Topic 2 – Introduction to Quantum Computing - Qubit, Operators, Gates and Circuits

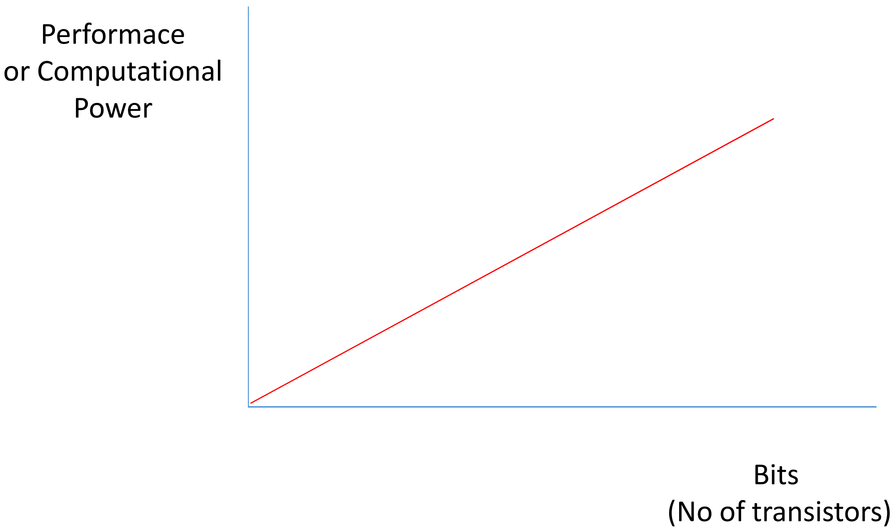
Quantum Machine Learning Course

Outline

- Qubit
- Quantum Gates.
- Quantum Circuits.

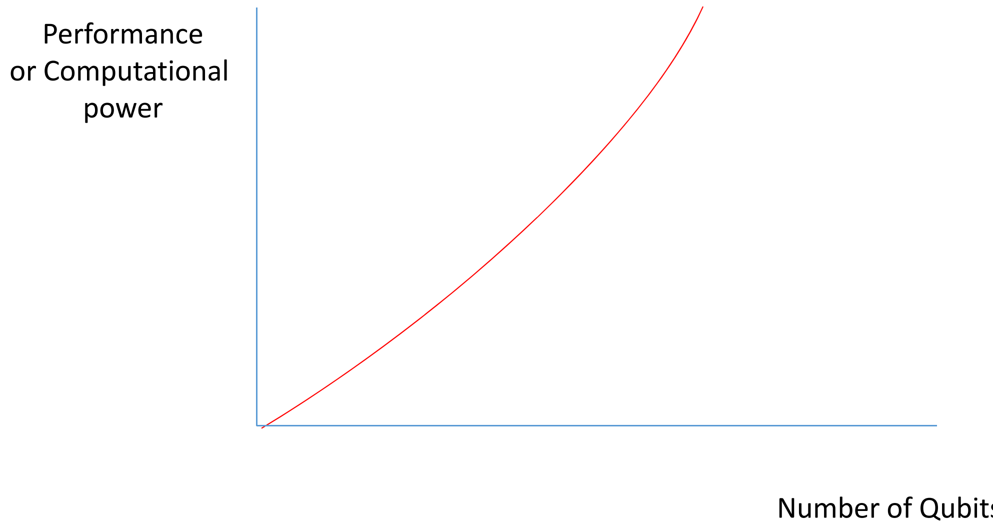
Classical Computing vs. Quantum Computing

Classical computing



- Tasks is for everyday use
- Have lower error rate
- Operates linearly

Quantum computing

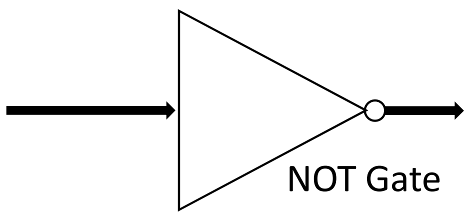


- Big tasks or complex calculations
- Have higher error rate
- Operates exponentially

Classical Computing vs. Quantum Computing

Classical computing

- Logic gates

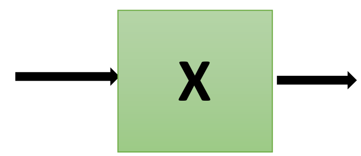


NOT Gate

Boolean Algebra

Quantum computing

- Quantum gates



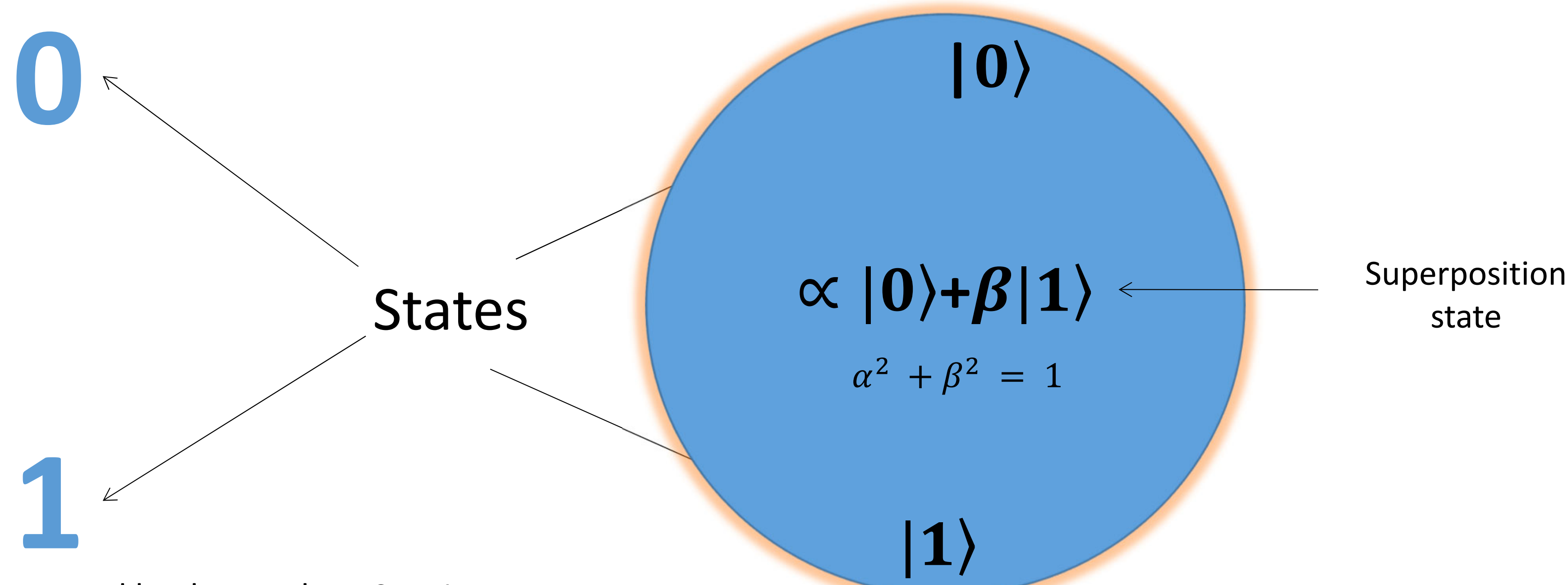
Pauli-X Gate

Linear Algebra

Qubit vs Classical Bit

Classical computing (digital bits)

Quantum computing (quantum bits/qubit)

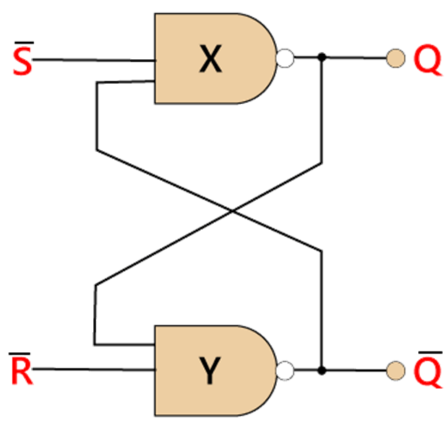


- Classical bit: Represented by the numbers 0 or 1.
- Qubit: Represented by a vector $|\psi\rangle$ in the two-dimensional Hilbert space $\mathcal{H}$.
- Qubit state can change with time.

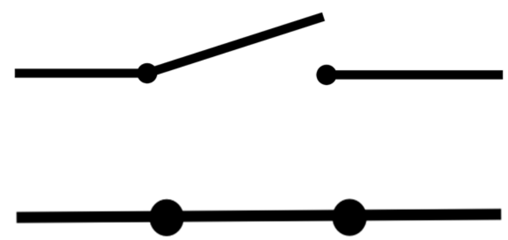
- α and β are known as probability amplitudes
- α and β represent the probabilities of finding $|\psi\rangle$ in states $|0\rangle$ and $|1\rangle$, respectively.
- $\alpha^2 + \beta^2 = 1$. (α and β can be complex numbers.)

Implementation of a bit

Two state of flip flop



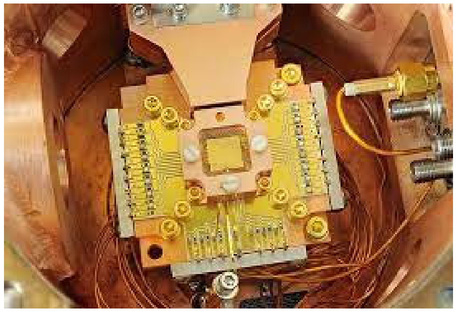
Two positions of an electric switch

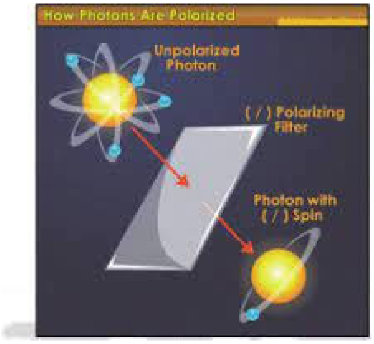


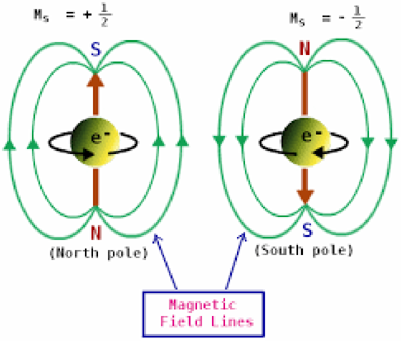
The Qubit

- Qubits (Quantum Bits): Two-level quantum mechanical systems represented by quantum states.

