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An improved sum-of-disjoint-products technique for the symbolic network reliability analysis with known minimal paths

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Abstract

Evaluating the network reliability is an important topic in the planning, designing, and control of systems. The sum-of-disjoint products technique (SDP) is a major fundamental tool for evaluating stochastic network reliability. In this study, a new SDP based on some intuitive properties that characterize the structure of minimal paths (MPs), and the relationships between MPs and subpaths are developed to improved SDP. The proposed SDP is easier to understand and implement, and better than the existing best- known SDP based algorithms under some special situation. The correctness of the proposed algorithm will be analyzed and proven. One bench example is illustrated to show how the network reliability with known MPs is determined using the proposed SDP.

Keywords: Network reliability; Minimal path (MP); Algorithm; Sum-of-disjoint products technique (SDP); The absorption law

1. Introduction

In recent years, network reliability theory has been applied extensively in many real-world systems such as computer and communication systems, power transmission and distribution systems, transportation systems, oil/gas production systems [1–4], etc. System reliability plays important roles in our modern society [1–19].

The evaluation of a stochastic network reliability is a NP-hard problem [19]. Reliability evaluation approaches exploit a variety of tools for system modeling and reliability index calculation. The network reliability, defined as the probability of connection of the source node s with the sink node t. Among the most popular tools are network-based algorithms founded in terms of either minimal cuts (MCs) or MPs [5,7–18]. Once the connection between nodes s and t is expressed in logical form after a MP "AND" or MC "OR" operation, computing the probability of the probability of this logical expression has been the object of extensive research studies (even in the simplified case in which the failure events of the edges are statistically independent). The most convenient techniques discussed in the literature at length are two different strategies which have clearly emerged:

(1) adopting a SDP, starting from Abraham [20,21].
(2) the binary decision diagram technique (BDD) first discussed in Bryant [22]. Bryant's paper is the top cited paper in the computer science area: see http://citeseer.ist.psu.edu/source.html. BDD is sometimes referred to as pivotal decomposition methods.

In this study, we focus only on the SDP in terms of the known MPs. The various versions of existing SDPs all involve a Boolean expansion and minimization after each disjoint product is formed [20–33]. Moreover, each MP/MC, say A_i, needs to compare with all other MPs/MCs A_j for all $j<i$ in each formed disjoint product. As expected, the running times for the above SDPs in finding a disjoint product increase exponentially as the number of MPs/MCs increases. This is a drastic limitation of "classical" SDPs.

The purpose of this paper is to develop a more efficient and intuitive algorithm than the existing SDPs. The proposed algorithm is based on some simple concepts derived from the adsorption law (discussed in Property 1 of

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Section 3). Instead of comparing MP A_i to other MPs A_j for all $j<i$, we only use the "shorter" paths between two pair of nodes in A_i. For A_1, A_2, and the longest MPs, the comparisons are not necessary. For MPs with the smallest or greatest index among all MPs with the same length (the number of edges), the comparisons are reduced. Furthermore, the simplification is executed during the formation of each disjoint product, not after each disjoint product is found. The computational complexity of the proposed algorithm is also analyzed and compared with the existing SDPs.

This paper is organized as follows. Section 2 describes the notations, nomenclature and assumptions required. Some important properties, lemmas and theorems are discussed in Section 3 to characterize the structure of MPs and the relationship between MPs and subpaths. Section 4 presents the proposed method for improving SDP using the known MPs in detail. The proposed algorithm is illustrated with the help of an example to show how to compute the reliability using known MPs in a network in Section 5. Concluding remarks are given in Section 6.

2. Notations, nomenclature and assumptions

2.1. Acronym

SDP	the sum-of-disjoint products technique
MC/MP	minimal cut/path

2.2. Notations

R	the network reliability
m	the number of all MPs between nodes s and t
$\bullet^c, [\bullet]^c$	the complementary of $\bullet$
$P(\bullet)$	the probability of $\bullet$
$P^c(\bullet)$	the probability of the complementary of $\bullet$
$G(V,E)$	a connected network with the node set V, the edge set E, and s, t are the specified source node and sink node, respectively. For example, Fig. 1, the so-called modified ARPANET [24–28], is a network with $V=\{s,t,1,2,3,4\}$ and $E=\{a,b,c,d,e,f,g,h,i\}$
$G(V,E-U)$	the residual subnetwork (not necessary to be connected) of $G(V,E)$ after removing U, where $U\subseteq E$. For example, the network in Fig. 2 is $G(V,E-U)$ after removing $U=\{a,d,g,i\}$ from Fig. 1
A_i	the ith MP between nodes s and t and $\|A_i\|\leqslant\|A_{i+1}\|$ for all i.
A_{ih}	the hth node in the path formed by MP A_i. Note that the 1st and last node in A_i are s and t, respectively
A_i-A_j	the difference between A_i andA_j, e.g. $A_1-A_2=\{a,d\}$ if $A_2=\{a,d,h\}$ and $A_3=\{b,e,h\}$
$a_{i\alpha\beta}$	the subpath between nodes α and β in MP A_i and no edge connected nodes α and β in MP A_i
$\Omega_{i\alpha\beta}$	$\{\Delta\|\Delta$ is a path between α and β with $\Delta\cap a_{i\alpha\beta}=\{\alpha,\beta\}$ in $G(V,E-A_i)\}$
$\Psi^{<}_{i\alpha\beta}$	$\{\Delta\in\Omega_{i\alpha\beta}\|\|\Delta\|<\|a_{i\alpha\beta}\|\}$
$\Psi^{\leqslant}_{i\alpha\beta}$	$\{\Delta\in\Omega_{i\alpha\beta}\|\|\Delta\|\leqslant\|a_{i\alpha\beta}\|\}$
$\Psi^{=}_{i\alpha\beta}$	$\{\Delta\in\Omega_{i\alpha\beta}\|\|\Delta\|=\|a_{i\alpha\beta}\|=A_h-A_i$ where A_h is an MP with $h<i\}$
$\binom{n}{k}$	$n!/(n-k)!k!=n(n-1)\ldots(n-k+1)/k!$ where n and k are positive integers
$\|\bullet\|$	the number of elements of $\bullet$, e.g., $\|V\|$ is the number of nodes in V

2.3. Nomenclature

Reliability	the probability of a live connection between the source node and the sink node
MP/MC	it is an edge set such that if any edge is removed from this path/cut, then the remaining set is no longer a path/cut. For example, $\{a,c,e,g,i\}$ is a MP between nodes s and t in Fig. 1. The MP/MC search is tied to the Graph Theory concepts and connectivity measures. Probabilistic techniques usually associate the connectivity level of a network with the availability and/or reliability of the communication paths between specific network node-pairs [5,7–18]
Fundamental product	a literal or a product of two or more literals in which no two literals involve the same variable
Disjoint products	each term obtained from the SDP

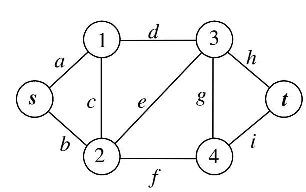

Fig. 1. An example network.

