Temiloluwa Omomuwasan
too2007
N13012494

Linear Algebra - Problem Set 1 - Spring 2023

To upload on Gradescope by ***Friday, February 3rd, 11.59 pm****.*

The below homework specifications will be enforced. If the specifications are not respected, points might be deducted, or the homework assignment may not be accepted for grading.

Guidelines for your work

- Write your name (as on the roster) and NetID on the first page.
- If you write on paper, use clean and new sheets of paper and take as much space as necessary.
- Number your pages in the top-right corner, such as 1/3, 2/3, 3/3.
- Use a draft and hand in your final version. Make sure that it
  - is clean and legible;
  - has each problem clearly indicated;
  - does not have anything crossed out or contain notes in the margins;
  - has solutions in which all steps are clearly shown and explained, including all steps of the computations;
  - has grammatically correct complete sentences, including punctuation and spelling;
  - is written using correct mathematical terminology and notation;
  - has final answers in exact forms (do not approximate unless otherwise stated).
- You may consult your classmates or other resources (including office hours) for ideas on the problems; however, the solutions you turn in must be in your own words and must reflect your own understanding. Your solutions and write-ups will be checked for textual similarities. You may not copy from, reword, or paraphrase another student's work or any other resource material; such conduct will be treated as a violation of academic integrity. Remember that you will not learn anything by simply copying, rewording or paraphrasing another person's work.

Guidelines for Gradescope

- You can either write on blank or lined paper, use a tablet, or type your assignment in LaTeX.
- Your work should be uploaded as a single PDF file (not as separate photos).
- If you write down on paper, scan your work using a scanner or an app such as Camscanner. Make sure that the scans are not blurry and are in portrait mode.
- When you upload this file, match each exercise with the corresponding pages. **You may lose 5 points for each problem that is not properly matched**.

Late work policy

Late homework is accepted according to the policy below. Please note that this is uniform, regardless of individual circumstances, and no additional extensions will be given.

- For your first late assignment within 12 hours after the deadline (as indicated on Gradescope), no point deductions.
- All subsequent assignments submitted within 12 hours after the deadline will convert to a zero at the end of semester.
- In all cases, work submitted 12 hours or more after the deadline will not be accepted.

Exercise I ($5 \times 5 = 25$ points)

True or False? In both cases, explain clearly.

1. The system of equations below does not have any solutions because the two lines are parallel.

$$3x_1 + 9x_2 = 15$$
$$x_2 = -3x_1 + 5$$

2. Let a be any real number. If $\vec{u} = (a, a, a)$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{v}$ is parallel to $\vec{w}$.
3. $\vec{s}$ is a linear combination of $\vec{v}$ and $\vec{w}$. of If $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is perpendicular to $\vec{s}$.
4. If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, then $||\vec{u} + 3\vec{v}|| = \sqrt{10}$.
5. The length of a unit vector in n dimensions is n.

Exercise II ($3 \times 10 = 30$ points)

Consider the following set of vectors in $\mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

1. Which pairs are orthogonal among the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$? Show all your work.
2. Find the angle between the pairs of vectors that are not orthogonal.
3. Construct a set of 3 unit vectors out of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Exercise III ($3 \times 5 = 15$ points)

Write each of the following vector equations as a matrix equation. That is, write it in the form $A\vec{x} = \vec{b}$. Specify what the matrix A, the vector $\vec{x}$, and the vector $\vec{b}$ are.

1. $x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$
2. $x_1 \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 11 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$
3. $x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

Exercise IV (20 points)

Consider the matrix below. Are the column vectors of this matrix linearly independent or linearly dependent? Show all explanations or computations.

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}.$$

Exercise V (10 points)

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$ and $\vec{v}_4$ is vector in $\mathbb{R}^n$. Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is also a linearly dependent set.

Exercise I (5 × 5 = 25 points)

True or False? In both cases, explain clearly.

1. The system of equations below does not have any solutions because the two lines are parallel.

$$3x_1 + 9x_2 = 15$$
$$x_2 = -3x_1 + 5$$

2. Let a be any real number. If $\vec{u} = (a, a, a)$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{v}$ is parallel to $\vec{w}$.
3. $\vec{s}$ is a linear combination of $\vec{v}$ and $\vec{w}$. of If $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is perpendicular to $\vec{s}$.
4. If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, then $||\vec{u} + 3\vec{v}|| = \sqrt{10}$.
5. The length of a unit vector in n dimensions is n.

① $3x_1 + 9x_2 = 15$

$3x_1 + x_2 = 5$

$8x_2 = 10$

$x_2 = \frac{10}{8}$

$3x_1 + \frac{10}{8} = 5$

$3x_1 = 5 - \frac{10}{8} = \frac{40 - 10}{8} = \frac{30}{8}$

$x_1 = \frac{30}{8} \times \frac{1}{3} = \frac{10}{8}$

Ans: False, because these equations have solutions

② True

③ True, because $\vec{s}$ is a linear combination of $\vec{v}$ and $\vec{w}$

④ False, because a unit vector means the length is 1.

