# MAT301H5Y

## Week 12 Lecture 1 Group action

### Tuesday Aug 2nd, 2023

3-5pm instructed by Marcu-Antone Orsoni

## Overview

We went over group actions again with some theorems and covered the class equation. Recall that group actions are not included in the final.

**Reminders: Group actions**

**Definition**: $G$ acts on a set $X$ if there is a function $G \times X \to X$ such that $(g, x) \to g \cdot x$ where the following hold true:

1. $\forall a, b \in G,\ a \cdot (b \cdot x) = (ab) \cdot x$
2. $\forall x \in X,\ e_G \cdot x = x$

**Proposition**: If $G$ acts on $X$ then there exists a homomorphism $\phi G \to S(X)$ defined by $g \to \sigma_g : x \to g \cdot x$

Converse, if $\phi : G \to S(X)$ is a homomorphism, then the actions defined by $g \cdot x \to \phi(g)(x)$ is an action

We also go over some examples as previous class:

**Examples**

1. $G$ acts on itself by left/right translation: $g \cdot h \to gh$
2. $G$ acts on itself by conjugation: $g \cdot h \to ghg^{-1}$
3. If $X = \{H \preceq G\}$, then $G$ can act by subgroup conjugation: $g \cdot H \to gHg^{-1}$
4. $S(X)$ can act on the set $X$ by the natural map $\sigma \cdot x \to \sigma(x)$
5. $D_n$ can act on the regular $n$-sided polygon $P_n$.

We recapped the orbit and stablizer as well

**Definition: Orbit and stabilizer**

1. $\omega(x) = \{g \cdot x : g \in G\} \subset X$
2. $G_x = \{g \in G : g \cdot x = x\} \preceq G$
3.

Proposition: orbit equivalence relation

The relation on $X$ defined by$x\ y \iff y \in w(x)$ is an equivalence relation. This means the orbits form a partition on $X$ as below:

$$X = \bigsqcup_{w \text{ is an orbit}} w$$

Stablizer and orbit relation

Given a group action $G$ on $X$, the following equation holds:

$$w(x) = [G : G_x] = \frac{|G|}{|G_x|}$$

*Outline of proof*: We try to show that the map $g \cdot x \to gG_x$ is well defined and a bijection

Corollary

$|X| = \sum w(x) = \sum \frac{G}{G_x}$