MAT301H5Y

Week 7 Lecture 1: Cosets revised and continued

Thursday Jul 6th, 2023

4-5pm instructed by Marcu-Antone Orsoni

Overview

In this lecture we revised cosets and coset properties.

Cosets revised

Reminders

Assuming that $H \preceq G$:

1. Left coset for some $g \in G$ of H is given by $gH = \{gh : \forall h \in H\}$. Same for right coset
2. In general $aH \neq Ha$
3. G is abelian then $gH = Hg$ for any $g \in G$
4. $aH = bH \iff a \in bH$

Remark: There are more properties of cosets in Proposition 6.3 of the book. From the previous lecture notes, find below a proposition on the equivalence relation which is once again brought up in class as a reminder:

Proposition: Equivalence relation of cosets

Define the crelations $\overset{r}{\sim}, \overset{l}{\sim}$ as right and left coset relations respectively on G for some subgroup $H \preceq G$ as below:

$$x \overset{l}{\sim} y \iff y \in xH \iff x^{-1}y \in H$$
$$x \overset{r}{\sim} y \iff y \in Hx \iff xy^{-1} \in H$$

From the properties of equivalence relations, we define the notations for the given equivalence relations as below:

- $G/\overset{l}{\sim} = G/H$ is defined as all the possible equivalence classes induced by the left coset relation. This tells us that xH forms a partition on G for all $x \in G$. In other words, $[a]_l \in G/\overset{l}{\sim}$ where $[a]_l = aH$
- $G/\overset{r}{\sim} = H\backslash G$ is defined as all the possible equivalence classes induced by the right coset relation. This tells us that Hx forms a partition on G for all $x \in G$. In other words, $[a]_r \in G/\overset{r}{\sim}$ where $[a]_r = Ha$

We covered a proposition in class which relates the cardinalities of left and right cosets.

Proposition: cardinality of cosets

Assume $H \preceq G$:

1. $\forall x \in G, |xH| = |Hx||H|$
2. $|G/H| = |H\backslash G|$

Proof:

(1) Let us show $|xH| = |H|$, we can then repeat similar steps for $|Hx|$. Consider the left shift map $T_x^l : H \to xH$ defined by $h \to xh$. We see by definition that it is surjective. Also realize it is an injection. Since we have found a bijection between H and xH, we may conclude that they have the same cardinality.

(2) Let us construct a bijection again from $G/H \to H\backslash G$. We choose our map as $\zeta : G/H \to H\backslash G$ defined by $gH \to Hg^{-1}$. Note that such a bijection between sets needs to be well defined. We will therefore show that $gH = g'H \implies \zeta(gH) = \zeta(g'H) = Hg^{-1} = H(g')^{-1}$ first. Start by assuming $gH = g'H$ we then see the following

$$\begin{aligned} gH = g'H &\iff g \in g'H \\ &\implies g = g'h \text{ for some } h \in H \\ &\implies g^{-1} = h^{-1}(g')^{-1} \\ &\implies g^{-1} \in H(g')^{-1} \\ &\iff Hg^{-1} = H(g')^{-1} \end{aligned}$$

We thus conclude that ζ is well defined. To show it is bijective we can construct an inverse. Define the inverse as $\alpha : H\backslash G \to G/H$ defined as $Hg \to g^{-1}H$. We need to show it is well defined but it follows the same steps as earlier. Let us show it is indeed an inverse

$$\begin{aligned} \alpha(\zeta(gH)) &= \alpha(Hg^{-1}) = gH \\ \zeta(\alpha(Hg)) &= \zeta(g^{-1}H) = Hg \end{aligned}$$

This shows that α is indeed an inverse of ζ, thus a bijection. It follows the two chosen sets have the same cardinality.

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Index

Definition: Index

Let $H \preceq G$, the index of H in G is given by $[G : H]$ is equal to the cardinality of G/H.

We considered some examples. For example take $G = \mathbb{Z}$ and $H = n\mathbb{Z}$, we see that $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. This can be understood by the implications of the following relation:

$$\begin{aligned} x \overset{l}{\sim} y &\iff x \in y(n\mathbb{Z})[\text{coset}] \iff x \in y + n\mathbb{Z}[\text{Additive coset notation}]. \\ &\implies y^{-1}x = xy^{-1} = x - y \in n\mathbb{Z} \\ &\implies x - y | n \iff x \equiv y \mod n \end{aligned}$$

Which gives better clarity as to why $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. Therefore $[\mathbb{Z} : n\mathbb{Z}] = n$

Now consider the example that $G = S_n$ and $H = A_n$. The conclusion is that $[S_n : A_n] = 2$. TThe reasoning for this is that we only have even and odd permutations in S_n. From this, we realize that the only possible cosets are A_n and τA_n where τ is some transposition.
We ended on lagranges theorem which will be covered in more detail in the next lecture.