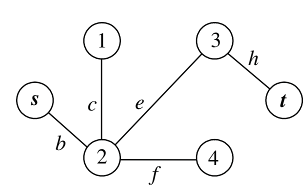

Fig. 2. An example network to explain $G(V,E-U)$.

2.4. Assumptions

The network satisfies the following assumptions:

1. Each node is perfectly reliable.
2. The graph is connected. Otherwise, the reliability is 0 if there is no path between nodes s and t, or the graph can be simplified by removing all of the disconnected subgraphs without including the nodes s and t.
3. The graph is free of self-loops. Otherwise, the self-loops can be removed.
4. Each edge has two states: working or failed. The edge states are statistically independent.
5. The degree of each node is greater than 2 except possibly nodes s and t.

3. Preliminaries

Before introducing the proposed algorithm, some useful properties and results will be described in this section. The following three properties directly follow from the Boolean Algebra.

Property 1 (*Absorption law*).

If $A_i \subseteq B \subseteq E$, then $P^{\mathrm{c}}(A_i) \cdot P^{\mathrm{c}}(B) = P^{\mathrm{c}}(A_i)$ [31]. (1)

Proof. Since $A_i \subseteq B$, let $B = A_i \cup C$ and $P(B) = P(A_i) \cdot P(C)$. We have

$$P^{\mathrm{c}}(A_i) \cdot P^{\mathrm{c}}(B)$$

$$= [P(A_i) + P(B)]^{\mathrm{c}} \text{ (De Morgan's law)} \tag{2}$$

$$= [P(A_i) + P(A_i) \cdot P(C)]^{\mathrm{c}} \tag{3}$$

$$= [P(A_i)(1 + P(C))]^{\mathrm{c}} \tag{4}$$

$$= [P(A_i)]^{\mathrm{c}} = P^{\mathrm{c}}(A_i). \qquad \square \tag{5}$$

Property 2.

$$P(A_i - A_j - [A_k - A_j]^{\mathrm{c}}) = P(A_i - A_j). \tag{6}$$

Property 3.

$$P^{\mathrm{c}}\left(\prod_{a \in A} a\right) P^{\mathrm{c}}\left(\prod_{b \in B} b\right) = P\left(\prod_{\gamma \in A \cap B} \gamma + \prod_{\gamma \in A \cap B} \gamma \cdot \left[\prod_{\alpha \in A-B} \alpha\right]^{\mathrm{c}} \cdot \left[\prod_{\beta \in B-A} \beta\right]^{\mathrm{c}}\right). \tag{7}$$

We now turn to the exact computation of R. The next property is implemented to calculate the exact reliability R. Basically; it is called the SDP and already used to evaluate the network reliability [20–22,24–33].

Property 4.

$$R = P(A_1) + P(A_2 \cap [A_1]^{\mathrm{c}}) + P(A_3 \cap [A_1]^{\mathrm{c}} \cap [A_2]^{\mathrm{c}}) + \cdots + P(A_m \cap [A_1]^{\mathrm{c}} \cap [A_2]^{\mathrm{c}} \cap \cdots \cap [A_{m-1}]^{\mathrm{c}}). \tag{8}$$

For general binary networks, computing Eq. (8) is NP-hard [20,21,23–33], since it involves a Boolean expansion and minimization. The following property is used to simplify each item in Eq. (8) [31].

Property 5.

$$P(A_i \cap [A_1]^{\mathrm{c}} \cap [A_2]^{\mathrm{c}} \cap \cdots \cap [A_{i-1}]^{\mathrm{c}}) = P(A_i) \cdot \prod_{j=1}^{i-1} P^{\mathrm{c}}(A_j - A_i). \tag{9}$$

Proof.

$$P(A_i \cap [A_1]^{\mathrm{c}} \cap [A_2]^{\mathrm{c}} \cap \cdots \cap [A_{i-1}]^{\mathrm{c}}) \tag{10}$$

$$= P(A_i) \cdot P(A_1 - A_i]^{\mathrm{c}} \cap [A_2 - A_i]^{\mathrm{c}} \cap \cdots \cap [A_{i-1} - A_i]^{\mathrm{c}}) \tag{11}$$

$$= P(A_i) \cdot P(A_{i1}) \cdot P([A_2 - A_i - A_{i1}]^{\mathrm{c}} \cap [A_3 - A_i - A_{i1}]^{\mathrm{c}} \cap \cdots \cap [A_{i-1} - A_i - A_{i1}]^{\mathrm{c}}) \quad (\text{let } A_{i1} = [A_1 - A_i]^{\mathrm{c}}) \tag{12}$$

$$= P(A_i) \cdot P(A_{i1}) \cdot P([A_2 - A_i]^{\mathrm{c}} \cap [A_3 - A_i]^{\mathrm{c}} \cap \cdots \cap [A_{i-1} - A_i]^{\mathrm{c}}) \quad (\text{by Property 2}) \tag{13}$$

$$= P(A_i) \cdot P(A_{i1}) \cdot P(A_{i2}) \cdot P([A_3 - A_i - A_{i2}]^{\mathrm{c}} \cap \cdots \cap [A_{i-1} - A_i - A_{i2}]^{\mathrm{c}}) \quad (\text{let } A_{i2} = [A_2 - A_i]^{\mathrm{c}}) \tag{14}$$

$$= P(A_i) \cdot P(A_{i1}) \cdot P(A_{i2}) \cdot P([A_3 - A_i]^{\mathrm{c}} \cap [A_4 - A_i]^{\mathrm{c}} \cap \cdots \cap [A_{i-1} - A_i]^{\mathrm{c}}) \quad (\text{by Property 2}) \tag{15}$$

$$= \cdots$$

$$= P(A_i) \cdot P(A_{i1}) \cdot P(A_{i2}) \cdots P(A_{i,i-1}) \tag{16}$$

$$= P(A_i) \cdot P^{\mathrm{c}}(A_1 - A_i) \cdot P^{\mathrm{c}}(A_2 - A_i) \cdots P^{\mathrm{c}}(A_{i-1} - A_i) \tag{17}$$

$$= P(A_i) \cdot \prod_{j=1}^{i-1} P^{\mathrm{c}}(A_j - A_i). \qquad \square \tag{18}$$