- Examples of Physical Implementation of Qubits:
  - Trapped ion (e.g., single calcium ion in an optical cavity).
  - Polarized photons.
  - Electron spin.
  - Superconducting Qubits



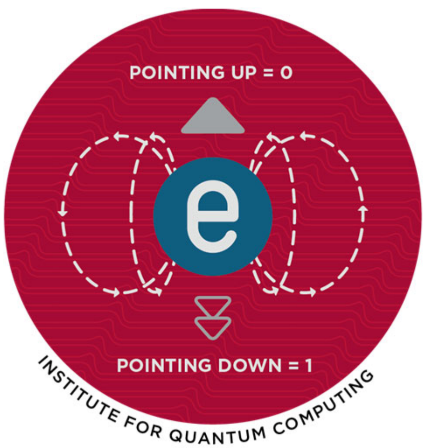




Implementation of a Qubit

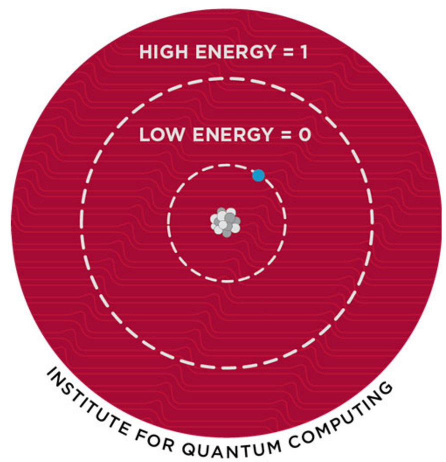
- A qubit is the fundamental unit of information in quantum computing
- Any quantum particle that can be measured in two discrete states could be used as a qubit

Spin



Superconducting qubits

Trapped Atoms and Ions



Polarized photons

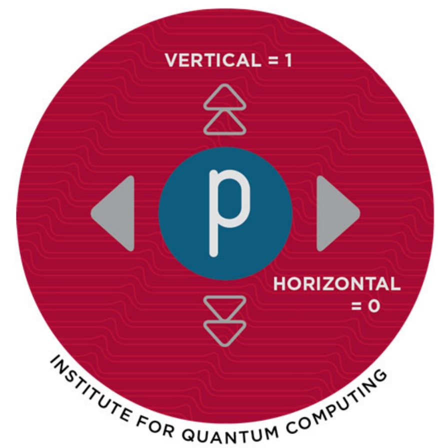


Image Credit – Institute for Quantum Computing

State of Qubit

- Dirac Notation (*Bra-Ket*) is used to describe a quantum state

ket $|\boldsymbol{a}\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ *Bra* $\langle \boldsymbol{b}| = [b_0 \quad b_1]$

$|\boldsymbol{a}\rangle = \langle \boldsymbol{a}|^{\dagger}$ (complex conjugate of $|\boldsymbol{a}\rangle$)

State of Qubit

- The states corresponding to 0 and 1 are represented by the two-dimensional vector space $|0\rangle$ and $|1\rangle$ where

$$|\mathbf{0}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |\mathbf{1}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- This vector space is also known as state space.

Skill Check

What does these matrix computation give?

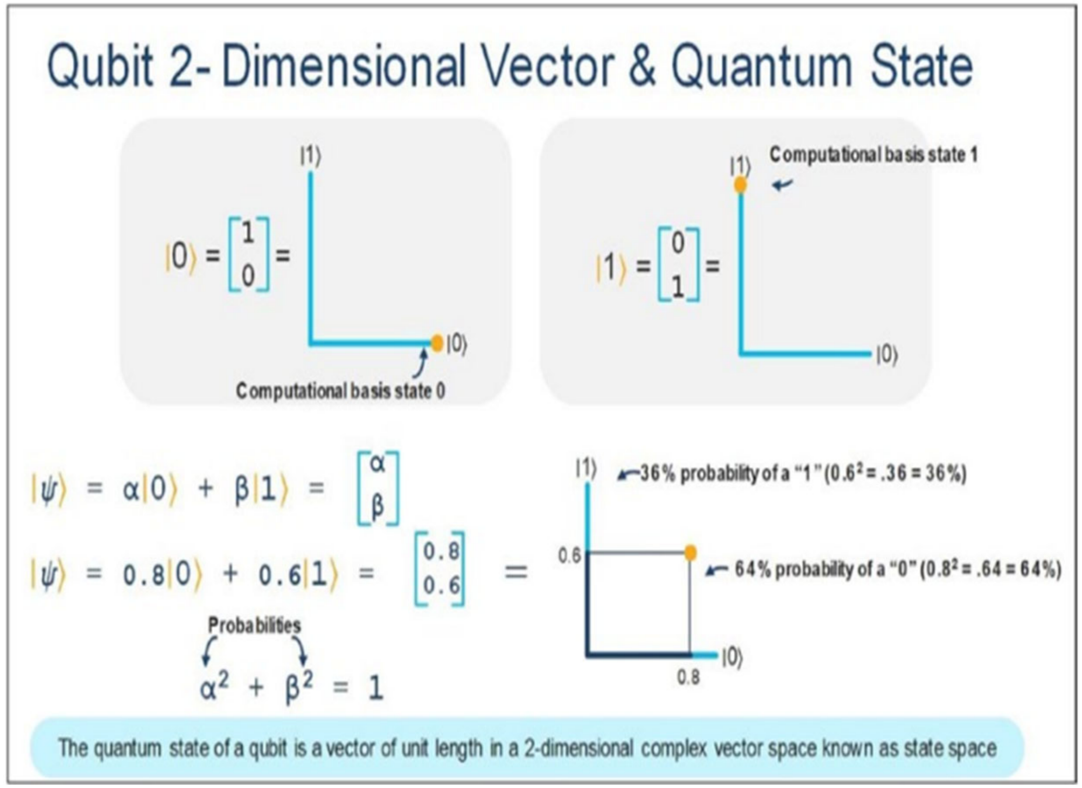
1. $|a\rangle\langle b| = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{bmatrix} a_0 b_0 & a_0 b_1 \\ a_1 b_0 & a_1 b_1 \end{bmatrix}$

2. $|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3. $|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

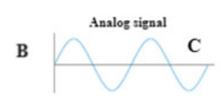
The Qubit State

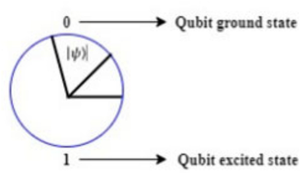
- The figure shows a two-dimensional representation of a qubit state.
- Top diagrams: Position of the vector for basis states $|0\rangle$ and $|1\rangle$.
- X-axis: Represents $|0\rangle$ state, Y-axis: Represents $|1\rangle$ state.



Qubits Vs. Bits Vs. Analogs

Comparison criteria	Bits	Classical analogs	Qubits
State	Two states only 0 or 1 (the state space size is n bits)	Signals (the state space size is n sliders)	Superposition of 0 and 1 states (the state space size is 2^n states)
Initialization	The initial value is 0 or 1	Signal	The initial value is 0.
Dimensionality	Binary digit	Vector space	Vector space
Adjustment	Logical gates	Logical or Analog processes	Quantum gates
Allocation	Determined allocation is 0 or 1	Determined allocation is 0 or 1 (however they can represent continuum of possibilities)	Equal probabilistic allocation states are 0 or 1





$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Quantum Gates

Quantum Gates

- Quantum gates are fundamental building blocks of quantum circuits.
  - Similar to classical logic gates are for conventional digital circuits
- Quantum gates manipulate the state of qubits to perform specific quantum operations.
- Quantum gates are logically reversible, like classical reversible gates.
- Quantum gates are more compact and universal than classical gates.
- A unitary operator acting on a two-dimensional quantum system or a qubit is called a one-qubit quantum gate
  - We can also have multi-qubit quantum gates acting on more than one qubit at a time.

Quantum Gates

- Quantum gates are represented by unitary matrices.
  - A unitary matrix A is a square matrix whose $A^*A = I$, where A^* represents the conjugate transpose of A.
  - $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & \frac{-1}{\sqrt{2}}i \end{bmatrix}$ $A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}i \end{bmatrix}$ $A^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{bmatrix}$ $A^*A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Unitarity implies
  - Reversibility of quantum evolution
  - Conservation of probability amplitudes.
  - Normalization condition
- Unitary operators acting on a qubit are called one-qubit quantum gates.
- Quantum gates can be represented by $2^n \times 2^n$ unitary matrices.
- The dimensions of the matrices depend on the number of qubits n that the gate acts on.
- Therefore single qubit Quantum gate are 2x2 matrices

Quantum Gates: The Pauli gates

- Set of one-qubit gates.
- Applied to quantum circuits, derived from Pauli operators.
- The Pauli gates span the vector space of one-qubit operators.
  - It is considered that there are four different Pauli operators, including the Identity operator

$$\sigma_I = \sigma_{0;}\ \sigma_X = \sigma_{1;}\ \sigma_Y = \sigma_{2;}\ \sigma_z = \sigma_3$$

- The corresponding Pauli gates are: I, X, Y, Z.
- Thus the Pauli group of matrices are a set of one qubit quantum operators that give rise to Pauli gates (applied to quantum circuits) and span the vector space formed by all one qubit operators.

Quantum Gates: The Pauli gates

- The Identity gate: $I = \sigma_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- The NOT gate $X = \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- The Y-axis rotation gate $Y = \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.
- The Z-axis rotation gate $Z = \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- The Pauli gates X, Y, and Z correspond to rotations about the X, Y, and Z axes of the Bloch sphere, respectively.

Quantum Gates: The Pauli gates

- The X flips $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.

- Example: $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 * 1 + 1 * 0 \\ 1 * 1 + 0 * 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$.

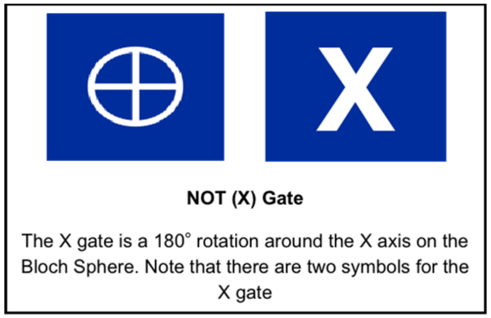
- Importance of Pauli Gates in quantum computing:
  - One-qubit unitary operators can be expressed as a linear combination of Pauli gates.
  - Enables manipulation and control of qubits in quantum circuits.