⑤ False, because the length of a unit vector is 1

Exercise II (3 × 10 = 30 points)

Consider the following set of vectors in $\mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

1. Which pairs are orthogonal among the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$? Show all your work.
2. Find the angle between the pairs of vectors that are not orthogonal.
3. Construct a set of 3 unit vectors out of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Answer:

1) $\vec{V}_1 \cdot \vec{V}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} = 4 + 0 - 8 + 0 = -4 \neq 0$

$\vec{V}_1 \cdot \vec{V}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = 1 - 2 + 2 - 1 = 0$ ☑

$\vec{V}_1 \cdot \vec{V}_4 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 + 2 - 2 + 1 = 2$

$\vec{V}_2 \cdot \vec{V}_3 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = 4 + 0 - 4 + 0 = 0$ ☑

$\vec{V}_2 \cdot \vec{V}_4 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4 + 0 + 4 + 0 = 8$

$\vec{V}_3 \cdot \vec{V}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 1 - 1 - 1 = -2$

$\therefore$ $\vec{V}_1, \vec{V}_3$ and $\vec{V}_2, \vec{V}_3$ are the pairs that are orthogonal

2) $\|V_1\| = \sqrt{1^2 + 2^2 + (-2)^2 + 1^2} = \sqrt{1 + 4 + 4 + 1} = \sqrt{10}$

$\|V_2\| = \sqrt{4^2 + 0^2 + 4^2 + 0} = \sqrt{16 + 16} = \sqrt{32}$

$\|V_3\| = \sqrt{1^2 + (-1)^2 + (-1) + (-1)^2} = \sqrt{4} = 2$

$\|V_4\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$

(a) $\vec{v_1} \cdot \vec{v_2} = ||v_1|| \, ||v_2|| \cos\theta$

$-4 = \sqrt{10}\sqrt{32} \cos\theta$

$-4 = \sqrt{320} \cos\theta$

$\frac{-4}{4\sqrt{20}} = \cos\theta$

$\cos^{-1}\left(\frac{-1}{2\sqrt{5}}\right) = \theta = 102.9204 = 102.9°$

(b) $\vec{v_1} \cdot \vec{v_4} = ||v_1|| \, ||v_4|| \cos\theta$

$2 = 2\sqrt{10} \; 1 \cos\theta$

$\frac{2}{2\sqrt{10}} = \cos\theta$

$\cos^{-1}\left(\frac{1}{\sqrt{10}}\right) = \theta = 71.5650 = 71.6°$

(c) $\vec{v_2} \cdot \vec{v_4} = ||v_2|| \, ||v_4|| \cos\theta$

$8 = \sqrt{32} \times 2 \cos\theta$

$\frac{8}{8\sqrt{2}} = \cos\theta$

$\cos^{-1} \frac{1}{\sqrt{2}} = \theta = 45°$

(d) $\vec{v}_3 \cdot \vec{v}_4 = \|v_3\| \|v_4\| \cos\theta$

$-2 = 4\cos\theta$

$\cos^{-1}(-1/2) = \theta = 120°$

(3) (a) $\vec{v}_1 = \frac{1}{\|v_1\|}\vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix}1\\2\\-2\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{10}\\2/\sqrt{10}\\-2/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$

(b) $\vec{v}_2 = \frac{1}{\|v_2\|}\vec{v}_2 = \frac{1}{4\sqrt{2}}\begin{bmatrix}4\\0\\4\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}$

(c) $\vec{v}_3 = \frac{1}{\|v_3\|}\vec{v}_3 = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\-1/2\end{bmatrix}$

Exercise III (3 × 5 = 15 points)

Write each of the following vector equations as a matrix equation. That is the matrix A, the vector $\vec{x}$, and the vector $\vec{b}$ are.

1. $x_1\begin{bmatrix}2\\1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}1\\2\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\1\\2\\1\end{bmatrix} + x_4\begin{bmatrix}0\\0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\5\end{bmatrix}$
2. $x_1\begin{bmatrix}2\\4\\0\end{bmatrix} + x_2\begin{bmatrix}5\\11\\1\end{bmatrix} + x_3\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\2\\3\end{bmatrix}$
3. $x_1\begin{bmatrix}3\\-2\\8\end{bmatrix} + x_2\begin{bmatrix}5\\0\\9\end{bmatrix} = \begin{bmatrix}2\\-3\\8\end{bmatrix}$

Exercise II (3 × 10 = 30 points)

Consider the following set of vectors in $\mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix}1\\2\\-2\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}4\\0\\4\\0\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

1. Which pairs are orthogonal among the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$? Show all your work.
2. Find the angle between the pairs of vectors that are not orthogonal.
3. Construct a set of 3 unit vectors out of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

(1) $A = \begin{bmatrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2\end{bmatrix}$ $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ $\vec{b} = \begin{bmatrix}0\\0\\0\\5\end{bmatrix}$

Ans: $\begin{bmatrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\5\end{bmatrix}$

(2) $A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 11 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

Ans: $\begin{bmatrix} 2 & 5 & 1 \\ 4 & 11 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

(3) $A = \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 8 & 9 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

Ans: $\begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

Exercise IV (20 points)

Consider the matrix below. Are the column vectors of this matrix linearly independent or linearly dependent? Show all explanations or computations.

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}.$$

(1) Column 1 is linearly independent because it is a non-zero vector.

(2) Column 2 is linearly independent because it is not a scalar multiple of Column 1

(3) Column 3 is linearly dependent because it is a linear combination of Columns 1 and 2.

5 x Column 1 + 4 x column 2 = Column 3

like this $\Rightarrow 5\begin{bmatrix}1\\-2\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}5\\-6\\8\end{bmatrix}$

Exercise V (10 points)

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$ and $\vec{v}_4$ is vector in $\mathbb{R}^n$. Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is also a linearly dependent set.

Since $\vec{V_1}$, $\vec{V_2}$ and $\vec{V_3}$ are linearly dependent, it means $\vec{V_2} = C_1\vec{V_1}$, $V_3 = C_2\vec{V_1}$ where C_1 & C_2 is some constant. So therefore $\vec{V_4}$ has to equal $C_3\vec{V_1}$ because all the other vectors are linearly dependent.