# MAT301H5Y

## Week 10 Lecture 2 External direct sum

Tuesday Jul 27th, 2023

4-5pm instructed by Marcu-Antone Orsoni

## Overview

In this class, we were introduced to external direct sums and their properties and how we can possibly "deconstruct" some groups.

## External direct sum

**Definition: External direct sum**

Let $(G,+), (G',*)$ be groups, the external direct sum of these groups is defined as below:

$$G \oplus G' = \{(g, g') : g \in G, g' \in G'\}$$

With the operation $(a,b)(c,d) = (a+b, c*d)$. (Same can be extended for a finite number of arbitrary groups, forming a set of $n$-tuple if you have $n$ groups which can be expressed as $G_1 \oplus G_2 \oplus G_3 \dots G_n = \bigoplus_{i=1}^{n} G_i$).

**Theorem: External direct sums are groups**

The external direct sum of an arbitrary collection of groups under the given operation form a group itself. In fact, $|G \oplus G'| = |G| \times |G'|$.

*Proof*: Instructor remarked the proof as can be seen in assignment 2 and some parts in the textbook.

**Example: External direct sum**

Find all elements of $U(8) \oplus \mathbb{Z}_3$.

*Solution*:

We can just list them out in the classic cartesian product format to obtain:

$$\begin{aligned} U(8) \oplus \mathbb{Z}_3 =& \{(1,0),(1,1),(1,2)\dots \\ & (7,0),(7,1),(7,2)\} \end{aligned}$$

**Theorem: Order of elements in direct sums**

Given an element of $\bigoplus_{i=1}^{n} G_i$ written as $(g_1, g_2, \dots, g_n)$, we have $|(g_1, g_2, \dots, g_n)| = \operatorname{lcm}(|g_1|, |g_2|, \dots |g_n|)$.

*Proof*: See Assignment 2.

**Proposition: isomorphism of direct sums**

Given $G, G'$ as groups then $G \oplus G' \cong G' \oplus G$.

*Proof*: Trivial(construct an inverse for the map $(a,b) \to (b,a)$ to shorten proof)

**Theorem: Cyclic direct sums**

if $G$ and $H$ are cyclic, then $G \oplus H$ is cyclic if and only if $\gcd(|G|, |H|) = 1$.

*Proof*: See assignment 2

**Corollary: Cyclic decomposition**

If $n = n_1 n_2 \dots n_k$ where $n_1, n_2 \dots, n_k$ are all coprime, then:

$$\mathbb{Z}_n \cong \bigoplus_{i=1}^{k} \mathbb{Z}_{n_i}$$

**Theorem: Normal subgroup quotient isomorphism**

If $H \trianglelefteq G$ and $K \trianglelefteq G'$ then $H \oplus K \trianglelefteq G \oplus G'$ and $G \oplus G' \Big/ H \oplus K \cong G/H \oplus G'/K$

*Proof*: Ran out of time.