Quantum Composer Demo

Pauli X-Gate

- The Pauli X-gate does a bit-flip which is a transformation that changes the state of a qubit from $|1\rangle$ to $|0\rangle$ and vice versa.
- It is the quantum analog to the classical NOT gate.
- For a state which is a general superposition of $|1\rangle$ to $|0\rangle$, X-gate is operation that is equivalent to a rotation around the X-axis of the Bloch sphere by angle π.
- X- gate is given as the matrix representation expressed in the computational basis as :

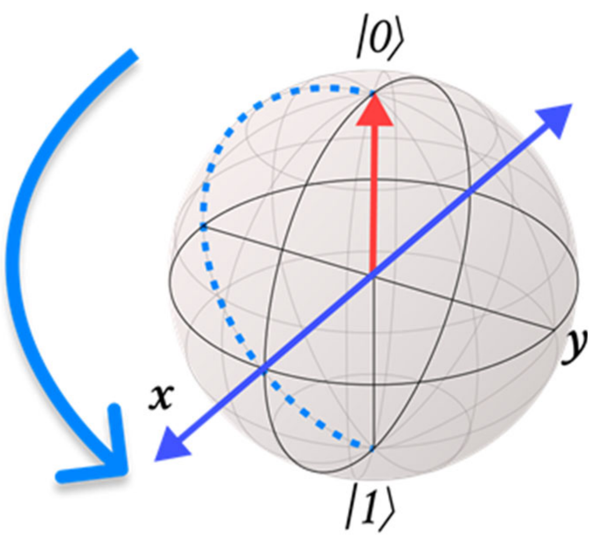
$$X = \sigma_X = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



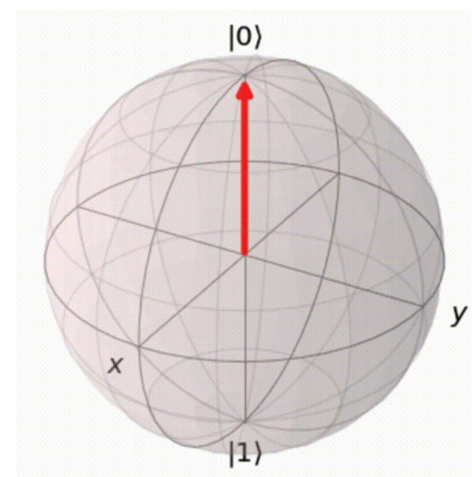
Quantum Circuit representation of X-Gate

Why is it called an X Gate?

The X gate does a **180° rotation around the X axis**.



In order for |0⟩ to turn to |1⟩ and vice versa, the state would have to rotate around the x axis.



How Pauli-X acts on Qubits

- Recall that $|0\rangle$ is given by the matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  and $|1\rangle$ is given by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- A X-gate acting on state $|0\rangle$ gives

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This corresponds to the $|1\rangle$ state

How Pauli-X acts on Qubits

- Recall that $|0\rangle$ is given by the matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  and $|1\rangle$ is given by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- A X-gate acting on state $|1\rangle$ gives

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This corresponds to the $|0\rangle$ state

How Pauli-X acts on Qubits

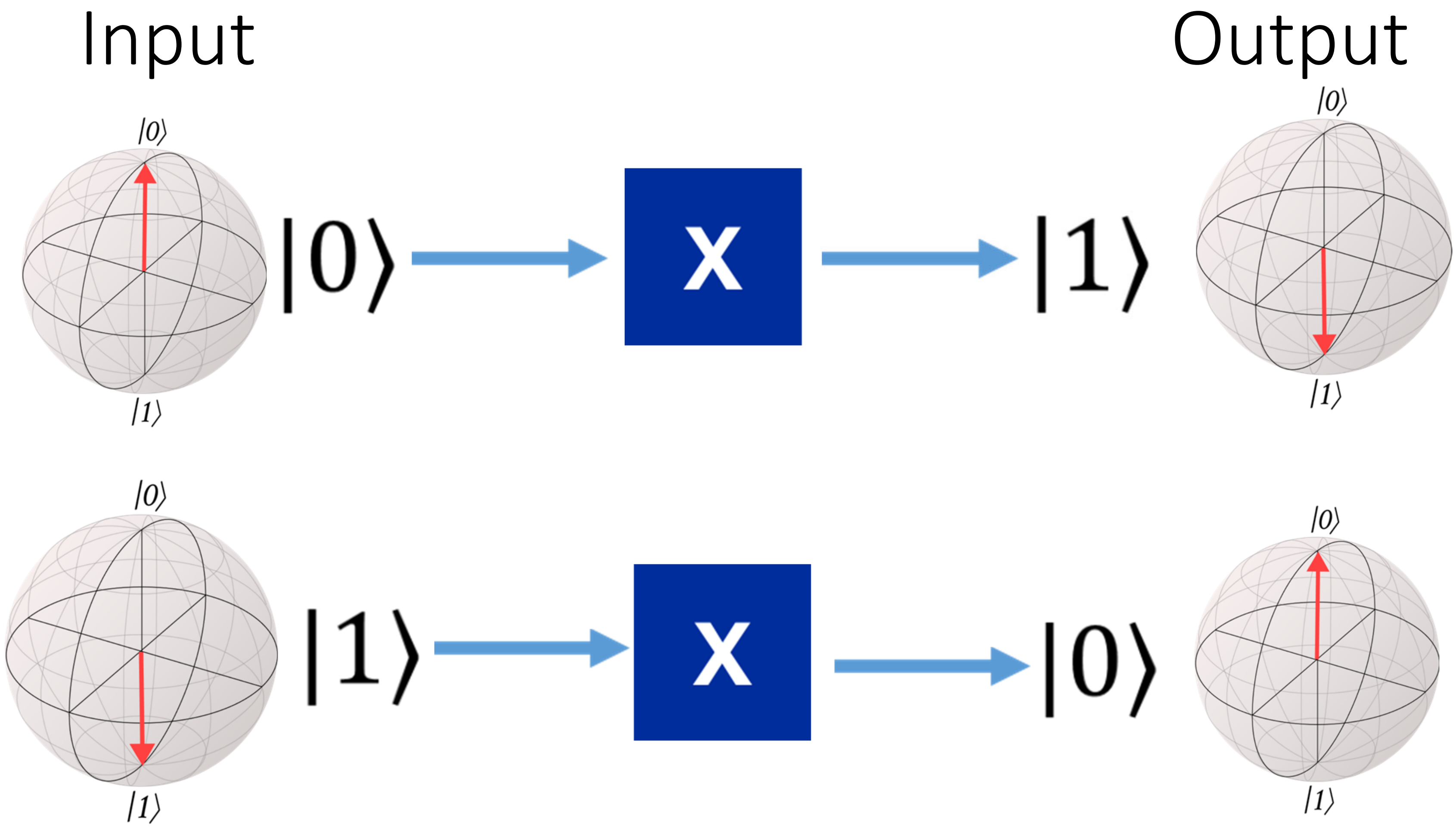
- Let $|v\rangle$ = **a** $|0\rangle$ + **b** $|1\rangle$

- A X-gate acting on state $|v\rangle$ gives

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$$

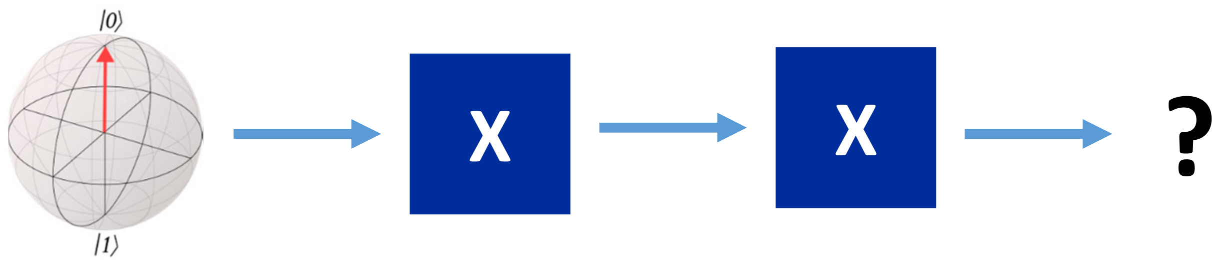
This corresponds to the **b** $|0\rangle$ + **a** $|1\rangle$ state

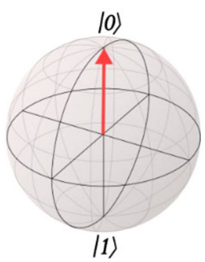
Where will the output Bloch sphere point to for each of the input states?



Practice: The X Gate

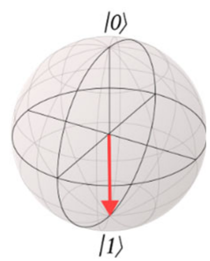
What do we get when we apply the following gate?



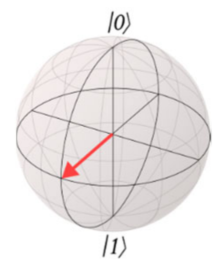


A

$|0\rangle$



B



C

Pauli Z-Gate

- The Z operator is also known as the **phase flip operator** as its effect is to rotate a state vector about the Z axis by π radians or 180 degrees.
- A phase flip is a transformation that changes the relative phase of the |1> state while leaving the |0> state unchanged. In this case, only the phase of the qubit state changes, not the actual basis states themselves.
- Z rotates the state |+⟩ and |-⟩

