógnitas

(15)



$$\boxed{\underline{u}_e = [\underline{\underline{N}}_e]\, \underline{\delta}_e} \qquad (16)$$

$$\underline{\underline{N}}_e = \left[\begin{array}{cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right]$$

$$\boxed{\begin{aligned} x &= \sum_{i=1}^{n} N_i \cdot x_i \\ y &= \sum_{i=1}^{n} N_i \cdot y_i \end{aligned}} \qquad (17)$$

CALCULO MATRIZ DEFORMACION.

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{I} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} \qquad (18)$$

I ~ MATRIZ UNITARIA AMPLIADA.

Por la regla de la cadena:

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

escribir

$$\boxed{\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \underline{\underline{J}}^e \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}}$$

Para la variable $\underline{u}$

(19)

Para la variable v es idéntico $\Rightarrow$

$$\begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \underline{J}^e \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}} \quad (20)$$

Sabemos que:

$$\underline{J}^e = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \quad (21)$$

$$\underline{J}^e = \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}}_{\underline{\partial N_e}} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \quad (22)$$

$\underline{\partial N_e}$ Matriz de gradientes de las funciones de forma o de interpolación respecto a las coordenadas naturales

$$\Rightarrow \underline{J}^e = \left[\underline{\partial N_e}\right] \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Sustituyendo las funciones de forma (9) tenemos:

$$\left[\underline{\partial N_e}\right] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \quad (23)$$

Falta calcular la inversa de la Jacobiana

Sabemos por (14) que:

$$\boxed{\left[J_e^{-1}\right] = \frac{1}{\det J^e} \cdot \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}} \quad (24)$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \left[J_e^{-1}\right] \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} \quad y \quad \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \left[J_e^{-1}\right] \begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \qquad (25)$$

=> en forma matricial podemos hacer:

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underbrace{\left[\begin{array}{c|cc} \left[J_e^{-1}\right] & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \left[J_e^{-1}\right] \end{array}\right]}_{\Gamma} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \qquad (26)$$

Debemos calcular.

Γ Matriz de Inversas del Jacobiano Ampliada.

↓ Estas son nuestras transformaciones

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}}_{\partial N^a} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} \qquad (27)$$

Matriz de derivadas de la funciones de Interpolación ampliada.

=> Al calcular la tenemos.

$$\boxed{\left[\partial N^a\right] = \frac{1}{4} \begin{bmatrix} -1+\eta & 0 & 1-\eta & 0 & 1+\eta & 0 & -1-\eta & 0 \\ -1+\xi & 0 & -1-\xi & 0 & 1+\xi & 0 & 1-\xi & 0 \\ 0 & -1+\eta & 0 & 1-\eta & 0 & 1+\eta & 0 & -1-\eta \\ 0 & -1+\xi & 0 & -1-\xi & 0 & 1+\xi & 0 & 1-\xi \end{bmatrix}} \qquad (28)$$

Sustituyendo (27) en (26) y este resultado en la deformación (18).

nos queda:

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\underline{\underline{I}}}_{(18)} \; \underbrace{\underline{\underline{\Gamma}}}_{(26)} \; \underbrace{[\partial \underline{\underline{N}}^a]}_{(27)} \; \underbrace{[\delta e]}_{(15)} \qquad (29)$$

Como $\underline{\underline{\varepsilon}} = [\underline{\underline{B}}e][\delta e] \Rightarrow$ (30)

$$\boxed{[\underline{\underline{B}}e] = \underline{\underline{I}}\,\underline{\underline{\Gamma}}\,[\partial N^a]}$$ Matriz de Deformación del elemento. (31)

Con todos estos datos calculamos la matriz de rigidez.

$$\boxed{\underline{\underline{K}}e = \int_{A_e} \underline{\underline{B}}e^T \cdot \underline{\underline{D}}\; \underline{\underline{B}}e\; t\, dA_e} \qquad (32)$$

que en función de ξ, η.

$$\boxed{\underline{\underline{K}}e = \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{B}}e^T\, \underline{\underline{D}}\; \underline{\underline{B}}e\, (\det J_e)\, t\, d\xi\, d\eta} \qquad (33)$$

donde $dA_e = (\det J_e)\, d\xi\, d\eta = dx\, dy$.