Each disjoint product, say A_i, obtained in Property 5 also involves a Boolean expansion and minimization. The Absorption Law plays a key role to simplify all disjoint products. It takes $O(i|A_i|^2)$ to form the disjoint product and $O(i^2|A_i|)$ to simplify the disjoint product after using the Absorption Law to reduce the number of items in A_i. Note that $i = 2^{|V|}$ in the worst case, i.e. it takes at most $O(2^{2|V|}|V|)$ to find a disjoint product using Property 5 and the Absorption Law. From the above discussion, it is a time-consuming and burdensome job even if Property 5 is used to calculate the network reliability. Hence, a more efficient and intuitive algorithm based on the rest properties is developed to improve the existing SDPs which are all based on Property 5. The first two items in Eq. (9) are simplified using following two properties in our algorithm.

Property 6.

$$P(A_2 \cap [A_1]^{\mathrm{c}}) = P(A_2) \cdot P^{\mathrm{c}}(A_1) \quad \text{if } A_1 \cap A_2 = \varnothing. \tag{19}$$

Proof.

$$P(A_2 \cap [A_1]^c) \quad (20)$$

$$= P(A_2) \cdot P^c(A_1 - A_2) \quad (21)$$

$$= P(A_2) \cdot P^c(A_1) \quad (\text{since } A_1 \cap A_2 = \varnothing). \qquad \square \quad (22)$$

Property 7.

$$P(A_1) + P(A_2 \cap [A_1]^c) = 1 - P^c(A_1) \cdot P^c(A_2)$$
$$\text{if } A_1 \cap A_2 = \varnothing. \quad (23)$$

Proof.

$$P(A_1) + P(A_2 \cap [A_1]^c) \quad (24)$$

$$= P(A_1) + P^c(A_1) \cdot P(A_2) \quad (\text{by Property 6}) \quad (25)$$

$$= 1 - P^c(A_1) \cdot P^c(A_2). \qquad \square \quad (26)$$

The following property is trivial and simple. However, it forms the basis of the proposed method to simplify the Eq. (9).

Property 8. Let α and β be two nodes in MP A_i. If Δ is a path between α and β in $G(V, E)$ and $\Delta \cap a_{i\alpha\beta} = \{\alpha, \beta\}$, then $A_i - a_{i\alpha\beta}$ is a MP.

The following comes directly from Property 8 and the Absorption Law.

Property 9. If $\{a_{k\alpha\beta}\} \subseteq A_k - A_i \neq A_k$, $a_{k\alpha\beta} \cap a_{i\alpha\beta} = \{\alpha, \beta\}$, and $A_h = A_i - a_{i\alpha\beta} \cup a_{k\alpha\beta}$ with $h < i$, then

$$P^c(A_h - A_i) \cdot P^c(A_k - A_i) = P^c(a_{k\alpha\beta}). \quad (27)$$

Proof. From Property 8, $A_h = A_i - a_{i\alpha\beta} \cup a_{k\alpha\beta}$ is a MP. Since $h < i$, we have $|A_h| \leqslant |A_i|$ and $|a_{k\alpha\beta}| \leqslant |a_{i\alpha\beta}|$. Moreover, $A_h - A_i = \{a_{k\alpha\beta}\} \subseteq A_k - A_i$, we have $P^c(A_h - A_i) \cdot P^c(A_k - A_i) = P^c(A_h - A_i) = P^c(a_{k\alpha\beta})$ from the Absorption Law. $\square$

The following two corollaries come directly from Property 9 and the Absorption Law.

Corollary 1. *Let α and β be two nodes in MP A_i. If Δ is a path between α and β with $|\Delta| < |a_{i\alpha\beta}|$ and $\Delta \cap a_{i\alpha\beta} = \{\alpha, \beta\}$, then*

$$\prod_k P^c(A_k - A_i) = P^c(\Delta) \quad \textit{for } \Delta \textit{ in all } A_k. \quad (28)$$

Corollary 2. *Let α and β be two nodes in MP A_i. If Δ is a path between α and β with $|\Delta| = |a_{i\alpha\beta}|$ and $\Delta \cap a_{i\alpha\beta} = \{\alpha, \beta\}$, and an MP $A_j = A_i - a_{i\alpha\beta}$ s.t. $j < i$, then*

$$\prod_k P^c(A_k - A_i) = P^c(\Delta) \quad \textit{for } \Delta \textit{ in all } A_k. \quad (29)$$

Proof. Since $\Delta \subseteq (A_k - A_i)$ for Δ in all A_k If there is a MP $A_j = A_i - a_{i\alpha\beta} \cup \Delta$ s.t. $j < i$, then $(A_j - A_i) = \Delta$. Hence, this corollary is true. $\square$

Obviously, if $(A_k - A_i) \neq A_k$, then $(A_k - A_i) = \{\Delta | \Delta \cap a_{i\alpha\beta} = \{\alpha, \beta\}\}$. Moreover, $|a_{kab}| < |a_{iab}|$ for at least one subpath $a_{kab} \in A_k - A_i$, or $|a_{k\alpha\beta}| = |a_{i\alpha\beta}|$ for all $a_{k\alpha\beta} \in (A_k - A_i)$. In the former case, Corollary 1 can be implemented to obtain the disjoint products. In the latter, Corollary 2 can be used to simplify the disjoint products. Originating from the above simple concepts discussed in Corollaries 1 and 2, the following theorem is utilized in the proposed algorithm to form and simplify each disjoint product using the Absorption Law.