$$Z = \sigma z = \sigma 3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

How Pauli-Z acts on Qubits

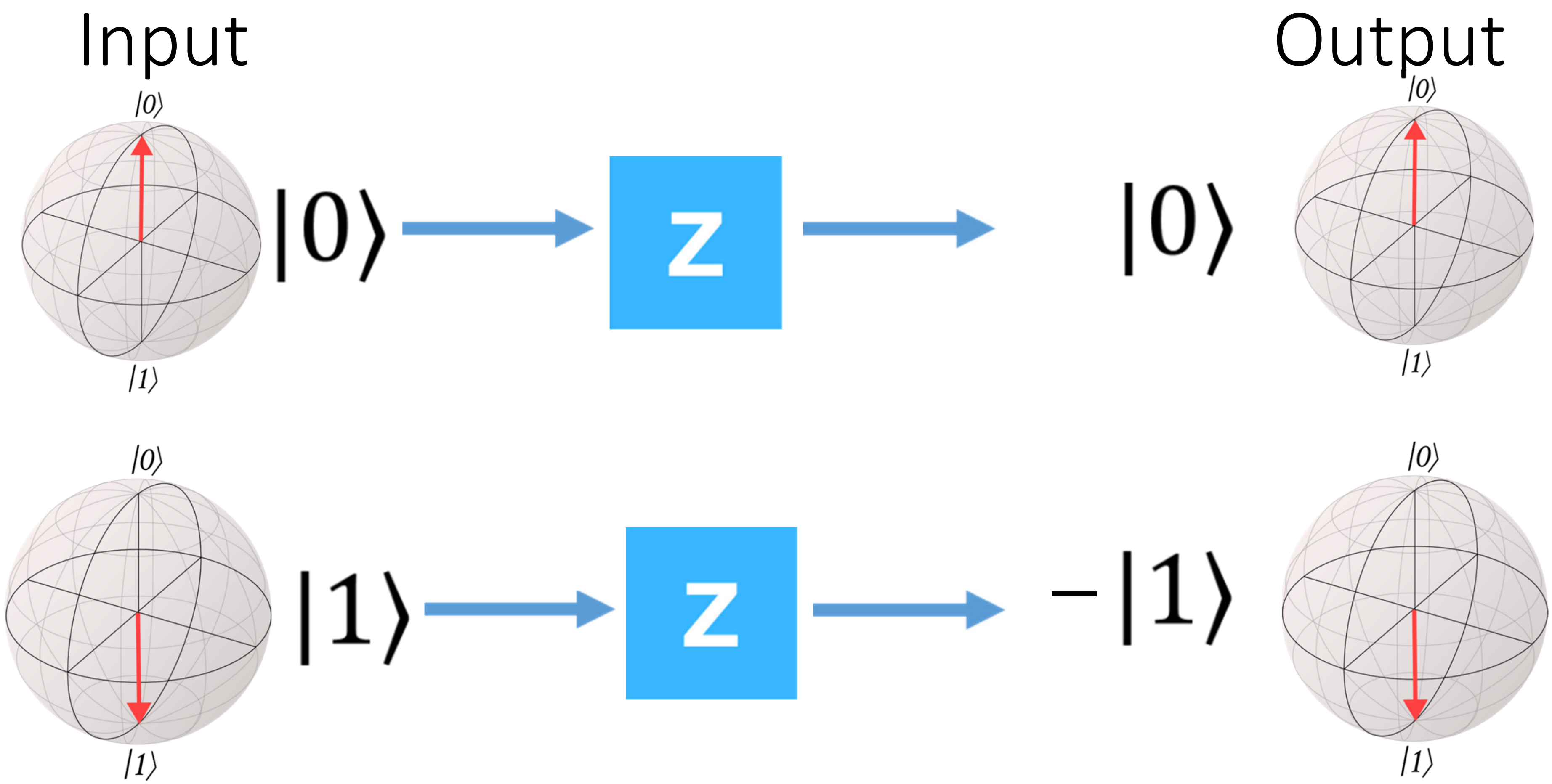
- Let |v⟩ = **a** |0⟩ + **b** |1⟩

- A Z-gate acting on state |v⟩ gives

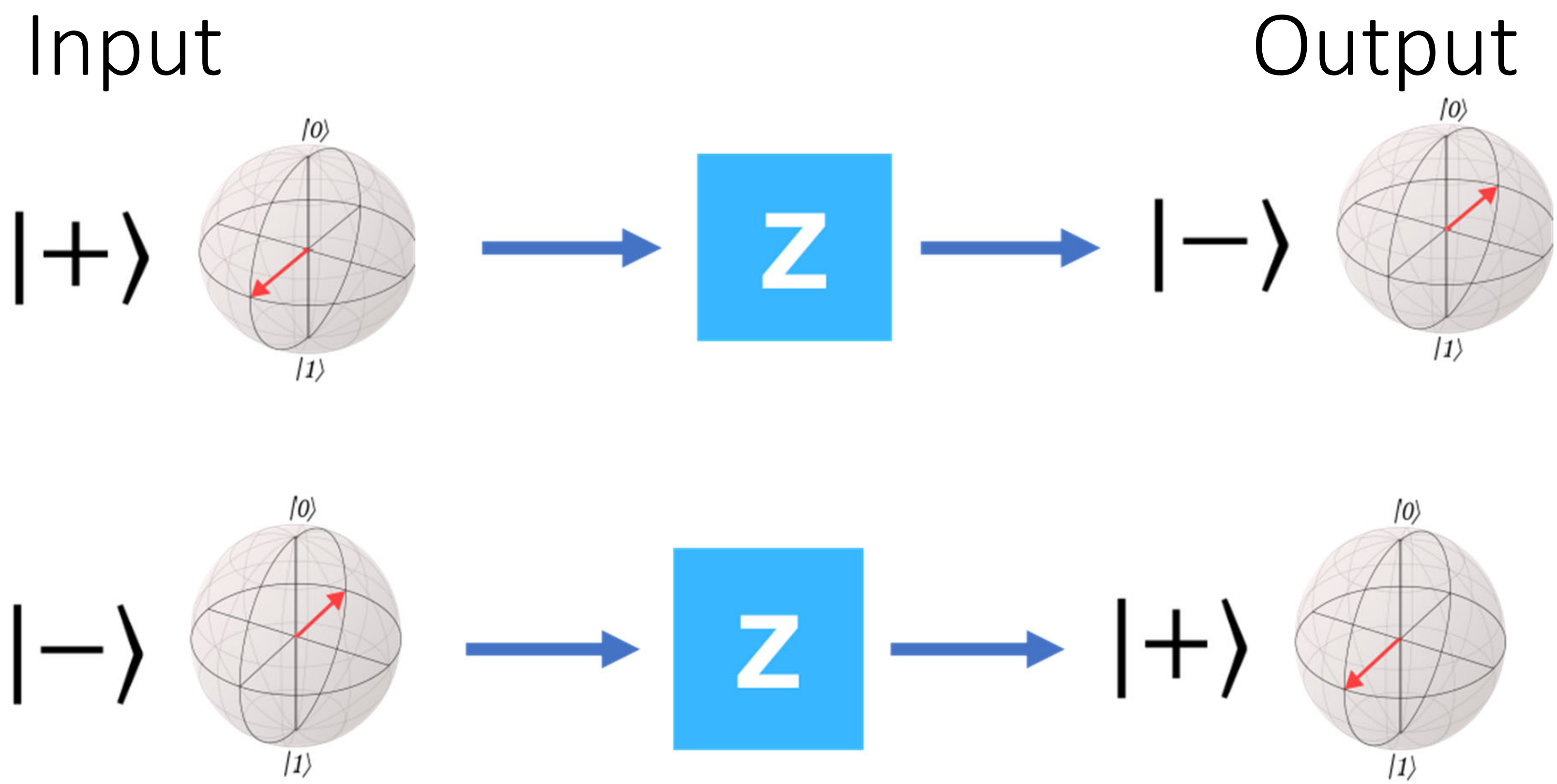
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$$

This corresponds to the **a** |0⟩ - **b** |1⟩ state

Where will the output Bloch sphere point to for each of the input states?



Where will the output Bloch sphere point to for each of the input states?



How Pauli-Z acts on Qubits States

Let's apply Z on $|+\rangle$

$$\mathbf{Z}|+\rangle = \mathbf{Z}(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{0}\rangle + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{1}\rangle) \quad \longrightarrow \quad \mathbf{Z}|+\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\mathbf{Z}|\mathbf{0}\rangle + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\mathbf{Z}|\mathbf{1}\rangle$$

Because Applying Z on $|0\rangle$ does nothing but applying on $|1\rangle$ adds a phase flip (negative sign), we have:

$$\mathbf{Z}|+\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{0}\rangle + -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{1}\rangle \quad \longrightarrow \quad \mathbf{Z}|+\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{0}\rangle - \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{1}\rangle$$

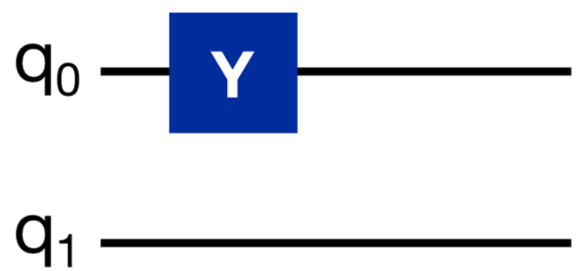
$\longrightarrow$ Since $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle$

$$\longrightarrow \quad \mathbf{Z}|+\rangle = |-\rangle$$

Pauli Y-Gate

- The Y operator is a single Qubit Quantum gate equivalent to a bit- and phase flip.
- This means it applies **both a bit-flip and a phase-flip** to the qubit.
- Pauli-Y gate not only changes the state of the qubit (like the Pauli-X gate), but it also changes the relative phase between the states |0> and |1> (like the Pauli-Z gate).
- It is a rotation through π radians around the y-axis on the bloch sphere
- Its matrix representation is given by:

$$Y = \sigma_Y = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$



Quantum Circuit representation Y-Gate

How Pauli-Y acts on Qubits

- A Y-gate acting on state $|0\rangle$ gives

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

How Pauli-Y acts on Qubits

- A Y-gate acting on state $|1\rangle$ gives

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix}$$

How Pauli-Y acts on Qubits

- Let |v⟩ = **a** |0⟩ **+ b** |1⟩
- A Y-gate acting on state |v⟩ gives

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \bullet \begin{bmatrix} a \\ b \end{bmatrix} = i \begin{bmatrix} -b \\ a \end{bmatrix}$$

Qiskit Demo

Topic 2 – Introduction to Quantum Computing - Qubit, Operators, Gates and Circuits

Quantum Machine Learning Course

Recap of Last Class Concepts

- Quantum Computing vs. Classical Computing
- Classical bit vs Qubit
- Qubit State
- Quantum Gates
- Pauli Gates
  - X Gate – changes the state of a qubit from $|1\rangle$ to $|0\rangle$ and vice versa.
  - Z Gate - Changes the relative phase of the $|1>$ state while leaving the $|0>$ state unchanged
  - Y Gate - changes the state of the qubit (like the Pauli-X gate), but it also changes the relative phase of the $|1>$ state.

Quantum Gates: The Phase Gates (S and T Gates)

- The phase shift operator or rotation gates causes $|0\rangle$ to remain unchanged but rotates $|1\rangle$ by a defined angle or phase θ.
  - $R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, where $e^{i\theta} = cos\,\theta + i * sin\,\theta = -1.$
- Using Euler's identity, If we substitute $\theta = \pi$ in R_θ, we get the Pauli Z gate $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- If we substitute $\theta = \pi/2$ in R_θ, we get $e^{i\theta}$ = i, which results in the S gate.
- The S gate rotates the original state by 90° or $\pi/2$ radians about the Z-axis.
- $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.

Quantum Gates: The Phase Gates

- If we substitute $\theta \ = \ \pi/4$ in R_θ, we get the T gate.
- The T gate rotates the original state by 45° or $\frac{\pi}{4}\ radians$ about the Z-axis.
- T gate is also known as the $\pi/8$ gate because the $e^{\frac{i\pi}{8}}$ can be factored out, leaving the diagonal components with an absolute phase of $|\pi/8|$.
- $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix} = e^{\frac{i\pi}{8}} \begin{pmatrix} e^{-\frac{i\pi}{8}} & 0 \\ 0 & e^{\frac{-\pi}{8}} \end{pmatrix}$.