Las submatrices serán:

$$\boxed{\underline{\underline{K}}_{ij}^e = \int_{-1}^{+1}\int_{-1}^{+1} [B_i]^T\, \underline{\underline{D}}\; \underline{\underline{B}}_j \cdot (\det J_e)\, t\, d\xi\, d\eta} \qquad (34)$$

Es decir un procedimiento será:

1) Hallar las submatrices de deformación

$$\boxed{[B_i] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}} \qquad (35)$$

2) Aplicando el concepto de Isoedromo

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = [\underline{J_e}] \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = [J_e^{-1}] \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \frac{1}{(\det J_e)} \cdot \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}} \quad (36)$$

$$\Rightarrow \frac{\partial N_i}{\partial x} = \frac{1}{\det J_e} \left(\frac{\partial y}{\partial \eta} \frac{\partial N_i}{\partial \xi} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial N_i}{\partial \eta} \right).$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{\det J_e} \left(\frac{\partial x}{\partial \xi} \frac{\partial N_i}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial N_i}{\partial \xi} \right)$$

$$\Rightarrow \underline{\underline{B_i}} = \frac{1}{\det J_e} \begin{bmatrix} \underbrace{\dfrac{\partial y}{\partial \eta} \cdot \dfrac{\partial N_i}{\partial \xi} - \dfrac{\partial y}{\partial \xi} \dfrac{\partial N_i}{\partial \eta}}_{a_i} & 0 \\ 0 & \underbrace{\dfrac{\partial x}{\partial \xi} \dfrac{\partial N_i}{\partial \eta} - \dfrac{\partial x}{\partial \eta} \dfrac{\partial N_i}{\partial \xi}}_{b_i} \\ \underbrace{\dfrac{\partial x}{\partial \xi} \dfrac{\partial N_i}{\partial \eta} - \dfrac{\partial x}{\partial \eta} \dfrac{\partial N_i}{\partial \xi}}_{b_i} & \underbrace{\dfrac{\partial y}{\partial \eta} \cdot \dfrac{\partial N_i}{\partial \xi} - \dfrac{\partial y}{\partial \xi} \dfrac{\partial N_i}{\partial \eta}}_{a_i} \end{bmatrix} \quad (37)$$

$$B_i = \frac{1}{\det J_e} \begin{bmatrix} a_i & 0 \\ 0 & b_i \\ b_i & a_i \end{bmatrix}.$$

$$\Rightarrow \underline{\underline{K_{ij}^e}} = \int_{-1}^{1} \int_{-1}^{1} \underbrace{\frac{1}{(\det J_e)} \begin{bmatrix} a_i & 0 & b_i \\ 0 & b_i & a_i \end{bmatrix}}_{B_i^T} \underbrace{\begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}}_{\underline{\underline{D}}} \overbrace{\frac{1}{(\det J_e)} \begin{bmatrix} a_j & 0 \\ 0 & b_j \\ b_j & a_j \end{bmatrix}}^{B_j} \cdot \underbrace{(\det J_e)\, t\, d\xi\, d\eta}_{t\, dA_e}$$

(38)

3) Nos quedará:

$$\boxed{K_{ij}^{e} = \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} d_{11}\, a_i a_j + d_{33}\, b_i b_j & d_{12} a_i b_j + d_{33} a_i b_j \\ d_{21}\, b_i a_j + d_{33} a_i b_j & d_{22} a_i a_j + d_{33} a_i a_j \end{bmatrix} \frac{t}{\det J_e}\, d\xi\, d\eta}$$

(39)

