MAT301H5Y

Week 4 Lecture 2: Symmetric groups continued and transposition

Thursday Jun 1st, 2023

4-5pm instructed by Marcu-Antone Orsoni

Overview

We continued exploring symmetry groups. In specific, we were introduced to transpositions and propositions related to them.

We started after some reminders as below:

Reminders

1. S_n called the symmetry group is the group of all bijections from $A = \{1, 2, \ldots, n\}$ to itself(also can be denoted by the permutation group $B_{ij}(A)$)
2. S_n is non abelian for $n \geq 3$
3. m-cycle is a tuple describing a bijection in the form $(a_1\ a_2\ a_3\ \ldots\ a_m)$ where $a_i \to a_{i+1}$ for $1 \leq i < m$ and $a_m \to a_1$
4. $\sigma \in S_n \implies \sigma = C_1 C_2 \ldots C_k$ where C_i is a cycle and disjoint from the others.
5. $\sigma \in S_n \implies |\sigma| = \text{lcm}(|C_1|, |C_2| \ldots |C_k|)$ where all the cycles are disjoint
6. Inverse of a cycle is just the cycle in reverse order. For example $(1\ 3\ 5)^{-1} = (5\ 3\ 1)$

Transpositions

Definition: transposition

A transposition is a 2-cycle. Sometimes transpositions are written as $\mathcal{T}$(used by instructor in class) or β(used in the course notes).

It turns out there is an interesting property of transpositions which can be summarized as in the below.

Proposition 4.12

Every permutation of S_n can be written as a product of transpositions(not necessarily disjoint).

Proof of this theorem was not given in class. However we considered the below example:

$$\begin{aligned}(1\ 4\ 5\ 7) &= (1\ 4)(4\ 5)(5\ 7)\\ &= (4\ 5)(5\ 7)(7\ 1)\\ &= (1\ 7)(1\ 5)(1\ 4)\end{aligned}$$

What followed in class were a series of theorems, definitions and propositions. Note that a "transposition" decomposition is just a decomposition of a permutation that consist of only transpositions.

Theorem: parity of transposition decomposition

For any permutation $\sigma \in S_n$ all the number of transpositions in the decomposition is either all odd or all even.

Remark: The above theorem just states that for some arbitrary permutation $\sigma \in S_n$, if we find an even number of transpositions in the transposition decomposition, then every other possible transposition decomposition must be even as well. Same argument extends for odd transposition decompositions.

Definition: Even/odd permutation

$\sigma \in S_n$ is said to be an even permutation if the number of transpositions in the transposition decomposition is even. Same applies for odd permutation.

Now that we can classify a permutation as even or odd. We can then define a new group as below.

Definition: Alternating group A_n

A_n is the set of all even permutations in S_n. In other words:

$$A_n = \{\sigma \in S_n : \sigma \text{ is an even permutation}\}$$

This is called the alternating group.

Remark: It is a proposition that $A_n \preceq S_n$. However, if we take the set of all permutations which are odd, it turns out not to be a subgroup of S_n. The reader is encouraged to think about why it isn't a subgroup.

Proposition: order of A_n

$|A_n| = \frac{n!}{2}$

Proof

One of the ways of measuring the order/cardinality of a set is by constructing bijections where helpful. Define O_n to be the set of all odd permutations. One can realize that A_n and O_n are disjoint and who's union is S_n. In other words: $A_n \sqcup O_n = S_n$(disjoint union where $A_n \cap O_n = \emptyset$). From this it is safe to conclude the following:

$$|A_n| + |O_n| = |S_n|$$

One can intuitively see where to go from here. If we can argue that $|A_n| = |O_n|$, we are mostly done. To do so, construct a bijection $f : A_n \to O_n$ such that $f(\sigma) = (1\ 2)\sigma$ for any $\sigma \in A_n$(realize that the choice of $(1\ 2)$ as a transposition was random). We see that this is indeed a bijection. From MAT102 we realize that this concludes that $|A_n| = |O_n|$ because there exists a bijection between them. We therefore have the following steps after realizing that $S_n = n!$

$$|A_n| + |O_n| = |S_n|$$

$$\iff$$

$$|A_n| + |A_n| = 2|A_n| = |S_n| = n!$$

$$\implies |A_n| = \frac{n!}{2}$$

$\blacksquare$

From here we did a couple examples in class. Consider the one below.

Exercise: transposition decomposition

Let $\alpha = (1\ 2\ 4\ 3)(3\ 4\ 5\ 6\ 1)$ and $\beta = (1\ 3\ 2)(5\ 4\ 2)$. Decompose α, β into transpositions and determine the parity of α, β.

Solution

First we decompose each of the individual cycles into transpositions pairwise and combine them for both α and β. They are done below:

$$
\begin{aligned}
\alpha &= (1\ 2\ 4\ 3)(3\ 4\ 5\ 6\ 1) \\
&= [(1\ 2)(2\ 4)(4\ 3)][(3\ 4)(4\ 5)(5\ 6)(6\ 1)] \\
&= (1\ 2)(2\ 4)\cancel{(4\ 3)}\cancel{(3\ 4)}(4\ 5)(5\ 6)(6\ 1) \text{ [Because we see that } (4\ 3)^{-1} = (3\ 4)] \\
&= (1\ 2)(2\ 4)(4\ 5)(5\ 6)(6\ 1) \\
\beta &= (1\ 3\ 2)(5\ 4\ 2) \\
&= [(1\ 3)(3\ 2)][(5\ 4)(4\ 2)] \\
&= (1\ 3)(3\ 2)(5\ 4)(4\ 2)
\end{aligned}
$$

From counting we see that parity of α is odd while parity of β is even.

Pairwise decomposition makes it easier for us to make conclusions about the parity of cycles as the below proposition.

Proposition: m-cycles

If we have an m cycle, we have a decomposition of $m - 1$ transpositions for our cycle pairwise. From this we can conclude that if we have a cycle of even length, then we have an odd permutation for that cycle. The same applies for cycles of odd length.

There was a final exercise in class, however due to lack of time and the professor being tired, we could not complete it. It is as below.

Exercise

Let $\alpha, \beta \in S_4$. $\beta\alpha = (1\ 3\ 4\ 2)$ $\alpha\beta = (1\ 3\ 2\ 4)$ find α and β