Cartesian Rotation Gates (Rx, Ry, Rz)

- The three Pauli-rotation gates Rx, Ry, and Rz
  - rotate the state vector by an arbitrary angle
  - about the corresponding axis in the Bloch sphere.
  - They are generated by taking exponentials of the Pauli operators.

$$Rx_\theta = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

RX

$$Ry_\theta = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

RY

$$Rz_\theta = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

RZ

These are commonly used rotation gates in Quantum Machine Learning

Quantum Gates: The Hadamard (H) Gate

- The H gate create superposition of states.

- The unitary matrix corresponding to the Hadamard Gate is as follows:
  - $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

- The Hadamard gate acts on a qubit in the state $|0\rangle$ or $|1\rangle$ and takes it to the equal superposition state as follows:

- $H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

- $H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Quantum Gates: The H Gate

- Hadamard gate is Unitary Matrix ($H^{*}H = I$)
  - $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ H^{T}= $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ H^{*}= $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ $H^{*}H = I$ = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- H gate is also Hermitian ($H = H^{*}$)
  - $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ H^{T}= $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ H^{*}= $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$
- Thu$s$ H gate is self-inverse: $H = H^{-1}$.
- $H\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] = |0\rangle$ and $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- $H\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] = \big|1\rangle$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
- If we apply the Hadamard gate twice in succession, the state of the qubit remains unchanged

The Hadamard Gate

Rule: the Hadamard gate creates superposition

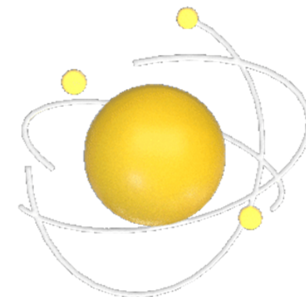
$$|0\rangle \longrightarrow \boxed{H} \longrightarrow |+\rangle$$

Applying the H gate to the $|0\rangle$ state creates the $|+\rangle$ state.

$$|1\rangle \longrightarrow \boxed{H} \longrightarrow |-\rangle$$

Applying the H gate to the $|1\rangle$ state creates the $|-\rangle$ state.

Unlike the X Gate, the H Gate does something *truly quantum*!

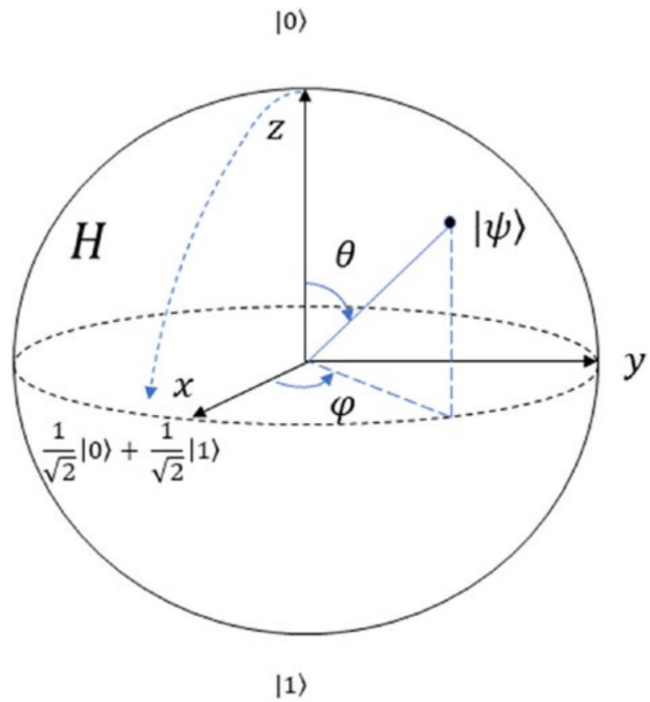


$|0\rangle$ **H** $|+\rangle$

$|1\rangle$ **H** $|-\rangle$

$\left.\begin{matrix}|+\rangle \\ |-\rangle\end{matrix}\right\}$ **Superpositions!**

The Hadamard Gate



Hadamard transform on $|0\rangle$ state

In terms of the Bloch sphere representation, the Hadamard gate takes the state $|0\rangle$ aligned along the z-axis to the state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ aligned along the positive x-axis.

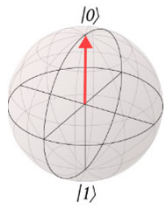
Skill Check

What does state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ corresponds to on the Bloch sphere?

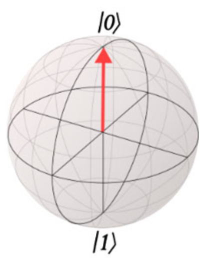
Ans = $|+\rangle$ and $|-\rangle$ respectively

Practice: The Hadamard Gate

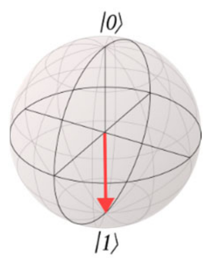
What do we get when we apply the following gate?



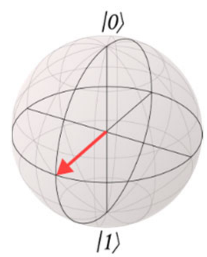
$|0\rangle \longrightarrow$ **H** $\longrightarrow$ **?**



a

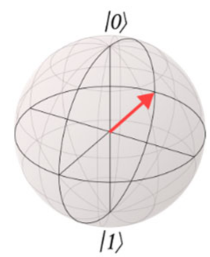


b



c

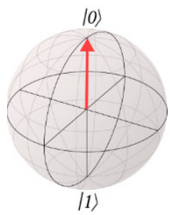
$|+\rangle$



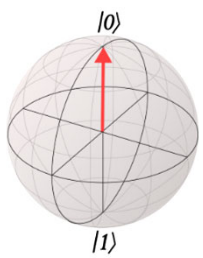
d

Practice: The Hadamard Gate

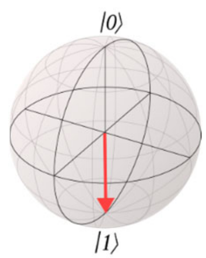
What do we get when we apply the following gates?



$|0\rangle \rightarrow$ **X** $\rightarrow$ **H** $\rightarrow$ **?**



a



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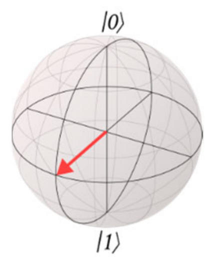
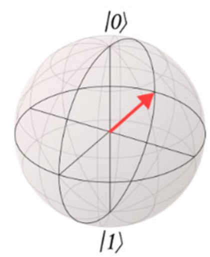


c

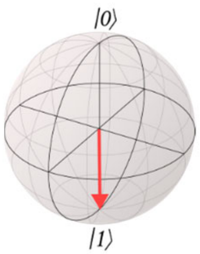


d

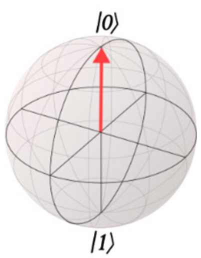
$|-\rangle$

Practice: The Hadamard Gate

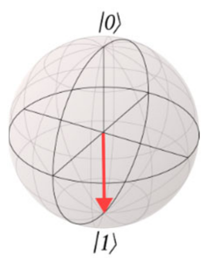
What do we get when we apply the following gates?



$|1\rangle$ → **X** → **H** → **?**



a



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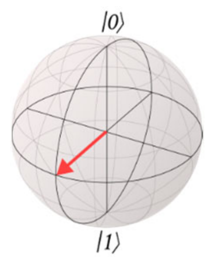
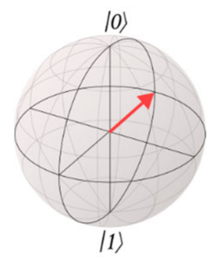


$\boxed{\text{c}}$

$|+\rangle$



d

Multiple Qubits

- 1 Qubit system has how many computational basis states?
- In general, an *n*-qubit system would have how many computational basis states?
- 2^n computational basis states, given by:

$$| x^1, x^2 .... x^n \rangle$$

- Each of the qubit's basis state variables x^1, x^2.... x^n can take up either of the two values 0,1,
- For example, a 1 qubit system will have the computation basis of 2^1 = 2 given by: $|0\rangle$ and $|1\rangle$
- A 2 qubit system will have the computation basis of 2^2 = 4, given by: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$
- One of the things to observe here is that with an increase in the number of qubits, we can get an exponential increase in the number of states. For *n* =500, the number of computational basis states 2^{500} exceeds the number of atoms in the universe.
    - Cleveland Clinic has 127 qubits quantum computer
    - Latest IBM's quantum computer has 1,121-qubit for computation

Skill check:

- A for 3 qubit system how may computational basis will it have and what will they be?

Ans: a 3 qubit system will have the computation basis of 2^3 = 8, given by: $|000\rangle$
, $|001\rangle$, $|010\rangle$, $|011\rangle$, $|100\rangle$, $|101\rangle$, $|110\rangle$, $|111\rangle$

These are all the possible combination of 1 and 0 in three ways

Multi-Qubit Gates

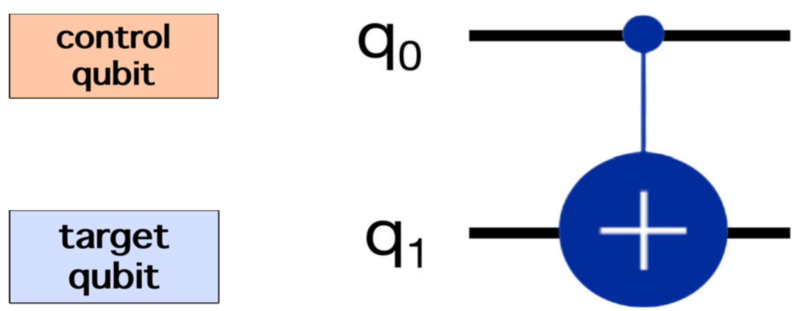
Two-Qubit System

- With two classical bits what are the possible states?
  - We can have four states: 00, 01, 10, and 11.
- With two qubits what are the possible states?
  - A quantum system with 2 qubits A and B can be in the superposition of the 4 states corresponding to the computational basis states 00, 01, 10, and 11
- We can represent the state of a two-qubit system as follows:
  - $|\psi\rangle_{AB} = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$
- In the computational basis state of the form $|ij\rangle$, *i* stands for the basis state of the first qubit, and *j* stands for the basis state for the second qubit.
- Hence, the probability amplitude α_{ij} stands for the joint state $|ij\rangle$
- These probability amplitudes belong to the complex plane, and the square of these amplitude magnitudes should sum to 1

$$|\alpha_{00}|^2+|\alpha_{01}|^2+|\alpha_{10}|^2+|\alpha_{11}|^2=1$$

Quantum Gates: The Controlled NOT (CNOT) Gate

- The $CNOT$ gate, also known as a controlled not gate, is a two-qubit gate.
- The $CNOT$ gate utilizes a control qubit as the first input and the second qubit as the target.
- The control qubit determines the action of the gate on the target qubit.
- If the control qubit is $|0\rangle$, the gate has no effect on the target qubit.
- If the control qubit is $|1\rangle$, the gate applies the X gate to the target qubit.



