MAT301H5Y

Week 8 Lecture 1: Quotient and factor groups

Thursday Jul 18th, 2023

3-5pm instructed by Marcu-Antone Orsoni

Overview

In this class we went over factor groups and some of the propositions that arise from such groups.

Reminders

The instructor glossed over brief reminders, most of them are in previous week's notes, however a few important ones which he noted are as below.

Proposition: homomorphism

If $\phi : G \to G'$ is a homomorphism, then $\ker(\phi) \trianglelefteq G$.

Theorem 7.8: image and pre-image of normal subgroups

If $\phi : G \to G'$ is a homomorphism then:

1. $K' \trianglelefteq G' \implies \phi^{-1}(K') \trianglelefteq G$
2. $K \trianglelefteq G \implies \phi(K) \trianglelefteq \phi(G)$ but not necessarily a normal subgroup of G'

Definition: simple group

A group is said to be simple if the only normal subgroups are itself and the trivial subgroup.

We then moved on to explore a couple examples as below

Examples: Simple groups

1. A_3 and A_n for all $n \geq 5$ are simple(not evident but no need to prove)
2. $\mathbb{Z}_p$ is simple
3. Groups of prime order.

We can see that (2) holds true because the only possible subgroups of $\mathbb{Z}_p$ are $\mathbb{Z}_p$ and $\{\epsilon\}$. The third point follows from the fact that groups of prime order are isomorphic to $\mathbb{Z}_p$.

Factor groups

Definition: Factor groups

Suppose we have a normal subgroup $H \trianglelefteq G$, then $G/H = H\backslash G$ is a group under the operation $(aH)(bH) = (ab)H$, this is called a factor or quotient group.

The proof of this fact is in the textbook. A remark is that if a subgroup is normal, then the binary operation as defined above is well-defined.

We also considered some examples.

Examples: factor groups

1. $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ is a quotient group of $\mathbb{Z}$ with $n\mathbb{Z}$
2. $G = \mathbb{Z}_{12}$, $H = \langle [4]_{12} \rangle$ then

$$G/H = \{[0] + \langle [4]_{12} \rangle, [1] + \langle [4]_{12} \rangle, [2] + \langle [4]_{12} \rangle, [3] + \langle [4]_{12} \rangle\}$$

The above given examples are indeed factor groups as the main group is abelian. Below is a theorem relating the center to the quotient of subgroups of the qutotient being cyclic.

Theorem

If $G/Z(G)$ is cyclic then G is abelian

Remark: One might find the contrapositive fairly useful in solving some problems.

Exercise: non abelian group's center

Show that if G is not abelian such that $|G| = pq$ for primes p, q then $Z(G) = \{\epsilon\}$

Solution We can consider cases for what $Z(G)$ can be. Applying lagrange's theorem we know that $|Z(G)| \in \{1, p, q, pq\}$. Since we know that the group is not abelian, $|Z(G)| \neq pq$. Now consider the case that $|Z(G)| = p$, we then see by lagrange's theorem that:

$$[G : Z(G)] = \frac{|G|}{|Z(G)|} = \frac{pq}{p} = q$$

Since the order of $G/Z(G)$ is prime, it must be cyclic, however, as per our previous theorem, this implies that G must be abelian which is not possible. We may apply similar reasoning for $|Z(G)| = q$ to see that $|Z(G)| \neq p, q$. This only leaves us with $|Z(G)| = 1$ which is the trivial subgroup.

We ended the class with Cauchy's theorem

Cauchy's theorem

If G is a group such that $2 \leq |G| < \infty$ and $p \Big| |G|$ for p prime then there exists element $g \in G$ such that $|g| = p$

The proof for this theorem is a challenging exercise in the book.