MAT301H5Y

Week 11 Lecture 1 Internal direct sum

Tuesday Aug 2nd, 2023

3-5pm instructed by Belal Abuelnasr

Overview

We went over some classification theorems and some extra theorems not covered in the textbook as well.

Classification theorems

Theorem: Group of order p^2

If $|G| = p^2$ then $G \cong \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$

Proof

If G is cyclic then trivially we see that $G \cong \mathbb{Z}_{p^2}$. Suppose that is not the case. From this by our assumption and using lagrange's theorem, we see that if $g \neq \epsilon$ then $|g| = p$. Since we know that $\langle g \rangle \prec G$, there exists some $g' \in G$ such that $g' \notin \langle g \rangle$. Observe from this that $\langle g \rangle \cap \langle g' \rangle = \{\epsilon\}$ as from lagrange we see that it is possible for $| \langle g \rangle \cap \langle g' \rangle | = p$, but this leads to them not being disjoint(not our assumption). Observe that the size of the two subgroups' product is

$$| \langle g \rangle \langle g' \rangle | = \frac{| \langle g \rangle || \langle g' \rangle |}{| \langle g \rangle \cap \langle g' \rangle |} = p^2 = |G|$$

Showing that all subgroups are normal is left as an exercise.

We then obtain the inner direct product that $G = \langle g \rangle \times \langle g' \rangle$ which we know is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

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Corollary: Abelian

If G is a group of size p^2 then G is abelian.

We are done with the textbook(good job)

Summary of classification theorems of groups:

Let G be a group

1. G cyclic $\begin{cases} G \cong \mathbb{Z} & |G| = \infty \\ G \cong \mathbb{Z}_n & |G| < \infty \end{cases}$

2. $|G| = 2p$(odd prime p) $\begin{cases} G \cong D_p & G \text{ not cyclic} \\ G \cong \mathbb{Z}_{2p} & \text{Otherwise} \end{cases}$

3. $|G| = p$ for p prime $\implies G \cong \mathbb{Z}_p$

4. $|G| = p^2$ then $\begin{cases} G \cong \mathbb{Z}_{p^2} & \text{if cyclic} \\ G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p & \end{cases}$

Summary for subgroups:

1. Fundamental theorem of cyclic subgroups
2. **Cauchy's theorem for abelian groups**: If $2 \leq |G| < \infty$ and $p||G|$ then there exists element of order p.

Finitely generated groups: Recall that finitely generated groups are groups which are generated from a finite set. Some examples are as below:

- D_n, S_n are finitely generated
- $\mathbb{Z}$ or $\mathbb{Z}^r = \bigoplus_{i=1}^r \mathbb{Z}$
- $\mathbb{Q}$ is not finitely generated.

We have a theorem for finitely generated subgroups which is new.

Fundamental theorem of finitely generated groups

If G is a finitely generated abelian group then

$$G \cong \mathbb{Z}^r \bigoplus_{i=1}^{s} \mathbb{Z}_{n_i}$$

1. $r, n_i \in \mathbb{Z}$, and $r \geq 0$, $n_i \geq 2$
2. $n_{i+1}|n_i$ for $1 \leq i \leq s-1$
3. The decomposition above is unique

Corollary: finite abelian groups

If G is finite and abelian then

$$G \cong \bigoplus_{i=1}^{s} \mathbb{Z}_{n_i}$$

Converse of lagrange theorem of finite abelian groups:

Converse of lagrange's theorem for finite abelian groups

If G is a finite abelian group and $n||G|$ then there exists $H \preceq G$ such that $|H| = n$

Proof

By previous corollary, $G \cong \bigoplus_{i=1}^{s} \mathbb{Z}_{n_i}$. From this we conclude $|G| = \prod_{i=1}^{s}$. We can then take a factor $k = n_1' n_2' \ldots n_s'$ in which $n_i'|n_i$. By FTGC it was left as an exercise to to complete the proof.

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