Theorem 1.

$$P(A_i) \cdot \prod_{j=1}^{i-1} P^c(A_j - A_i) = P(A_i) \cdot \prod_{\omega^<} P^c(\omega^<) \cdot \prod_{\omega^=} P^c(\omega^=), \quad (30)$$

where $\omega^< \in \Psi^<_{i\alpha\beta}$ and $\omega^= \in \Psi^=_{\alpha\alpha\beta}$ for all nodes α and β in A_i.

Theorem 1 is an essential point in this study. It explores some special characteristics between the subpaths of a MP and the Absorption Law. There are $|A_i| + 1$ nodes and $\binom{|A_i|+1}{2} (< |A_i|^2 < |V|^2)$ subpaths in MP A_i. In the worst case, there are at most

$$\sum_{n=2}^{|a_{i\alpha\beta}|} \binom{|V|}{n} (< O(|V|^{|A_i|} \leqslant 2^{|V|}))$$

paths between two nodes α and β in A_i with lengths not greater than $|a_{i\alpha\beta|}$. Therefore, it takes $O(|A_i|^2 |V|^{|A_i|})$ to form and simplify the disjoint product for A_i using Theorem 1. Hence, the following corollary is true.

Corollary 3. *If $|V|^{|A_i|} |A_i|^2 \leqslant i|A_i|^2 + i^2|A_i|$, i.e. $|V|^{|A_i|}|A_i| \leqslant i|A_i| + i^2$, the proposed Theorem 1 is better than the existing SDPs which needs $O(i|A_i|^2 + i^2|A_i|)$ for finding the corresponding disjoint products for MP A_i.*

Some special cases that improve Theorem 1 further are considered next:

Corollary 4. *If $|A_i| < |A_{i+1}|$, i.e. A_i is the one with the greatest index number among other MPs with the same length. Eq. (30) can then be simplified as follows:*

$$P(A_i) \cdot \prod_{j=1}^{i-1} P^c(A_j - A_i) = P(A_i) \cdot \prod_{\omega^\leqslant} P^c(\omega^\leqslant), \quad (31)$$

where $\omega^\leqslant \in \Psi^\leqslant_{i\alpha\beta}$ for all α and β are two nodes in A_i.

Proof. It is trivial that $(\Psi^<_{i\alpha\beta} \cap \Psi^=_{i\alpha\beta}) \subset \Psi^\leqslant_{i\alpha\beta}$. Let α and β be two nodes in A_i. If $\Delta \neq a_{i\alpha\beta}$ is a path between α and β with $|\Delta| \leqslant |a_{i\alpha\beta|}$ and $\Delta \cap a_{i\alpha\beta} = \{\alpha, \beta\}$, then $A_j = A_i - a_{i\alpha\beta} \cup \Delta$ is a MP and $|A_j| \leqslant |A_i|$. Since A_i is the one with the greatest index number among other MPs with the same length, we have $j < i$. Hence, $\Psi^\leqslant_{i\alpha\beta} \subset (\Psi^<_{i\alpha\beta} \cup \Psi^=_{i\alpha\beta})$.Thus, $\Psi^\leqslant_{i\alpha\beta} = \Psi^<_{i\alpha\beta} \cup \Psi^=_{i\alpha\beta}$, i.e. Eq. (30) in Theorem 1 can be reduced to Eq. (31) if $|A_i| < |A_{i+1}|$. $\square$

Corollary 5. *If $|A_{i-1}| < |A_i|$, i.e. A_i is the one with the smallest index number among other MPs with the same* *length. Eq.* (30) *can then be simplified as follows*:

$$P(A_i) \cdot \prod_{j=1}^{i-1} P^c(A_j - A_i) = P(A_i) \cdot \prod_{\omega^<} P^c(\omega^<), \tag{32}$$

where $\omega^< \in \Psi^<_{i\alpha\beta}$ *for all* α *and* β *are two nodes in* A_i *and* $a_{i\alpha\beta}$ *is not the shortest path between nodes* α *and* β *in* $G(V,E)$.

Proof. If $a_{i\alpha\beta}$ is the shortest path between nodes α and β in $G(V,E)$, then $\Psi^<_{i\alpha\beta} = \varnothing$. Let $\Delta \neq a_{i\alpha\beta}$ be a path between nodes α and β with $\Delta \cap a_{i\alpha\beta} = \{\alpha,\beta\}$, and MP $A_k = A_i - a_{i\alpha\beta} \cup \Delta$ with $k<i$. If $|\Delta| = |a_{i\alpha\beta}|$, then $|A_j| = |A_i|$ which contradicts the fact that $|A_i|>|A_k|$ for all $k<i$. Hence, Eq. (30) in Theorem 1 can be reduced to Eq. (32). □

The next corollary improves Theorem 1 to $O(|E|)$ if the MP is one of the longest MPs. This is a significant improvement.

Corollary 6. *If* A_i *is one of the longest MP, i.e.* $|A_k| \leqslant |A_i|$ *for all k, then Eq.* (30) *can be simplified as follows*:

$$P(A_i) \cdot \prod_{j=1}^{i-1} P^c(A_j - A_i) = P(A_i) \cdot \prod_{e} P^c(\{e\}), \tag{33}$$

for all $e \in E - A_i$.

Proof. According to the assumption that the degree of each node is greater than 2 except possibly nodes s and t, there is at least a path or an edge from nodes $A_{i,j}$ to $A_{i,j+2}$ in $G(V, E - A_i)$ for all $A_{i,j}$ and $A_{i,j+2}$ in A_i. Let Δ be a path from nodes $A_{i,j} = \alpha$ to $A_{i,j+2} = \beta$ in $G(V, E - A_i)$, i.e. $1 \leqslant |\Delta|$. If $|\Delta|>2$, then $A_k = A_i \cup \Delta - a_{i\alpha\beta}$ is a MP and $|A_k|>|A_i|$ which contradicts that A_i is not the longest path. Hence, $|\Delta| \in \{1,2\}$. If $|\Delta| = 2$, say from nodes α to γ to β, then $A_i \cup \{$the edge between α to γ in $\Delta\} \cup \{$ any path between γ to $A_{k,i+1}$ in $G(V, E - A_i)\} - a_{i\alpha\beta}$ is also a MP with longer length than A_i. Thus, $|\Delta| = 1$ and Eq. (30) in Theorem 1 can be reduced to Eq. (33) from Corollaries 1 and 2. □

The next theorem improves Theorem 1 furthermore while no path is from nodes s to t after removing any subpath in a MP. It also implements in the proposed SDP to reduce the calculation time for Eqs. (30)–(32).