Quantum Gates: The CNOT Gate

- The possible input states to a $CNOT$ gate are as follows: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.
- The action of the $CNOT$ gate on these states are as follows:
  - $|00\rangle \rightarrow |00\rangle$.
  - $|01\rangle \rightarrow |01\rangle$.
  - $|10\rangle \rightarrow |11\rangle$.
  - $|11\rangle \rightarrow |10\rangle$.
- CNOT is self-adjoint; applying it for a second time reverses its effect.

Input		**Output**	
Control bit	Target bit	Control bit	Target bit
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Quantum Gates: The CNOT Gate

- The matrix representation of the $CNOT$ gate is given by:

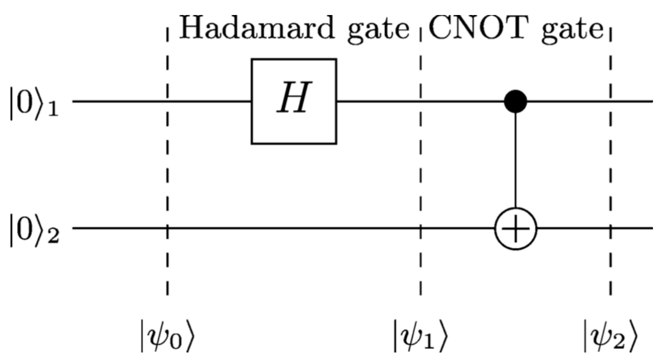
$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

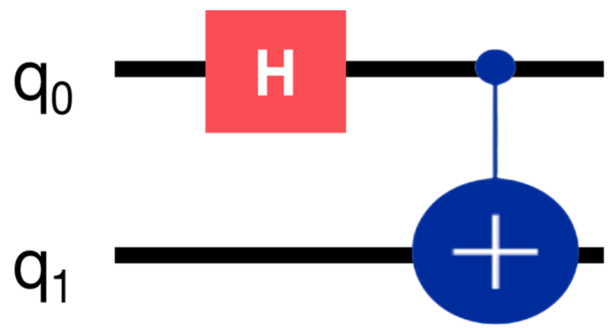
CNOT (CX) Gate

- A controlledgate in quantum computing involves using a control, or C, qubit as the first input qubit and the second qubit as the target qubit.
- The first input to the CNOT gate acts as the control qubit: If the control qubit is in a state $|0\rangle$, then the gate does not do anything to the target qubit
- But if the state of the control qubit is $|1\rangle$, then the gate applies the NOT or X operator to the target Qubit
- The $CNOT$ gate induces entanglement between the two qubits.
  - Entanglement is a phenomenon unique to quantum computing.
  - Entangled qubits cannot be described independently.

How CNOT acts on qubits

- Applying a Hadamard gate on the first qubit and a CNOT gate with the first qubit as control and the second qubit as the target creates an entanglement





Using two qubits to generate an entanglement state, also called Bell state

$|\psi\rangle = |0\rangle \otimes |0\rangle = |00\rangle$ $\quad$ $|\psi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle$

$|\psi\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$ $\quad$ $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Skill check

Complete the CNOT mapping

1. $|00\rangle \rightarrow |00\rangle$

2. $|10\rangle \rightarrow |11\rangle$

3. $|01\rangle$ Maps to?
Ans: $|01\rangle$

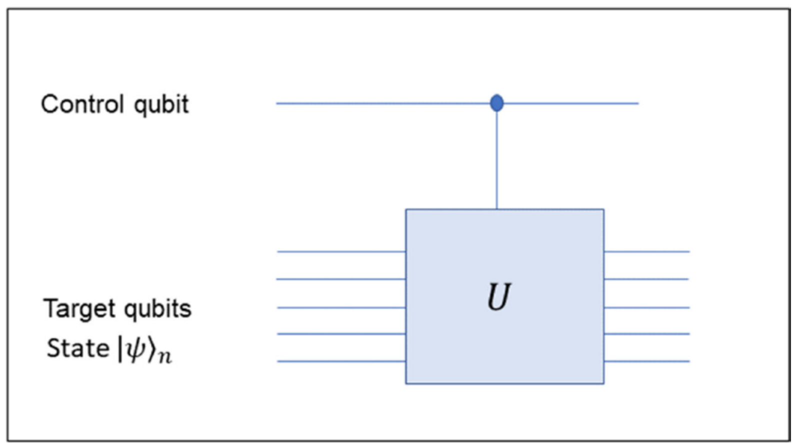
4. $|11\rangle$ Maps to?
Ans: $|10\rangle$

The SWAP Gate

- The $SWAP$ gate is a two-qubit gate.
- A SWAP operator reverses the states of bits in an input qubit state.
- For example, it takes the state $|10\rangle$ and reverse it to $|01\rangle$.
- SWAP operators are represented by the following matrix:
- $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Controlled U-Gate

- Controlled-U gate applies the unitary operator U on the system of n target qubits based on the state of a control qubit.
  - When the control qubit is in $|0\rangle$ state, no transformation is applied on the system of n target qubit
  - When the control qubit is in state $|1\rangle$, the unitary operator is applied on the system of n target qubits
- The CNOT gate is a special case of controlled-U gate, where the unitary operator is the single-qubit X gate.



Quantum circuit representation of a Controlled U-gate

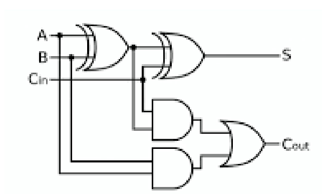
A list of Common Quantum Gates

Operator	Gate(s)		Matrix
Pauli-X (X)	X		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)	Y		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)	Z		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)	H		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)	S		$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$ (T)	T		$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Controlled Not (CNOT, CX)			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Controlled Z (CZ)	Z		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
SWAP			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Toffoli (CCNOT, CCX, TOFF)			$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

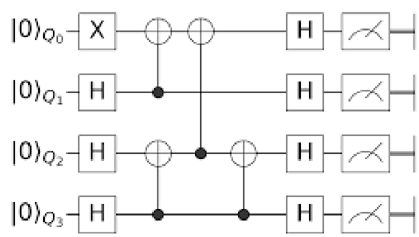
In Class Activity – Math Exercises

Quantum Circuits

- A sequence of quantum gates applied to qubits.
- Analogous to classical circuits in classical computing.



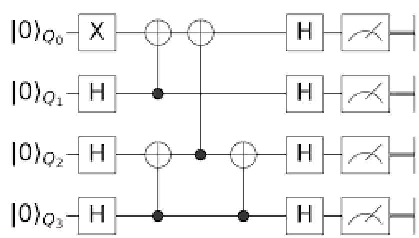
Classical Digital Circuit Example



Quantum Circuit Example with Hadamard and CNOT Gates

Quantum circuits

- A quantum circuit is a model for quantum computation in which a computation is **a sequence of quantum gates with n-qubit register** linked by “wires”
- Quantum Circuits have three main components:
  - **Qubits**: The fundamental unit of quantum computation.
  - **Gates**: The functions we apply to change and manipulate the qubits
  - **Measurement**: a crucial last step in the circuit. It gives us our circuit’s output!

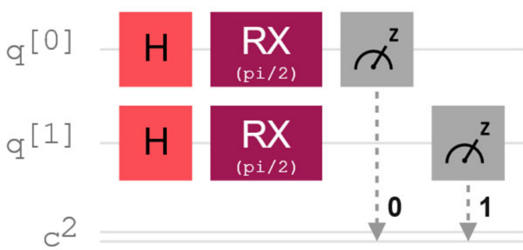


Importance of Quantum Circuits

- Quantum circuits are the foundation of quantum computing.
- Enable quantum operations and algorithms.
- Key to harnessing the power of quantum information processing.

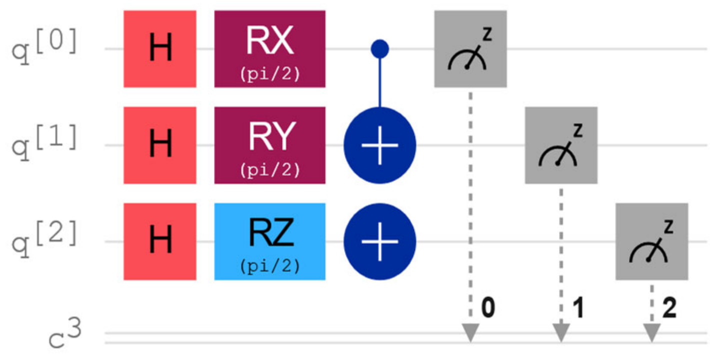
Quantum Circuits Example

- The circuit consists of two qubits, two H gates, two RX_θ gates, and two measurements.
- First, the H gates create superposition on the qubits.
- Then, the RX_θ gates rotates the qubits around the X-axis.
- Finally, the qubits are measured and collapsed to classical output and stored in the classical registers C_i.



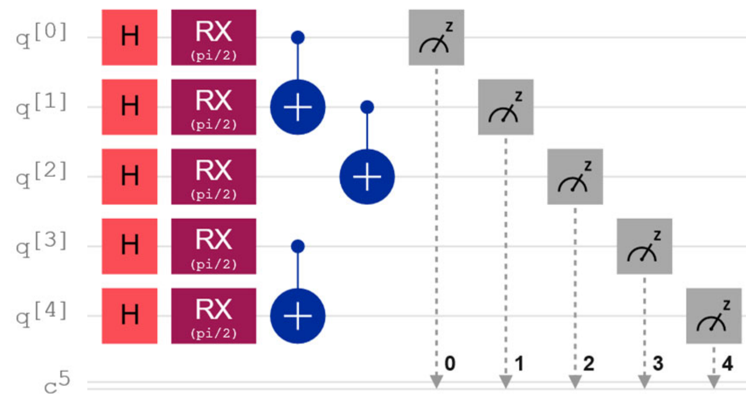
Quantum Circuit Example

- The circuit consists of three qubits, three H gates, one RX_θ gate, one RY_θ gate, one RZ_θ gate, one $CNOT$ gate, one X gate, and three measurements.
- The H gates create superposition on the qubits.
- The RX_θ, RY_θ, and RZ_θ gates represents rotations on the qubits.
- The qubits are measured and collapsed to classical output and stored in the classical registers C_i.



