MAT301H5Y

Week 6 Lecture 1: Autmorphisms and inner automorphisms

Tuesday Jun 13th, 2023

3-5pm instructed by Marcu-Antone Orsoni

Overview

In this class we were introduced to automorphisms and inner automorhpisms as groups and their respective propositions. Assume in the rest of these notes that the identity element as ϵ unless specified otherwise for our groups.

Automorphism

Definition: Automorphism and Aut(G)

An isomorphism from a group to itself is an automorphism. The set of all automorphisms for a group G is denoted by $\text{Aut}(G)$.

It turns out that the set of all automorphisms are a group under function composition as per the proposition below:

Proposition: Aut(G) a group

For some group G, $\text{Aut}(G)$ is a group under function composition.

The proof was left as an exercise to the reader. We considered an example of an automorphism that isn't the trivial map(identity).

Example: non-trivial automorphism

Assume G is abelian, the map $\phi : G \to G$ defined by the map $x \to x^{-1}$ is an automorphism.

The proof was not covered and was also left as an exercise to the reader. That is mostly all in regards to the definition of automorphism. We then moved on to inner automorphism.

Inner automorphisms

Definition: Inner automorphism

Choose some $g \in G$, the map denoted by $\phi_g : G \to G$ given by $x \to gxg^{-1}$ is a special automorphism called an inner automorphism.

Proof:

We have to prove each of the properties of an automorphism. Namely, that it is a well defined isomorphism from $G \to G$.

- **Well defined:** We see by definition and closure under binary operator on G that $g, x \in G \implies \phi_g(x) = gxg^{-1} \in G$ and thus ϕ_g is well defined.

- **Is a homomorphism**: In order to show that ϕ_g is a homomorphism, assume $x, y \in G$. Observe the following steps:

$$\begin{aligned}\phi_g(xy) &= gxyg^{-1} \\ &= gx\ \epsilon\ yg^{-1} \\ &= gx(g^{-1}g)yg^{-1} \\ &= (gxg^{-1})(gyg^{-1}) \\ &= \phi_g(x)\phi_g(y)\end{aligned}$$

  Which shows it is a homomorphism.

- **Bijectivity**: We now need to show bijectivity, in order for it to qualify as an isomorphism. We can do this by constructing an inverse. Realize that we can choose our inverse as $\phi_{g^{-1}}(x) = (g^{-1})x(g^{-1})^{-1} = g^{-1}xg$(notice that all we did was replace g with g^{-1} as per the definition of ϕ_g). We want to show that $\phi_{g^{-1}}$ is indeed a valid inverse for our function:

$$\begin{aligned}\phi_g \circ \phi_{g^{-1}}(x) &= \phi_g(g^{-1}xg) = gg^{-1}xgg^{-1} = x \\ &= \epsilon(x)\end{aligned}$$

$$\begin{aligned}\phi_{g^{-1}} \circ \phi_g(x) &= \phi_{g^{-1}}(gxg^{-1}) = g^{-1}gxg^{-1}g = x \\ \\ &= \epsilon(x)\end{aligned}$$

  Since obtain the identity element upon composition, both ways, we have found a valid inverse of ϕ_g, making ϕ_g an isomorphism.

Having shown all the properties of an automorphism, we can conclude that the inner automorphism is indeed an automorphism.

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It turns out the set of all inner automorphisms form a subgroup of $\text{Aut}(G)$ as per the proposition below:

Proposition

The set of all inner automorphisms denoted by $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$.

Proof

We need to show the properties of a subgroup below:

- **Subset**: $\text{Inn}(G) \subseteq \text{Aut}(G)$: We see this follows from the previous proposition which states that inner automorphims are indeed automorphisms themselves.

- **Non-empty**: $\text{Inn}(G)$ is non-empty as the identity transformation $\epsilon(x) = x$ is an inner automorphism($\phi_\epsilon(x) = \epsilon x \epsilon^{-1} = x = \epsilon(x)$) and thus non-empty.

- **Closure**: Assume $g, h \in G$, and take their inner automorphisms $\phi_g, \phi_h \in \text{Inn}(G)$. Observe the following:

$$\begin{aligned}\phi_g \circ \phi_{h^{-1}}(x) &= \phi_g(h^{-1}xh) \\ &= gh^{-1}xhg^{-1} = (gh^{-1})x(gh^{-1})^{-1} \\ &= \phi_{gh^{-1}} \\ \implies \phi_g \circ \phi_{h^{-1}} &\in \text{Inn}(G)\end{aligned}$$

Having shown the above and applying proposition 2.21, we see that $\text{Inn}(G) \preceq \text{Aut}(G)$.

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Remark: If G is abelian, realize that the only possible inner automorphism is the identity ϵ. The reason for this is that if choose $g \in G$ assuming it is abelian we see that $\phi_g = gxg^{-1} = xgg^{-1} = x\epsilon = x = \epsilon(x)$.

We then moved on to an interesting map that maps an element from G to its inner automorphism. This map is defined as below and turns outuu to be a group homomorphism.

Φ homomorphism to Aut(G)

The map $\Phi : G \to \text{Aut}(G)$ defined by $g \to \phi_g$ is a group homomorphism.

Proof:
Choose $g, h \in G$, see that:

$$\begin{aligned}\Phi(gh)(x) &= \phi_{gh}(x) = ghx(gh)^{-1} \\ &= ghxh^{-1}g^{-1} = \phi_g(hxh^{-1}) = \phi_g \circ \phi_h \\ &= \Phi(g)\Phi(h)\end{aligned}$$

Therefore, our given map Φ is a group homomorphism.

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Since we have a homomorphism, we can try to compute its kernel and image, we see this in the given proposition.

Proposition: kernel and image of Φ

Given our definition of $\Phi : G \to \text{Aut}(G)$, the following holds true:

(1) $\ker(\Phi) = Z(G)$

(2) $\text{Im}(\Phi) = \text{Inn}(G)$

Proof for (1):
We need to show double inclusion of the two sets $\ker(G)$ and $Z(G)$, however we can do this with equivalences. Let $x \in G$ be arbitrary and choose some $g \in \ker(G)$ the following steps follow:

$$\begin{aligned}g \in \ker(G) \iff \Phi(g) = \epsilon(x) \\ &\iff \phi_g(x) = gxg^{-1} = x \\ &\iff gx = xg \\ &\iff g \in Z(G)\end{aligned}$$

Therefore we have shown that $\ker(G) = Z(G)$.
Proof for (2) is trivial by definition of inner automorphisms.

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As per the proposition above we have the following corollary:

Corollary: injectivity and surjectivity of Φ

With the previous proposition, the following results follow:

- Φ is surjective iff $\text{Inn}(G) = \text{Aut}(G)$
- Φ is injective iff $Z(G) = \{\epsilon\}$

Remark: Note that if G is abelian then $Z(G) = G = \ker(\Phi)$ and $\text{Im}(\Phi) = \{\epsilon\}$.

We then moved on to Cayley's theorem.

Theorem: Cayley's Theorem

Let G be an arbitrary group, G is isomorphic to some subgroup of $S(G) = B_{ij}(G)$(The permutation group of G).

A proof outline was given to this theorem and not a complete proof. Choose some $g \in G$, define the left shift map $T_g : G \to G$ as $x \to gx$. Observe that T_g is bijective(but not a homomorphism). From this we see that $T_g \in S(G)$.

Define another map $T : G \to S(G)$ as $g \to T_g$. Show that T is a homomorphism. This is sufficient to show it is isomorphic to a subgroup as T is injective(it is left as an exercise to show this).