Theorem 2. *If the removal of all edges and nodes in* $a_{iuv}(u = A_{ij}$ *and* $v = A_{ik})$ *will disconnect nodes s and t, then* $\Omega_{i\alpha\beta} = \varnothing$ *for all* $\alpha = A_{ip}$ *with* $p<j$ *and* $\beta = A_{iq}$ *with* $q \geqslant k$.

Proof. Since the removal of all edges and nodes in a_{iuv} will disconnect nodes s and t, and $\Omega_{i\alpha\beta} = \{\Delta \mid \Delta$ is a path between α and β with $\Delta \cap a_{i\alpha\beta} = \{\alpha,\beta\}$ in $G(V, E - A_i)\}$. Hence, this theorem is true. □

4. The proposed algorithm

This section outlines the proposed algorithm, which is simpler and more efficient than the existing SDPs for computing the reliability in $G(V,E)$. Theorem 1 is used repeatedly together with Corollaries 3–6 in the proposed algorithm to reduce the calculation. The detail of the proposed algorithm is described in the following steps:

Algorithm: Find the network reliability.

Input: A connected graph $G(V,E)$ with node set V, edge set E, a source node s, a sink node t, and all MPs $A_1, A_2, \ldots, A_m$ with $|A_\gamma| \leqslant |A_{\gamma+1}|$.

Output: The network reliability R.

Step 0: Let $\gamma = 3$.

Step 1: If $A_1 \cap A_2 = \varnothing$, then $R = 1 - P^c(A_1) \cdot P^c(A_2)$. Otherwise, let $R = P(A_1) + P(A_2) \cdot P^c(A_1 - A_2)$.

Step 2: If A_γ is one of the longest MPs, then let $R_\gamma = P(A_\gamma) \cdot \prod_e P^c(\{e\})$ for all $e \in E - A_\gamma$, and go to Step 13.

Step 3: Let $R_\gamma = P(A_\gamma)$, $j = 1$ and $k = 3$.

Step 4: Let Δ be the shortest path between nodes $A_{\gamma j}$ and $A_{\gamma k}$ in $G(V, E - A_\gamma)$. If $\Delta = \varnothing$, $|\Delta|>k-j$, or $|\Delta| = k-j$ while $|A_{\gamma-1}|<|A_\gamma|$, then go to Step 8.

Step 5: If $|A_{\gamma-1}|<|A_\gamma|$, then let $R_\gamma = R_\gamma \cdot \prod_{\omega^<} P^c(\omega^<)$ for all $\omega^< \in \Psi^<_{\gamma jk}$ and go to Step 8.

Step 6: If $|A_\gamma|<|A_{\gamma+1}|$, then let $R_i = R_i \cdot \prod_{\omega^\leqslant} P^c(\omega^\leqslant)$ for all $\omega^\leqslant \in \Psi^\leqslant_{\gamma jk}$ and go to Step 8.

Step 7: Let $R_\gamma = R_\gamma \cdot \prod_{\omega^<} P^c(\omega^<) \cdot \prod_{\omega^=} P^c(\omega^=)$ for all $\omega^< \in \Psi^<_{\gamma jk}$ and $\omega^= \in \Psi^\leqslant_{\gamma jk}$.

Step 8: If $j>|A_\gamma|-2$, then go to Step 12.

Step 9: If the removal of all edges and nodes in the subpath between nodes $A_{\gamma j+1}$ and $A_{\gamma k}$ in A_γ will disconnect nodes s and t, then go to Step 11.

Step 10: If $k \leqslant |A_\gamma|$, then let $k = k+1$ and go to Step 4.

Step 11: Let $j = j+1$ and $k = j+2$, and go to Step 4.

Step 12: Implement Properties 2 and 3 to simplify R_γ if it is necessary.

Step 13: Let $R = R + R_\gamma$.

Step 14: If $\gamma<m$, then let $\gamma = i+1$ and go to Step 2. Otherwise, R is the network reliability and halt.

Sometimes the Adsorption Law maybe not enough to completely simplify the joint products. Therefore, Properties 2 and 3 are implemented in Step 12, if it is necessary. The correctness of the proposed algorithm follows directly from Theorems 1–2 and Corollaries 4–6. The main complexity of the proposed algorithm is based on Corollary 3. Hence, we have the following theorem.

Theorem 3. *The above algorithm locates the corresponding disjoint product with time complexity* $O(|A_i|^2|V|^{|A_i|})$ *for* A_i *for all i.*

Theorem 3 and Corollary 3 result in the proposed algorithm being better than the existing SDPs for finding and simplifying disjoint products for MP A_i for $|V|^{|A_i|}|A_i| \leqslant i|A_i| + i^2$, and is also the best for finding the disjoint products for A_1, A_2, and the longest MPs.

5. An example

To enumerate all of the MPs in a network is a NP-hard problem [6–19]. It possesses a computational difficulty that, in the worse case, grows exponentially with network size. Owing to this inherent problem, instead of presenting practically large network systems, a moderate sized benchmark network called the modified ARPANET, which is the most frequently cited example [24–28] shown in Fig. 1, was selected to demonstrate this methodology. The following are MPs known in advance:

$A_1 = \{b, f, i\}$, $A_2 = \{a, d, h\}$, $A_3 = \{b, e, h\}$, $A_4 = \{a, d, g, i\}$, $A_5 = \{a, c, f, i\}$, $A_6 = \{b, f, g, h\}$, $A_7 = \{b, c, d, h\}$, $A_8 = \{a, c, e, h\}$, $A_9 = \{b, e, g, i\}$, $A_{10} = \{b, c, g, d, i\}$, $A_{11} = \{a, d, e, f, i\}$, $A_{12} = \{a, c, f, g, h\}$, and $A_{13} = \{a, c, e, g, i\}$.

Consider the corresponding disjoint product for $\gamma = 4$, i.e. A_4, as follows:

Step 1: Let $\gamma = 4$ and $R = 0$.

Step 2: Since A_γ is not one of the longest MPs, go to Step 3.