Quantum Circuit Example

- The circuit consists of five qubits, five H gates, five RX_{θ} gates, three $CNOT$ gate, and five measurements.
- The H gates create superposition on the qubits.
- The RX_{θ} gates represents rotation of the qubits around the X-axis.
- The qubits are measured and collapsed to classical output and stored in the classical registers C_i.

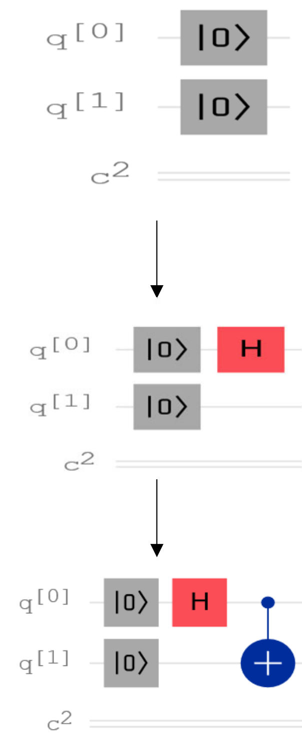


Quantum Circuit Example: The Bell Circuit

- The Bell states are entangled states that exhibit unique correlations between qubits.
- The Bell circuit is a specific quantum circuit used to create Bell States.
- The resulting states exhibit non-classical behavior and cannot be represented as independent states of individual qubits.
- Measuring one qubit of a Bell State provides information about the other qubit, regardless of their physical separation.

Quantum Circuit Example: The Bell Circuit

- Building a Bell circuit:
  - *Start with two qubitsSo, the system has two qubits:* $|00\rangle = |0\rangle\,|0\rangle$.
  - *initialized to the* $|0\rangle$ *state.*
  - *Apply a* H *gate to the first qubit to put it in a superposition of |0⟩ and |1⟩:* $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.
  - *So, you will have the following two qubit system:*
  - $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$.
  - *Apply a* $CNOT$ *gate with the first qubit as the control and the second* qubit as the target.
  - This flips $|10\rangle$ to $|11\rangle$: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.



Quantum Circuits: The Bell Circuit

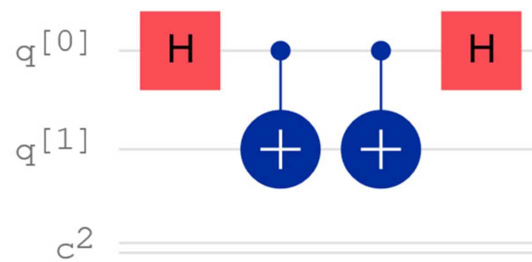
- The generated Bell states:
  - $B|00\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.
  - $B|01\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$.
  - $B|10\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$.
  - $B|11\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$.

Quantum Circuits: The Reverse Bell Circuit

- The reverse Bell circuit reverses the action of the Bell circuit.
- It enables the extraction of useful information from entangled qubits.
- By applying the reverse sequence of gates, the entanglement is undone, and the initial states can be retrieved.

Quantum Circuits: The Bell Circuit

- The reverse Bell circuit works as follows:
- If we send a pair of qubits through the circuit, the first thing that happens is that the H gate is applied, and then we apply the $CNOT$ gate.
- The $CNOT$ action is immediately undone by the second $CNOT$ gate.
- Finally, the second H gate undoes the action done by the first H gate.
- The result is that the circuit doesn't change anything.
- The qubits output are identical to the qubits that entered.



Quantum Circuits: The Bell Circuit

- Applications of Bell states:
  - Bell States are used in various quantum information processing tasks. For Example.
    - Quantum teleportation,
    - Quantum cryptography, and
    - Quantum error correction

In Class Activity Exercise 2

- How can we obtain X gate from and H? Implement X gate on both IBM quantum composer and qiskit using only Z and H gates and show that the obtained final state obtained is equal to the final state when X only is used. Use the bloch sphere visualization to visualize the final state of both operations.

In Class Activity Exercise 3

- a) Implement a quantum circuit in Qiskit that prepares an initial state $|1\rangle$ and applies a Hadamard gate followed by a rotation around the x-axis by an angle θ. Define a variable θ and set it to $\pi/6$.
- b) Visualize the final state on the Bloch sphere.
- c) Measure the qubit in the computational basis and then draw the circuit

In Class Activity Exercise 4

- Implement a circuit using Qiskit that prepares the state $|\psi\rangle = (i|1\rangle - i|0\rangle)/\sqrt{2}$. Verify the state using the state vector simulator.

Summary

- Quantum Computing vs. Classical Computing
- Classical bit vs Qubit
- Qubit State
- Quantum Gates
  - Pauli Gates
  - Phase Gates
  - Cartesian Rotation Gates (Rx, Ry, Rz)
  - Hadamard Gates
  - C-NOT Gates
  - Controlled U-Gate
  - SWAP Gate
- Quantum Circuits
  - Qubits; Gates and Measurement
  - IBM Quantum Composer and Qiskit
  - Bell Circuit

Next Week Topics

- Topic 2 - Introduction to Quantum Computing – Gates; Operators and Circuits
  - Read Text Book (QML- An Applied Approach - Ganguly) – Ch. 1
  - Read Ref. Book 1 (Intro to QC –Bernhardt) – pp. 118-140

Appendix

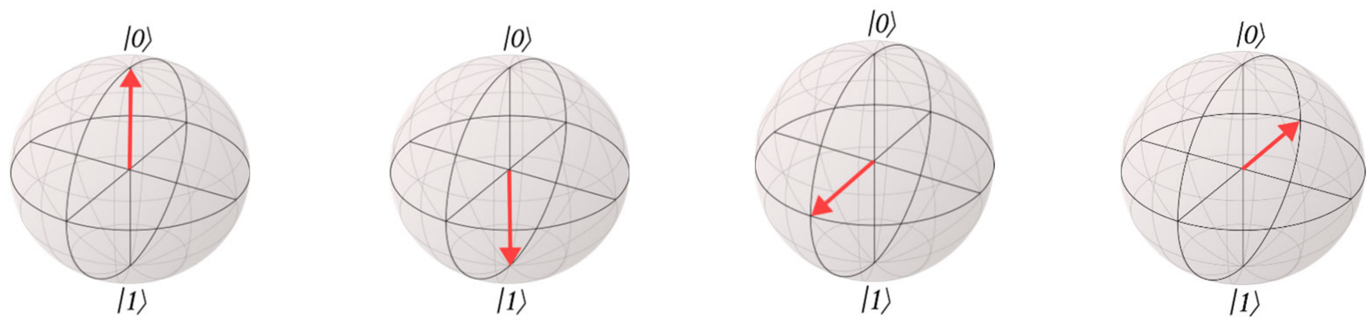
INDUSTRY APPLICATIONS

- MEDCINE AND MATERIALS
- CRYPTOGRAPHY
- OPTIMIZATION
- FINANCE
- SEARCHING BIG DATA

Key Lecture Takeaways

Here is what you need to know from today's lecture:

We use the Bloch Sphere to visualize qubits:



|0>　　|1>　　|+>　　|->

Basis States　　**Superpositions**

Key Lecture Takeaways

Here is what you need to know from today's lecture:

- Quantum gates change the state of qubits
- Each quantum gate can be visualized as a rotation on the Bloch Sphere
- Today, we learned the X and Hadamard gates:

X

180° rotation around X axis

|0> becomes |1>

|1> becomes |0>

H

Creates superposition

|0> becomes |+>

|1> becomes |->

Key Lecture Takeaways

Here is what you need to know from today's lecture:

- We also Learned the Y gate and the Z gate

Z

180° rotation around Z axis

|+> becomes |->

|-> becomes |+>

Y

180° rotation around Y axis

|0> becomes *i*|1>

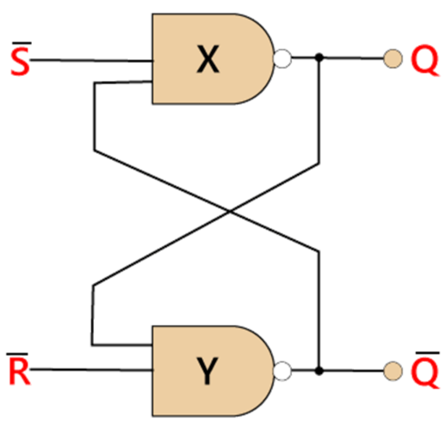
|1> becomes -*i*|0>

Challenges:

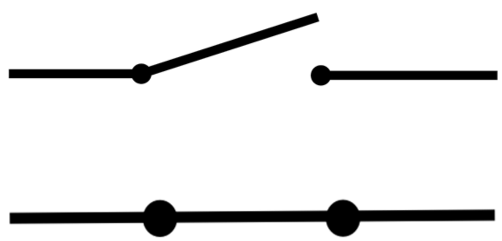
What is the smallest number of gates you need to apply to get the |-> state?

Implementation of a bit

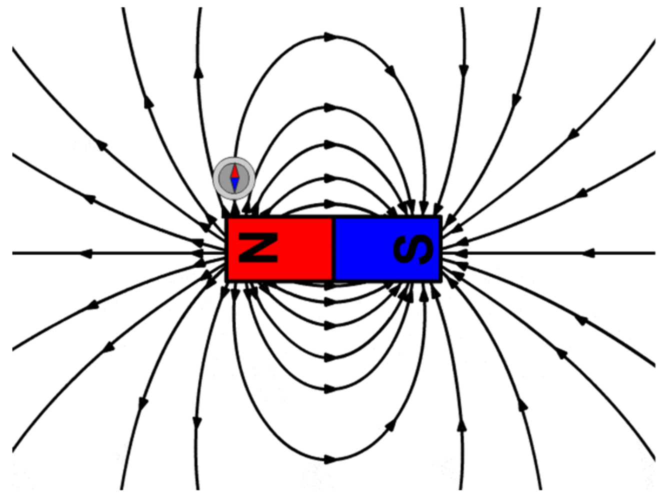
Two state of flip flop



Two positions of an electric switch



Two direction of magnetization



Quantum Gates

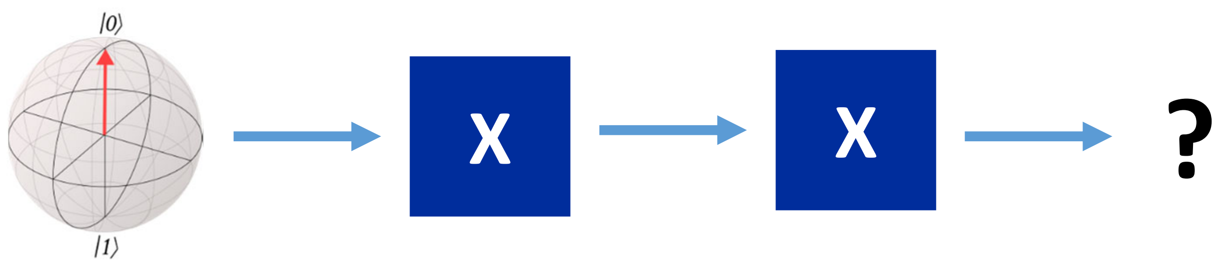
- A unitary operator produces identity matrix when multiplied by its own adjoint: $UU^\dagger = I$.
- Has inverse equal to its adjoint: $U^{-1} = U^\dagger$, which means that $UU^{-1} = I$.
- $UU^\dagger = U^\dagger U = I$.

