MAT301H5Y

Week 12 Lecture 1 Group action

Tuesday Aug 2nd, 2023

3-5pm instructed by Marcu-Antone Orsoni

Overview

In this class we began with group actions, this is not coming in the final exam.

Group actions

Reminders

1. Let X be a set, then $S(X)$ is the group of bijections from X to itself which is also called the permutations of X.

Definition

Let G be a group and X be a set. An action of G on X is an application : $\cdot : G \times X \to X$ defined as $(g, x) \to g \cdot x$ which obeys the following:

1. $e_g \cdot x = x, \ \forall x \in X$
2. $g \cdot (h \cdot x) = gh \cdot x, \ \forall x \in X, \forall g, h \in G$

Consider some examples below

Examples

1. Any group G and any set X, G acts on X trivially: $g \cdot x = x$
2. Let X be a set and $S(X)$ be the permutations then: $g \cdot x = g(x) \forall x \in X, \forall x \in S(X)$, cross checking the properties
   (a) $e_{S(X)} \cdot x = Id \cdot x = x$ and thus
   (b) $\sigma(\alpha \cdot x) = \sigma(\alpha(x)) = \sigma \cdot \alpha(x) = (\sigma\alpha) \cdot x$
3. G can act on itself using left or right translation: $\forall g, h \in G$, $g \cdot h = gh$:
   - $\forall g \in G$ $e_G \cdot g = e_G g = g$
   - $\forall g, g' \in G$, see that $g \cdot (g' \cdot h) = gg'h = (gg') \cdot h$
4. G can act on itself by conjugation: $\forall g, h \in G$, $g \cdot h = ghg^{-1}$
   - $\forall h \in G$, observe that $e_G \cdot h = h$
   - $\forall g, g' \in G$, $h \in G$ $g \cdot (g' \cdot h) = gg'hg'^{-1}g = (gg')h(gg')^{-1} = (gg') \cdot h$
5. $G = GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$. Same thing than 2).
6. Take $H \trianglelefteq G$, G can act on G/H by left translation: $\forall g \in G, \forall kH \in G/H$: $g \cdot kH = gkH$
   - $\forall kH \in G/H$, we see that $e_G \cdot kH = kH$
   - $\forall g, h \in G, kH \in G/H, \quad g \cdot (h \cdot kH) = ghkH = (gh)kH$
7. Let X be the set of all subgroup of G we can have G act on X by subgroup conjugation $g \cdot H = gHg^{-1}$

Definition

1. We say that $x \in X$ is fixed under an action $\cdot$ of G if $\forall g \in G$ $g \cdot x = x$
2. We say subset $Y \subseteq X$ is stable under the action of G if $\forall g \in G, g \cdot Y \subset Y$

Proposition:

If G acts on X, then we can define a group homomorphism $\phi : G \to S(X)$ where $g \to \sigma_g : x \to g \cdot x$.
Conversely if $\phi : G \to S(X)$ is a group homomorphism, we can define the group action $g \cdot x = \phi(g)(x)$

Proof:
Assume that G acts on X. Let's prove that $\phi : g \to \phi$ is a group homomorphism. First let us begin by showing that ϕ is well defined. Given that σ_g is a bijection, the inverse is $\sigma_{g^{-1}}$, this means that $\phi(g) \in S(X)$. for all $g \in G$, making it well defined. This is demonstrated below:

$$\sigma_g \sigma_{g^{-1}}(x) = g \cdot (g^{-1} \cdot x) = (gg^{-1})(x) = x = \sigma_{g^{-1}}\sigma_g = (g^{-1}g)(x)$$

Suppose $a, b \in G$, observe the following steps:

$$\phi(ab)(x) = \sigma_{ab}(x) = ab \cdot x = a \cdot (b \cdot x) = \sigma_a \cdot \sigma_b(x) = (\phi(a) \cdot \phi(b))(x)$$

Therefore, ϕ is a group homomorphism. The converse is left as an exercise.

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Definition: orbit

- Let G act on X. The orbit of $x \in X$ is $G \cdot x = \{g \cdot x, g \in G\}$.
- The stabilizer of x is $G_x = \{g \in G, g \cdot x = x\} \subseteq G$.

We can We say that G acts faithfully on X if $g \cdot x = x \implies g = e_G$.
The action is said to be transitive if $\forall x, y \in X, \exists g \in G$ such that $g \cdot x = y$

Examples

1. For group action (3) left translation is a faithful action.

Theorem: orbit-stabilizer theorem

$|G \cdot x| = [G : G_x]$