MAT301H5Y

Week 6 Lecture 2: Cosets

Thursday Jun 15th, 2023

4-5pm instructed by Marcu-Antone Orsoni

Overview

We went over an example regarding inner automorphisms in class. However, most of the lecture was related to the topic of cosets. To avoid confusion, let $\text{Id} : G \to G$ be the identity automorphism and $\epsilon \in G$ be the identity element of group G unless specified otherwise.

Example where $\text{Inn}(D_n)$ not equal to $\text{Aut}(D_n)$

We covered an example in the previous class regarding the cases where $\text{Inn}(G) \prec \text{Aut}(G)$ and this is the continuation of that example. We want to show the following.

Example: Showing $\text{Inn}(D_n)$ not equal to $\text{Aut}(D_n)$

Consider D_n for odd n and $n > 3$. Show that $\text{Inn}(D_n) \prec \text{Aut}(D_n)$. You may use without proof that $|\text{Aut}(D_n)| = n\phi(n)$[where ϕ is the euler totient function].

Solution:

Consider the special homomorphism discussed in the previous lecture defined as $\Phi : G \to \text{Aut}(G)$ where $g \to \phi_g$(where $\phi_g(x) = gxg^{-1}$). We'll take this homomorphism for $\Phi : D_n \to \text{Aut}(D_n)$. We learned as a proposition last class that $Z(G) = \ker(\Phi)$, observe that $Z(D_n) = \{\epsilon\} = \ker(\Phi)$ as n is odd(basically there is only one center element, this result is an exercise in the course notes).

Since the kernel is trivial, we may conclude that Φ is injective. Knowing that the image is inner automorphisms of $D_n(\text{Im}(\Phi) = \text{Inn}(D_n))$. We conclude that it is *bijective* to $\text{Im}(\Phi)$ and thus an isomorphism between D_n and $\text{Inn}(D_n) \iff D_n \cong \text{Inn}(D_n)$.

We learned in class that $|D_n| = 2n$, since it is isomorphic to $\text{Inn}(D_n)$, we conclude that $|\text{Inn}(D_n)| = |D_n| = 2n$. Knowing that $|\text{Aut}(D_n)| = n\phi(n)$ and that $n > 3$, the following inequality holds

$$\begin{aligned}
&|\text{Aut}(D_n)| = n\phi(n) > 2n = |\text{Inn}(D_n)| \\
\iff &|\text{Aut}(D_n)| > |\text{Inn}(D_n)| \\
\implies &\text{Aut}(D_n) \succ \text{Inn}(D_n)
\end{aligned}$$

Realize that the inequality $n\phi(n) > 2n$ arrises from the fact that $\phi(n) > 2$ for $n > 2$ as per the nature of the euler totient function.

After this we moved on to cosets which should be chapter 6 in the course notes.

Cosets

Definition: Coset

Suppose G is a group and $H \preceq G$. The types of cosets are as below:

- **Left cosets**: all the left cosets are defined as $aH = \{ah : \forall h \in H\}$ for all $a \in G$
- **Right cosets**: all the right cosets are defined as $Ha = \{ha : \forall h \in H\}$ for all $a \in G$.
- **Conjugation**: all conjugation sets are defined as $gHg^{-1} = \{ghg^{-1} : \forall h \in H\}$ for all $g \in G$

Remark: Note that above we describe *all* possible cosets. While if you just take a single element a, a possible left coset of some $H \preceq G$ is $aH = \{ah : \forall h \in H\}$.

Addtional remarks

- We sometimes use the addition symbol for cosets for subgroups have operation is standard addition. In other words, if the operation for a subgroup H is addition and $g \in G$, then we write $gH = g + H = \{g + h : \forall h \in H\}$.
- Observe that if G is abelian then $aH = Ha$ for all cosets.
- The notation $n\mathbb{Z}$ is *not* a coset notation
- aH is a subgroup if an only if $a \in H$

Proposition: Equivalence relation of cosets

Define the relations $\overset{r}{\sim}, \overset{l}{\sim}$ as right and left coset relations respectively on G for some subgroup $H \preceq G$ as below:

$$x \overset{r}{\sim} y \iff y \in xH \iff x^{-1}y \in H$$
$$x \overset{l}{\sim} y \iff y \in Hx \iff xy^{-1} \in H$$

From the properties of equivalence relations, we define the notations for the given equivalence relations as below:

- $G/ \overset{l}{\sim} = H\backslash G$ is defined as all the possible equivalence classes induced by the left coset relation. This tells us that xH forms a partition on G for all $x \in G$. In other words, $[a]_l \in G/ \overset{l}{\sim}$ where $[a]_l = aH$
- $G/ \overset{r}{\sim} = G/H$ is defined as all the possible equivalence classes induced by the right coset relation. This tells us that Hx forms a partition on G for all $x \in G$. In other words, $[a]_r \in G/ \overset{r}{\sim}$ where $[a]_r = Ha$

Remark: The reader is encourage to see other properties of cosets in proposition 6.3 in the course notes. However, the instructor remarked that a lot of the properties listed followed from the fact that the cosets induce an *equivalence* relation and those have nice properties.

Additional remarks

- In general $aH \neq Ha$
- If $aH \neq bH$, then $aH \cap bH = \emptyset$(because of partitions). However, it is not necessary that as a result, $aH \cap Hb = \emptyset$

We then covered an example in class regarding cosets considering the subgroups of $\mathbb{Z}$.

Example: coset relation on Z with nZ subgroups

Observe how congruence modulo relation relates to cosets of $\mathbb{Z}$ with the subgroup $n\mathbb{Z}$

Solution

Taking our group $G = \mathbb{Z}$ and $n\mathbb{Z} \preceq \mathbb{Z}$ for non-negative n. Knowing that $\mathbb{Z}$ is abelian, we see that $a + n\mathbb{Z} = n\mathbb{Z} + a$ for all $a \in \mathbb{Z}$. Therefore, consider our relation on $\mathbb{Z}$ we see that for any $x, y \in \mathbb{Z}$ that $x \overset{r}{\sim} y \iff x \overset{l}{\sim} y \iff xy^{-1} \in n\mathbb{Z} \iff x - y \in n\mathbb{Z}$. By definition of $n\mathbb{Z}$, we see that $x - y \in n\mathbb{Z} \iff n|x - y$. Using our modulo relation, this means that $x \equiv y \mod n$.

Remark: it is now in the course where we finally realize that $\mathbb{Z}/\overset{r}{\sim} = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. All we are doing is partitioning $\mathbb{Z}$ into it's equivalence classes on congruence modulo n when constructing $\mathbb{Z}_n$.