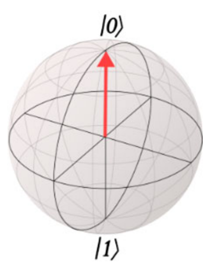
Quantum Gates

- Reversibility and Quantum Operations:
- Quantum operations are reversible, except for measurements.
- Reversibility is due to closed systems.
- Irreversible classical computation can be simulated by reversible classical computation.
- The same holds for quantum computation.
- These matrices have orthonormal rows.

Practice: The X Gate

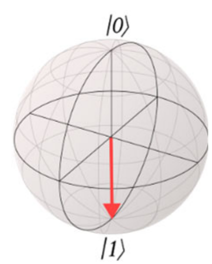
What do we get when we apply the following gate?



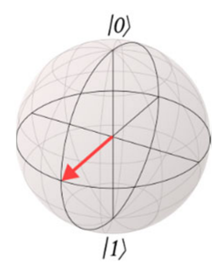


A

$|0\rangle$



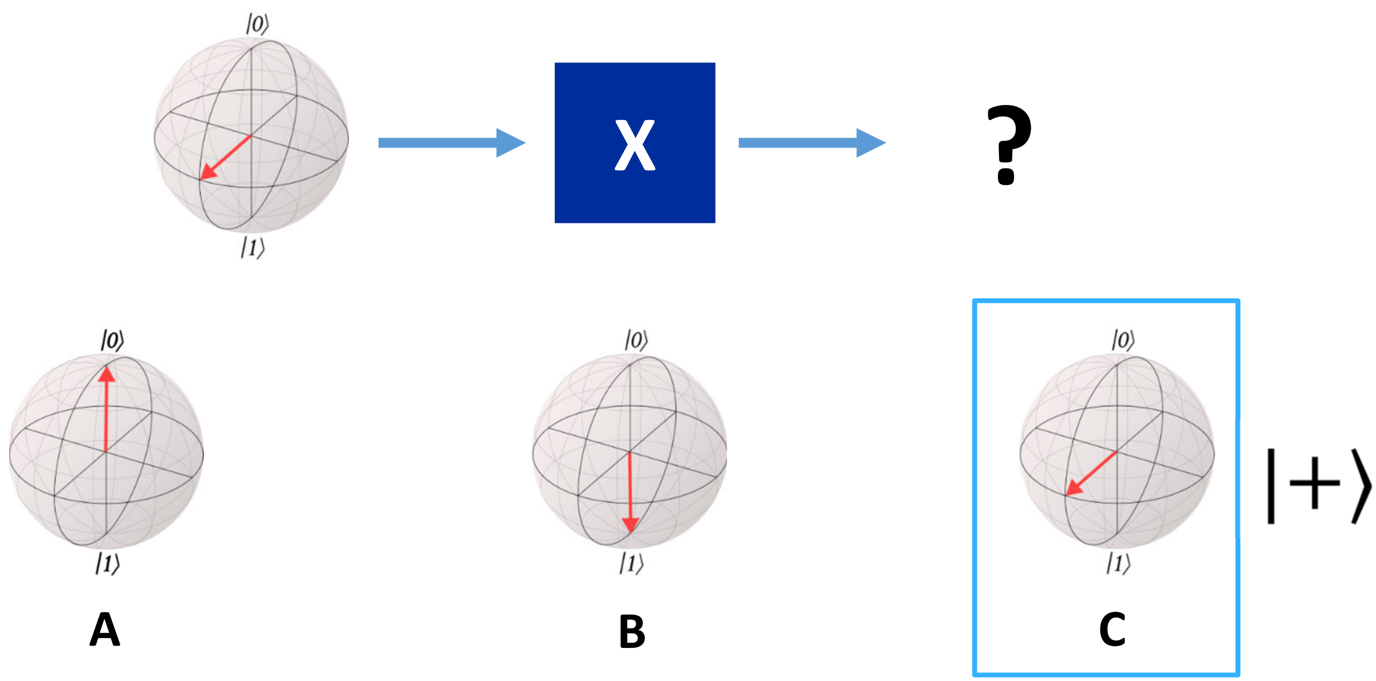
B



C

Practice: The X Gate

What do we get when we apply the following gate?



How Pauli-Y acts on Qubits

- Applying Y Gate to $|0\rangle$ state changes the qubit state to the state

Ans = $i|1\rangle$

- Applying Y Gate to $|1\rangle$ state changes the qubit state to the state and also does a phase flip.

Ans = $-i|0\rangle$

Note: Since qubits can exist in superpositions, the Pauli-Y gate can also transform states that are combinations of $|0\rangle$ and $|1\rangle$

Skill check:

- Show how the Pauli-Z operator changes the qubit superpostion state $|-\rangle$
- How does the Pauli-Y operator change the superposition state $|-\rangle$ and $|+\rangle$?

Hadamard Gate

- The unitary matrix corresponding to the Hadamard Gate is as follows:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- The Hadamard gate acts on a qubit in the state $|0\rangle$ and takes it to the equal superposition state of $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- It also transforms the state $|1\rangle$ to the state $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$
- In terms of the Bloch sphere representation, the Hadamard gate takes the state $|0\rangle$ aligned along the z-axis to the state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ aligned along the positive x-axis.
- Hadamard gate creates superposition of states

How Hadamard Gate acts on Qubits

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

- One thing to take note of is that if we apply the Hadamard gate twice in succession, the state of the qubit remains unchanged. This is because the square of the Hadamard matrix H2 is equal to the identity *I*

Phase Gates (S and T)

- The phase shift operator or rotation gate causes the state $|0\rangle$ to remain unchanged but rotates the $|1\rangle$ state by a defined angle or phase θ.

$$R_\theta \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \quad \text{---- (1)}$$

- Using Euler's identity, if we set $\theta = \pi$ in equation 1, we recover the Pauli Z gate because $e^{i\theta} = \cos(\pi) + i.\sin(\pi) = -1$
- If we substitute $\theta = \pi/2$ in equation 1, we get $e^{\wedge}i\theta = i$, which in turn gives another operator called S.

$$S \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Phase Gates (S and T)

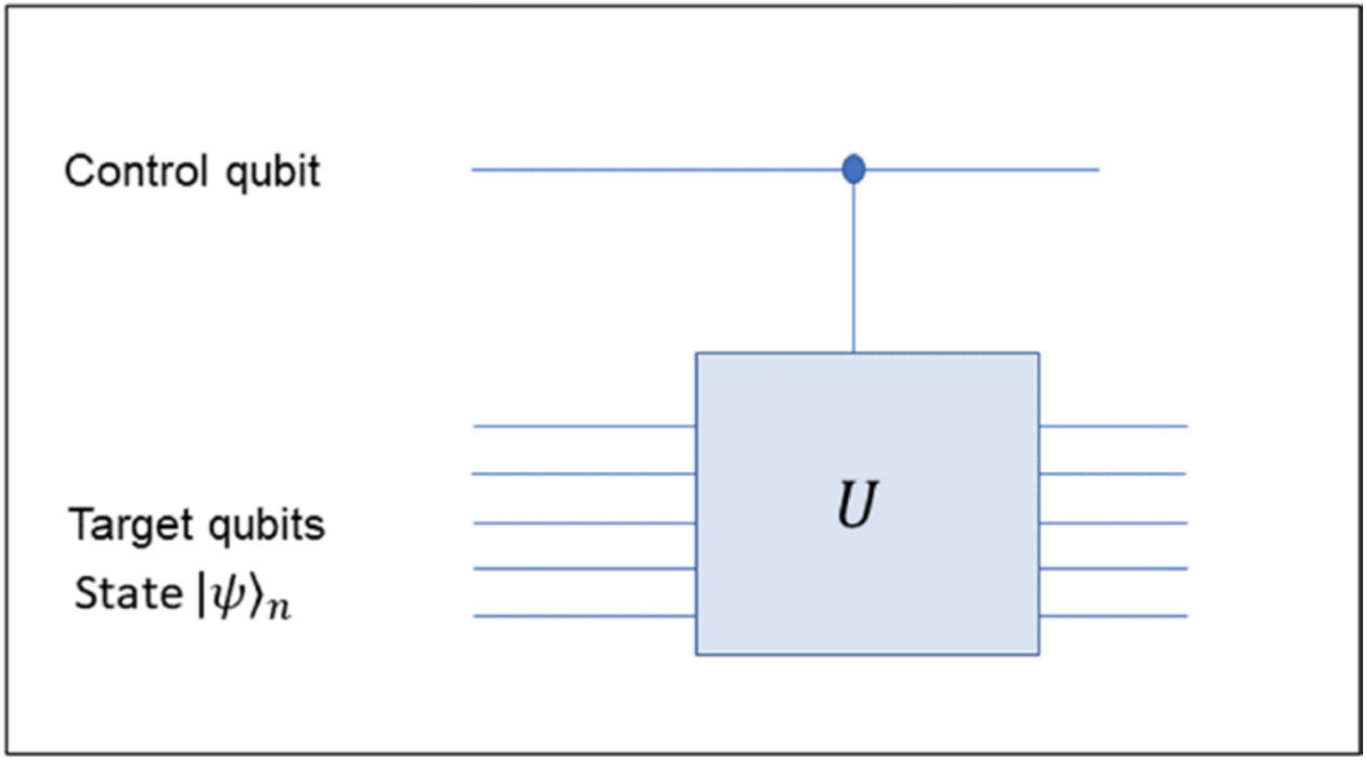
- S operator rotates the original state by 90° or $\frac{\pi}{2}$ radians about the Z axis.
- The T operator rotates the original state by 45° or $\frac{\pi}{4}$ radians about the Z axis.

$$T \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

- The T gate is also known as the $\frac{\pi}{8}$ because the $e^{i\frac{\pi}{8}}$ can be factored out, leaving the diagonal components with an absolute phase of $|\pi/8|$

$$T \equiv e^{i\frac{\pi}{8}} \begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$$

Controlled U-Gate (Cont.)



Quantum circuit representation of a Controlled U-gate