Step 3: Let $R_\gamma = P(A_\gamma) = P(\{a, d, g, i\}), j = 1$, and $k = 3$.

Step 4: Since $\Delta = \{b, e\}$ is the only one shortest path between nodes $A_{4j} = s$ and $A_{4k} = 3$ in $G(V, E - A_4)$, and $|\Delta| = k - j = 2$ while $|A_{\gamma-1}| = 3 < |A_\gamma| = 4$, go to Step 8.

Step 8: Since $j = 1 \leqslant |A_\gamma| - 2 = 2$, go to Step 9.

Step 9: Since the removal of all edges and nodes in the subpath between nodes $A_{4,j+1} = 1$ and $A_{4k} = 3$ in A_4, i.e. $\{d\}$, will not disconnect nodes s and t, go to Step 10.

Step 10: Since $k = 3 \leqslant |A_\gamma| = 4$, then let $k = k + 1 = 4$ and go to Step 4.

Step 4: Since $\Delta = \{b, f\}$ is the only one shortest path between nodes $A_{4j} = s$ and $A_{4k} = 4$ in $G(V, E - A_4)$, and $|\Delta| < k - j = 3$, go to Step 5.

Step 5: Since $|A_{\gamma-1}| = 3 < |A_\gamma| = 4$, let $R_\gamma = R_\gamma \cdot \prod_{\omega^<} P^c(\omega^<) = P(\{a, d, g, i\}) \cdot P^c(\{b, f\})$ and go to Step 8.

Step 8: Since $j = 1 \leqslant |A_\gamma| - 2 = 2$, go to Step 9.

Step 9: Since the removal of all edges and nodes in the subpath between nodes $A_{4,j+1} = 1$ and $A_{4k} = 4$ in A_4, i.e. $\{d, g\}$, will disconnect nodes s and t, go to Step 11.

Step 11: Let $j = j + 1 = 2$ and $k = j + 2 = 4$, and go to Step 4.

Step 4: Since $\Delta = \{c, f\}$ is the only one shortest path between nodes $A_{4j} = 1$ and $A_{4k} = 4$ in $G(V, E - A_4)$, and $|\Delta| = k - j = 2$ while $|A_{\gamma-1}| = 3 < |A_\gamma| = 4$, go to Step 8.

Step 8: Since $j = 1 \leqslant |A_\gamma| - 2 = 2$, go to Step 9.

Step 9: Since the removal of all edges and nodes in the subpath between nodes $A_{4,j+1} = 3$ and $A_{4k} = 4$ in A_4, i.e. $\{g\}$, will disconnect nodes s and t, go to Step 11.

Step 11: Let $j = j + 1 = 3$ and $k = j + 2 = 5$, and go to Step 4.

Step 4: Since $\Delta = \{h\}$ is the only one shortest path between nodes $A_{4j} = 3$ and $A_{4k} = t$ in $G(V, E - A_4)$, go to Step 5.

Table 1
Complete analysis of the modified ARPANET with the proposed SDP

γ	A_γ	j	k	$A_{\gamma j} \to A_{\gamma k}$	Δ	$P^c(\Delta)$	Remarks	R_γ
1	$\{b, f, i\}$							
2	$\{a, d, h\}$							$1 - [adh]^c \cdot [bfi]^c$
3	$\{b, e, h\}$	1	3	$\{b, e\}$[a]	$\{a, d\}$	$[ad]^c$	$\lvert\Delta\rvert = 2\&\lvert A_\gamma\rvert < \lvert A_{\gamma+1}\rvert$	$beh \cdot [fi]^c \cdot [ad]^c$
		2	4	$\{e, h\}$	$\{f, i\}$	$[fi]^c$	$\lvert\Delta\rvert = 2\&\lvert A_\gamma\rvert < \lvert A_{\gamma+1}\rvert$	
4	$\{a, d, g, i\}$	1	3	$\{a, d\}$	$\{b, e\}$		$\lvert\Delta\rvert = 2\&\lvert A_{\gamma-1}\rvert < \lvert A_\gamma\rvert$	$adgi \cdot h^c \cdot [bf]^c$
			4	$\{a, d, g\}$[a]	$\{b, f\}$	$[bf]^c$	$\lvert\Delta\rvert < 3$	
		2	4	$\{d, g\}$[a]	$\{c, f\}$		$\lvert\Delta\rvert = 2\&\lvert A_{\gamma-1}\rvert < \lvert A_\gamma\rvert$	
		3	5	$\{g, i\}$	$\{h\}$	h^c	$\lvert\Delta\rvert < 3$	
5	$\{a, c, f, i\}$	1	3	$\{a, c\}$[a]	$\{b\}$	b^c	$\lvert\Delta\rvert < 2$	$acfi \cdot b^c \cdot [dg]^c \cdot [dh]^{c\#}$
		2	4	$\{c, f\}$	$\{d, g\}$	$[dg]^c$	$A_4 = A_p = \{a, d, g, i\}\&p < \gamma$	
			5	$\{c, f, i\}$	$\{d, h\}$	$[dh]^c$	$\lvert\Delta\rvert < 3$	
		3	5	$\{f, i\}$	$\{e, h\}$		$A_8 = A_p = \{a, d, g, i\}\&p > \gamma$	
6	$\{b, f, g, h\}$	1	3	$\{b, f\}$	$\{a, d, g\}$		$\lvert\Delta\rvert > 2$	$bfgh \cdot e^c \cdot i^c \cdot [ad]^c$
			4	$\{b, f, g\}$[a]	$\{a, d\}$	$[ad]^c$	$\lvert\Delta\rvert < 3$	
		2	4	$\{f, g\}$[a]	$\{e\}$	e^c	$\lvert\Delta\rvert < 2$	
		3	5	$\{g, h\}$	$\{i\}$	i^c	$\lvert\Delta\rvert < 2$	
7	$\{b, c, d, h\}$	1	3	$\{b, c\}$[a]	$\{a\}$	a^c	$\lvert\Delta\rvert < 2$	$bcdh \cdot a^c \cdot e^c \cdot [fg]^c \cdot [fi]^{cb}$
		2	4	$\{c, d\}$	$\{e\}$	e^c	$\lvert\Delta\rvert < 2$	
					$\{f, g\}$	$[fg]^c$	$A_6 = A_p = \{b, f, g, h\}\&p < \gamma$	
			5	$\{c, d, h\}$	$\{f, i\}$	$[fi]^c$	$\lvert\Delta\rvert < 3$	
		3	5	$\{d, h\}$	$\{c, f, i\}$		$\lvert\Delta\rvert < 2$	
8	$\{a, c, e, h\}$	1	3	$\{a, c\}$[a]	$\{b\}$	b^c	$\lvert\Delta\rvert < 2$	$aceh \cdot b^c \cdot d^c \cdot [fi]^c$
		2	4	$\{c, e\}$[a]	$\{d\}$	d^c	$\lvert\Delta\rvert < 2$	
		3	5	$\{e, h\}$	$\{f, i\}$	$[fi]^c$	$A_5 = A_p = \{a, d, g, i\}\&p < \gamma$	
9	$\{b, e, g, i\}$	1	3	$\{b, e\}$[a]	$\{a, d\}$	$[ad]^c$	$\lvert\Delta\rvert = 2\&\lvert A_\gamma\rvert < \lvert A_{\gamma+1}\rvert$	$begi \cdot h^c \cdot f^c \cdot [ad]^c$
		2	4	$\{e, g\}$[a]	$\{f\}$	f^c	$\lvert\Delta\rvert < 2$	
		3	5	$\{g, i\}$	$\{h\}$	h^c	$\lvert\Delta\rvert < 2$	
10	$\{b, c, g, d, i\}$							$bcdgi \cdot a^c \cdot e^c \cdot f^c \cdot h^c$
11	$\{a, c, e, g, i\}$							$acegi \cdot b^c \cdot d^c \cdot f^c \cdot h^c$
12	$\{a, c, f, g, h\}$							$acfgh \cdot b^c \cdot d^c \cdot e^c \cdot i^c$
13	$\{a, d, e, f, i\}$							$adefi \cdot b^c \cdot c^c \cdot g^c \cdot h^c$