Para n/ejemplo al inicio suponiendo Tensión plana: $E = 2\times10^{6}\,(t/m^2)$

$\nu = 0.2$

$a = 0.4\,m$ } cuadrado.
$b = 0.4\,m$ }

$t = 0.1\,m.$

$N_1 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta).$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta).$ con $x = \sum N_i x_i = \frac{1}{2}(1+\xi)a.$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta).$ $y = \sum N_i y_i = \frac{1}{2}(1+\eta)a.$

$$\Rightarrow \underline{\underline{K_e}} = \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{B_e}}^{T} \cdot \underline{\underline{D}}\; \underline{\underline{B_e}}\, (\det J_e) \cdot t\, d\xi\, d\eta.$$

$$\underline{\underline{B_e}} = \underline{\underline{I}} \cdot \underline{\underline{\Gamma}} \cdot \underline{\underline{\partial N}}^{a}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \qquad [J_e] = \begin{bmatrix} a/2 & 0 \\ 0 & a/2 \end{bmatrix}$$

$$\det J_e = \frac{a}{2}\cdot\frac{a}{2} = \frac{a^2}{4} = 0.04.$$

$$\underline{\Gamma} = \frac{2}{a}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = 5.\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \underline{\underline{\partial N}}^{a} = \frac{1}{4}\begin{bmatrix} -1+\eta & 0 & 1-\eta & 0 & 1+\eta & 0 & -1-\eta & 0 \\ -1+\xi & 0 & -1-\xi & 0 & 1+\xi & 0 & 1-\xi & 0 \\ 0 & -1+\eta & 0 & 1-\eta & 0 & 1+\eta & 0 & -1-\eta \\ 0 & -1+\xi & 0 & -1-\xi & 0 & 1+\xi & 0 & 1-\xi \end{bmatrix}$$

$$\underline{\underline{B_e}} = 1.25 \begin{bmatrix} -1+\eta & 0 & 1-\eta & 0 & 1+\eta & 0 & -1-\eta & 0 \\ 0 & -1+\xi & 0 & -1-\xi & 0 & 1+\xi & 0 & 1-\xi \\ -1+\xi & -1+\eta & -1-\xi & 1-\eta & 1+\xi & 1+\eta & 1-\xi & -1-\eta \end{bmatrix}$$

$$\underline{\underline{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{2\times10^6}{1-0.2^2}\begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & \frac{1-0.2}{2} \end{bmatrix}$$

$$\underline{\underline{D}} = 10^6 \begin{bmatrix} 2.0833 & 0.4167 & 0 \\ 0.4167 & 2.0833 & 0 \\ 0 & 0 & 0.8333 \end{bmatrix}.$$

de aquí calculamos $\underline{\underline{K}}^e$

$$\underline{\underline{K}}^e = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\boxed{\underline{\underline{B_e}}^T \, \underline{\underline{D}} \, \underline{\underline{B_e}} \, (\det J_e) . t}}_{A^e} \, d\xi \, d\eta .$$

Si usamos integración numérica

$$\underline{\underline{K_e}} = \sum_i \sum_j W_i W_j \left[A^e\right]_{\substack{\xi_i \\ \eta_i}}$$

Utilizando 4 ptos de Gauss. ($n=2$ en ξ, $n=2$ en η).

La integración es exacta siempre que el grado de los polinomios sea igual o menor a:

$$2n-1 = 2.2-1 = 3.$$

lo que sucede en este caso.—

$\xi_A = -1/\sqrt{3}$ | $\eta_A = -1/\sqrt{3}$ | $\xi_C = 1/\sqrt{3}$ | $\xi_D = 1/\sqrt{3}$

$\xi_B = -1/\sqrt{3}$ | $\eta_B = 1/\sqrt{3}$ | $\eta_C = -1/\sqrt{3}$ | $\eta_D = 1/\sqrt{3}$.

$$W_i = W_j = 1.$$

$$\underline{K_e} = \left[A(\xi_A, \eta_A)\right] + \left[A(\xi_B, \eta_B)\right] + \left[A(\xi_C, \eta_C)\right] + \left[A(\xi_D, \eta_D)\right]$$