MAT301H5Y

Week 11 Lecture 1 Internal direct sum

Tuesday Aug 2nd, 2023

3-5pm instructed by Belal Abuelnasr

Overview

In this class, we began internal direct sums and their properties after we studied the projection map for external direct sum.

External direct sum continued

We studied the projection map for external direct sums

Definition: projection map for external direct sums

The map $\Phi_i : \bigoplus_{i=1}^{n} G_i \to G_i$ defined by $(g_1, g_2, \ldots, g_i \ldots, g_n) \to g_i$

Proposition: projection homomorphism

Given groups G, H, the projection map $\Phi_G : G \oplus H \to G$ and $\Phi : G \oplus H \to H$ are both surjective homomorphisms and moreover:

$$\ker(\Phi_G) = \{\epsilon_G\} \oplus H$$
$$\ker(\Phi_H) = G \oplus \{\epsilon_H\}$$

Proof:

Suffices to show for Φ_G without loss of generality. Surjectivity is almost immediate. Let $g \in G$, we can then those $(g, \epsilon_H) \in G \oplus H$ such that $\Phi_G((g, \epsilon_H)) = g$, showing surjectivity.

We now need to show it is indeed a group homomorphism. To do so, let $(g_1, h_1), (g_2, h_2) \in G \oplus H$ and observe the following steps:

$$\begin{aligned}\Phi_G((g_1, h_1)(g_2, h_2)) =& \Phi_G((g_1 g_2, h_1 h_2)) \\ &= g_1 g_2 \\ &= \Phi_G((g_1, h_2))\Phi_G((g_2, h_2))\end{aligned}$$

To show the kernel, let us do our double subset inclusion as shown below:

$$\begin{aligned}x = (g, h) \in \ker(\Phi_G) &\iff \Phi_G(x) = g = \epsilon_G \\ &\iff x = (\epsilon_G, h) \in \epsilon_G \oplus H\end{aligned}$$

Which shows that $\ker(\Phi_G) = \epsilon_G \oplus H$ as needed.

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Proposition: normal subgroup direct product

Given $H_1 \trianglelefteq G_1, H_2 \trianglelefteq G_2$. We have that $H_1 \oplus H_2 \trianglelefteq G_1 \oplus G_2$ and $G_1 \oplus G_2/H_1 \oplus H_2 \cong G_1/H_1 \oplus G_2/H_2$.

Proof:

Given the assumptions of our groups, it is left as an exercise to show that $H_1 \oplus H_2 \preceq G_1 \oplus G_2$. To show normality, let $(h_1, h_2) \in H$ and $(g_1, g_2) \in G$. Observe that

$$\begin{aligned}(g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1} &= (g_1, g_2)(h_1, h_2)(g_1^{-1}, g_2^{-1}) \\ &= (g_1 h_1 g_1^{-1}, g_2 h_2 h_2^{-1})\end{aligned}$$

We know that $g_1 h_2 g_1^{-1} \in H_1$ and same for the other entry because they both are normal subgroups. It therefore directly follows that $(g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1} \in H_1 \oplus H_2$, making it a normal subgroup.
We now need to show the given isomorphism. To do so, let us try and apply the first isomorphism theorem we have learned earlier. To begin, let us construct a surjective homomorphism from $\phi : G_1 \oplus G_2 \to G_1/H_1 \oplus G_2/H_2$. Choose the map as $(g_1, g_2) \to (g_1 H_1, g_2 H_2)$. It follows fairly trivially that this map is surjective. To show it is a homomorphism, choose $(a, b), (c, d) \in G_1 \oplus G_2$ and observe the following:

$$\begin{aligned}\phi((a, b)(c, d)) &= \phi((ac, bd)) \\ &= (acH_1, bdH_2) \\ &= ([aH_1][cH_1], [bH_2][dH_2]) \\ &= (aH_1, bH_2)(cH_1, dH_2) \\ &= \phi((a, b))\phi((c, d))\end{aligned}$$

Which shows it is indeed a homomorphism. The kernel turns out to be $H_1 \oplus H_2$, we can show this as below through double subset inclusion:

$$\begin{aligned}x = (a, b) \in \ker(\phi) &\iff \phi(a, b) = (aH_1, bH_2) = \epsilon = (H_1, H_2) \quad [\because \text{The identity is the subgroup} \\ &\qquad \text{tuple itself as one can notice}] \\ &\iff aH_1 = H_1, bH_2 = bH_2 \\ &\iff a \in H_1, b \in H_2 \\ &\iff (a, b) = x \in H_1 \oplus H_2\end{aligned}$$

From which we conclude indeed that the kernel is $H_1 \oplus H_2$. From this, we apply the first isomorphism theorem to conclude that:

$$G_1 \oplus G_2 / \ker(\phi) = G_1 \oplus G_2 / H_1 \oplus H_2 \cong \phi(G_1 \oplus G_2) = G_1/H_1 \oplus G_2/H_2$$

Which is what we needed to show.

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Internal direct product

Definition: internal direct product

G is said to be an internal direct product of $H_1 \times H_2 \times \cdots \times H_n$ if the following conditions apply:

1. $H_i \trianglelefteq G, \forall i \leq n$
2. $G = H_1 H_2 H_3 \ldots H_n$
3. $(H_1 H_2 H_3 \ldots H_{i-1}) \cap H_i = \{\epsilon\} \, \forall i \leq n$

Corollary

If G is an internal direct product of $H_1, H_2, \ldots H_n$ then $H_i \cap H_j = \{\epsilon\}$ for all $i \neq j$.

Theorem: internal and external direct product isomorphism

If G is an internal direct product of $H_1, H_2, \ldots, H_n$ then $G \cong \bigoplus_{i=1}^{n} H_i$

Proof of above theorem is fairly long and is recommended to take a look in the textbook.