[a]the result with the same value i but with greater value j are all removed according to Theorem 2.
[b]the Property 2 is needed to implement to simplify the corresponding R_γ.

Table 2
Final results of example 1

γ	A_γ	$\lvert A_\gamma \rvert$	R_γ
1,2	$\{b,f,i\},\{a,d,h\}$	3	$1-[adh]^c\cdot[bfi]^c$
3	$\{b,e,h\}$	3	$beh\cdot[fi]^c\cdot[ad]^c$
4	$\{a,d,g,i\}$	4	$adgi\cdot h^c\cdot[bf]^c$
5	$\{a,c,f,i\}$	4	$acfi\cdot b^c\cdot[dg]^c\cdot[dh]^c=acfi\cdot b^c\cdot(d^c+d[gh]^c)$
6	$\{b,f,g,h\}$	4	$bfgh\cdot e^c\cdot i^c\cdot[ad]^c$
7	$\{b,c,d,h\}$	4	$bcdh\cdot a^c\cdot e^c\cdot[fg]^c\cdot[fi]^c=bcdh\cdot a^c\cdot e^c\cdot(f^c+f\cdot[gi]^c)$
8	$\{a,c,e,h\}$	4	$aceh\cdot b^c\cdot d^c\cdot[fi]^c$
9	$\{b,e,g,i\}$	4	$begi\cdot h^c\cdot f^c\cdot[ad]^c$
10	$\{b,c,g,d,i\}$	5	$bcdgi\cdot a^c\cdot e^c\cdot f^c\cdot h^c$
11	$\{a,c,e,g,i\}$	5	$acegi\cdot b^c\cdot d^c\cdot f^c\cdot h^c$
12	$\{a,c,f,g,h\}$	5	$acfgh\cdot b^c\cdot d^c\cdot e^c\cdot i^c$
13	$\{a,d,e,f,i\}$	5	$adefi\cdot b^c\cdot c^c\cdot g^c\cdot h^c$

Table 3
Comparison between seven preprocessing techniques [28] and the proposed SDP

Methods	Random	H	L	C	C+H	C+L	SODA	Yeh
Sequences of paths	$\{a,d,h\}$	$\{a,d,h\}$	$\{a,c,e,g,i\}$	$\{b,f,i\}$	$\{a,d,h\}$	$\{a,d,h\}$	$\{a,d,h\}$	$\{b,f,i\}$
	$\{a,d,g,i\}$	$\{b,c,d,h\}$	$\{a,c,e,h\}$	$\{b,e,h\}$	$\{b,e,h\}$	$\{b,e,h\}$	$\{b,f,i\}$	$\{a,d,h\}$
	$\{a,d,e,f,i\}$	$\{a,c,f,g,h\}$	$\{a,c,f,g,h\}$	$\{a,d,h\}$	$\{b,f,i\}$	$\{b,f,i\}$	$\{b,e,h\}$	$\{b,e,h\}$
	$\{a,c,e,h\}$	$\{a,c,e,h\}$	$\{a,c,f,i\}$	$\{b,f,g,h\}$	$\{a,d,g,i\}$	$\{a,c,e,h\}$	$\{a,c,f,i\}$	$\{a,d,g,i\}$
	$\{a,c,e,g,i\}$	$\{a,d,e,f,i\}$	$\{a,d,e,f,i\}$	$\{b,e,g,i\}$	$\{a,c,f,i\}$	$\{a,c,f,i\}$	$\{a,d,g,i\}$	$\{a,c,f,i\}$
	$\{a,c,f,g,h\}$	$\{a,d,g,i\}$	$\{a,d,g,i\}$	$\{b,c,d,h\}$	$\{b,e,g,i\}$	$\{a,d,g,i\}$	$\{b,c,d,h\}$	$\{b,f,g,h\}$
	$\{a,c,f,i\}$	$\{b,f,g,h\}$	$\{a,d,h\}$	$\{a,c,f,i\}$	$\{a,c,e,h\}$	$\{b,c,d,h\}$	$\{b,f,g,h\}$	$\{b,c,d,h\}$
	$\{b,c,d,h\}$	$\{b,e,h\}$	$\{b,c,d,g,i\}$	$\{a,c,e,h\}$	$\{b,c,d,h\}$	$\{b,e,g,i\}$	$\{a,c,e,h\}$	$\{a,c,e,h\}$
	$\{b,c,d,g,i\}$	$\{b,c,d,g,i\}$	$\{b,c,d,h\}$	$\{a,d,g,i\}$	$\{b,e,g,h\}$	$\{b,f,g,h\}$	$\{b,e,g,i\}$	$\{b,e,g,i\}$
	$\{b,e,h\}$	$\{a,c,f,i\}$	$\{b,e,g,i\}$	$\{b,c,d,g,i\}$	$\{a,d,e,f,i\}$	$\{a,c,e,g,i\}$	$\{a,c,e,g,i\}$	$\{b,c,g,d,i\}$
	$\{b,e,g,i\}$	$\{a,c,e,g,i\}$	$\{b,e,h\}$	$\{a,c,f,g,h\}$	$\{a,c,e,g,i\}$	$\{a,c,f,g,h\}$	$\{a,c,f,g,h\}$	$\{a,c,e,g,i\}$
	$\{b,f,g,h\}$	$\{b,f,i\}$	$\{b,f,g,h\}$	$\{a,c,e,g,i\}$	$\{a,c,f,g,h\}$	$\{a,d,e,f,i\}$	$\{a,d,e,f,i\}$	$\{a,c,f,g,h\}$
	$\{b,f,i\}$	$\{b,e,g,i\}$	$\{b,f,i\}$	$\{a,d,e,f,i\}$	$\{b,d,f,g,i\}$	$\{b,c,d,g,i\}$	$\{b,c,d,g,i\}$	$\{a,d,e,f,i\}$
Product number	23	22	26	16	16	17	15	14

