MAT301H5Y

Week 10 Lecture 1: Isomorphism theorems

Tuesday Jul 25th, 2023

3-5pm instructed by Marcu-Antone Orsoni

Overview

In this class we went over the isomorphism theorems and some of their proofs

Isomorphism theorems

Consider the first isomorphism theorem as per the textbook.

First isomorphism theorem

Suppose $\phi : G \to G'$ is a group homomorphism. The map $\tilde{\phi} : G/H \to G'$ for $H \preceq \ker(\phi)$ defined as $\tilde{\phi}(xH) = \phi(x)$ is a group homomorphism. Moreover, $\tilde{\phi}$ is injective if and only if $H = \ker(\phi)$. It then follows that $G/\ker\phi \cong \phi(G)$

Proof:

First, let us show that $\tilde{\phi}$ is well defined. W.T.S $aH = bH \implies \tilde{\phi}(aH) = \tilde{\phi}(bH)$. Assume $aH = bH$, we know from properties of cosets that $b^{-1}a \in H$, since $H \subseteq \ker(\phi)$ we see that $b^{-1}a \in \ker(\phi)$.
From this it follows that:

$$\begin{aligned} \phi(b^{-1}a) &= \phi(b)^{-1}\phi(a) = \epsilon_G \\ &\implies \phi(a) = \phi(b) \\ &\implies \phi(aH) = \phi(bH) \end{aligned}$$

We now proceed to show that $\tilde{\phi}$ is a group homomorphism. Let $x, y \in G$, the following steps demonstrate why it is a group homomorhism:

$$\begin{aligned} \tilde{\phi}((xH)(yH)) &= \tilde{\phi}(xyH) \\ &= \phi(xy) \\ &= \phi(x)\phi(y) = \tilde{\phi}(xH)\tilde{\phi}(yH) \end{aligned}$$

Which is what we needed to show for group homomorphism.

We then need to show that $\tilde{\phi}$ is an injective homomorhpism if and only if $H = \ker(\phi)$. Starting with the reverse direction, let us assume that $H = \ker(\phi)$. Observing the kernel of $\tilde{\phi}$ by assuming $x \in \ker(\tilde{\phi})$:

$$\begin{aligned} \tilde{\phi}(xH) &= \phi(x) = \epsilon_{G'} \\ &\implies x \in \ker(\phi) \\ &\iff x \in H \\ &\iff xH = H \\ &\implies \ker(\tilde{\phi}) \subseteq \{H\} \end{aligned}$$

We can also follow the proof backwards to show that $\ker(\tilde{\phi}) = \{H\}$ which means that the kernel is trivial(since H is the identity in G/H). Having shown this, we see

indeed that $\tilde{\phi}$ is injective.
To complete the reverse direction, assume that $\tilde{\phi}$ is injective, and therefore the kernel is trivial. We see that $\ker(\tilde{\phi}) = \{H\}$ for that reason. Let $x \in \ker(\phi)$, this means that:

$$\begin{aligned}
\phi(xH) = \phi(x) = \epsilon_{G'} \\
\implies xH \in \ker(\tilde{\phi}) \\
\iff xH \in H \implies xH = H \\
\iff x \in H \\
\implies \ker(\phi) \subseteq H
\end{aligned}$$

The reverse direction follows similarly to show that $H = \ker(\phi)$. Lastly, it is trivial to show that $\tilde{\phi}(G/H) = \phi(G)$,making $\tilde{\phi}$ surjective to $\phi(G)$. From this we see that $\tilde{\phi} : G/\ker(\phi) \to \phi(G)$ is an isomorphism and therefore $G/\ker(\phi) \cong \phi(G)$.

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Corollary

Suppose $\phi : G \to G'$ is a group homomorphism then the following equality holds true:

$$|G| = |\ker(\phi)||\text{Im}(\phi)|$$

Proof: The proof is trivial by lagrange's theorem and applying the first isomorphism theorem.

Corollary

Suppose $\phi : G \to G'$ is a homomorphism, then $\phi(G)|\gcd(|G|, |G'|)$.

Proof:

From the previous corollary we see that $\phi(G)\Big||G|$ and from lagranges theorem we see that $\phi(G)\Big||G'|$ because $\phi(G)$ is a subgroup. From this we conclude that $\phi(G)|\gcd(|G|, |G'|)$.

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Corollary

$H \preceq G \implies N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$.

Proof

Define the homomorhpism from $N_G(H) \to \text{Aut}(H)$ such that $n \to \phi_n$ where ϕ_n is the conjugate map. Note that it is important to show that ϕ_n is indeed well defined within the defined co-domain. The reasoning for this is that it is possible that ϕ_n is not an automorphism in H but in G instead. It is sufficient to show that $\phi_n(H) \subseteq H$ as this means all the automorphisms are contained within H and thus result an automorphism of H. Observe that $\phi_n(H) = nHn^{-1} = H$ because $n \in N_G(H)$, therefore showing that $\phi_n(H) \subseteq H$, making ϕ a well defined function within the domain.

W.T.S. that $\ker(\phi) = C_G(H)$. Let $n \in \ker(\phi)$, we observe that:

$$\begin{aligned} n \in \ker(\phi) &\implies \phi(n) = \phi_n = \mathrm{Id} \ \ [\text{Identity in } \mathrm{Aut}(H)] \\ &\iff \phi_n(x) = \mathrm{Id}(x) \iff nxn^{-1} = x \\ &\iff nx = xn \\ &\iff n \in C_G(H) \ \ [\because x \in H] \end{aligned}$$

Which ends up showing us that $\ker(\phi) = C_G(H)$. Using the first isomorphism theorem we know that $G/\ker(\phi) = G/C_G(H) \cong \phi(N_G(H)) \preceq \mathrm{Aut}(H)$.

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Second isomorphism theorem

$N \trianglelefteq G$, $H \preceq G \implies H \cap N \trianglelefteq H$ and $NH/N \cong H/H \cap N$.

Proof: check the course notes

Third isomorphism theorem

If $N, H \trianglelefteq G$ then the following holds true

$$G/N \Big/ H/N \cong G/H$$

Proof was left as an exercise.