MAT301H5Y

Week 8 Lecture 2: Cosets revised and continued

Thursday Jul 13th, 2023

4-5pm instructed by Marcu-Antone Orsoni

Overview

We covered normal subgroups in this class and moreover, the instructor discussed these key questions that can be answered with the help of normal subgroups:

1. When is $G/H = H\backslash G$?(left cosets equal to right cosets)?
2. When is G/H a group?
3. How do we obtain an isomorphism from a homomorphism?

Turns out the answer to all the questions above are normal subgroups.

Normal subgroups

Definition: Normal subgroup

A subgroup $H \preceq G$ is said to be normal if $\forall g \in G, gH = Hg$ which is equivalent to the following:

- $gHg^{-1} = H$
- $H \trianglelefteq G$

A few corollaries of normal subgroups are as below:

Corollaries of normal subgroups

- $H \trianglelefteq G \implies G/H = H\backslash G$
- Recall the normalizer: $N_G(H) = \{g \in G : gHg^{-1} = H\}$. Normal subgroups have the following equivalence:

$$H \trianglelefteq G \iff N_G(H) = G$$

Proposition: normal subgroup equivalence

If we have a subgroup $H \preceq G$ then the followingg equivalence holds:

$$H \trianglelefteq G \iff \forall g \in G, \forall h \in H \quad ghg^{-1} \in H$$

We also c overed some examples of some normal subgroups.

Example: normal subgroups

- Trivial examples: $\{\epsilon\}, G \preceq G$
- All subgroups of an abelian group
- Subgroups of $Z(G)$
- If $H \preceq G$ and $N \trianglelefteq G$ then $HN \trianglelefteq G$

Proposition

If $[G : H] = 2$ then H is normal.

Remark: The converse is false.

Proposition: kernel

If $\phi : G \to G'$ is a homomorphism then $\ker(\phi)$ is a normal subgroup.

Definition: Simple group

A group G is said to be simple if the only normal subgroups of G are G and $\{\epsilon\}$.

We will later see that if H is a normal subgroup then $G \cong G/H \times H$.

We ended by noting that transitivity usually holds for subgroups e.i. $H \preceq K \preceq G \implies H \preceq G$ but it may not be the case for normal subgroups. However, it is the case when K is cyclic.