Step 5: Since $|A_{\gamma-1}|=3<|A_\gamma|=4$, let $R_\gamma=R_\gamma\cdot\prod_{\omega^<}P^c(\omega^<)=P(\{a,d,g,i\})\cdot P^c(\{b,f\})\cdot P^c(\{h\})$ and go to Step 8.

Step 8: Since $j=3>|A_\gamma|-2$, go to Step 12.

Step 12: Since $R_\gamma=P(\{a,d,g,i\})\cdot P^c(\{b,f\})\cdot P^c(\{h\})$ is already simplified, go to Step 13.

Step 13: Let $R=R+R_\gamma$.

Hence, $R_4=P(adgi)\cdot P^c(bf)\cdot P^c(h)$. Table 1 lists the complete results obtained by applying the proposed SDP to Fig. 1. The final result for computing the modified ARPANET reliability using the proposed Algorithm is summarized in Table 2.

As we can see, the proposed SDP yields 12 disjoint products. In consequence, the formula expressing the system's reliability:

$$\begin{aligned}R=\;&acegi\cdot b^c\cdot d^c\cdot f^c\cdot h^c+acfgh\cdot b^c\cdot d^c\cdot e^c\cdot i^c\\&+adefi\cdot b^c\cdot c^c\cdot g^c\cdot h^c+bcdgi\cdot a^c\cdot e^c\cdot f^c\cdot h^c\\&+begi\cdot h^c\cdot f^c\cdot[ad]^c+aceh\cdot b^c\cdot d^c\cdot[fi]^c\\&+bcdh\cdot a^c\cdot e^c\cdot(f^c+f\cdot[gi]^c)+bfgh\cdot e^c\cdot i^c\cdot[ad]^c\\&+acfi\cdot b^c\cdot(d^c+d[gh]^c)+adgi\cdot h^c\cdot[bf]^c\\&+beh\cdot[fi]^c\cdot[ad]^c+(1-[adh]^c\cdot[bfi]^c).\end{aligned}$$

Soh and Rai provided experimental results to compare the performance of six different preprocessing techniques applied with MPs using the computer aided reliability evaluator (CAREL) algorithm [24,25] to obtain the disjoint product. Chatelet et al. describe an alternative procedure, the so-called semi-optimal disjunction approach (SODA), using proper initial ordering of MPs based on a complex criterion to reduce the number of disjoint products and overall computation time [27].

Table 3 reports a comparison between these seven methods and the proposed SDP. The meaning of the acronyms used in this table are as follows: H: decreasing Hamming ordering for CAREL, L: lexicographic ordering for CAREL, and C: increasing cardinality for CAREL, respectively (more details can be found in the cited papers).

Based on the time complexity needed by the proposed SDP discussed in Section 4, the proposed SDP is more attractive than the existing SDPs. As for this example, the proposed SDP appears as the best SDP to reduce the number of terms in the SDP.

6. Conclusion

The SDP plays a very important role in the network reliability analysis. Basically, the existing SDPs are implemented in a straightforward enumeration procedure. Hence, the entire enumeration of all possible disjoint products is quite huge and tedious.

In this article, a new SDP for generating disjoint products for finding the network reliability in terms of the known MPs with a time complexity $O(|V|^{|A_i|}|A_i|)$ for MP, A_i was developed. The proposed method is an improvement on the other SDP methods known by the authors in following three ways: (1) the Absorption Law is merged in the procedure used to form each disjoint product. It is better than SDPs that implement the Absorption Law only after each disjoint product is formed, (2) by characterising the structure of the MPs and the relationship between MPs and subpaths, the time complexity for forming and simplifying the disjoint products of A_i is better than that for the known SDPs if $|V|^{|A_i|}|A_i| \leqslant i|A_i| + i^2$, (3) the time complexity for forming each disjoint product from the longest MP-flow is faster than that for any existing SDPs. Moreover, in the classical test example, the proposed SDP produces shorter formulae than the other known SDPs with less computation effort for each disjoint product.

From the time complexity, the proposed SDP is efficient than any existing SDPs. However, as mentioned in the paper [22] cited in Section 2, all of these SDP based approaches, including algorithms and heuristics, cannot be compared from a general and theoretical point of view. They should be compared experimentally with respect to (at least) the following criteria:

- The size of the results,
- The computation time,
- The form of the resulting formulae.

In this article, only the first criterion is considered. Further applications are needed to establish the superiority of the proposed SDP. Moreover, it is well-known (see Section 2) that running times for BDDs are by orders of magnitude smaller than those of other methods. As a consequence, the computation of the probability of any formula is much more efficient using BDDs than any other methods. Therefore, we are convinced that the future of SDP methods depends on their ability to compete fairly with Binary Decision Diagram techniques. Such a comparative study remains to be done.

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